Unipotent Blocks of Finite General Linear Groups in Non-defining Characteristic

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Abstract

In representation theory of finite groups, there have been an important conjecture due to M. Broué [4], [5]. In [9], Chuang and Kessar showed that for any weight w smaller than p, the characteristic of coefficient field, there exists a p-block of symmetric groups with p-weight w such that this is Morita equivalent to the principal block of $\mathfrak{S}_p \wr \mathfrak{S}_w$. Here, $\mathfrak{S}_n$ denotes the symmetric group on n letters.

In representation theory of finite groups of Lie type, there have been an important conjecture due to G.D. James [30],[21]. Moreover, R. Dipper and G.D. James constructed a lot of important results on modular representation theory of finite general linear groups. The author wish to classify the unipotent blocks of finite groups of Lie type, up to Morita equivalence, and describe a process recovering its structure as algebras from some blocks of local groups.

In this paper we deal some special cases of finite general linear groups concerning problems by showing an analogy of J. Chaung and R. Kessar's work for symmetric groups and comparing the labels of simple modules of a global group with that of a local group. More special cases of this paper had been done by [37].

Moreover, the label of simple modules due to R. Dipper and G.D. James [28], [12],[10] is quite important in order to compare a block whose structure is well-known with another one. We will be able to calculate the decomposition numbers for unipotent characters lying in these special blocks explicitly by using these labels, equivalences of blocks, and its images of simple modules.

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Chapter 1

Introduction

Let ℓ be a prime number. Let G be a finite group and let $(K, \mathcal{O}, \mathbf{k})$ be an ℓ-modular system for G.

In representation theory of finite groups, one of the most important problems is to settle a conjecture given by M. Broué [4],[5]. Broué has conjectured the following:

Conjecture 1.0.1. (**M. Broué**) *Let B be a block of $\mathcal{O}G$ with abelian defect group D. Then, B and its Brauer correspondent in $\mathcal{N}_G(D)$ are derived equivalent.*

This conjecture means that the structure of B as an algebra is recovered from its Brauer corrspondent by Rickard's Morita theory [52],[51],[53]. The author learned this meaning from a lot of T. Okuyama's works [48]. This is our first motivation, say, "recovering the structure of blocks as algebras from blocks of some small groups". (Note that more precise and deep conjecture for finite reductive groups in non-defining characteristic is found in [6].)

In this article we would like to investigate representations of finite general linear groups in non-defining characteristic and show that the above conjecture is true for blocks with some special properties. In order to state the other motivations we recall briefly the history of representations of finite general linear groups in non-defining characteristic and some important conjectures in this subject.

Let e be the multiplicative order of q in $\mathbb{F}_\ell$. P. Fong and B. Srinivasan showed that each of blocks of finite general linear groups $GL_n(q)$ is indexed by semisimple ℓ'-element s and a set of cores of partitions [18]. In particular, each of the blocks which a unipotent character lies in is called a unipotent block and indexed by an e-core of paritions and semisimple (Lie) rank n of $GL_n(q)$. In this way we denote a unipotent block of the finite general linear group $GL_n(q)$ by $B_{1_n,\nu}$ indexed by the identity element 1_n of $GL_n(q)$ and e-core ν. The positive integer $(n-|\nu|)/e$ is called e-weight of $B_{1_n,\nu}$. So, in this meaning we write $B_{w,\nu}$ instead of $B_{1_n,\nu}$ to see an analogy of blocks in symmetric groups clearly. (Moreover, they used Deligne-Lusztig generalized characters, and constructed a fundation of modular representation theory of finite groups of Lie type in non-defining characteristic.)

In 80's R. Dipper and G.D. James have constructed a lot of results concerning $GL_n(q)$; the identification problems of the irreducible modules of $GL_n(q)$ [12], a lot of investigations of modular representations of Iwahori-Hecke algebras of type A [13],[14], introducing q-Schur algebras , reductions for computing whole the decomposition matrix for $GL_n(q)$ to computing the decomposition matrices for q^i-Schur algebras with degree less or equal to n

where i runs over all the non-negative integers i dividing $\ell - 1$. (In particular, the decomposition matrix of q-Schur algebra describe that of unipotent characters.) [15]. A naturally reformulated q-Schur algebras in the sense of J. A. Green, now called, Dipper-James q-Schur algebras $\boldsymbol{\mathcal{S}}_{R,q}(n,r)$ where R is an intergral domain with an invertible element q [16].($\boldsymbol{\mathcal{S}}_{R,q}(n,r)$ is Morita equivalent to the original q-Schur algebra defined in [15])

G.D. James constructed all the cuspidal modules of finite general linear groups and "Specht"like modules by using the idempodents of (degenerate) Gelfand-Graev characters [28],[29], and gave explicit tables of all the decomposition matrices of the q-Schur algebras for $GL_n(q)$ where $n \leq 10$ and conjectured James' conjecture [30](see also M. Geck's interpretation [21]).

Conjecture 1.0.2. (**G.D. James**) *Let $\overline{q}$ be $q \cdot 1_{\mathbf{k}}$ whose multiplicative order e. Let ζ be a primitive e-th root of unity in the complex field $\mathbb{C}$. If $\ell > n$, then the decomposition matrix of $\boldsymbol{\mathcal{S}}_{\mathbf{C},\zeta}(n,r)$ is equal to the decomposition matrix of $\boldsymbol{\mathcal{S}}_{\mathbf{k},\overline{q}}(n,r)$.*

Note that the decomposition matrix of $\boldsymbol{\mathcal{S}}_{\mathbb{C},\zeta}(n,r)$ is computable by some recursive algorithm. (This is originally due to Lascoux-Leclerc-Thibon [38] for Iwahori-Hecke algebras of type A at roots of unity. See [40],[59] and [20] for the details and a lot of improvements.)

So, computing the decomposition numbers for $\boldsymbol{\mathcal{S}}_{\mathbf{k},\overline{q}}(n,r)$ with a positive characteristic coefficient field $\mathbf{k}$ and comparing it with characteristic zero case are quite interesting. This is James' second step. That is not only for modular representation theory of finite general linear groups in non-defining characteristic, but also for this meaning we want to calculate the decompositon matrices of Dipper-James q-Schur algebras with a positive characteristic coefficient field explicitly. This is our second motivation, say "to find a connection between characteristic zero cases and positive characteristic cases". We will show that the decomposition matrices for some special unipotent blocks for given weights are computable and describe its label explicitly in Chapter 7.

There is a concept of J. Scopes' $[w:k]$-pair. J. Scopes found such notion in blocks of symmetric groups [55] and T. Jost developed it for finite general linear groups in non-defining characteristic [34]. p-Blocks of symmetric groups are indexed by p-weight w and p-core ν as $B^{w,\nu}$ (see for example [31]). J. Scopes showed that if p-cores λ and μ form a $[w:k]$-pair with $k \geq w$ then $B^{w,\lambda}$ and $B^{w,\mu}$ are Morita equivalent by a p-permutation bimodule (i.e. Puig equivalent). For a fixed prime power q T. Jost investigated $[w_f : k_f]$-pair between $B_{s,\lambda}$ and $B_{\overline{s},\mu}$ for some elementary divisor f of s, and showed an analogy of Scopes' result [55]. Therefore, to investigate the connection between blocks $B^{w,\lambda}$ and $B^{w,\mu}$ of symmetric groups (resp. blocks $B_{s,\lambda}$ and $B_{\overline{s},\mu}$ of general linear groups) which form a $[w:k]$-pair with $w > k$ is important.

By C. Bonnafé and R. Rouquier's forthcoming quite strong results [3], any non-isolated block of finite reductive groups $\mathbf{G}(\mathbb{F}_q)$ are Morita equivalent to some unipotent block of $\mathbb{F}_q$-fixed point of the dual group of centralizer of some semisimple ℓ'-element in reductive groups $\mathbf{G}^*$. This means for finite general linear groups that we may just concentrate the blocks in which principal series characters lie. Moreover, we may only use T. Jost's result for these blocks and can use his result quite similar to J. Scopes'.

Towards the classification of blocks for finite reductive groups, up to Morita equivalence, one must compare blocks of groups defined by various finite fields $\mathbb{F}_q$'s. (i.e. We must consider and move infinitly many q.) By observing a lot of phenomenon, for example [37], the author wish to prove the following conjecture:

Conjecture 1.0.3. *Suppose that finite reductive groups $\mathbf{G}(\mathbb{F}_q)$ and $\mathbf{G}(\mathbb{F}_{q'})$ should have the same root data and Weyl group and satisfy that $e(q) = e(q')$ and $r(q) = r(q')$. Here, ℓ is a prime number coprime to q, $e(q)$ is the multiplicative order of q in $\mathbb{F}_\ell$ and $r(q)$ is the maximal integer i such that ℓ^i devides $q^{e(q)} - 1$. Then for any unipotent block B of $\mathbf{G}(\mathbb{F}_q)$ there exists a unipotent block B' of $\mathbf{G}(\mathbb{F}_{q'})$ such that B and B' are Morita equivalent. This equivalence is given by an exact ℓ-permutation (B, B')-bimodule, preserves the natural indices of simple modules.*

This is our third motivation, that is, "independence of q". Moreover, if $e(q) = e(q')$ and James' conjecture is ture, then $\boldsymbol{\mathcal{S}}_{\mathbf{k},\overline{q}}(n,r)$ and $\boldsymbol{\mathcal{S}}_{\mathbf{k},\overline{q'}}(n,r)$ should have the same decomposition matrix. By M. Takeuchi [58], q-Schur algebra is realized as a quotient of the unipotent blocks of finite general linear groups. By Dipper-James theory $e(q) = e(q')$ and $r(q) = r(q')$ means that the decompositon matrix of the unipotent blocks of $GL_n(q)$ coincides that of $GL_n(q')$ if the decomposition matrices of q-Schur algebra and q'-Schur algebra are the same.

In main results of this paper we deal quite special case of unipotent blocks of finite general linear groups and assume a quite strong assumption that these blocks have abelian defect. But the author optimisitically hope that this assumption is suitable in the meaning that the knowelge of unipotent blocks with abelian defect by obtaining the structure as algebras in the sense of M. Broué's conjecture should play a role to investigate unipotent blocks with non-abelian defect and James' conjecture.

Our methods in this paper are quite combinatrial. In Chapter 2 we introduce some notation relating to partitions. In particular, in Definition 2.2.1 we describe a special e-cores ρ for finite general linear groups which label our target "good" block for any e-weight w by just changing p (in [9]) to e. (We will call this e-core a Chuang-Kessar-Rouquier (relative to given e-weight w).)

In Chapter 4 we recall terminologies of representation theory of finite general linear groups in non-defining characteristic.

