BROWN REPRESENTABILITY FOR THE DUAL

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Abstract. Let $\mathcal{T}$ be the homotopy category of all spectra. Brown proved that a homological functor $H : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$ is representable if it takes coproducts to products. That is, the functors $[-, h]$ may be characterised as the homological functors taking coproducts to products.

In this article, we will prove the dual. A covariant functor $H : \mathcal{T} \longrightarrow \mathcal{A}b$ which takes products to products is representable; it is of the form $[h, -]$.

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0. Introduction

Let $\mathcal{T}$ be the homotopy category of spectra. One of the great foundational theorems of homotopy theory is the theorem of Brown [3]. It asserts that representable functors $[-, h]$ can be characterised as the cohomological functors taking coproducts to products. It is a theorem with a tremendous range of applications. One can use it to construct any number of useful spectra.

Brown's proof decidedly does not work in the dual of the category of spectra. The proof is based on the existence of "enough" finite spectra, and by a theorem of Boardman [2] it is known that the dual of the category of spectra has no objects which are the analogues of finite spectra. None at all; certainly not enough. We know plenty about the dual of the category of spectra, most of it negative. The dual of the category of spectra has some very nasty properties.

It is therefore quite surprising to learn, as we will here, that the dual of the category of spectra nevertheless *does satisfy Brown representability.* The representable functors $[h, -]$ can be characterised as the homological functors that take products to products.

Needless to say, there has to be a new ingredient in the proof. The proof is really an application of work done independently by Christensen and Strickland [5] and the author [6]. All three of us looked at the cohomology theories one gets by restricting representable functors $[-, h]$ from the category $\mathcal{T}$ of all spectra to the full subcategory $\mathcal{T}^c$ of finite spectra. And we all independently concluded that the cohomological functors on $\mathcal{T}^c$ can be characterised as the objects in the abelian category of all functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$ which are of finite homological dimension. There is a discussion of the work, as well as older backgound, in Section 1.

What led Christensen and Strickland to their results was quite different from what motivated the present author. Christensen and Strickland were interested in properties of phantom maps, while the author studied generalisations of the theorem of Brown and Adams [1]. Neither the author nor Christensen nor Strickland was motivated by the problem of characterising $[h, -]$. It seems remarkable that the result, aside from its two original applications, now has a totally unrelated third.

It should be noted that the theorem generalises somewhat. The category $\mathcal{T}$ may be replaced by any triangulated category with enough compact objects, satisfying the theorem of Brown and Adams. I chose not to state the theorem as generally as possible, mostly for simplicity.

Finally a notational point. A functor $H : \mathcal{T} \longrightarrow \mathcal{A}b$ is homological if it takes triangles to long exact sequences. A homological functor $H : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$ is called a cohomological functor on $\mathcal{T}$. Because we will be dealing with homological functors both on $\mathcal{T}$ and its dual $\mathcal{T}^{op}$, this could get very confusing. If I explicitly give the variance, for instance saying the functor is $H : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$, I will use only the adjective homological. When I say the functor H is cohomological, this will mean as a functor on $\mathcal{T}$ or on a subcategory. Thus H is a contravariant functor on a subcategory of $\mathcal{T}$, taking triangles to long exact sequences.

1. Review

The main result of this article is an application of work that has not yet appeared; we owe the reader a brief review of that work.

Let $\mathcal{T}$ be the homotopy category of spectra, $\mathcal{T}^c$ the full subcategory of finite spectra. The category $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$ is defined to be the abelian category of all functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. Inside the large, abelian category $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$ lies a full subcategory $\mathcal{H}ol\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$. The objects of $\mathcal{H}ol\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$ are the homological functors; they take triangles in $\mathcal{T}^c$ to long exact sequences.

There are some obvious objects in the category

$$\mathcal{H}ol\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big) \qquad \subset \qquad \mathcal{C}at\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big).$$

Let x be any object of $\mathcal{T}$. Then the representable functor $\mathcal{T}(-,x) = [-,x]$ is a homological functor $\mathcal{T}^{op} \longrightarrow \mathcal{A}b$. The restriction of this functor to $\mathcal{T}^c \subset \mathcal{T}$, which by rights ought to be denoted something like

$$\mathcal{T}(-,x)|_{\mathcal{T}^c}$$

is a homological functor $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$. Unless serious confusion is likely to arise, we will follow the convention of homotopy theory and denote this functor $[-,x]$. That is, we will confuse the representable functor on $\mathcal{T}$ with its restriction to $\mathcal{T}^c$.

