Conjugacy classes in maximal parabolic subgroups of general linear groups

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We compute conjugacy classes in maximal parabolic subgroups of the general linear group. This computation proceeds by reducing to a "matrix problem". Such problems involve finding normal forms for matrices under a specified set of row and column operations. We solve the relevant matrix problem in small dimensional cases. This gives us all conjugacy classes in maximal parabolic subgroups over a perfect field when one of the two blocks has dimension less than 6. In particular, this includes every maximal parabolic subgroup of $\mathrm{GL}_n(k)$ for $n < 12$ and k a perfect field. If our field is finite of size q, we also show that the number of conjugacy classes, and so the number of characters, of these groups is a polynomial in q with integral coefficients.

Key Words: conjugacy classes, parabolic subgroup, general linear group

1. INTRODUCTION

A great deal of progress has been made recently towards describing the representation theory of reductive algebraic groups. For example, the study of representations of finite reductive groups was greatly advanced by the work of Deligne and Lusztig [3] and has been an active field of research. Conjugacy classes in reductive groups have been investigated by Springer and Steinberg [23]. In comparison, little is known for solvable algebraic groups [8]. Even less is known about groups which are neither reductive nor solvable.

The parabolic subgroups of the general linear group are among the simplest such "mixed" groups. Each is a semidirect product of the unipotent radical (which is a solvable normal subgroup) with a Levi complement (which is a reductive group). Representations of the Levi complement can be inflated to the maximal parabolic—this is vital to the inductive step of classifications of representations of the general linear group. Drozd [4] generalized these inflated representations to a much larger class of mixed groups and showed they are Zariski dense in the set of irreducible representations. Almost nothing is known about the other representations of the maximal parabolics.

For parabolic subgroups of a reductive group, the conjugacy classes contained in the unipotent radical were first investigated by Richardson, Röhrle, and Steinberg [16]. A series of papers on this subject have been written by Hille, Jürgens, Popov, and Röhrle [6, 7, 9, 14, 15, 17, 18, 19, 20]. Some of their results use matrix problems similar to the ones discussed here.

In this paper, we describe conjugacy classes in maximal parabolic subgroups. This can be considered as a step towards a better understanding of the representation theory of parabolic subgroups of reductive algebraic groups. A maximal parabolic subgroup

$$G = P^{(m,n)} = \begin{pmatrix} \mathrm{GL}_m(k) & \mathrm{M}_{m,n}(k) \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}$$

has *unipotent radical*

$$U = \begin{pmatrix} I_m & \mathrm{M}_{m,n}(k) \\ 0 & I_n \end{pmatrix}$$

and *Levi complement*

$$L = \begin{pmatrix} \mathrm{GL}_m(k) & 0 \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}.$$

Multiplication in the unipotent radical is given by

$$\begin{pmatrix} I_m & v \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & w \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & v+w \\ 0 & I_n \end{pmatrix}.$$

Hence U can be identified with $M_{m,n}(k)$, the additive group of $m \times n$ matrices The Levi subgroup acts on the unipotent radical in the natural manner:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} 1 & AvB^{-1} \\ 0 & 1 \end{pmatrix}.$$

The following lemma describes the conjugacy classes in a semidirect product with abelian normal subgroup. This is analogous to the description of the characters proved by Clifford theory [2, section 11B].

LEMMA 1.1. *Let the group G be a semidirect product $U \rtimes L$ with U abelian. Then, for every h in L, the conjugacy classes in G intersecting Uh are in one-to-one correspondence with the orbits of $C_L(h)$ on $C^U(h) = U/[U,h]$.*

This is proved by taking $u, v \in U$ and $h, k \in L$ and then rearranging $(vk)(uh)(vk)^{-1}$ to be a product of an element of U with an element of L.

This lemma provides us with a procedure for finding the conjugacy classes. In Section 3 we describe the generalized Jordan normal form, which gives us a set of conjugacy class representatives for L. Then, for every such representative h, we compute the centralizer $C_L(h)$ in Sections 4 and 5, and the cocentralizer $C^U(h)$ in Section 6. Note that Sections 3, 4 and 6 each have two subsections: in the first we consider matrices with rational eigenvalues, in the second we show that for an irrational separable eigenvalue we get essentially the same thing, but over the extension of k with the eigenvalue adjoined. If you are only interested in algebraic closed fields, you need only read the first subsection in each section.

Finding orbits of the centralizer on the cocentralizer turns out to be a "matrix problem". Such problems involve finding normal forms for matrices under a specified set of row and column operations. They have been extensively studied by the Kiev school founded by Nazarova and Roĭter [12]. A good reference on matrix problems and their applications to representations of algebras is [5]. There is a classification of matrix problems: finite type problems have finitely many orbits whose representatives can be independent of the field; while for infinite type, the number of orbits depends on the field and is infinite whenever the field is. Infinite type problems can further be divided into tame type, where an explicit solution is known; and wild type, which reduces to the classical unsolved problem of finding a normal form for a pair of noncommuting matrices. The Brauer-Thrall conjecture, proved in [13], shows that every matrix problem is either finite, tame or wild.

In Section 7, we show that the matrix problems associated with $P^{(m,n)}$ are all finite type if, and only if, m or n is less than 6. In particular, this includes every maximal parabolic subgroup of $\mathrm{GL}_n(k)$ for $n < 12$ and k a perfect field. If our field is finite of size q, it follows from our proof that the number of conjugacy classes, and so the number of characters, of these groups is a polynomial in q with integral coefficients. Finally in Section 8 we recompute the conjugacy classes of the affine general linear groups.

2. NOTATION

This section explains some of the notational conventions used in this paper. We use standard group theoretic notation as in [1]. We use the equality symbol to denote natural isomorphism as well as strict equality.

Throughout this paper k is a field. Many of our results work for perfect fields or separable field extensions only. We denote the set of $m \times n$ matrices over a k-algebra R by $\mathrm{M}_{m,n}(R)$ and write $\mathrm{M}_n(R)$ for $\mathrm{M}_{n,n}(R)$. We frequently consider the *multiplicative group* $R^\times$ of a k-algebra; for example, the *general linear group* $\mathrm{GL}_n(k)$ is just $\mathrm{M}_n(k)^\times$. The n-dimensional space of column vectors is k^n. We write A^t for the *transpose* of the matrix A. We also apply this operation to sets of matrices, for example, $(k^n)^t$ is the space of row vectors over k. The algebra direct sum of n copies of k is written $k^{\oplus n}$ and is identified with the algebra of *diagonal matrices* in $\mathrm{M}_n(k)$.

A *composition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of natural numbers whose sum is $n = |\lambda|$. We call λ_i the ith part of λ. A *partition* is a composition whose parts are in decreasing order. Often we find it convenient to write the partition λ in the form $(r^{l_r}, \ldots, 2^{l_2}, 1^{l_1})$ where l_j is the number of times the part j occurs.

We consider algebras or groups consisting of matrices with different kinds of entries in different positions. These are denoted by matrices whose entries are the appropriate sets of possible entries. A matrix whose entries are also matrices is called a *block matrix*, and is identified in the obvious manner with a matrix of larger dimension. For example, we have defined the elements of $P^{(n,m)}$ as 2×2 matrices of matrices, but we generally consider them as $(n+m) \times (n+m)$ matrices over k.