In Chapter 5 we will see that Chuang-Kessar's argument [9] works also unipotent blocks for general linear groups; For any e-weight $w < \ell$ we show that $B_{w,\rho}$ is Morita equivalent to the principal block of $GL_e(q) \wr \mathfrak{S}_w$ by essentially the Green correpondence with respect to $(G, \Delta(D), H)$ where G is $GL_{ew+|\rho|}(q)$ D is a defect group of $B_{w,\rho}$ and H is a subgroup of G which contains $N_G(D)$ and $GL_e \wr \mathfrak{S}_w$. So, for our first motivation we can settle Broué's abelian defect conjecture by using A. Marcus' result [43]. However, in [9] the correspondents of simple modules in the principal block of $\mathfrak{S}_p \wr \mathfrak{S}_w$ is not known. In order to compare the other blocks with this special block and calculate decomposition matrices explicitly it is quite important to determine this correspondents. In Chapter 6 we will label the simple $B_0(GL_e(q) \wr \mathfrak{S}_w)$-modules fitting to that of $B_{w,\rho}$ and find trivial source modules associated with e-weight w and Chuang-Kessar-Rouquier e-core ρ. Our main result is to dertemine the Green correspondents of simple modules of some local subgroup of a general linear group. In Chapter 7 we will see that this is determined explicitly. So, the decompositon matrix of $B_{w,\rho}$ is described by the knowelge of the principal block of $GL_e(q) \wr \mathfrak{S}_w$ explicitly for our second motivation and developments of representation theory of finite general linear groups. As an application we can describe some improvement of our third motivation in Chapter 8. In Chapter 9 we will make Morita and derived equivalent classes of blocks which can be known a little bit wider in the sense of our first and third motivation by using and mimicking results of J. Scopes [56] and J. Chuang [8] for p-weight 2 blocks of symmetric groups. One can know that if $B_{w,\lambda}$ and $B_{w,\mu}$ form a $[w : w-1]$-pair then these are derived equivalent once one knows their results. In Chapter 10 we will calculate the decomposition numbers for unipotent characters lying in unipotent blocks $B_{w,\rho}$ which is Morita equivalent to the pricipal block of $GL_e(q) \wr \mathfrak{S}_w$ by using the results of Chapter 5 and 7, and the label which will be described in Chapter 6. In Chapter 11 we will conjecture some relations between radical series of Specht modules for $B_{w,\rho}$ given by James and a canonical basis of v-deformed Fock space for Kac-Moody algebra $U_v(\widehat{\mathfrak{gl}}_e)$ given by Leclerc and Thibon. This conjecture is based on James' conjecture , Leclcerc-Rouquier-Thibon conjecture on Jantzen filtrations of Weyl modules of Dipper-James Schur algebra at roots of unity in characteristic zero, and Andersen's comment on tilting modules concerning representation theory of reductive groups over $\overline{\mathbb{F}_p}$ in the defining characteristic in 1998. In Chapter 12 we will discuss about the decomposition numbers and filtration of Steinberg modules, and the decomposition numbers for the projective cover of the trivial module as an appendix.

Chapter 2

Combinatrics relating to partitions

2.1 Notation

We would like to introduce some combinatrics relating to partitions. A *composition* λ of the non-negative integer n is a sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative integers λ_i whose sum is n. If, in addition, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots$, then λ is called a *partition* of n. Moreover, "$\lambda \models n$" (resp. "$\lambda \vdash n$") means that λ is a composition (resp. partition) of n. We will use abbreviations like $(1^m) = (1, 1, \ldots, 1, 0, 0, \ldots)$ for partitions. For a natural number r, $\lambda \vdash n$ is said to be *r-singular* if there exists $i \geq 1$ such that

$$\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+r} > 0.$$

Otherwise, λ is called *r-regular*. If $\lambda \models n$, then define $\lambda' = (\lambda'_1, \lambda'_2, \ldots) \vdash n$ by letting λ'_i equal to the cardinality of $\{\lambda_j \mid \lambda_j \geq i\}$. The *dominance order*

$\trianglerighteq$ on the set of compositions of m is defined by $\lambda \trianglerighteq \mu$ if, for all j,

$$\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \mu'_i.$$

For $\lambda, \mu \vdash n$, we also write $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$. The *lexicographic order* $>$ on the set of partitions of m is defined by $\lambda > \mu$ if, for some j, $\lambda_j > \mu_j$, while $\lambda_i = \mu_i$ for all $i < j$. We also write $\lambda \geq \mu$ if $\lambda > \mu$ or $\lambda = \mu$. Note that if $\lambda \trianglerighteq \mu$ then $\lambda \geq \mu$. For natural numbers n, d and k with $n = dk$, and for $\lambda \models k$, we define $d\lambda \models n$ by $(d \cdot \lambda)_i = d \cdot (\lambda_i)$. For partitions $\alpha \vdash m, \beta \vdash n$ and $\gamma \vdash m+n$, we denote by $LR^{\gamma}_{\alpha,\beta}$ the Littlewood-Richardson coefficient. Conventionally, we define $LR^{\gamma}_{\alpha,\beta} = 0$ unless $|\alpha| + |\beta| = |\gamma|$. Moreover, $LR^{\emptyset}_{\emptyset,\emptyset} = 1$.

Fix an integer e. Removing rim e-hooks from the Young diagram of a partition λ as possible, we get the diagram of a partition which is called *e-core* of λ.; it is independent of the sequence of rim hooks removed. The number of rim hooks removed from λ is called the *e-weight*. In order to treat partitions relating to e-cores, abacus configurations with e-runners is useful, due to Gordon James [31, pp.77-78]. For a parition λ, we denote its *e-quotient* $\overline{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(e-1)})$. It depends on the number of beads used in the abacus configuration, but there exists a bijection between the set of partitions whose lengths are not more than the number of beads, whose e-cores are the same, and whose e-weight are the same w, and the set of e-tuples $(c^{(0)}, \dots, c^{(e-1)})$ of partitions satisfying $|c^{(0)}| + \cdots |c^{(e-1)}| = w$.

Definition 2.1.1. *1. For a non-negative integer m let $\mathcal{P}(m)$ be the set of* *partitions of m.*

2. *For a k-core τ and a non-negative integer w, let $\Lambda_{k,w,\tau}$ be the set of paritions of $kw + |\tau|$ whose k-core is τ.*

3. *For a non-negative integer d let $\Lambda(d, n)$ be the set of compostitions of n with at most d non-zero parts; say,*

$$\Lambda(d,n) = \{\ \lambda \models n \mid \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\}\,.$$

4. *Let $\mathcal{MP}(m, w)$ be the set of m-tuple multipartitions of w; say,*

$$\mathcal{MP}(m,w) = \{\ \boldsymbol{\lambda}\ \mid\ \boldsymbol{\lambda}_i \vdash t_i, \sum_{i=1}^{m} t_i = w, t_i \in \mathbb{N} \cup \{0\}\}.$$

2.2 An e-core ρ

Let ℓ be an odd prime. Let w be a non-negative integer smaller than ℓ.

Definition 2.2.1. (**Chuang-Kessar-Rouquier**[*9*, §4]) *Let ρ be an e-core which satisfies the following property: ρ has an abacus configuration in which each runner other than the leftmost one (the 0-th runner) has at least $w-1$ more beads than the runners to its immediate left. We call these ρ's Chuang-Kessar-Rouquier e-cores (CKR e-cores for short) relative to w.*

Remark

We may also choose $e = p$ for some prime p and consider some blocks of symmetric groups in characteristic p as in [8],[9]. In [8], Joseph Chuang wrote some comments like the following:

> Raphaël Rouquier conjectured that the block $B^{2,\rho}$ of the symmetric group $\mathfrak{S}_{2w+|\rho|}$ on $2w+|\rho|$ letters with p-weight 2 and p-core ρ is Morita equivalent to the principal block of $\mathfrak{S}_p \wr \mathfrak{S}_2$ based on his observation of examples of matching decomposition matrices.

We may choose such ρ whose size as small as possible. In through this paper, we may use and fix this ρ. But, we can see the result is also true for any CKR e-core based on $[w:k]$-pairs.

Lemma 2.2.2 (Chuang-Kessar).

1. *Let λ be a partition with CKR e-core ρ and weight $v \leq w$. Suppose that in the (m-th beads) abacus configulation of λ there is a beads on the s-th runner in the t-th row. Then on any runner to the right of the s-th runner there are beads in all rows above the t-th row. If $v < w$, then on any runner to the right of the s-th runner there are beads in all rows on or above the t-th row.*

2. *Let λ be a partition with CKR e-core ρ and weight $v \leq w$, and let μ be a partition with CKR e-core ρ and weight $v-1$. If $\mu \subseteq \lambda$ then there exists s with $0 \leq s \leq e-1$ such that $\mu^{(i)} = \mu^{(i)}$ for $i \neq s$ and $\mu^{(s)} \subseteq \lambda^{(s)}$ with $|\mu^{(s)}| = |\lambda^{(s)}|-1$. Moreover the complement of the Young diagram of μ in that of λ is the Young diagram of the hook partition $(s+1, 1^{e-s-1})$.*

Chapter 3

Modular representations of finite groups

We would like to recall some important terminologies of modular representation theory of abstruct finite groups which are required to read this article. For the other terminologies we refer, for example, [1],[2],[46].

3.1 Brauer homomorphism

Let M be an $\mathcal{O}G$-module and D an ℓ-subgroup of G. The *Brauer homomorphism*

$$M(D) = M^E / \left(\sum_{E<D} \mathrm{Tr}_E^D(M^E) + J(\mathcal{O})M^D \right)$$

is defined by taking the quotient of the D-fixed points of M by relative traces from all proper subgroups E of D of the R-fixed points and then passing to the residue field.

$M(D)$ is a $\mathbf{k}\mathcal{N}_G(D)$-module. If M is a ℓ-permutation module, then $M(D) \neq 0$ if and only if M has an indecomposable direct summand which has a vertex containing D. In the case that $M = \mathcal{O}G$ and G acts by conjugation, we have $\mathcal{O}G(D) = \mathbf{k}C_G(D)$, and the natural quotient map

$$\mathrm{Br}_D^G : (\mathcal{O}G)^D \to \mathbf{k}C_G(D),$$

which may be given by the rule

$$\mathrm{Br}_D^G \left(\sum_{g \in G} a_g g \right) = \sum_{g \in C_G(D)} \overline{a_g} g,$$

is a homomorphism of $\mathcal{O}$-algebras, called the *Brauer homomorphism* with respect to D. If H is a subgroup of G containing D, and $j \in (\mathcal{O}G)^H$ is an idempodent, then $\mathcal{O}Gj$ is a direct summand of $\mathcal{O}G$ as an $\mathcal{O}[G \times H]$-module, we have that $(\mathcal{O}Gj)(\Delta D)$ is isomorphic to $\mathbf{k}C_G(D)\,\mathrm{Br}_D^G(j)$ as an $\mathbf{k}\,\mathcal{N}_{G\times H}(\Delta D)$-modules.