The classical theorem of Brown [3] and Adams [1] asserts that every object of $\mathcal{H}ol\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$ is isomorphic to $[-,x]$ for some $x \in \mathcal{T}$, and that any natural transformation $[-,x] \longrightarrow [-,y]$ of functors in $\mathcal{H}ol\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$ comes from a (non–unique) morphism $x \longrightarrow y$ in $\mathcal{T}$. The non–uniqueness is measured by the *phantom maps*. A map $x \longrightarrow y$ in $\mathcal{T}$ is called phantom if the induced natural transformation $[-,x] \longrightarrow [-,y]$ vanishes.

It is standard that if x is a finite spectrum, that is $x \in \mathcal{T}^c$, then the functor $[-,x]$ is a projective object in the abelian category $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$. It is also possible to use x to define an injective objects in $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$. Let us recall how this is done.

If x is a finite spectrum, then the functor $[x,-]$ is homological and commutes with coproducts. We study the functor

$$\mathcal{A}b\Big([x,-], \mathbb{Q}/\mathbb{Z}\Big).$$

It is the functor which takes a spectrum $y \in \mathcal{T}$ to maps of abelian groups

$$[x,y] \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

The functor $\mathcal{A}b\Big([x,-], \mathbb{Q}/\mathbb{Z}\Big)$ clearly takes coproducts to products. But $[x,-]$ takes triangles to long exact sequences, and the group $\mathbb{Q}/\mathbb{Z}$ is injective. Thus the functor

$$\mathcal{A}b\Big([x,-], \mathbb{Q}/\mathbb{Z}\Big)$$

takes triangles to long exact sequences. It is cohomological.

By the classical theorem of Brown [3] the functor is representable. There is an object in $\mathcal{T}$, which we denote Ix, so that

$$[-,Ix] \qquad = \qquad \mathcal{A}b\Big([x,-],\mathbb{Q}/\mathbb{Z}\Big).$$

The object Ix is classical; it is the Brown–Comenetz dual of the Spanier–Whitehead dual of x. See [4]. It is an exercise to show that if $H(-):\{\mathcal{T}^c\}^{op}\longrightarrow\mathcal{A}b$ is any object of $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$, then maps of the form

$$H(-)\longrightarrow[-,Ix]$$

correspond 1–1 to elements of

$$\mathcal{A}b\Big(H(x),\mathbb{Q}/\mathbb{Z}\Big).$$

This shows immediately that $[-,Ix]$ is an injective object of $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$. Furthermore, in the case where $H(-)=[-,z]$ is a homological functor, we have identified the maps

$$[-,z]\longrightarrow[-,Ix]$$

with

$$\mathcal{A}b\Big([x,z],\mathbb{Q}/\mathbb{Z}\Big) \qquad = \qquad [z,Ix].$$

In other words, there are no phantom maps $z\longrightarrow Ix$.

Now we are ready to state the main recent results of Christensen and Strickland, and independently the author. The discussion in Christensen and Strickland's [5] is nicer and the results sharper, so I will quote them. First we have, by Proposition 1.4 on page 2 in Christensen and Strickland:

Proposition 1.4 of [5]. *Let H be an object of the abelian category $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$. The following are equivalent:*

- **(i):** *H has finite projective dimension.*
- **(ii):** *H has projective dimension at most one.*
- **(iii):** *H is homological; that is, $H\in\mathcal{H}ol\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$.*
- **(iv):** *H has injective dimension at most one.*
- **(v):** *H has finite injective dimension.*

The other result we want to quote is

Proposition 1.6 of [5].