We define the Jordan block

$$J_n(A) = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ I_d & A & 0 & \cdots & 0 \\ 0 & I_d & A & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & I_d & A \end{pmatrix}$$

where A is a $d \times d$ matrix and n is the number of times A appears. We also write $J_\lambda(A) = \bigoplus_{i=1}^s J_{\lambda_i}(A)$ for a composition λ

3. GENERALIZED JORDAN NORMAL FORM

We need a set of conjugacy class representatives for the Levi complement L, so that we can apply Lemma 1.1. Since this group is just $\mathrm{GL}_m(k) \oplus \mathrm{GL}_n(k)$, it suffices to give representatives of the similarity classes of invertible matrices. For our purposes, these representatives should be rational (ie. defined over k) but also as close to diagonal as possible. The generalized Jordan normal form has these properties. A proof that any matrix over a perfect field is similar to a matrix in this form can be found in [11].

Corresponding to every direct sum decomposition of a matrix, there is a decomposition of the underlying vector space into a direct sum of subspaces invariant under that matrix. An element of $\mathrm{M}_n(k)$ which is not similar to a direct sum of smaller square matrices over k is called *indecomposable*. Every square matrix is a direct sum of indecomposables.

Fix an $n \times n$ matrix A. Given p, a monic irreducible polynomial over k, the subspace $V_p = \{v \in k^n : p(A)^m v = 0 \text{ for some natural number } m\}$ is easily seen to be A-invariant. If this subspace is nonzero, we say p is a *generalized eigenvalue* of A and V_p is the corresponding *generalized eigenspace*. In fact, k^n is a direct sum of the generalized eigenspaces, so we get a corresponding decomposition of A. In particular, an indecomposable matrix has a unique generalized eigenvalue whose generalized eigenspace is the entire underlying vector space.

Rational case

Suppose every generalized eigenvalue of A is of the form $p(t) = t - \alpha$, for some α in k. Then each such α is also an eigenvalue. So A is similar to $\bigoplus_\alpha A_\alpha$ where A_α has unique eigenvalue α. The indecomposables with eigenvalue α are just the Jordan blocks $J_m(\alpha)$. Hence A_α is similar to $J_{\lambda_\alpha}(\alpha)$ for some partition λ_α and A is similar to $\bigoplus_\alpha J_{\lambda_\alpha}(\alpha)$ where α runs over the eigenvalues of A and $n = \sum_\alpha |\lambda_\alpha|$. Of course, this is just the ordinary Jordan normal form.

Irrational case

Now suppose the generalized eigenvalues of A are arbitrary monic, separable, irreducible polynomials.

Let p be such a polynomial and write $K = k(\alpha)$, where α is a root of p in the algebraic closure $\bar{k}$. Then K is a separable field extension of k and multiplication by α induces a k-linear transformation $K \to K$. The matrix of this transformation with respect to the k-basis $\{1, \alpha, \ldots, \alpha^{d-1}\}$ is just the companion matrix of p, denoted C_p. This is an indecomposable matrix with generalized eigenvalue p. More generally, every indecomposable matrix with generalized eigenvalue p is similar to the matrix $J_m(C_p)$ for some natural number m.

So the matrix A is similar to $\bigoplus_p J_{\lambda_p}(C_p)$, where p runs over the generalized eigenvalues of A and the λ_p are partitions with $n = \sum_p |\lambda_p| \deg(p)$. This is the *generalized Jordan normal form* of A.

4. CENTRALIZERS IN GENERAL LINEAR GROUPS

We describe the centralizers in general linear groups. Our description is explicit provided that all the generalized eigenvalues of our matrix are separable over k. We first compute the centralizer in $M = \mathrm{M}_n(k)$, then use this to find the centralizer in $G = \mathrm{GL}_n(k)$.

Let $A = \bigoplus_p J_{\lambda_p}(C_p)$ be an element of G in generalized Jordan normal form. Corresponding to this decomposition of A is a decomposition of k^n into a direct sum of generalized eigenspaces V_p. Further, a matrix B that centralizes A also centralizes $p(A)$, and so V_p is invariant under B.

Hence $C_M(A) = \bigoplus_p C_{\mathrm{M}_{n_p(k)}}(J_{\lambda_p}(C_p))$ where $n_p = |\lambda_p| \deg(p)$. We may now assume, without loss of generality, that A has a single generalized eigenvalue p with corresponding partition $\lambda = \lambda_p$.

Rational case

First consider $p(t) = t - \alpha$, ie. the matrix has rational eigenvalue α. Now $A = J_\lambda(\alpha) = \alpha I_n + J_\lambda(0)$ and αI_n is in the center of M, so $C_M(A) = C_M(J_\lambda(0))$. We write J_λ for $J_\lambda(0)$ and $k[x]_n$ for $k[x]/(x^n)$. With $\lambda = (n)$ there is an isomorphism $C_M(J_n) \to k[x]_n$ taking J_n to x. The natural action of $C_M(J_n)$ on k^n is equivalent to the regular action of $k[x]_n$ on itself. We generalize this to an arbitrary partition.

Take a matrix B centralizing J_λ and write it in block form

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1s} \\ \vdots & \ddots & \vdots \\ B_{s1} & \cdots & B_{ss} \end{pmatrix},$$

where B_{ij} is a $\lambda_i \times \lambda_j$ matrix. Then $BJ_\lambda = J_\lambda B$ implies $B_{ij}J_{\lambda_j} = J_{\lambda_i}B_{ij}$ for all i and j. If we write $B_{ij} = (b_{l,m})$, this becomes $b_{l,m+1} = b_{l-1,m}$ and $b_{1\lambda_j} = b_{l,\lambda_j} = b_{\lambda_i,m} = 0$ for $l = 2, \ldots, \lambda_i$ and $m = 1, \ldots, \lambda_j - 1$. Hence B_{ij} is

$$\left(\begin{array}{cccc|c} b_0 & 0 & \cdots & 0 & \\ b_1 & b_0 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \\ b_{\lambda_i} & \cdots & b_1 & b_0 & \end{array}\right) \quad \text{or} \quad \begin{pmatrix} & & 0 & \\ \hline b_{\lambda_j-\lambda_i} & 0 & \cdots & 0 \\ b_{\lambda_j-\lambda_i+1} & b_{\lambda_j-\lambda_i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{\lambda_j} & \cdots & b_{\lambda_j-\lambda_i+1} & b_{\lambda_j-\lambda_i} \end{pmatrix}$$

for $\lambda_i \le \lambda_j$ or $\lambda_i \ge \lambda_j$ respectively. The appearance of the full matrix B is illustrated by Figure 1 for $\lambda = (5, 3, 3, 2)$, where each line represents entries which are equal and blank spaces represent zero entries. Define $X^a_{c\times d}$ to be the $c \times d$ matrix whose (i, j)-entry is 1 if $i = j + a$ and 0 otherwise. Then B_{ij} can be written as $\sum_{a=0}^{\lambda_i} b_a X^a_{\lambda_i\times\lambda_j}$ or $\sum_{a=\lambda_j-\lambda_i}^{\lambda_j} b_a X^a_{\lambda_i\times\lambda_j}$ respectively. It is easily checked that $X^a_{c\times d}X^b_{d\times e} = X^{a+b}_{c\times e}$ for any nonnegative integers a, b, c, d, e. This identity gives us an algebra homomorphism

$$\begin{pmatrix} k[x] & x^{\lambda_1-\lambda_2}k[x] & \cdots & x^{\lambda_1-\lambda_s}k[x] \\ k[x] & k[x] & \cdots & x^{\lambda_2-\lambda_s}k[x] \\ \vdots & \vdots & \ddots & \vdots \\ k[x] & k[x] & \cdots & k[x] \end{pmatrix} \to C_M(A)$$

which takes x^a in the (i, j)-entry to $X^a_{\lambda_i\times\lambda_j}$ in the (i, j)-block. The matrices $X^a_{c\times d}$ are linearly independent for $a = 0, \ldots, \min(c, d)$ and zero for $a >$