3.2 Stable equivalence

We refer [35] to define some terminologies on stable categories. Let A and B are block algebras over $\mathbf{k}$ which have no semisimple factor. (A and B are finite dimentional symmetric $\mathbf{k}$-algebras.) Following Broué[5, Sect.5], we define the following:

Definition 3.2.1. *We denote the category of finitely generated right A-modules by* mod-A. *We say that a (A, B)-bimodule M is exact if M is projective as a left A-module and a right B-module. Let M be an exact (A, B)-bimodule. The module M induces a stable equivalence of Morita type between A and B*

if

$$M \otimes_B M^\vee \cong A \oplus \text{ projective } (A,A)\text{-bimodules and}$$
$$M^\vee \otimes_A M \cong A \oplus \text{ projective } (B,B)\text{-bimodules}$$

Then, there is a useful theorem due to Linckelmann[41] as follows (see also[35, p.201],[54]):

Theorem 3.2.2. (**Linckelmann**) *Suppose that (A,B)-bimodule M induces stable equivalence of Morita type between A and B, and M has no projective direct summand . Then, the following holds:*

(1) $S \otimes_A M$ is indecomposable for any simple A-module S.

(2) If $S \otimes_A M$ is simple for any simple A-module S, then M induces Morita equivalence between A and B. i.e.

$$M \otimes_B M^\vee \cong A, \quad M^\vee \otimes_A M \cong B.$$

3.3 Rickard's Morita theory

We would like to recall some terminologies of derived categories of block algebras. J. Rickard parametrized equivalences of derived categories of module categories mod-R and mod-Q for given two rings R and Q similar to Morita.([52],[35].) For finte group algebras in [53] J. Rickard introduced a notion of **splendid** . Let A and B be block $\mathbf{k}$-algebras of some finite groups.

Let C be a bounded complex of (A,B)-bimodules such that each term of C is exact.

Definition 3.3.1. (**Rickard**) *The complex C induces a derived equivalence between* mod-A *and* mod-B *if*

$$\begin{array}{rcl} C \otimes_B C^\vee & \sim & A \quad (\textit{ homotopy equivalent }), \\ C^\vee \otimes_A C & \sim & B \quad (\textit{ homotopy equivalent }). \end{array}$$

Here, $C^\vee$ is the **k***-dual of C. In addition to the above condition, if A and B have a common defect group D and each term of C is a direct summand of some ℓ-permutation module induced from $\Delta(D)$, C is called a splendid Rickard complex and we say that A and B splendid Rickard equivalent. Moreover, we say that A and B are Puig equivalent, if A and B are splendid Rickard and Morita equivalent by an exact bimodule C.*

Remark

The above definition is Rickard's theorem. Originally, derived equivalence means the derived categories of module categories for given two rings are equivalent as triangulated categories. Moreover, Puig equivalence means the source algebras of two block algebras with defect group D of finite groups are isomorphic as interior D-algebras. For example, see [43, Section 1 Theorem] for the above reason.

Chapter 4

Representations of $GL_n(q)$

Representation theory of finite general linear group $GL_n(q)$ in $\mathbb{C}$ was established by J. A. Green [23] in 1955. In this article we deal ℓ-modular representation theory of $GL_n(q)$. We would like to recall some terminologies and known results on this field. Moreover, by Bonnafé and Rouquier's theorem, we may concentrate upon unipotent block of $GL_n(q)$. So, we would like to proceed on a short course of ℓ-representation theory of $GL_n(q)$ in order to study problems written in Introduction.

The terminologies and results that we will recall come mainly from Fong and Srinivasan [18], Dipper and James [15],[29],[30], and T. Jost [34], and will be quite essential and basic parts to investigate representations of $GL_n(q)$. Moreover, we might, all through this article, use these terminologies and results without stated.

We would like to define some notation. Let q be a prime power and let ℓ be a prime coprime to q. We use the following notation:

Definition 4.0.2. *For a prime power q and a prime number ℓ with $\ell \nmid q$,*

1. *let* $e_\circ$ *be the least positive integer* i *such that* ℓ *divides* $1+q+q^2+\cdots+q^{i-1}$*,*

2. *let* $e(q)=e$ *be the least positive integer* i *such that* ℓ *divides* q^i-1*.*

Note that $e=e_\circ$ *unless* ℓ *divides* $q-1$*, in which case* $e=1$ *and* $e_\circ=\ell$*. Moreover,* $1<e_\circ\leq\ell$ *and* e *divides* $\ell-1$*.* e *is the multiplicative order of* $q\cdot 1_{\mathbb{F}_\ell}$*.*

Let $(K,\mathcal{O},\mathbf{k})$ be a splitting ℓ-modular system for all subgroups of G where the characteristic of K (resp. $\mathbf{k}$) is 0 (resp. ℓ).

Let $G=GL_n(q)$ be a finite general linear group over the finite field $\mathbb{F}_q$ and let ℓ be a prime number coprime to q.

For an indeterminate x let $\mathcal{F}$ be the set of all monic irreducible polynomials in $\mathbb{F}_q[x]$ except a monomial x, and let $\mathcal{F}'$ be the subset of $\mathcal{F}$ of all polynomials whose roots have order prime to ℓ.

4.1 The theory of Fong-Srinivasan

In 1982 P. Fong and B. Srinivasan [18] classified the ℓ-blocks of $GL_n(q)$ for odd prime ℓ.

Let $R\in\{\mathcal{O},\mathbf{k}\}$. By the argument similar to parametrizing the K-characters of G, the ℓ-blocks of $R[G]$ may be indexed by pairs $(s,\boldsymbol{\lambda})$, where s is a semisimple ℓ'-element with primary decomposition $s=\prod_{f\in\mathcal{F}'}s_f$ and a multipartition $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_f)_{f\in\mathcal{F}}$ such that the partition $\boldsymbol{\lambda}_f$ is the $e(q^{\deg(f)})$-core of a partition of the multiplicity of f as an elementary divisor of s. For $R = \mathbf{k}$, we denote it by $B_{s,\boldsymbol{\lambda}}$. Moreover, let $\mathcal{O}B_{s,\boldsymbol{\lambda}} := \mathcal{O} \otimes_{\mathbf{k}} B_{s,\boldsymbol{\lambda}}$.

We concentrate unipotent blocks of $R[G]$. We denote $B_{1_n,\boldsymbol{\lambda}}$ by $B_{w,\lambda}$ if w and λ are such that

$$\boldsymbol{\lambda}_f = \begin{cases} \lambda & \text{if } f = x - 1, \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$w = \frac{n - |\lambda|}{e}.$$

Let

$$w = \sum_{i \geq 0} a_i \ell^i, \quad 0 \leq a_i < \ell$$

be the ℓ-adic expansion of w.

Let T be a Coxeter torus of $GL_e(q)$, and let T_ℓ be the Sylow ℓ-subgroup of T.

For a non-negative integer i, let W_i be the group of permutation matrices of degree ℓ^i, and K_i be a Sylow ℓ-subgroup of W_i. Here, K_i is isomorphic to the wreath product $C_\ell \wr \cdots \wr C_\ell$ of i copies of a cyclic group of order ℓ.

Then, by Fong-Olsson-Srinivasan [18, §3,§5], we know the defect group of $B_{w,\lambda}$.

Theorem 4.1.1. *The defect group D of unipotent block $B_{w,\lambda}$ is conjugate to*

$$I_{|\lambda|} \prod_{i \geq 0} (T_\ell \wr K_i)^{\times a_i}.$$

Here, I_m means the identity matrix of $GL_m(q)$.

Theorem 4.1.2 (Fong-Srinivasan). [*18*, (7.*A*)] *The character $\chi_{s,\boldsymbol{\nu}}$ belongs to the unipotent ℓ-block $B_{w,\lambda}$ if and only if*

1. *$s \sim_G y$ for some $y \in D$ and*

2. *for every $f \in \mathcal{F}$ except $x-1$, $\boldsymbol{\nu}_f$ has $e(q^{\deg(f)})$-core $\emptyset$.*

3. *$\boldsymbol{\nu}_{x-1}$ has e-core λ.*

Remark

In [18, (7.A)] Fong and Srinivasan dealt more general situation, say, arbitrary blocks of G indexed by some semisimple ℓ'-element of G where ℓ is odd. The case $\ell = 2$ in this context was done by M. Broué and L. Puig. Moreover, R. Dipper and G.D. James gave anothor proof of Fong-Srinivasan Theorem(Nakayama conjecture for $GL_n(q)$) [15, 7.13 Corollary].

4.1.1 Harish-Chandra induction and restriction

Let $G := GL_n(q)$ and P be a parabolic subgroup of G with Levi decomposition $P = LU_P$.

We denote the Harish-Chandra induction (resp. restriction) operator by R_L^G (resp. ${}^*R_L^G$).

Let $\chi_{s,\boldsymbol{\lambda}}$ and $\chi_{t,\boldsymbol{\mu}}$ be irreducible characters of G and L respectively. By [18, (1 B) and (1 C)] for example, we know the following:

Theorem 4.1.3. *1. if $s \not\sim_G t$, then*

$$\left(\chi_{s,\boldsymbol{\lambda}},\ R_L^G(\chi_{t,\boldsymbol{\mu}})\right)_G = \left({}^*R_L^G(\chi_{s,\boldsymbol{\lambda}}),\ \chi_{t,\boldsymbol{\mu}}\right)_L = 0 \textit{ and}$$

2. *if $s = t$ then*

$$\left({}^*R_L^G(\chi_{s,\boldsymbol{\lambda}}),\ \chi_{t,\boldsymbol{\mu}}\right)_L = \left({}^*R_{C_L(s)}^{C_G(s)}(\chi_{\boldsymbol{\lambda}}),\ \chi_{\boldsymbol{\mu}}\right)_{C_L(t)}.$$

3.

$$\left({}^*R_L^G(\chi_\lambda), \chi_\mu\right)_L = \left(\mathrm{Res}_{W_L}^{W_G}(\phi_\lambda), \phi_\mu\right)_{W_L}$$

where W_G and W_L are Weyl groups of G and L respectively and the characters ϕ_λ and ϕ_μ of W_G and W_L correspond to unipotent characters χ_λ and χ_μ respectively.