- **(2):** *Any spectrum X sits in a triangle $P\to Q\to X\to\Sigma P$ with $[-,P]$ and $[-,Q]$ projectives in $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$.*
- **(3):** *Any spectrum X sits in a triangle $X\to J\to K\to\Sigma X$ with $[-,J]$ and $[-,K]$ injectives in $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$.*

Actually, it will be convenient to have a slight refinement. In Lemma 3.1 of [6] it is proved that in (2) above, the objects P and Q in the triangle $P \to Q \to X \to \Sigma P$ may be chosen to be coproducts of $x \in \mathcal{T}^c$. That is, the projectives $[-, P]$ and $[-, Q]$ may be chosen to be the coproducts of $[-, x]$ with x a finite spectrum. The functors $[-, x], x \in \mathcal{T}^c$ are the projectives we discussed above. By a very similar trick, the triangle $X \to J \to K \to \Sigma X$ above may be chosen so that J and K are products of $Ix, x \in \mathcal{T}^c$.

2. Brown representability in the dual

The main theorem of this section is

Theorem 2.1. *Let H be a (covariant) homological functor $H : \mathcal{T} \longrightarrow \mathcal{A}b$. Suppose H respects products. That is, for every small set $\{x_\lambda, \lambda \in \Lambda\}$ of objects of $\mathcal{T}$, the natural map*

$$H\left(\prod_{\lambda \in \Lambda} x_\lambda\right) \longrightarrow \prod_{\lambda \in \Lambda} H(x_\lambda)$$

is an isomorphism. Then H is representable; there is a natural isomorphism

$$H(-) \qquad = \qquad [h, -]$$

for some $h \in \mathcal{T}$.

Proof: Let U be the set of all pairs (x, α) with $x \in \mathcal{T}^c$ and $\alpha \in H(Ix)$. Put

$$X = \prod_{(x,\alpha) \in U} Ix.$$

Because H respects products, the canonical map is an isomorphism

$$H(X) \qquad = \qquad \prod_{(x,\alpha) \in U} H(Ix).$$

Furthermore, the product of the elements $\alpha \in H(Ix)$ gives an element $\overline{\alpha} \in H(X)$, so that the projection

$$H(X) \qquad = \qquad \prod_{(x,\alpha)\in U} H(Ix) \xrightarrow{\begin{array}{c}\text{projection to}\\ (x,\alpha)\text{–factor}\end{array}} H(Ix)$$

takes $\overline{\alpha} \in H(X)$ to $\alpha \in H(Ix)$.

Now define the set V to consist of all pairs (x, f) with $x \in \mathcal{T}^c$ and $f : X \longrightarrow Ix$ so that under the map $H(X) \longrightarrow H(Ix)$ the image of

$\overline{\alpha} \in H(X)$ vanishes. To repeat,

$$V \quad = \quad \left\{ (x,f) \;\middle|\; \begin{array}{l} x \in \mathcal{T}^c,\ f : X \longrightarrow Ix \\ \overline{\alpha} \in H(X) \text{ maps to zero under} \\ H(f) : H(X) \longrightarrow H(Ix) \end{array} \right\}.$$

Define Y to be the product

$$Y \quad = \quad \prod_{(x,f)\in V} Ix.$$

For each (x,f) we have a map $f : X \longrightarrow Ix$, and these maps combine to give a map to the product $\overline{f} : X \longrightarrow Y$. Since H respects products, we have

$$H(Y) \quad = \quad \prod_{(x,f)\in V} H(Ix),$$

and since the image of $\overline{\alpha}$ vanishes under f for every $(x,f) \in V$, we have that the map $H(X) \longrightarrow H(Y)$ takes $\overline{\alpha}$ to zero.

By Yoneda's lemma, the element $\overline{\alpha} \in H(X)$ corresponds to a natural transformation

$$[X,-] \longrightarrow H(-),$$

and the vanishing of the image of α under $H(X) \longrightarrow H(Y)$ gives that the composite

$$[Y,-] \xrightarrow{[\overline{f},-]} [X,-] \longrightarrow H(-)$$

must vanish. By construction, we have arranged that for any $x \in \mathcal{T}^c$, the sequence

$$[Y,Ix] \xrightarrow{[\overline{f},Ix]} [X,Ix] \longrightarrow H(Ix) \longrightarrow 0$$

is exact.