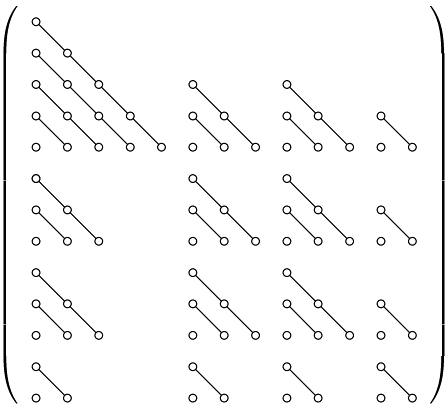

FIG. 1. An element of the centralizer

$\min(c, d)$. So this homomorphism is surjective and its kernel is

$$\begin{pmatrix} (x^{\lambda_1}) & (x^{\lambda_1}) & \cdots & (x^{\lambda_1}) \\ (x^{\lambda_2}) & (x^{\lambda_2}) & \cdots & (x^{\lambda_2}) \\ \vdots & \vdots & \ddots & \vdots \\ (x^{\lambda_s}) & (x^{\lambda_s}) & \cdots & (x^{\lambda_s}) \end{pmatrix}.$$

Hence $C_M(J_\lambda)$ is isomorphic to the quotient algebra

$$k[x]_\lambda = \begin{pmatrix} k[x]_{\lambda_1} & x^{\lambda_1-\lambda_2}k[x]_{\lambda_1} & \cdots & x^{\lambda_1-\lambda_s}k[x]_{\lambda_1} \\ k[x]_{\lambda_2} & k[x]_{\lambda_2} & \cdots & x^{\lambda_2-\lambda_s}k[x]_{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ k[x]_{\lambda_s} & k[x]_{\lambda_s} & \cdots & k[x]_{\lambda_s} \end{pmatrix}.$$

Next we consider the natural action of this algebra. The left action of $C_M(J_\lambda)$ on k^n is identical to its action on its own first column. This is easily seen to be isomorphic to the action of $k[x]_\lambda$ on its first column which is

$$k[x]_{\lambda\times -} = \begin{pmatrix} k[x]_{\lambda_1} \\ \vdots \\ k[x]_{\lambda_s} \end{pmatrix}.$$

We also need the right action on row vectors. There is an isomorphism $D : k[x]_\lambda \to {k[x]_\lambda}^t$ given by

$$\begin{pmatrix} a_{11} & x^{\lambda_1-\lambda_2}a_{12} & \cdots & x^{\lambda_1-\lambda_s}a_{1s} \\ a_{21} & a_{22} & \cdots & x^{\lambda_2-\lambda_s}a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ x^{\lambda_1-\lambda_2}a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x^{\lambda_1-\lambda_s}a_{s1} & x^{\lambda_2-\lambda_s}a_{s2} & \cdots & a_{ss} \end{pmatrix}.$$

We consider $k[x]_\lambda$ to act naturally on $k[x]_{-\times\lambda} = (k[x]_{\lambda\times -})^t$ via this isomorphism.

We now turn to the multiplicative group of $k[x]_\lambda$, which is isomorphic to $C_G(A) = C_M(A)^\times$. Now $B \in k[x]_\lambda$ can be written

$$B = B_0 + (xI)B_1 + (xI)^2B_2 + \cdots + (xI)^{\lambda_s-1}B_{\lambda_s-1},$$

where the entries of each B_i are in k and xI is the matrix with x in each diagonal entry and zero elsewhere. But $(xI)^{\lambda_s} = 0$, so xI is nilpotent and B is invertible exactly when B_0 is. Writing λ as $(r^{l_r}, \dots, 2^{l_2}, 1^{l_1})$, we see that B_0 is block lower triangular of the form

$$B_0 = \begin{pmatrix} B_{rr} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ B_{1r} & \cdots & B_{11} \end{pmatrix},$$

where B_{ij} is an $l_i \times l_j$ matrix. Hence B is invertible if, and only if, all the matrices B_{ii} are invertible, and we have described ${k[x]_\lambda}^\times$.

Irrational case

Finally we tackle the case where p is nonlinear and separable. Let $\alpha = \alpha_1, \alpha_2, \dots, \alpha_d$ be the distinct roots of p in the algebraic closure $\bar{k}$. We take A to be $J_\lambda(C_p)$. Recall that C_p is the k-matrix of multiplication by α on $K = k(\alpha)$ with respect to the basis $\{1, \alpha, \dots, \alpha^{d-1}\}$. Hence $C_{\mathrm{M}_d(k)}(C_p)$ is isomorphic to the centralizer of α in $\mathrm{End}_k(K)$, which is just $\mathrm{End}_K(K) = K$. On the other hand, the Jordan normal form of C_p is $D_p = \mathrm{diag}(\alpha_1, \dots, \alpha_d)$, so $tD_pt^{-1} = C_p$ for some t in $GL_d(\bar{k})$. The centralizer of D_p in $\mathrm{M}_d(\bar{k})$ is the algebra of diagonal matrices $\bar{k}^{\oplus d}$. Hence

$$C_{\mathrm{M}_d(\bar{k})}(C_p) = t(C_{\mathrm{M}_d(\bar{k})}(D_p))t^{-1} = t(\bar{k}^{\oplus d})t^{-1},$$

so $C_{\mathrm{M}_d(k)}(C_p) \cong K$ is the set of elements of $t(\bar{k}^{\oplus d})t^{-1}$ defined over k.

Turning now to $J_\lambda(C_p)$, we see it is conjugated to $J_\lambda(D_p)$ by $t^{\oplus n}$. Further, $J_\lambda(D_p)$ is conjugated to $J_\lambda(\alpha_1) \oplus \cdots \oplus J_\lambda(\alpha_d)$ by the obvious permutation of basis elements. From the rational case, we know that the centralizer in $\mathrm{M}_n(\bar{k})$ of this last matrix is isomorphic to ${\bar{k}[x]_\lambda}^{\oplus d}$. Reversing these conjugations we get $C_{\mathrm{M}_n(\bar{k})}(J_\lambda(D_p)) \cong (\bar{k}^{\oplus d})[x]_\lambda$ and $C_{\mathrm{M}_n(\bar{k})}(J_\lambda(C_p)) \cong \big(t(\bar{k}^{\oplus d})t^{-1}\big)[x]_\lambda$. So $C_M(J_\lambda(C))$ is just the set of elements of this algebra defined over k, which is $K[x]_\lambda$. The natural action and multiplicative group can now be computed as in the rational case.