4.2 The theory of Dipper-James

Definition 4.2.1. *Let R be an integral domain , and let v be an unit in R. The symmetric group on k letters is denoted by $\mathfrak{S}_k$. Then, the* Iwahori-Hecke algebra *$\mathcal{H}_{R,v,k} = \mathcal{H}_{R,v}(\mathfrak{S}_k)$ of type $\mathbf{A}_{k-1}$ is a free R-module with basis $\{T_w \mid w \in \mathfrak{S}_k\}$ with the multiplication being defined by the rule that if $t = (i, i+1)$ is a basic transposition in $\mathfrak{S}_k$ and $w \in \mathfrak{S}_k$, then*

$$T_w T_t = \begin{cases} T_{wt} & \text{if } l(wt) > l(w), \\ vT_{wt} + (v-1)T_w & \text{otherwise }. \end{cases}$$

Here, $l(x)$ denotes the length function of $\mathfrak{S}_k$.

For $\lambda \models k$, let $\mathfrak{S}_\lambda$ be the standard parabolic subgroup of $\mathfrak{S}_k$, and let

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w, \quad y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-v)^{-l(w)} T_w$$

Let $S_{R,v}^\lambda$ be the Specht $\mathcal{H}_{R,v}(\mathfrak{S}_n)$-module corresponding to $\lambda \vdash n$. If R is a field and $v = 1_R$ then we simply denote it by S_R^λ.

Theorem 4.2.2 (James/Gelfand). *[29] Let $G := GL_d(q)$ and $\mathbf{G} := GL_d(\overline{\mathbb{F}_q})$. Let U be the upper unitriangular group of G, and $\mathbf{T}_0$ be the group of diagonal matrices of $\mathbf{G}$ (a maximally split F-stable maximal torus of $\mathbf{G}$). Let w be the longest element of $\mathfrak{S}_d$. Let $\mathbf{T}_w$ be a torus of $\mathbf{G}$ obtained from $\mathbf{T}_0$ by twisting with w. ($\mathbf{T}_w^F$ is a Coxeter torus of G.) Take $\theta \in \mathbf{T}_w^F$ as in generic position. Let φ^+ be a non-trivial linear character of the additive group $\mathbb{F}_q$. We can obtain a non-trivial linear character φ of U by*

$$\varphi(u) := \varphi^+\left(\sum_{i=1}^{d-1} u_{i,i+1}\right)$$

where $u_{i,i+1}$ is the $(i, i+1)$-entry in $u \in U$. Let

$$e_{U,\varphi} = \frac{1}{|U|}\sum_{u\in U}\varphi(u^{-1})u.$$

Moreover, let G^-_{d-1} be the subgroup of $GL_d(q)$ consisting of elements whose $(1,1)$-entry is 1, and whose $(i,1)$-entry is 0 for $i > 1$. Then, for any $g, h \in G$,

$$\frac{1}{|U|}\sum_{x\in G^-_{d-1}} R^{\mathbf{G}}_{\mathbf{T}_w}(\theta)(e_{U,\varphi}gx)R^{\mathbf{G}}_{\mathbf{T}_w}(\theta)(x^{-1}he_{U,\varphi}) = R^{\mathbf{G}}_{\mathbf{T}_w}(\theta)(gh\ e_{U,\varphi}).$$

Let $1 = g_1, g_2, \ldots, g_N$ be right coset representatives for U in G^-_{d-1} where $N = (q-1)(q^2-1)\cdots(q^{d-1}-1)$. So, $\{[U]g_i\}$ is a basis of $\mathbf{k}_U \uparrow^{G^-_{d-1}}$. Then, the space $\mathbf{k}_U \uparrow^{G^-_{d-1}}$ is a simple $\mathbf{k}G$-module by the following G-action:

$$[U]g_jg = \sum_{i=1}^{N}\overline{R^{\mathbf{G}}_{\mathbf{T}_w}(\theta)(g_jgg_i^{-1}e_{U,\varphi})}[U]g_i$$

for $g \in G(1 \leq j \leq N)$. Following James, let $M_{\mathbf{k}}(s,(1))$ be the above simple $\mathbf{k}G$-module where s is a root of the minimal polynomial of $\theta \in G^* \cong G$ other than 1.

Theorem 4.2.3 (James). *[29] $M_{\mathbf{k}}(s,(1)) \cong M_{\mathbf{k}}(r,(1))$ if and only if ℓ-regular parts of s and r have the same minimal polynomial over $\mathbb{F}_q$.*

Definition 4.2.4. *Supose that $n = dk$ and s have degree d over $\mathbb{F}_q$. Let*

$$M_R(s,(1^k)) := R_{L_{(d^k)}}^{GL_n(q)}(X) := \mathrm{Ind}_{P_{(d^k)}}^{GL_n(q)} \mathrm{Inf}_{L_{(d^k)}}^{P_{(d^k)}}(X)$$

where $X = M_R(s,(1))^{\boxtimes k}$ is a cuspidal $R[GL_d(q)^{\times k}]$-module.

Theorem 4.2.5 (Dipper). *Suppose that $R \in \{K, \mathcal{O}, \mathbf{k}\}$ and $n = dk$ and s has degree d. Then,*

$$\mathrm{End}_{R[GL_n(q)]}(M_R(s,(1^k))) \cong \mathcal{H}_{R,q^d}(\mathfrak{S}_k)$$

Definition 4.2.6 (James). *[29] For $\lambda \models k$ and $R \in \{K, \mathbf{k}\}$, let*

$$M_R(s,\lambda) := x_\lambda M_R(s,(1^k)), \quad S_R(s,\lambda) := M_R(s,\lambda)E_R(d\lambda')R[GL_n(q)]$$

where $E_R(\mu)$ is the idempodent of the (degenerate) Gelfand-Graev representation over R corresponding to the unipotent class C_μ.

We also denote $S_R(s,\lambda)$ by $S_R(\lambda)$ if $s = 1$.

By the argument similar to constructing Specht modules of the symmetric groups, James proved the following:

Theorem 4.2.7 (James). *[29] Let G be $GL_{dk}(q)$. Suppose that s has degree d over $\mathbb{F}_q$ and $\lambda, \mu \vdash k$.*

1. *$S_K(s,\lambda)$ is a simple KG-module and $S_{\mathbf{k}}(s,\lambda)$ is an ℓ-modular reduction.*

2. $S_{\mathbf{k}}(s,\lambda)$ *has a unique maximal submodule* $S_{\mathbf{k}}(s,\lambda)^{\max}$.

3. $D_{\mathbf{k}}(s,\lambda) := S_{\mathbf{k}}(s,\lambda)/S_{\mathbf{k}}(s,\lambda)^{\max}$ *is simple.*

4. $D_{\mathbf{k}}(s,\lambda) \cong D_{\mathbf{k}}(s,\mu)$ *if and only if* $\lambda = \mu$.

5. *There exist non-negative integers* $d^{s}_{\lambda,\mu}$ *such that* $S_{\mathbf{k}}(s,\lambda)$ *and* $\bigoplus_{\mu} d^{s}_{\lambda,\mu} D_{\mathbf{k}}(s,\mu)$ *have the same composition factors. Moreover, these numbers have the property* $d^{s}_{\lambda,\lambda} = 1$ *and* $d^{s}_{\lambda,\mu} \neq 0$ *only if* $\mu \trianglerighteq \lambda$.

6. $H(S_{\mathbf{k}}(s,\lambda)) = S^{\lambda}_{\mathbf{k},\overline{q}}$. *Here* H *is the Hecke functor.*

Definition 4.2.8. *Let* $\mathbf{D}^{s}_{k,\overline{q},\ell}$ *be the* $\mathcal{P}(n)$ *by* $\mathcal{P}(n)$ *matrix* $(d^{s}_{\lambda,\mu})$ *of decomposition numbers of* $GL_n(q)$ *which appears in Theorem 4.2.7.6. Here,* $\mathcal{P}(n)$ *is the cardinality of the set of partitions of* n*. Moreover, we write* $d_{\lambda,\mu}$ *instead of* $d^{1}_{\lambda,\mu}$.

Theorem 4.2.9 (Dipper-James). [15, *Theorem* 6.1, 6.2]
Suppose that $dk = n$.

1. *If* s_1 *and* s_2 *are non-zero roots of monic irreducible polynomials of degree* d *over* $\mathbb{F}_q$*, then* $\mathbf{D}^{s_1}_{k,\overline{q},\ell} = \mathbf{D}^{s_2}_{k,\overline{q},\ell}$.

2. *If* s *is non-zero roots of a monic irreducible polynomial of degree* d *over* $\mathbb{F}_q$*, then* $\mathbf{D}^{s}_{k,\overline{q},\ell} = \mathbf{D}^{1}_{k,\overline{q^d},\ell}$.

The above theorem means that to determine the decomposition numbers of unipotent characters of finite general linear groups becomes an essential part.

In order to deal decomposition matrices, James' conjecture, and our hope 11.2.3, we would like to define Dipper-James q-Schur algebra.

Definition 4.2.10 (Dipper-James). [15], [16] *Suppose that R is an integral domain with a unit v and $\mathcal{H} = \mathcal{H}_{R,v}(\mathfrak{S}_n)$. Let*

$$\boldsymbol{\mathcal{S}}_{\mathbf{k}}(n,r) = \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda\,\mathcal{H}\,.\right)$$

We call $\boldsymbol{\mathcal{S}}_{R,v}(n,r)$ Dipper-James v-Schur algebra for $\Lambda(n,r)$ over R with a parameter v. It is known that if $n \geq r$, then $\boldsymbol{\mathcal{S}}_{R,v}(n,r)$ is Morita equivalent to $\mathrm{End}_{\mathcal{H}}\,(\bigoplus_{\lambda\vdash r} x_\lambda\,\mathcal{H})$.

Theorem 4.2.11 (Dipper-James). [15], [16] *Let $R \in \{K, \mathcal{O}, \mathbf{k}\}$, $G := GL_{dk}(q)$ and $M := M_R(s,(1^k))$. Then,* $\mathrm{End}_{RG}(\bigoplus_{\lambda\vdash k} x_\lambda M)$ *is Morita equivalent to $\boldsymbol{\mathcal{S}}_{R,q^d}(k,k)$.*

In order to explain James'conjecture we described decomposition numbers for Dipper-James algebras and their relation with $\mathbf{k}GL_n(q)$.

Theorem 4.2.12 (Dipper-James). [15] *Keep the notation in Definition 4.2.10. Then,*

$$d^s_{\lambda,\mu} = [W^{\lambda'} : L^{\mu'}] = \tilde{d}_{\lambda',\mu'}$$

for any $\lambda,\mu \vdash k$. Here, $W^{\lambda'}$ is a Weyl module corresponding to λ', $L^{\mu'}$ is simple module corresponding to μ', and $\tilde{d}_{\lambda',\mu'}$ is its decomposition number. (The details are also described in [45, Chapter4].)

4.3 Scopes' $[w{:}k]$-pairs

In this section we would like to describe a notion of Scopes' pairs [55].