But now we recall the definition of Ix. It was defined as the object in $\mathcal{T}$ representing a cohomology theory. Precisely, for every $z \in \mathcal{T}$,

$$\mathcal{A}b\Big([x,z], \mathbb{Q}/\mathbb{Z}\Big) \quad = \quad [z, Ix].$$

The map

$$[Y,Ix] \xrightarrow{[\overline{f},Ix]} [X,Ix]$$

is therefore identified with

$$\mathcal{A}b\Big([x,Y], \mathbb{Q}/\mathbb{Z}\Big) \xrightarrow{\mathcal{A}b\left([x,\overline{f}],\mathbb{Q}/\mathbb{Z}\right)} \mathcal{A}b\Big([x,X], \mathbb{Q}/\mathbb{Z}\Big).$$

For each $x \in \mathcal{T}^c$ define $K(x)$ to be the kernel of the map

$$[x,Y] \xleftarrow{[x,\overline{f}]} [x,X],$$

so that is there is an exact sequence

$$[x,Y] \xleftarrow{[x,\overline{f}]} [x,X] \longleftarrow K(x) \longleftarrow 0.$$

Recall that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. Applying the exact functor $\mathcal{A}b(-,\mathbb{Q}/\mathbb{Z})$ to the exact sequence above, we have an exact sequence

$$\begin{array}{ccccc} \mathcal{A}b\Big([x,Y],\mathbb{Q}/\mathbb{Z}\Big) & \xrightarrow{\mathcal{A}b\left([x,\overline{f}],\mathbb{Q}/\mathbb{Z}\right)} & \mathcal{A}b\Big([x,X],\mathbb{Q}/\mathbb{Z}\Big) & \longrightarrow & \mathcal{A}b(K(x),\mathbb{Q}/\mathbb{Z}) \\ & & & & \big\downarrow \\ & & & & 0. \end{array}$$

This identifies, naturally in $x \in \mathcal{T}^c$, the groups

$$\mathcal{A}b(K(x),\mathbb{Q}/\mathbb{Z}) \qquad = \qquad H(Ix).$$

Now let $r \to s \to t \to \Sigma r$ be a triangle in $\mathcal{T}^c$. We have a long sequence

$$K(\Sigma r) \longrightarrow K(t) \longrightarrow K(s) \longrightarrow K(r) \longrightarrow K(\Sigma^{-1}t)$$

and on applying the functor $\mathcal{A}b(-,\mathbb{Q}/\mathbb{Z})$ we get the exact sequence

$$H(\Sigma^{-1}It) \longrightarrow H(Ir) \longrightarrow H(Is) \longrightarrow H(It) \longrightarrow H(\Sigma Ir).$$

But the homology of the chain complex obtained from

$$K(\Sigma r) \longrightarrow K(t) \longrightarrow K(s) \longrightarrow K(r) \longrightarrow K(\Sigma^{-1}t)$$

by applying the exact functor $\mathcal{A}b(-,\mathbb{Q}/\mathbb{Z})$ is the functor $\mathcal{A}b(-,\mathbb{Q}/\mathbb{Z})$ applied to the homology. And for a non–zero abelian group G, $\mathcal{A}b(G,\mathbb{Q}/\mathbb{Z}) \neq 0$. We deduce that the sequence

$$K(\Sigma r) \longrightarrow K(t) \longrightarrow K(s) \longrightarrow K(r) \longrightarrow K(\Sigma^{-1}r)$$

is exact; the functor $K(-)$ is cohomological on $\mathcal{T}^c$.

In other words, there is an exact sequence

$$[-,Y] \xleftarrow{[-,\overline{f}]} [-,X] \longleftarrow K(-) \longleftarrow 0$$

of functors $\{\mathcal{T}^c\}^{op} \longrightarrow \mathcal{A}b$, with K homological and $[-,X]$ and $[-,Y]$ injective. The reason $[-,X]$ and $[-,Y]$ must be injective is that they are the products of injective objects $[-,Ix]$ in $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op},\mathcal{A}b\big)$; see Section 1. But by Proposition 1.4 of [5] the homological functor $K(-)$ has injective dimension no more than one; the functor $[-,Y]$ must split as the direct sum of injective functors $I(-)$ and $J(-)$ so that the sequence

$$0 \longleftarrow I(-) \longleftarrow [-,X] \longleftarrow K(-) \longleftarrow 0$$

is short exact. There is an idempotent $e : [-, Y] \longrightarrow [-, Y]$ which is the identity on $I(-)$ and zero on $J(-)$. By Adam's [1] there is a map $\overline{e} : Y \longrightarrow Y$ inducing e, and by Section 1 the map is even unique; there are no phantom maps into Ix, and Y is a product of Ix's. Since $\overline{e}^2$ is another lifting of e, we deduce $\overline{e}^2 = \overline{e}$. Thus $\overline{e} : Y \longrightarrow Y$ is idempotent.