An example should make this process clearer. Suppose $k = \mathbb{Q}$, $p(t) = t^2 - 3$, and $\lambda = (2)$. Then $K = \mathbb{Q}(\sqrt{3})$, $\alpha_1 = \sqrt{3}$ and $\alpha_2 = -\sqrt{3}$. Hence $J_\lambda(C_p)$, $J_\lambda(D_p)$, and $J_\lambda(\alpha_1) \oplus J_\lambda(\alpha_2)$ are

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -3 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right), \left(\begin{array}{cc|cc} \sqrt{3} & 0 & 1 & 0 \\ 0 & -\sqrt{3} & 0 & 1 \\ \hline 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right), \text{ and } \left(\begin{array}{cc|cc} \sqrt{3} & 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ \hline 0 & 0 & -\sqrt{3} & 1 \\ 0 & 0 & 0 & -\sqrt{3} \end{array}\right)$$

respectively. The centralizers of $J_\lambda(\alpha_1) \oplus J_\lambda(\alpha_2)$ and $J_\lambda(D_p)$ consist of matrices of the form

$$\left(\begin{array}{cc|cc} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ \hline 0 & 0 & c & d \\ 0 & 0 & 0 & c \end{array}\right) \text{ and } \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & c & 0 & d \\ \hline 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \end{array}\right)$$

respectively. Finally $C_M(J_\lambda(C_p)) \cong K[x]_\lambda = K[x]_2$.

5. GENERATORS FOR THE CENTRALIZERS

In order to find the orbits of the centralizers on the cocentralizers, we need a generating set for the centralizers. The generators we use are analogous to the elementary matrices of linear algebra—thus finding the orbits becomes a matrix problem.

Using the notation of the previous section, $C_G(A)$ is isomorphic to the group $K[x]_\lambda{}^\times$, which we write in block form as

$$\begin{pmatrix} \mathrm{M}_{l_r}(K[x]_r)^\times & \cdots & \mathrm{M}_{l_r l_2}\big(x^{r-2}K[x]_r\big) & \mathrm{M}_{l_r l_1}\big(x^{r-1}K[x]_r\big) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{M}_{l_2 l_r}(K[x]_2) & \cdots & \mathrm{M}_{l_2}(K[x]_2)^\times & \mathrm{M}_{l_2 l_1}(xK[x]_2) \\ \mathrm{M}_{l_1 l_r}(K) & \cdots & \mathrm{M}_{l_1 l_2}(K) & \mathrm{M}_{l_1}(K)^\times \end{pmatrix}$$

with $\mathrm{M}_l(K[x]_r)^\times = \mathrm{GL}_l(K) + \mathrm{M}_l(xK[x]_r)$.

Define the following matrices in $K[x]_\lambda{}^\times$:

- $M_{i,a}(l)$, for $i = 1, \ldots, r$, $a \in k[x]_{l_i}{}^\times$, and $l = 1, \ldots, l_i$: diagonal entries all 1, except for the (l,l)-entry in the (i,i)-block which is equal to a; off-diagonal entries all 0.

TABLE 1.

Regular action of $K[x]_\lambda{}^\times$

Matrix	Row operation	Column operation
$M_{i,a}(l)$	$R_{i,l} \to a \cdot R_{i,l}$	$C_{i,l} \to C_{i,l} \cdot a$
$E_i(l,m)$	$R_{i,l} \leftrightarrow R_{i,m}$	$C_{i,l} \leftrightarrow C_{i,m}$
$A_{i\leq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + ax^{i-j} \cdot R_{j,m}$	$C_{j,m} \to C_{j,m} + C_{i,l} \cdot ax^{i-j}$
$A_{i\geq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + a \cdot R_{j,m}$	$C_{j,m} \to C_{j,m} + C_{i,l} \cdot a$

- $E_i(l,m)$, for $i = 1,\dots,r$, and $l,m = 1,\dots,l_i$: entries all 1 and off diagonal entries all 0, except in the (i,i)-block where the (l,l) and (m,m)-entries are 0 and the (l,m) and (m,l)-entries are 1.
- $A_{i\leq j,a}(l,m)$, for $i,j = 1,\dots,r$ with $i \leq j$, $a \in k[x]_i$, $l = 1,\dots,l_i$ and $m = 1,\dots,l_j$: diagonal entries all 1; off diagonal entries all zero, except in the (i,j)-block where the (l,m)-entry is $x^{i-j}a$.
- $A_{i\geq j,a}(l,m)$, for $i,j = 1,\dots,r$ with $i \geq j$, $a \in k[x]_j$, $l = 1,\dots,l_i$ and $m = 1,\dots,l_j$: diagonal entries all 1; off diagonal entries all zero, except in the (i,j)-block where the (l,m)-entry is a.

Denote the lth row in the ith block by $R_{i,l}$, and the lth column in the ith block by $C_{i,l}$. Then these matrices act on $K[x]_\lambda{}^\times$ as in Table 1.

In order to prove that these matrices generate $K[x]_\lambda{}^\times$, it suffices to show that we can reduce any matrix in this group to the identity using these row and column operations. We proceed by induction on the number of parts of λ (ie. the dimension of our matrices). The result is clear if λ has one part. Now take a matrix

$$B = B_0 + (xI)B_1 + (xI)^2B_2 + \cdots + (xI)^{r-1}B_{r-1}$$

in $K[x]_\lambda{}^\times$ and write

$$B_0 = \begin{pmatrix} B_{rr} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ B_{1r} & \cdots & B_{11} \end{pmatrix}$$

as is Section 4. Since B_0 is invertible, we know B_{rr} is invertible, and so we can use row and column operations within the first block to get the (1,1)-entry of B to be one. Now, by adding multiples of the top row to the other rows, we can make every other entry in the first column zero. We can also add multiples of the left-most column of B to a column in the ith block, as long as we also multiply by x^{r-i}. This is not a problem since an entry in the ith block of the top row must be a multiple of x^{r-i} anyway.

TABLE 2.

Natural action of $K[x]_\lambda{}^\times$

Matrix	Row operation	Column operation
$M_{i,a}(l)$	$R_{i,l} \to a \cdot R_{i,l}$	$C_{i,l} \to C_{i,l} \cdot a$
$E_i(l,m)$	$R_{i,l} \leftrightarrow R_{i,m}$	$C_{i,l} \leftrightarrow C_{i,m}$
$A_{i\leq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + ax^{i-j} \cdot R_{j,m}$	$C_{i,l} \to C_{i,l} + C_{j,m} \cdot ax^{i-j}$
$A_{i\geq j,a}(l,m)$	$R_{i,l} \to R_{i,l} + a \cdot R_{j,m}$	$C_{i,l} \to C_{i,l} + C_{j,m} \cdot a$

We can now ignore the first row and column and reduce the rest of the matrix to the identity by induction. Hence we are done.

In subsequent sections we study the natural action of these matrices on the spaces $K[x]_{\lambda\times -}$ and $K[x]_{-\times\lambda}$. These actions are slightly different from the regular action, because the right action is via the isomorphism D of Section 4. The row and column operations for the natural action are in Table 2.

6. COCENTRALIZERS

Now that we have the centralizer and its generators in terms of algebras, we find a similar description for the cocentralizer and the action of the generators on it.