Definition 4.3.1 (Scopes). *For a positive integer m and an m-core $\tau = (\tau_1, \ldots, \tau_r)$, let Γ be the r-element β-set for τ, and suppose that when Γ is diplayed on an abacus with m-runners there are k more than beads in the i-th column than in the $(i-1)$-th column. Let m-core $\overline{\tau}$ be displayed by an r-element β-set $\overline{\Gamma}$ satisfying*

$$\begin{aligned} \overline{\Gamma}_j &= \Gamma_j \qquad \textit{for } j \neq i, i-1 \\ \overline{\Gamma}_i &= \Gamma_{i-1} \\ \overline{\Gamma}_{i-1} &= \Gamma_i, \end{aligned}$$

where Γ_j is the number of beads on the j-th runner in the abacus configulation for Γ. In these situation, we shall say that $\Lambda_{m,w,\tau}$ and $\Lambda_{m,w,\overline{\tau}}$ form a Scopes $[w:k]$-pair.

Scopes proved the following:

Theorem 4.3.2 (Scopes). *[55] If $\Lambda_{p,w,\tau}$ and $\Lambda_{p,w,\nu}$ form a $[w:k]$-pair with $k \geq w$, then p-blocks $B^{w,\tau}$ and $B^{w,\nu}$ of symmetric groups are Puig equivalent.*

By Jost we also know the following:

Theorem 4.3.3 (Jost). *[34] If $\Lambda_{e,w,\tau}$ and $\Lambda_{e,w,\nu}$ form a $[w:k]$-pair with $k \geq w$, then unipotent ℓ-blocks $B_{w,\tau}$ and $B_{w,\nu}$ are Puig equivalent.*

Remark

Jost proved the above theorem in more general setting up, say, for any elementary divisor of semisimple ℓ-regular elements. Since we concentrate upon investigations of unipotent blocks in this article, we need not require his general version. He also discussed blocks of Iwahori-Hecke algebras of type **A** and **B**.

Moreover, Kessar has also discussed p-blocks of Weyl groups of type **B** and **D** in this context. Recently, Hiss and Kessar also discussed ℓ-blocks of classical groups in this context.

Moreover, Scopes pairs on quantized Fock space for Kac-Moody algebra $U_v(\widehat{\mathfrak{gl}})$ were also discussed by Leclerc.

Chapter 5

Chuang-Kessar type theorem

Let G be $GL_{ew+|\rho|}(q)$. For $i = 0, \ldots, w$, let G_i be the standard Levi subgroup corresponding to the composition $(e^i, e(w-i) + |\rho|)$.

Let $\mathcal{F}$ be the set of monic irreducible polynomials $f = f(x)$ in the polynomial ring $\mathbb{F}_q[x]$ excepting x.

We choose one root of each $f \in \mathcal{F}$ and call this set of roots $\mathcal{C}$. We can order $\mathcal{C}$ suitably.

Let $\mathcal{C}_{b_0}$ be the set of ℓ-elements a in $\mathcal{C}$ such that a is a root of an elementary divisor of a semisimple element t with $\chi_{t,\lambda}$ lying in KG_0b_0 for some λ.

Proposition 5.0.4. *If $\ell > w$, then every element of $\mathcal{C}_{b_0}$ except 1 has degree e. (i.e. If $a \in \mathcal{C}_{b_0}$, then $a \in \mathbb{F}_{q^e}$.)*

proof

Take an element a of $\mathcal{C}_{b_0} \backslash \{1\}$. Then, there exists an $n = e\ell^i > 0$ for some i such that $a \in (\mathbb{F}_{q^n})^*$. Since b_0 is a block of $GL_{ew+|\rho|}(q)$, a is a root of an elementary divisor of a semisimple element t whose rank is $ew + |\rho|$. If

G-conjugate of t has a canonical form

$$(s_1)^{\times n_1} \times (s_2)^{\times n_2} \times \cdots \times (s_N)^{\times n_N} \tag{5.1}$$

where $\deg(s_1)n_1 + \cdots \deg(s_N)n_N = ew + |\rho|$, $s_1 = 1, s_2, \ldots$ are pairwise different elements of $\mathcal{C}$ and (s_j) is the companion matrix of the minimum polynomial of s_j over $\mathbb{F}_q$ (in fact a is just one of s_j's), then $n_1 \geq |\rho|$. Indeed, for any character $\chi_{t,\lambda}$ lying in KG_0b_0, e-core of $\lambda^{(1)}$ is ρ. Therefore, the rank of any (s_i) is less than ew. So, we may assume that $a \in (\mathbb{F}_{q^n})^*$ with $n \leq ew$. Hence, $i = 0$.

□

We assume that $char(\mathbf{k}) = \ell > w$. Choose a prime power q with $e(q) = e$. Just mimicking Chuang and Kessar's setting up, we define the following groups and blocks:

$G := GL_{ew+|\rho|}(q)$.

$B^{w,\rho} :=$ the unipotent ℓ-block of $\mathbf{k}G$ with e-weight w and e-core ρ.

$D :=$ a defect group of $B^{w,\rho}$.

$N_w := GL_e(q) \wr \mathfrak{S}_w \supset D$.

$L_{(e^w,|\rho|)} := GL_e(q) \times \cdots \times GL_e(q) \times GL_r(q)$.

$H_w := (GL_e(q) \wr \mathfrak{S}_w) \times GL_{|\rho|}(q) \supset \mathcal{N}_G(D)$.

$\mathcal{O}H_wf$:=the Brauer correspondent of $B_{w,\rho}$ in H_w.

Definition 5.0.5. *Let e_{w-i} be the block idempotent of $\mathcal{O}B_{w-i,\rho}$. Let a be the principal block idempotent of $\mathcal{O}GL_e(q)$. Let*

$$b_i := \overbrace{a \boxtimes \cdots \boxtimes a}^{i} \boxtimes e_{w-i} \in \mathcal{O}[\overbrace{GL_e(q) \times \cdots \times GL_e(q)}^{i} \times GL_{e(w-i)+|\rho|}(q)].$$

By the lifting theorem of idempodent, for an idempotent $b \in \mathcal{O}G$ for some finite group we may also use its reduction by $b\mathbf{k}G$. Moreover, for a block algebra A we also denote its block idempotent by 1_A.

By [18, §3,4], we have the following statement which is analogous to [9, Lemma 5]:

Lemma 5.0.6. *(1) For $i = 0, \ldots, w$, D is a defect group of $\mathcal{O}G_ib_i$.*

(2) For $i = 0, \ldots, w$, $\mathrm{Br}_D^G(b_i) = 1_{\mathbf{k}D} \otimes e_0$.

(3) H_w stabilizes b_w, and $\mathcal{O}H_wb_w$ is a block which is Morita equivalent to the principal block of $\mathcal{O}[(GL_e(q)) \wr \mathfrak{S}_w]$.

(4) $\mathcal{O}H_wf$ is indecomposable as an $\mathcal{O}[H_w \times L_{(e^w,|\rho|)}]$- module with vertex $\Delta(D)$.

(5) $\mathrm{rank}_{\mathcal{O}}(\mathcal{O}H_wb_w) = w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}L_{(e^w,|\rho|)}b_w)$.

proof

(1): Clear by Theorem 4.1.1.

(2): Clear by the proof of [18, p.126, Theorem (3C)].

(3): $H_w \cong (GL_e(q) \wr \mathfrak{S}_w) \times GL_{|\rho|}(q)$ and $\mathcal{O}H_w b_w \cong \mathcal{O}B_0(GL_e(q) \wr \mathfrak{S}_w) \boxtimes \mathcal{O}B_{0,\rho}$. Here, $B_0((GL_e(q)) \wr \mathfrak{S}_w)$ is the principal block of N_w. $B_{0,\rho}$ is a defect zero block of $GL_{|\rho|}(q)$. So, we are done.

(4) and (5): Copy the proof of [9, Lemma 5].

□

Let $\hat{\boldsymbol{X}}_w(q)$ be the Green correspondent of $\mathcal{O}Gb_0$ with respect to $(G \times G, \Delta(D), G \times H_w)$.

Theorem 5.0.7. (**An analogy of Chuang-Kessar theorem**)
$\hat{\boldsymbol{X}}_w(q)$ induces a Morita equivalence between $\mathcal{O}B_{w,\rho}$ and $(\mathcal{O}B_0(H_w) \otimes \mathcal{O}B_{0,\rho})$.

Corollary 5.0.8. *If there exists a sequence of e-cores $(\rho = \tau^0, \tau^1, \tau^2, \cdots, \tau^s = \upsilon)$ for some intger s such that τ^j and τ^{j+1} form a $[w : k_j]$-pair with $k_j \geq w$, then Broué's conjecture holds for $B_{w,\upsilon}$.*

By [43],[34] and Theorem 5.0.7 this immediately follows.

Proof of Theorem 5.0.7

Let $\boldsymbol{Y}(q) := \mathcal{O}G(q)b_0 \cdots b_w$. By the argument similar to [9, p.6-7], it suffices to show that

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_{G(q)}(\hat{\boldsymbol{X}}_w(q))) \leq \mathrm{rank}_{\mathcal{O}}(\mathrm{End}_{G(q)}(\boldsymbol{Y}(q))) \leq w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}L_{(e^w,|\rho|)}b_w). \tag{5.2}$$

As in [9, p.8 (4) and (5)], we know

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_{G(q)}(\boldsymbol{Y}(q))) = \sum_{\chi \in \mathrm{Irr}(KG_0 b_0)} \chi(b_w \cdots b_0)^2 \tag{5.3}$$

and

$$\chi(b_w \cdots b_0) = b_w {}^*R_{G_w}^{G_{w-1}}(\cdots b_2 {}^*R_{G_2}^{G_1}(b_1 {}^*R_{G_1}^{G_0}(\chi)))(1). \tag{5.4}$$

For $i = 0, \ldots, w$ let $\mathbb{S}_i$ be the set of semisimple ℓ-elements s whose canonical rational forms are

$$(s_1)^{\times n_1} \times (s_2)^{\times n_2} \times \cdots \times (s_N)^{\times n_N}$$

such that $s_j \in \mathcal{C}_{b_0}, s_1 = 1, \sum_{j=1}^{N} \deg(s_j)n_j = e(w-i) + |\rho|, s_1 < s_2 < \cdots < s_N$, and $n_1 \geq |\rho|$. Note that by [18, p.147, Theorem (7A)], or [15, p.47, 7.11.Theorem] and Proposition 5.0.4

$$\{\chi_{s,\lambda} \mid s \in \mathbb{S}_i, e\text{-core of } \lambda_1 \text{ is } \rho\}$$

is a complete set of inequivalent irreducible $e_{w-i}KGL_{|\rho|+e(w-i)}(q)$-characters.

For $\zeta \in \mathcal{C}_{b_0}\backslash\{1\}$, we write, a character of $GL_e(q)$, ν_ζ as follows:

$$\nu_\zeta := \chi_{(\zeta),(1)}$$

where (ζ) is the companion matrix of the minimum polynomial of ζ.