It follows that Y splits as $Y = i \oplus j$ with $I(-) = [-, i]$ and $J(-) = [-, j]$. The composite $X \longrightarrow Y \longrightarrow i$ induces a map $[-, X] \longrightarrow [-, i]$ and an exact sequence of functors on $\mathcal{T}^c$

$$0 \longleftarrow [-, i] \longleftarrow [-, X] \longleftarrow K(-) \longleftarrow 0.$$

Applying the functor $\mathcal{A}b(-, \mathbb{Q}/\mathbb{Z})$ we have that, for every $x \in \mathcal{T}^c$ the sequence below is exact

$$0 \longrightarrow [i, Ix] \longrightarrow [X, Ix] \longrightarrow H(Ix) \longrightarrow 0.$$

In any case, the composite

$$[i, -] \longrightarrow [Y, -] \longrightarrow [X, -] \longrightarrow H(-)$$

vanishes; but the exactness is only guaranteed on objects of the form $Ix, x \in \mathcal{T}^c$.

Now complete $X \longrightarrow i$ to a triangle

$$h \longrightarrow X \longrightarrow i \longrightarrow \Sigma h.$$

We are given a natural transformation $[X, -] \longrightarrow H(-)$, which by Yoneda is the same as an element $\overline{\alpha} \in H(X)$. The composite

$$[i, -] \longrightarrow [X, -] \longrightarrow H(-)$$

vanishes, which means that the image of $\overline{\alpha} \in H(X)$ via $H(X) \longrightarrow H(i)$ vanishes. But H is homological; the sequence

$$H(h) \longrightarrow H(X) \longrightarrow H(i)$$

is exact. Hence there is an element $\beta \in H(h)$ mapping to $\overline{\alpha} \in H(X)$. By Yoneda, this says the map $[X, -] \longrightarrow H(-)$ factors through $[h, -]$.

For $x \in \mathcal{T}^c$, we have an exact sequence

$$0 \longrightarrow [i, Ix] \longrightarrow [X, Ix] \longrightarrow H(Ix) \longrightarrow 0.$$

But we also have the exact sequence for the triangle

$$[i, Ix] \longrightarrow [X, Ix] \longrightarrow [h, Ix] \longrightarrow [i, \Sigma Ix] \longrightarrow [X, \Sigma Ix]$$

from which it easily follows that the map $[h, Ix] \longrightarrow H(Ix)$ must be an isomorphism for all $x \in \mathcal{T}^c$. We wish to conclude that $[h, z] \longrightarrow H(z)$ is an isomorphism for every $z \in \mathcal{T}$. In any case, it is clear that $[h, z] \longrightarrow H(z)$ is an isomorphism if z is a product of objects $Ix, x \in \mathcal{T}^c$.

Now recall that Proposition 1.6 of [5]. It tells us that every object $z \in \mathcal{T}$ fits in a triangle

$$z \longrightarrow K \longrightarrow L \longrightarrow \Sigma z$$

with $[-, K]$ and $[-, L]$ injective as objects of $\mathcal{C}at\big(\{\mathcal{T}^c\}^{op}, \mathcal{A}b\big)$. The refinement I left to the reader, which slightly simplifies the proof we are about to give, said we may assume K and L are products of objects of the form $Ix, x \in \mathcal{T}^c$. But then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
[h, \Sigma^{-1}K] & \longrightarrow & [h, \Sigma^{-1}L] & \longrightarrow & [h, z] & \longrightarrow & [h, K] & \longrightarrow & [h, L] \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \downarrow \wr \\
H(\Sigma^{-1}K) & \longrightarrow & H(\Sigma^{-1}L) & \longrightarrow & H(z) & \longrightarrow & H(K) & \longrightarrow & H(L)
\end{array}$$

and the 5–lemma tells us that $[h, z] \longrightarrow H(z)$ is an isomorphism. $\square$

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