Let $G = P^{(m,n)} = U \rtimes L$. The Levi complement is $L = \mathrm{GL}_m(k) \oplus \mathrm{GL}_n(k)$ and we can identify the unipotent radical U with the additive group of $\mathrm{M}_{m,n}(k)$. Note that $\mathrm{M}_{m,n}(k) = k^m \otimes (k^n)^t$ where we are, as always, tensoring over k. Let $h = A \oplus B$, where $A \in \mathrm{GL}_m(k)$ and $B \in \mathrm{GL}_n(k)$ are both in generalized Jordan normal form. The action of L on U is given by $A \oplus B \cdot v = AvB^{-1}$. We wish to describe $C^U(h) = U/[U,h]$ as a $C_L(h)$-module over k. Using transfer of structure and the fact that $I \oplus B^{-1}$ is in the center of $C_L(h)$, this module is isomorphic to $(I \oplus B^{-1}) \cdot C^U(h) = U/((1 \oplus B^{-1}) \cdot [U, A \oplus B])$. Finally,

$$\begin{aligned}(I \oplus B^{-1}) \cdot [U, A \oplus B] &= \{(I \oplus B^{-1}) \cdot (v - A \oplus B \cdot v) : v \in U\} \\ &= \{vB - Av : v \in U\}.\end{aligned}$$

We know that $C_L(h)$ is a direct sum of $C_{\mathrm{GL}_m(k)}(A)$ and $C_{\mathrm{GL}_n(k)}(B)$, each of which is a direct sum of centralizers corresponding to the generalized eigenvalues of A and B. So a matrix v in $(I \oplus B^{-1})[U, A \oplus B]$ can be broken up into blocks corresponding to these centralizers and we can treat each block separately. Hence we may assume, without loss of generality, that A and B each have a single generalized eigenvalue.

Take $A = J_\mu(C_p)$ and $B = J_\nu(C_q)$ where p and q are monic, separable, irreducible polynomials. Let $K = k(\alpha)$ and $K' = k(\beta)$ with $\alpha, \beta \in \bar{k}$ the roots of p and q respectively. We identify $C_L(A \oplus B) = C_{\mathrm{GL}_{m(k)}}(A) \oplus C_{\mathrm{GL}_{n(k)}}(B)$ with $K[x]_\mu{}^\times \oplus K'[y]_\nu{}^\times$. This allows us to identify $U = k^m \otimes (k^n)^t$ with

$$K[x]_{\mu\times -} \otimes K'[y]_{-\times\nu} = \begin{pmatrix} K[x]_{\mu_1} \\ \vdots \\ K[x]_{\mu_s} \end{pmatrix} \otimes \begin{pmatrix} K'[y]_{\nu_1} & \cdots & K'[y]_{\nu_t} \end{pmatrix} = \begin{pmatrix} R'_{11} & \cdots & R'_{1t} \\ \vdots & \ddots & \vdots \\ R'_{s1} & \cdots & R'_{st} \end{pmatrix}$$

where

$$R'_{ij} = K[x]/(x^{\mu_i}) \otimes K'[y]/(y^{\nu_j}) = K \otimes K'[x,y]/(x^{\mu_i}, y^{\nu_j}).$$

Now the action of A on k^m corresponds to the action of $(\alpha + x)I_m$ on $K[x]_{\mu\times -}$ and the action of B on $(k^n)^t$ corresponds to $(\beta+y)I_n$ on $K'[y]_{-\times\nu}$. Hence $(I \oplus B^{-1}) \cdot [U, A \oplus B]$ is identified with the set of elements of the form

$$v(\beta + y)I_m - (\alpha + x)I_n v = (\beta - \alpha + y - x)v$$

for $v \in K[x]_{\mu\times -} \otimes K'[y]_{-\times\nu}$. Hence the the (i,j)-entry of $C^U(h) \cong U/(I \oplus B^{-1}) \cdot [U, A \oplus B]$ is identified with

$$R_{ij} = R'_{ij}/(\beta - \alpha + y - x) = K \otimes K'[x,y]/(x^{\mu_i}, y^{\nu_j}, \beta - \alpha + y - x).$$

Rational case

Suppose that α and β are both in k. Then $K = K' = k$ and

$$R = R_{ij} = k[x,y]/(x^{\mu_i}, y^{\nu_j}, \beta - \alpha + y - x),$$

If $\alpha \neq \beta$, then $\operatorname{rad} R$ contains x, y and $\beta - \alpha = x - y$. So the head of R, $R/\operatorname{rad} R$, maps onto $k/(\beta - \alpha) = 0$ and hence $R = 0$.

So we can assume $\alpha = \beta$. Then $x = y$ in R, so $R = k[x]/(x^{\mu_i}, x^{\nu_j}) = k[x]_{l_{ij}}$ where l_{ij} is the minimum of μ_i and ν_j. Hence $C^U(h)$ becomes

$$k[x]_{\mu\times\nu} = \begin{pmatrix} k[x]_{l_{11}} & k[x]_{l_{12}} & \cdots & k[x]_{l_{1s}} \\ k[x]_{l_{21}} & k[x]_{l_{12}} & \cdots & k[x]_{l_{2s}} \\ \vdots & \vdots & \ddots & \vdots \\ k[x]_{l_{r1}} & k[x]_{l_{r2}} & \cdots & k[x]_{l_{rs}} \end{pmatrix}, \quad l_{ij} = \min(\mu_i, \nu_j).$$

Since x and y are identified, we have $C_L(h) \cong k[x]_\mu{}^\times \oplus k[y]_\nu{}^\times = k[x]_\mu{}^\times \oplus k[x]_\nu{}^\times$ acting on $C^U(h) \cong k[x]_{\mu\times\nu}$.

Irrational case

Now consider arbitrary monic, separable, irreducible polynomials p and q. Since $K' = k[u]/(q(u))$, we have

$$\begin{aligned} R = R_{ij} &= K[x,y,u]/(q(u), x^{\mu_i}, y^{\nu_j}, u-\alpha+y-x) \\ &= K[x,y]/(q(x-y+\alpha), x^{\mu_i}, y^{\nu_j}). \end{aligned}$$

Over the field K, $q(u) = (u-\alpha)^\varepsilon f(u)$, where ε is 1 or 0 depending on whether p and q are equal or unequal. In either case $f(\alpha) \neq 0$. So we have $q(x-y+\alpha) = (x-y)^\varepsilon f(x-y+\alpha)$ and $\operatorname{rad}((x-y)^\varepsilon, f(x-y+\alpha)) = K[x,y]$ as it contains $x-y$ and so also contains $f(\alpha)$, which is a unit. Hence $((x-y)^\varepsilon, f(x-y+\alpha)) = K[x,y]$ and, by the Chinese Remainder theorem [10, Section III.2],

$$R = K[x,y]/((x-y)^\varepsilon, x^{\mu_i}, y^{\nu_j}) \oplus K[x,y]/(f(x-y+\alpha), x^{\mu_i}, y^{\nu_j}).$$

The second summand is trivial since its radical contains x and y, so its head maps onto $K/(f(\alpha)) = 0$. The first summand is trivial for $\varepsilon = 0$ and is $K[x]_{l_{ij}}$ for $\varepsilon = 1$. Hence $C^U(h)$ is trivial for $p \neq q$ and is isomorphic to $K[x]_{\mu\times\nu}$ for $p = q$. Once again $C_L(h)$ can be identified with $K[x]_\mu{}^\times \oplus K[x]_\nu{}^\times$.