Similarily for $1 \in \mathcal{C}_{b_0}$ and $j = 0, \ldots, e-1$, we write $\nu_{(1,j)}$ as follows:

$$\nu_{(1,j)} := \chi_{1_e,(j+1,1^{e-j-1})}.$$

Let $\mathrm{Irr}(KG_ib_i)$ be the set of irreducible characters of KG_i lying in KG_ib_i.

Define a map u from $\mathbb{S}_0$ to $\{0, 1, \ldots, w\}$ as follows:

$$u(s) := \frac{n_1 - |\rho|}{e} \text{ if } s = (1)^{\times n_1} \times (s_2)^{\times n_2} \times \cdots \times (s_N)^{\times n_N}$$

where $1, s_2, s_3, \dots, s_N$ are pairwise different elements of $\mathcal{C}_{b_0}$.

Let $\mathrm{Irr}(KG_ib_i, s)$ be the set of characters in $s_{G_i} \cap \mathrm{Irr}(KG_ib_i)$.

For a semisimple ℓ-element $s = (1)^{\times|\rho|+eu(s)} \times (s_2)^{\times n_2} \times \cdots \times (s_N)^{\times n_N} \in \mathbb{S}_0$, let I_s be the set of multicompositions $c = (c^{(1)}, \dots, c^{(e)})$ of w satisfying

$c^{(1)} = (c_{(1,0)}, c_{(1,1)}, \dots, c_{(1,e-1)}), c^{(i)} = (c_{s_i}), c_{s_i} = n_i$ for $i = 2, \dots, N$, $\sum_j c_j^{(1)} = u(s)$, and $\sum_k |c^{(k)}| = w$.

Let $\mathbb{I}_1 := \{(1,j) \mid j = 0, \dots e-1\}, \mathbb{I}_e := \mathcal{C}_{b_0} \backslash \{1\}$, and $\mathbb{I} := \mathbb{I}_1 \cup \mathbb{I}_e$.

For $i = 0, \dots, w$, semisimple ℓ-element s of G and $c \in I_s$, we define $\mathrm{Irr}(KG_ib_i, s, c)$ to be the set of characters $\nu_{x_1} \times \cdots \times \nu_{x_i} \times \chi_{\overline{s}, \boldsymbol{\lambda}} \in \mathrm{Irr}(KG_ib_i, s)$ such that for $x = (1,j) \in \mathbb{I}_1$,

$$| \boldsymbol{\lambda}_1^{(j)} | + \sharp\{\gamma \mid x_\gamma = x\} = c_x$$

and for $x \in \mathbb{I}_e$,

$$|\boldsymbol{\lambda}_x| + \sharp\{\gamma \mid x_\gamma = x\} = c_x$$

Here, for $s \in \mathcal{C}$ and multipartition $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}_s$ denotes the partition of s-part of $\boldsymbol{\lambda}$.

For $i = 0, \dots, w$ we have a disjoint union

$$\mathrm{Irr}(KG_ib_i) = \coprod_s \coprod_{c \in I_s} \mathrm{Irr}(KG_ib_i, s, c).$$

We mention that for a fixed $c \in I_s$, there are

$$\frac{w!}{\prod_{x \in \mathbb{I}} c_x!}$$

characters in $\mathrm{Irr}(KG_wb_w, s, c)$ and all have the same degree.

Suppose that $0 \leq i \leq w-1$ and $\chi = \nu_{x_1} \times \cdots \times \nu_{x_i} \times \chi_{t,\boldsymbol{\lambda}} \in \mathrm{Irr}(KG_i b_i)$. Then, using [18, p.117, Proposition (1B)], the Littlewood-Richardson rule (see, for example, [31, p.93, Theorem 2.8.13]) and [9, Lemma 4], we get

$$b_{i+1}{}^{*}R^{G_i}_{G_i+1}(\chi)$$

$$= \sum_{x \in \mathbb{I}_1} \sum_{\boldsymbol{\mu},\ \bar{t} \in \mathbb{S}_{w-i-1}, (1_e \times \bar{t}) = t} \nu_{x_1} \times \cdots \times \nu_{x_i} \times \nu_x \times \chi_{\bar{t},\boldsymbol{\mu}}$$

$$+ \sum_{x \in \mathbb{I}_e} \sum_{\boldsymbol{\mu},\ \bar{t} \in \mathbb{S}_{w-i-1}, ((x) \times \bar{t}) \sim t} \nu_{x_1} \times \cdots \times \nu_{x_i} \times \nu_x \times \chi_{\bar{t},\boldsymbol{\mu}} \qquad (5.5)$$

where $\boldsymbol{\mu}$ runs over all multipartitions:

$x = (1,k) \in \mathbb{I}_1$, e-core of $\boldsymbol{\mu}_1$ is ρ, $\boldsymbol{\mu}_1^{(j)} = \boldsymbol{\lambda}_1^{(j)}$ for $j \neq k$, $\boldsymbol{\mu}_1^{(k)} \subseteq \boldsymbol{\lambda}_1^{(k)}$ with $|\boldsymbol{\mu}_1^{(k)}| = |\boldsymbol{\lambda}_1^{(k)}| - 1$, and $\boldsymbol{\mu}_s = \boldsymbol{\lambda}_s$ for any $s \in C_{b_0}$

or,

$x \in \mathbb{I}_e$, $\boldsymbol{\mu}_s = \boldsymbol{\lambda}_s$ for any $s \neq x$ and $\boldsymbol{\mu}_x \subseteq \boldsymbol{\lambda}_x$ with $|\boldsymbol{\mu}_x| = |\boldsymbol{\lambda}_x| - 1$.

Note that if $\chi \in \mathrm{Irr}(KG_i b_i, s, c)$ then every irreducible component of $b_{i+1}{}^{*}R^{G_i}_{G_i+1}(\chi)$ is in $\mathrm{Irr}(KG_{i+1}b_{i+1}, s, c)$ by [18, p.117 Proposition (1C)].

Fix $\chi_{s,\boldsymbol{\lambda}} \in \operatorname{Irr}(KG_0b_0, s, c)$. For each $\chi_{s,\boldsymbol{\mu}} \in \operatorname{Irr}(KG_wb_w, s, c)$, we would apply equation (5.5) to determine that the multiplicity of $\chi_{s,\boldsymbol{\mu}}$ as an irreducible component of

$$b_w{}^*R_{G_w}^{G_{w-1}}(\cdots b_2{}^*R_{G_2}^{G_1}(b_1{}^*R_{G_1}^{G_0}(\chi_{s,\boldsymbol{\lambda}}))\cdots)$$

is

$$\prod_{i=0}^{e-1} f(\boldsymbol{\lambda}_1^{(i)}) \cdot \prod_{x\in\mathbb{I}_e} f(\boldsymbol{\lambda}_x).$$

Here, for any partition α of k, $f(\alpha)$ means the dimention of the irreducible character χ^α of $\mathfrak{S}_k$. We also knows that

$$\sum_{\alpha\vdash k} f(\alpha)^2 = k!.$$

So, we know

$$\chi_{s,\boldsymbol{\lambda}}(b_w\cdots b_0) = \prod_{i=0}^{e-1} f(\boldsymbol{\lambda}_1^{(i)}) \cdot \prod_{x\in\mathbb{I}_e} f(\boldsymbol{\lambda}_x) \cdot \sum_{\chi_{s,\boldsymbol{\mu}}\in\operatorname{Irr}(KG_wb_w,s,c)} \chi_{s,\boldsymbol{\mu}}(1). \tag{5.6}$$

$$\begin{aligned}
&\sum_{\chi_{s,\boldsymbol{\lambda}}\in\operatorname{Irr}(KG_0b_0,s,c)} \chi_{s,\boldsymbol{\lambda}}(b_w\cdots b_0)^2 \\
&= \sum_{\chi_{s,\boldsymbol{\lambda}}\in\operatorname{Irr}(KG_0b_0,s,c)} \prod_{i=0}^{e-1} f(\boldsymbol{\lambda}_1^{(i)})^2 \cdot \prod_{x\in\mathbb{I}_e} f(\boldsymbol{\lambda}_x)^2 \cdot \left(\sum_{\chi_{s,\boldsymbol{\mu}}\in\operatorname{Irr}(KG_wb_w,s,c)} \chi_{s,\boldsymbol{\mu}}(1)\right)^2 \\
&= \prod_{i=0}^{e-1}\left(\sum_{\alpha_i\vdash c_{(1,i)}} f(\alpha_i)^2\right)\cdot\prod_{x\in\mathbb{I}_e}\left(\sum_{\boldsymbol{\lambda}_x\vdash w} f(\boldsymbol{\lambda}_x)^2\right)\cdot\frac{w!}{\prod_{x\in\mathbb{I}} c_x!}\cdot\left(\sum_{\chi_{s,\boldsymbol{\mu}}\in\operatorname{Irr}(KG_wb_w,s,c)} \chi_{s,\boldsymbol{\mu}}(1)^2\right) \\
&= w!\cdot\sum_{\chi_{s,\boldsymbol{\mu}}\in\operatorname{Irr}(KG_wb_w,s,c)} \chi_{s,\boldsymbol{\mu}}(1)^2.
\end{aligned}$$

Summing the above equation over all $s \in \mathcal{C}$ and $c \in I_s$, using equation (5.2), we get

$$\mathrm{rank}_{\mathcal{O}}(\mathrm{End}_{G(q)}(\boldsymbol{Y}(q))) = w! \cdot \sum_{\varphi \in \mathrm{Irr}(KG_w b_w)} \varphi(1)^2 = w! \cdot \mathrm{rank}_{\mathcal{O}}(\mathcal{O}L_{(e^w, |\rho|)} b_w).$$

$\square$

Chapter 6

Irreducible modules of $GL_e(q) \wr \mathfrak{S}_w$

6.1 Indices of the simple $B_0(GL_e(q) \wr \mathfrak{S}_w)$-modules

Let $N_w := GL_e(q) \wr \mathfrak{S}_w$. The principal block $B_0(GL_e(q))$ has e non-isomorphic irreducible modules $\{ D_{\mathbf{k}}(\mu) \mid \mu$ is a hook partition of $e\}$. Fix $R \in \{K, \mathbf{k}\}$. Let $\boldsymbol{n}$ be an e-tuple non-negative integer of w. i.e. $\sum_{i=1}^{e} \boldsymbol{n}_i = w$. $S_R(\boldsymbol{n}) := \displaystyle\boxtimes_{i=1}^{w} (S_R(\nu_i)^{\boxtimes \boldsymbol{n}_i})$ is an $R[GL_e(q)^{\times w}]$-module. In particular, $S_K(\boldsymbol{n})$ is a simple $K[GL_e(q)^{\times w}]$-module. The parabolic subgroup $\mathfrak{S}_{\boldsymbol{n}}$ act on $S_R(\boldsymbol{n})$.