So we have reduced our problem to finding the orbits of $K[x]_\mu{}^\times \oplus K[x]_\nu{}^\times$ on $K[x]_{\mu\times\nu}$, for appropriate fields K. Further, this action is given by the row and column operations of Table 2. So we have reduced to a matrix problem, which we also denote $K[x]_{\mu\times\nu}$.

7. SOLVING THE MATRIX PROBLEM FOR SMALL DIMENSIONS

We solve the matrix problem $k[x]_{\mu\times\nu}$ described in the previous sections for an arbitrary field k and either $|\mu|$ or $|\nu|$ less than 6. In particular this gives us the conjugacy classes in maximal parabolics of the general linear group of dimension less than 12 over a perfect field.

Let $\mu = (r^{m_r}, \ldots, 2^{m_2}, 1^{m_1})$ and $\nu = (s^{n_s}, \ldots, 2^{n_2}, 1^{n_1})$ be a pair of partitions with $m = |\mu|$ and $n = |\nu|$. We wish to find a normal form for

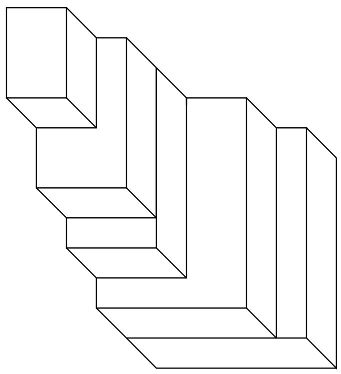

FIG. 2. A three dimensional array

matrices in

$$k[x]_{\mu\times\nu} = \begin{pmatrix} \mathrm{M}_{m_r n_s}\big(k[x]_{\min(r,s)}\big) & \cdots & \mathrm{M}_{m_r n_2}(k[x]_2) & \mathrm{M}_{m_r n_1}(k) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{M}_{m_2 n_s}(k[x]_2) & \cdots & \mathrm{M}_{m_2 n_2}(k[x]_2) & \mathrm{M}_{m_2 n_1}(k) \\ \mathrm{M}_{m_1 n_s}(k) & \cdots & \mathrm{M}_{m_1 n_2}(k) & \mathrm{M}_{m_1 n_1}(k) \end{pmatrix}$$

under the row and column operations of Table 2. I find it useful to visualize such a matrix as a three dimensional array of elements of k, with rows and columns as usual, and levels corresponding to the powers of x. This array is not rectangular since the number of levels depends on which row and column you are in. Figure 2 illustrates such an array for $\mu = (6, 5^2, 4^2, 3, 2)$ and $\nu = (5^2, 4, 2^2, 1)$. So, for example, multiplying a row by $1 + x^a$ takes every level in that row and adds its entries i levels higher up in the same row. Note that to add a column to another column i blocks to the left, we also have to move i levels up. We don't have this complication when adding to a column on the right, although we cannot add to a lower level. Similar we need to move to a higher level when adding a row to another row above it.

We now prove that our matrix problem can be infinite type when $m = n = 6$.

THEOREM 7.1. *The matrix problem $k[x]_{(4,2)\times(4,2)}$ is infinite type.*

Proof. Consider matrices in $k[x]_{(4,2)\times(4,2)}$ of the form

$$\begin{pmatrix} \alpha x^2 + \cdots & \beta x + \cdots \\ \gamma x + \cdots & \delta + \cdots \end{pmatrix}$$

for $\alpha, \beta, \gamma, \delta$ in $k^\times$. It is easily checked that every allowable row or column operation leads to another matrix of the same form and preserves the value of $\alpha\beta^{-1}\gamma^{-1}\delta$. Hence there are at least as many orbits as elements of $k^\times$, and the problem is infinite type. ∎

Next we prove our main theorem, showing that all smaller matrix problems are finite type.

THEOREM 7.2. *The matrix problem $k[x]_{\mu\times\nu}$ is finite type for ν arbitrary and μ of the form $(2^{m_2}, 1^{m_1})$, $(r, 1^{m_1})$ or $(3,2)$. In particular, $k[x]_{\mu\times\nu}$ is finite type whenever $|\mu| < 6$.*

Proof. Our basic approach is to solve the 0th level using the permissable row and column operations, then to solve the 1st level using only those operations which preserve the 0th level, and so on. It is a general property of finite type matrix problems that solutions can be found with every entry either 0 or 1. We call positions with a 1 entry *pivots*. These pivots can be used to "kill" other positions (ie. make them 0 with a row or column operation).

The proof is in four cases:

1. First we consider $\mu = (2^{m_2}, 1^{m_1})$. The solution for the 0th level is shown in the cutaway diagram of Figure 3a. In this diagram $\nu = (5^{m_5}, \ldots, 1^{m_1})$ but the general case is easily seen to be similar. The blocks are divided by solid lines. Each square containing a diagonal line is an identity matrix (of course, they are not all actually the same size). Now we can use the pivots in the 0th level to kill everything in the 1st level, except for the shaded blocks

The shaded blocks of the 1st level are redrawn in Figure 3b. We can add columns to blocks on the left, but not on the right. Also we can add rows to blocks below but not above, because, when adding to a block below, the damage done by one pivot can be repaired by a column operation from another pivot. This level can now be solved as shown.

2. Next we consider $\mu = (r)$, which is shown in Figure 4. Find the first nonzero block starting in the bottom left as shown. We can use row operations to put a pivot at the left hand end of this block and then kill the other entries indicated by the arrows. Now ignore all the entries marked with an arrow or a zero, and repeat the same process with what remains.

3. The case $\mu = (r, 1)$ is illustrated in Figure 5. We start by solving the 2nd row: find the first nonzero block, make a pivot in that block and kill the rest of the row. Then, ignoring the shaded part, we solve the rest of the 1st row as with $\mu = (r)$. We can now use a column multiplication to ensure that the shaded positions contains a single 1, followed by a row

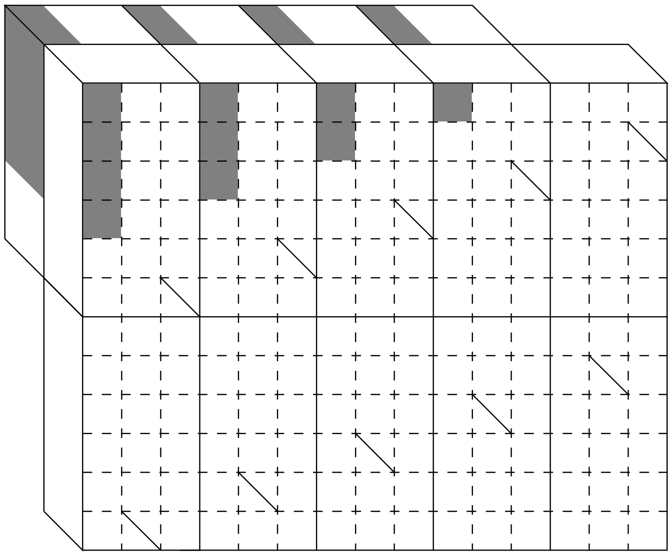

FIG. 3a. $\mu = (2^{m_2}, 1^{m_1})$, Level 0

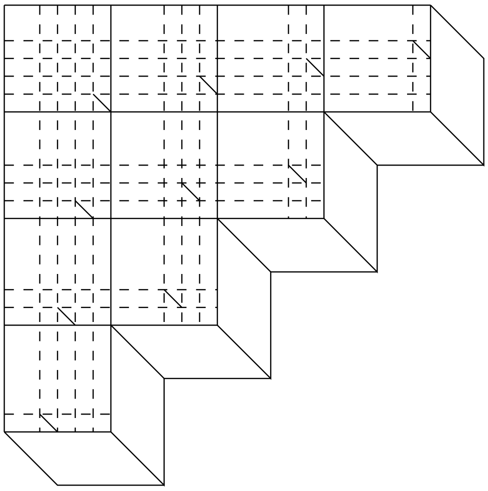

FIG. 3b. $\mu = (2^{m_2}, 1^{m_1})$, Level 1

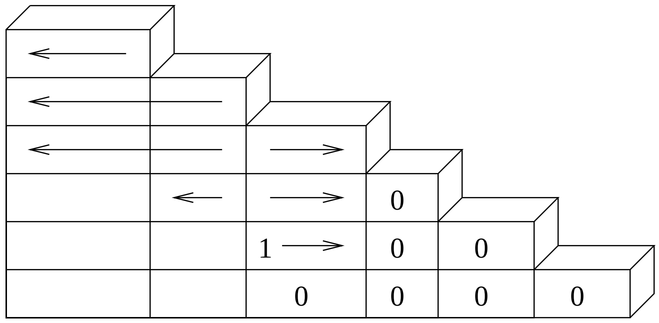

FIG. 4. $\mu = (r)$

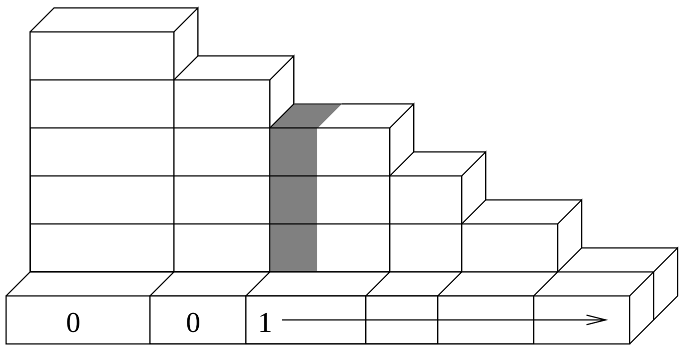

FIG. 5. $\mu = (r, 1)$

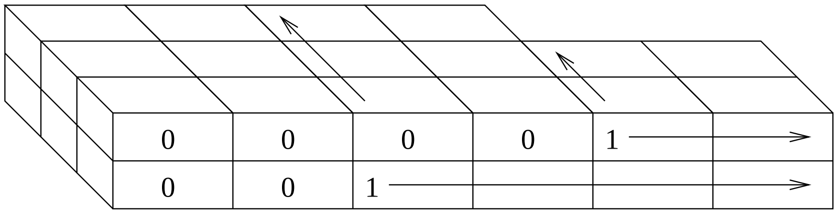

FIG. 6a. $\mu = (3, 2)$, Level 0

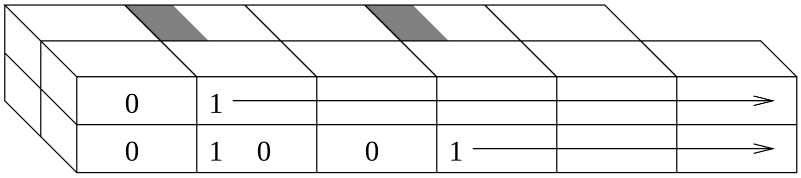

FIG. 6b. $\mu = (3, 2)$, Level 1

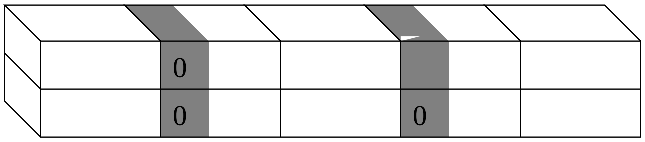

FIG. 6b. $\mu = (3, 2)$, Level 2

multiplication to repair any damage this does to the pivot in the second row.

The solution for $\mu = (r, 1^{m_1})$ is easily seen to be similar, except that there can be more than one shaded column.

4. Finally we turn to $\mu = (3, 2)$. First we solve the 0th level as in Figure 6a, and remove the two pivotal columns. Now, in the second level, all column operations are allowed, and row multiplication is allowed, because the damage it does to the pivots can be repaired by a column multiplication. However, the rows cannot be added to each other. This level is solved as in Figure 6b. For level 2 we get the same row and column operations as in level 1, except that the shaded columns cannot be added to other columns. However this does not cause a problem, since all but one entry in these columns has already been killed by a pivot on level 1.

The final claim follows because all partitions of a number less than 6 are of one of these three forms. ▮

Corollary 7.1. *Computing the conjugacy classes in $P^{(m,n)}$ over a perfect field reduces to matrix problems of finite type if, and only if, either $m < 6$ or $n < 6$. In particular, computing conjugacy classes in the maximal parabolics of the general linear group over a perfect field reduces to finite type problems if, and only if, the dimension is less than 12.*

Proof. The previous theorem, together with the results of Sections 4 to 6, show that we get finite type problems if $m < 6$. By symmetry, this is also true for $n < 6$. By Theorem 7.1, $P^{(6,6)}$ involves a problem of infinite type and it follows easily that $P^{(m,n)}$ does whenever $m, n \geq 6$. ▮

When our field is finite we get the following result.

Corollary 7.2. *Suppose that k is finite of size q and either $m < 6$ or $n < 6$. Then the number of conjugacy classes, and therefore the number of irreducible characters, of $P^{(m,n)}$ is a polynomial in q with integral coefficients.*

Proof. The number of solutions of the relevant matrix problems is independent of q. Hence this result follows immediately from the well known fact that the number of characters of $\mathrm{GL}_n(q)$ is a polynomial in q with integral coefficients. ▮

Note that the proof of the Theorem 7.2 also provides a procedure for solving these finite type problems, so this section gives an implicit description of all conjugacy classes in the parabolic subgroups mentioned in Corollary 7.1. For example, suppose the original eigenvalue is α, $\mu = \nu = (4, 2)$, and our orbit representative in $k[x]_{\mu\times\nu}$ is

$$\begin{pmatrix} \beta x^2 & x \\ x & 1 \end{pmatrix}.$$

Then our conjugacy class representative in $P^{(6,6)}$ is

$$\left(\begin{array}{cccccc|cccccc}
\alpha & & & & & & & & & & & \\
1 & \alpha & & & & & & & & & & \\
 & 1 & \alpha & & & & \beta & & & & & \\
 & & 1 & \alpha & & & & \beta & & & 1 & \\
 & & & & \alpha & & & & & & 1 & \\
 & & & & 1 & \alpha & 1 & & & & & 1 \\
\hline
 & & & & & & \alpha & & & & & \\
 & & & & & & 1 & \alpha & & & & \\
 & & & & & & & 1 & \alpha & & & \\
 & & & & & & & & 1 & \alpha & & \\
 & & & & & & & & & & \alpha & \\
 & & & & & & & & & & 1 & \alpha
\end{array}\right),$$

where blank entries are zero.