So, $S_R(\boldsymbol{n})$ is an $R[L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}]$-module. $\mathrm{Ind}_{L_{(e^w)}}^{L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}} S_R(\boldsymbol{n})$ is decomposed into $\displaystyle\bigoplus_{\boldsymbol{\lambda} \vdash \boldsymbol{n}} (S_R(\boldsymbol{n}) \otimes_R (\dim_R S_R^{\boldsymbol{\lambda}}) \cdot S_R^{\boldsymbol{\lambda}})$ where S_R^{μ} means the Specht module of $R[\mathfrak{S}_{|\mu|}]$ corresponding to μ, $S_R^{\boldsymbol{\lambda}} = \displaystyle\boxtimes_i S_R^{\boldsymbol{\lambda}_i}$ and $S_R(\boldsymbol{n}) \otimes_R S_R^{\boldsymbol{\lambda}}$ is the inner tensor product of $R[L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}]$-modules $S_R(\boldsymbol{n})$ and $S_R^{\boldsymbol{\lambda}}$. (Since $L_{(e^w)} \lhd L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}$, every module for $R[\mathfrak{S}_{\boldsymbol{n}}]$ may be regarded as a $R[L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}]$-module by allowing $L_{(e^w)}$ to act trivially.) Let

$$T_R^{\boldsymbol{\lambda}_i} = \begin{cases} S_R^{\boldsymbol{\lambda}_i} & \text{if } i+e: \text{ even} \\ S_R^{\boldsymbol{\lambda}_i'} & \text{if } i+e: \text{ odd.} \end{cases}$$

Here, $\boldsymbol{\lambda}_i'$ is the conjugate partition of $\boldsymbol{\lambda}_i$. Let $T_R^{\boldsymbol{\lambda}} = \boxtimes {}_i T_R^{\boldsymbol{\lambda}_i}$.

For $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ let $U_R(\boldsymbol{\lambda})$ be $\mathrm{Ind}_{L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}}^{GL_e(q) \wr \mathfrak{S}_w} \left(S_R(\boldsymbol{n}) \otimes T_R^{\boldsymbol{\lambda}} \right)$, and Let $U_{\mathbf{k}}(\boldsymbol{\lambda})^\rho$ be the $R[H_w]$-module $U_R(\boldsymbol{\lambda}) \boxtimes_R S_R(\rho)$.

Moreover, one can construct a module by using

$$\{ \ D_{\mathbf{k}}(\mu) \mid \mu \text{ is a hook partition of } e\}$$

instead of $\mathbf{k}[GL_e(q)]$-modules $\{ \ S_{\mathbf{k}}(\mu) \mid \mu$ is a hook partition of $e\}$. We denote it by $V_{\mathbf{k}}(\boldsymbol{\lambda})^\rho$.

6.2 Trivial source modules

For $\lambda \vdash w$ define a partition ρ_λ of $ew + |\rho|$ as follows:

$$(\rho_\lambda)_i = \rho_i + e\lambda_i \text{ for } i = 1, \cdots, ew + |\rho|.$$

Note that the e-core of ρ_λ is ρ. In particular, $\rho_{(2)} = [e]$ and $\rho_{(1^2)} = [e, e]$.

Example

Let $w = 4, e = 2, \rho = (3, 2, 1)$. $\{\lambda \mid \lambda \vdash 4\} = \{(4), (3, 1), (2^2), (2, 1^2), (1^4)\}$.

$$\begin{array}{ccc} (4) & \leftrightarrow & (11,2,1) \\ (3,1) & \leftrightarrow & (9,4,1) \\ (2^2) & \leftrightarrow & (7,6,1) \\ (2,1^2) & \leftrightarrow & (7,4,3) \\ (1^4) & \leftrightarrow & (5,4,3,2) \end{array}$$

The Young module $X(\mu)$ is defined as an indecomposable direct summand of $M_{\mathbf{k}}(\mu)$ which contain $S_{\mathbf{k}}(\mu)$ as a submodule. (This is a unique direct summand of $M_{\mathbf{k}}(\mu)$.) Moreover, $D_{\mathbf{k}}(\mu)$ and $M_{\mathbf{k}}(\mu)$ are selfdual, then so is $X_{\mathbf{k}}(\mu)$. Hence, if $D_{\mathbf{k}}(\mu) = S(\mu)$, then $D_{\mathbf{k}}(\mu) = S_{\mathbf{k}}(\mu) = X(\mu)$.

By q-Jantzen-Schaper theorem [33, (4.33) Theorem] or [30, p.251, 6.21. Theorem],

for $\lambda \vdash w$ $S_{\mathbf{k}}(\rho_\lambda)$ is a simple $\mathbf{k}GL_{ew+|\rho|}(q)$-module. Hence,

$$D_{\mathbf{k}}(\rho_\lambda) = S_{\mathbf{k}}(\rho_\lambda) = X(\rho_\lambda)$$

By these observation $D_{\mathbf{k}}(\rho_\lambda)$ has trivial source. In particular, its Green correspondent is simple by [47, p.302, Lemma 2.2.].

Recall that $L_{(e^w,|\rho|)} = GL_e(q)^{\times w} \times GL_{|\rho|}(q), H_w = \mathcal{N}_G(L_{(e^w,|\rho|)}) \subset GL_{ew}(q) \times GL_{|\rho|}(q)$.

For $\lambda \vdash w$, let $K_\lambda := (GL_e(q)^{\times w}) \rtimes \mathfrak{S}_\lambda \subset GL_{ew}(q)$ and $H_{w,\lambda} := K_\lambda \times GL_{|\rho|}(q)$.

Take a parabolic subgroup P_ρ of $GL_{|\rho|}(q)$. Then, $K_{(w)} \times P_\rho \subset P_{\rho_{(w)}}$, take a Sylow ℓ-subgroup E_{ρ_λ} of P_{ρ_λ}. (In fact, $\mathcal{N}_{P_{\rho_\lambda}}(D) \subset K_{(w)} \times P_\rho$.)

$\mathbf{k}_{P_{\rho_\lambda}}$ is an indecomposable direct summand of $\mathbf{k}_{K_{(w)} \times P_\rho} \uparrow^{P_{\rho_\lambda}}$ since $\mathbf{k}_{P_{\rho_\lambda}}$ is the Scott module $\mathrm{Scott}(P_{\rho_\lambda}, E_{\rho_\lambda})$.

Similary, we can choose compositions $\rho^*_\lambda \vDash ew + |\rho|$ and $\rho^* \vDash |\rho|$ such that

$$(\rho^*_\lambda)' = \rho'_\lambda, (\rho^*)' = \rho', E_{\rho^*_\lambda} \subset K_\lambda \times P_{\rho^*} \subset P_{\rho^*_\lambda}$$

for some Sylow ℓ-subgroup $E_{\rho^*_\lambda}$ of $P_{\rho^*_\lambda}$.

$k_{P_{\rho_\lambda}}$ is an indecomposable direct summand of $\mathbf{k}_{K_\lambda \times P_{\rho^*}} \uparrow^{P_{\rho^*_\lambda}}$ since $\mathbf{k}_{P^*_{\rho_\lambda}}$ is the Scott module $\mathrm{Scott}(P_{\rho^*_\lambda}, E_{\rho^*_\lambda})$.

Since $M_{\mathbf{k}}(\rho_\lambda) \cong M_{\mathbf{k}}(\rho_{\lambda^*})$, we get the following proposition:

Proposition 6.2.1. *$M_{\mathbf{k}}(\rho_\lambda)$ is a direct summand of $\mathbf{k}_{K_\lambda \times P_{\rho^*}} \uparrow^{GL_{ew+|\rho|}(q)}$ for* *any $\lambda \vdash w$. In particular, $D_{\mathbf{k}}(\rho_\lambda)$ is an indecomposable direct summand of $\mathbf{k}_{K_\lambda \times P_{\rho^*}}\uparrow^{GL_{ew+|\rho|}(q)}$.*

Next, we would like to consider $\mathbf{k}H_{w,\lambda}$-modules induced from $K_\lambda \times P_{\rho^*}$. Clearly, we know

$$\mathbf{k}_{K_\lambda \times P_{\rho^*}} \uparrow^{H_{w,\lambda}} \cong \mathbf{k}_{K_\lambda} \boxtimes_{\mathbf{k}} M_{\mathbf{k}}(\rho).$$

So,

$$\mathbf{k}_{K_\lambda \times_{\mathbf{k}} P_{\rho^*}} \uparrow^{H_w} \cong M^\lambda \boxtimes_{\mathbf{k}} M_{\mathbf{k}}(\rho).$$

Here, M^λ denotes $M_{\mathbf{k}}((e))^{\otimes w} \uparrow^{K_{(w)}}$. Note that M^λ is semisimple and

$$b_w \cdot M^\lambda \otimes_{\mathbf{k}} M_{\mathbf{k}}(\rho) = M^\lambda \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(\rho).$$

Therefore, we have the following proposition:

Proposition 6.2.2. *$S^\lambda_{\mathbf{k}} \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(\rho) \uparrow^{GL_{ew+|\rho|}(q)}$ is a direct summand of $\mathbf{k}_{K_\lambda \times P_{\rho^*}} \uparrow^{GL_{ew+|\rho|}(q)}$. Here, $S^\lambda_{\mathbf{k}}$ is $S^{\boldsymbol{\lambda}}_{\mathbf{k}}$ where $(\boldsymbol{\lambda})_e = \lambda$ and $\boldsymbol{\lambda}_j = \emptyset$ for $j \neq e$.*

Using induction on $(\lambda, >)$ and the property of the multiplicity of $D_{\mathbf{k}}(\rho_\lambda)$ in $M_{\mathbf{k}}(\rho_\lambda)$, we deduce the following:

Lemma 6.2.3. *The Green correspondent of $D_{\mathbf{k}}(\rho_\lambda)$ is $S^\lambda_{\mathbf{k}} \boxtimes D_{\mathbf{k}}(\rho)$ with respect to $(GL_{ew+|\rho|}(q), D, H_w)$. Here, $S^\lambda_{\mathbf{k}}$ is $S^{\boldsymbol{\lambda}}_{\mathbf{k}}$ where $\boldsymbol{\lambda}_e = \lambda$ and $\boldsymbol{\lambda}_j = \emptyset$ for $j \neq e$.*

Chapter 7

Green correspondents of simple $B_{w,\rho}$-modules

In this chapter we would like to determine the Green correspondents of the simple $B_{w,\rho}$-modules uniquely. Recall the following:

1. For $i = 0, \dots, w$, let G_i be the standard Levi subgroup corresponding to the composition $(e^i, e(w-i) + |\rho|)$.