8. THE AFFINE GENERAL LINEAR GROUP

We apply the results of the previous section to the affine general linear groups [21, 22]. The representation theory of these well known groups is computed in [24]. The affine general linear group of degree n over the field k, $\mathrm{AGL}_n(k)$, is the semidirect product of $\mathrm{GL}_n(k)$ and the row space $(k^n)^t$. It can be realized as the subgroup

$$\begin{pmatrix} 1 & (k^n)^t \\ 0 & \mathrm{GL}_n(k) \end{pmatrix}$$

of $\mathrm{GL}_{n+1}(k)$. The generalized Jordan normal form provides a set of conjugacy class representatives for $\mathrm{GL}_n(k)$. Each can be written in the form $N \oplus E$ where N has no eigenvalues equal to 1 and E has eigenvalue 1. There is a partition $\lambda = (r^{l_r}, \ldots, 2^{l_2}, 1^{l_1})$ so that

$$E = J_\lambda(1) = \bigoplus_{i=1}^{r} E_i$$

where $E_i = J_i(1)^{\oplus l_i}$.

THEOREM 8.1. *A set of conjugacy class representatives for* $\mathrm{AGL}_n(k)$ *is given by the matrices*

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline & N & 0 \\ & 0 & E \end{array}\right) \quad \textit{and} \quad \left(\begin{array}{c|cccccc} 1 & 0 & 0 & \cdots & e & \cdots & 0 \\ \hline & N & 0 & \cdots & 0 & \cdots & 0 \\ & 0 & E_1 & & & & 0 \\ & \vdots & & \ddots & & & \vdots \\ & 0 & & & E_i & & 0 \\ & \vdots & & & & \ddots & \vdots \\ & 0 & 0 & \cdots & 0 & \cdots & E_m \end{array}\right),$$

where $e = (1, 0, 0, \dots)$.

Proof. Let $Z = kI_{n+1}$ be the center of $\mathrm{GL}_{n+1}(k)$. Then $P^{(1,n)} = Z{\cdot}\mathrm{AGL}_n(k)$ and so the conjugacy classes in $\mathrm{AGL}_n(k)$ are just the noncentral conjugacy classes in $P^{(1,n)}$ intersected with $\mathrm{AGL}_n(k)$. Hence we need the matrices given by the proof of Theorem 7.2 with $\mu = (1)$ and 1 in the first summand of $L = \mathrm{GL}_1(k) \oplus \mathrm{GL}_n(k)$. The result is now immediate. ▮

Let k be a finite field. We denote by c_n the number of conjugacy classes in $\mathrm{GL}_n(k)$ and use the convention that $\mathrm{GL}_0(k)$ is the trivial group. For $d = 0, 1, \dots, n$, we consider the conjugacy class representatives of $\mathrm{AGL}_n(k)$ with an e above a Jordan block of size d. Then $A = N \oplus E$ is an arbitrary conjugacy class representative of $\mathrm{GL}_n(k)$, except that it must have a least one Jordan block of size d and eigenvalue 1. If you remove one such block of size d from A, you get an arbitrary conjugacy class representative of $\mathrm{GL}_{n-d}(k)$, of which there are c_{n-d}. So the total number of conjugacy class representatives of $\mathrm{AGL}_n(k)$ is

$$\sum_{d=0}^{n} c_{n-d} = c_n + c_{n-1} + \cdots + c_0.$$

This agrees with the count of the number of irreducible characters of $\mathrm{AGL}_n(k)$ gotten by Zelevinsky [24].

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REFERENCES

1. J. L. Alperin and Rowen B. Bell. *Groups and representations.* Springer-Verlag, New York, 1995.
2. Charles W. Curtis and Irving Reiner. *Methods of representation theory. Vol. I.* John Wiley & Sons Inc., New York, 1990.
3. P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976.
4. Yu. A. Drozd. Matrix problems, small reduction and representations of a class of mixed Lie groups. In *Representations of algebras and related topics (Kyoto, 1990)*, pages 225–249. Cambridge Univ. Press, Cambridge, 1992.
5. P. Gabriel and A. V. Roiter. *Representations of finite-dimensional algebras.* Springer-Verlag, Berlin, 1997. Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation.
6. L. Hille and G. Röhrle. A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical. *Transform. Groups*, 4(1):35–52, 1999.
7. Lutz Hille and Gerhard Röhrle. On parabolic subgroups of classical groups with a finite number of orbits on the unipotent radial. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(5):465–470, 1997.
8. I. M. Isaacs and Dikran Karagueuzian. Conjugacy in groups of upper triangular matrices. *J. Algebra*, 202(2):704–711, 1998.
9. U. Jürgens and G. Röhrle. Algorithmic modality analysis for parabolic groups. *Geom. Dedicata*, 73(3):317–337, 1998.
10. Serge Lang. *Algebra.* Addison-Wesley Publishing Co., Reading, Mass., third edition, 1993.
11. A. I. Mal'cev. *Foundations of linear algebra.* W. H. Freeman & Co., San Francisco, Calif.-London, 1963.
12. L. A. Nazarova and A. V. Roĭter. Representations of partially ordered sets. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 28:5–31, 1972.
13. L. A. Nazarova and A. V. Roĭter. *Kategornye matrichnye zadachi i problema Brauèra-Trèlla.* Izdat. "Naukova Dumka", Kiev, 1973.
14. Vladimir Popov and Gerhard Röhrle. On the number of orbits of a parabolic subgroup on its unipotent radical. In *Algebraic groups and Lie groups*, pages 297–320. Cambridge Univ. Press, Cambridge, 1997.
15. Vladimir L. Popov. A finiteness theorem for parabolic subgroups of fixed modality. *Indag. Math. (N.S.)*, 8(1):125–132, 1997.
16. Roger Richardson, Gerhard Röhrle, and Robert Steinberg. Parabolic subgroups with abelian unipotent radical. *Invent. Math.*, 110(3):649–671, 1992.
17. Gerhard Röhrle. Parabolic subgroups of positive modality. *Geom. Dedicata*, 60(2):163–186, 1996.
18. Gerhard Röhrle. Maximal parabolic subgroups in classical groups are of modality zero. *Geom. Dedicata*, 66(1):51–64, 1997.
19. Gerhard Röhrle. A note on the modality of parabolic subgroups. *Indag. Math. (N.S.)*, 8(4):549–559, 1997.
20. Gerhard Röhrle. On the modality of parabolic subgroups of linear algebraic groups. *Manuscripta Math.*, 98(1):9–20, 1999.

21. Louis Solomon. On the affine group over a finite field. In *Representation theory of finite groups and related topics (Proc. Sympos. Pure Math., Vol. XXI, Univ. Wisconsin, Madison, Wis., 1970)*, pages 145–147. Amer. Math. Soc., Providence, R.I., 1971.

22. Louis Solomon. The affine group. I. Bruhat decomposition. *J. Algebra*, 20:512–539, 1972.

23. T. A. Springer and R. Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, pages 167–266. Springer, Berlin, 1970. Lecture Notes in Mathematics, Vol. 131.

24. Andrey V. Zelevinsky. *Representations of finite classical groups*. Springer-Verlag, Berlin, 1981. A Hopf algebra approach.