2. $N_w = GL_e(q) \wr \mathfrak{S}_w \subset GL_{ew}(q)$.

3. $H_w := N_w \times GL_{|\rho|}(q)$.

4. $H_w = \mathcal{N}_{GL_{ew+|\rho|}(q)}(L_{(e^w,|\rho|)})$.

7.1 Main theorem

Let $\boldsymbol{X}_w(q)$ be the Green correspondent of $\mathbf{k}Gb_0$ with respect to $(G\times G, \Delta(D), G\times H_w)$. Let $\boldsymbol{X}_w(q)^\vee$ be $\mathrm{Hom}_{B_{w,\rho}}(\boldsymbol{X}_w(q), B_{w,\rho})$. Now, we can state our main theorem of this paper as follows:

Theorem 7.1.1. *For any* $\lambda \in \Lambda_{e,w,\rho}$, $D_{\mathbf{k},q}(\lambda) \otimes_{B_{w,\rho}} \boldsymbol{X}_w(q) \cong V_{\mathbf{k},q}(\overline{\lambda})^\rho$. *Here* $\overline{\lambda}$ *is the e-quotient of* λ. *Moreover,* $d_{\lambda,\mu} = d_{\overline{\lambda},\overline{\mu}} = [U_{\mathbf{k}}(\overline{\lambda})^\rho : V_{\mathbf{k}}(\overline{\mu})^\rho]$ *for any* $\lambda, \mu \in \Lambda_{e,w,\rho}$.

Proof of Theorem 7.1.1

Note that $\mathrm{Res}^{H_w}_{G_w} \circ (- \otimes_{B_{w,\rho}} \boldsymbol{X}_w(q)) = b_w{}^*R^{G_{w-1}}_{G_w} \circ \cdots \circ b_2{}^*R^{G_1}_{G_2} \circ b_1{}^*R^{G_0}_{G_1}$ as a functor by the proof of theorem 5.0.7.

In §5 we have considered the correspondence between

$$\{\mathrm{Irr}(K[G_0]b_0, s, c) \mid s \in \mathcal{C}_{b_0}, c \in I_s\} \text{ and } \{\mathrm{Irr}(K[G_w]b_w, s, c) \mid s \in \mathcal{C}_{b_0}, c \in I_s\}.$$

In particular, $\boldsymbol{X}_w(q)$ induces a correspondence between

$$\{\mathrm{Irr}(K[G_0]b_0, 1_{G_0}, c) \mid c \in I_1\} \text{ and } \{\mathrm{Irr}(K[G_w]b_w, 1_{G_w}, c) \mid c \in I_1\}.$$

For the proof of Theorem 7.1.1 we consider not only character but also its modules. For a module M we denote by $[M]$ the corresponding element in the Grothendieck group. By Theorem 4.2.7, we can construct a module $S_{\mathbf{k}}(\lambda)$ (resp. $S_K(\lambda)$) such that $[S_{\mathbf{k}}(\lambda)] = \overline{\chi_\lambda}$ (resp. $[S_K(\lambda)] = \chi_\lambda$) in a suitable Grothendiek group. So, for a fixed $c \in \mathbb{I}_1$ we deduce that for any $\chi_\lambda \in \mathrm{Irr}(K[G_0]b_0, 1_{G_0}, c)$, $[\mathrm{Res}^{H_w}_{G_w}(S_{\mathbf{k}}(\lambda) \otimes_{B_{w,\rho}} \boldsymbol{X}_w(q))] \in \mathrm{Irr}(K[G_w]b_w, 1_{G_w}, c)$.

For $c \in \mathbb{I}_1$ let $\mathrm{Irr}(K[H_w]b_w, c)$ be the set of characters $\chi^{\boldsymbol{\lambda}}$ lying in $k[H_w]b_w$ such that $\mathrm{Res}^{H_w}_{G_w}(\chi^{\boldsymbol{\lambda}}) \in \mathrm{Irr}(K[G_w]b_w, 1_{G_w}, c)$. In this setting up, for a fixed $c \in I_1$, $\boldsymbol{X}_w(q)$ induces a bijection between $\mathrm{Irr}(K[G_0]b_0, 1, c)$ and $\mathrm{Irr}(K[H_w]b_w, c)$. In particular, for $\chi_\lambda \in \mathrm{Irr}(K[G_0]b_0, 1, c)$ we can deduce that $[S_{\mathbf{k}}(\lambda) \otimes_{B_{w,\rho}}$

$\boldsymbol{X}_w(q)] = \overline{\chi^{\boldsymbol{\mu}}}$ with $\chi^{\boldsymbol{\mu}} \in \mathrm{Irr}(K[H_w]b_w, c)$ for some e-tuple multipartition $\boldsymbol{\mu}$ of w. So, this means that

$$[S_{\mathbf{k}}(\lambda) \otimes_{B_{w,\rho}} \boldsymbol{X}_w(q)] = [U_{\mathbf{k}}(\boldsymbol{\mu})^{\rho}] \text{ for some } e\text{-tuple multipartition } \boldsymbol{\mu} \text{ of } w \text{ with } |\lambda^{(i-1)}| = |\boldsymbol{\mu}_i| \text{ for } i = 1, \dots, e. \tag{7.1}$$

We claim that

$$\lambda^{(i-1)} = \boldsymbol{\mu}_i \text{ for } i = 1, \dots, e \tag{7.2}$$

where χ_λ corresponds to $\chi^{\boldsymbol{\mu}}$ by the isometry induced by $\hat{\boldsymbol{X}}_w(q)$.

In order to show this claim, first we shall investigate the unipotent blocks with e-core ρ and e-weight 2 and show the claim inductively in the next subsection, and second we will show this claim by induction on e-weight w.

7.2 The unipotent block with e-core ρ and e-weight 2

Lemma 7.2.1. (*Compare with* [8, p.126, *Lemma* 4.3.])

(1)(*Dominance order for partitions of $r + 2e$ with e-core ρ.*)

For $1 \le i \le j \le e$ and $1 \le i' \le j' \le e$,

$[i, j] \trianglelefteq [i', j']$ *if and only if* $i \le i'$ *and* $j \le j'$,

$[i, j] \trianglelefteq [j']$ *if and only if* $j \le j'$,

$[i] \trianglelefteq [i', j']$ *if and only if* $1 < i \le i'$ *and* $i' \le j'$, *or* $1 = i$ *and* $2 \le j'$,

$[i] \trianglelefteq [i']$ *if and only if* $i \le i'$.

(2) *For* $1 \leq i < j \leq e$,

$$\partial([i,j]) = j - i, \quad \partial([i]) = 0, \quad \textit{and} \quad \partial([i,i]) = 0.$$

Using [50, Theorem 4.4] and [30, p.251, Corollary 6.19.] we get

Proposition 7.2.2. (*Decomposition numbers for unipotent characters lying in* $B_{2,\rho}$).

For $1 \leq i < j \leq e$ *and* $j - i \geq 2$,

$$\overline{\chi_{[i,j]}} = \varphi_{[i,j]} + \varphi_{[i+1,j]} + \varphi_{[i,j+1]} + \varphi_{[i+1,j+1]};$$

for $2 \leq i \leq e$,

$$\overline{\chi_{[i-1,i]}} = \varphi_{[i-1,i]} + \varphi_{[i]} + \varphi_{[i,i]} + \varphi_{[i-1,i+1]} + \varphi_{[i,i+1]};$$

for $1 \leq i \leq e$,

$$\overline{\chi_{[i]}} = \varphi_{[i]} + \varphi_{[i+1,i+1]} + \varphi_{[i,i+1]};$$

and for $1 \leq i \leq e$,

$$\overline{\chi_{[i,i]}} = \varphi_{[i,i]} + \varphi_{[i+1]} + \varphi_{[i,i+1]}.$$

Definition 7.2.3. *For* $1 \leq i < j \leq e$, *let* $\{i,j\}$ *be the e-tuple multipartition* $\boldsymbol{\lambda}$ *of* 2 *such that* $\boldsymbol{\lambda}_i = (1), \boldsymbol{\lambda}_j = (1)$ *and the other part is zero partition* $\emptyset$.

For $1 \leq i \leq e$, *let* $\{i\}$ *be the e-tuple multipartition* $\boldsymbol{\lambda}$ *of* 2 *such that* $\boldsymbol{\lambda}_i = (2)$ *and the other part is the zero partition* $\emptyset$.

For $1 \leq i \leq e$, *let* $\{i,i\}$ *be the e-tuple multipartition* $\boldsymbol{\lambda}$ *of* 2 *such that* $\boldsymbol{\lambda}_i = (1^2)$ *and the other part is the zero partition* $\emptyset$.

As in [8, p.127] we describe some irreducible characters of $GL_e(q) \wr \mathfrak{S}_2$. Note the following:

$$\chi^{\{i\}} + \chi^{\{i,i\}} = \mathrm{Ind}_{L_{(e^2)}}^{GL_e(q)\wr\mathfrak{S}_2}(\chi_{(i+1,1^{e-i-1})} \cdot \chi_{(i+1,1^{e-i-1})}) \quad (0 \le i < e).$$

$$\chi^{\{i,j\}} = \mathrm{Ind}_{L_{(e^2)}}^{GL_e(q)\wr\mathfrak{S}_2}(\chi_{(i+1,1^{e-i-1})} \cdot \chi_{(j+1,1^{e-j-1})}) \quad (0 \le i < j < e).$$

By the construction of $K[GL_e(q)\wr\mathfrak{S}_w]$- and $\mathbf{k}[GL_e(q)\wr\mathfrak{S}_w]$-irreducible modules as in §6 and the Brauer tree of $B_0(GL_e(q))$, we get decomposition numbers corresponding to these modules as follows:

Proposition 7.2.4. (*Decomposition numbers for $B_0(GL_e(q) \wr \mathfrak{S}_2)$*). *For $1 \le i < j \le e$ and $j - i \ge 2$,*

$$\overline{\chi^{\{i,j\}}} = \varphi^{\{i,j\}} + \varphi^{\{i+1,j\}} + \varphi^{\{i,j+1\}} + \varphi^{\{i+1,j+1\}};$$

for $2 \le i \le e$,

$$\overline{\chi^{\{i-1,i\}}} = \varphi^{\{i-1,i\}} + \varphi^{\{i\}} + \varphi^{\{i,i\}} + \varphi^{\{i-1,i+1\}} + \varphi^{\{i,i+1\}};$$

for $1 \le i \le e$,

$$\overline{\chi^{\{i\}}} = \varphi^{\{i\}} + \varphi^{\{i+1,i+1\}} + \varphi^{\{i,i+1\}};$$

and for $1 \le i \le e$,

$$\overline{\chi^{\{i,i\}}} = \varphi^{\{i,i\}} + \varphi^{\{i+1\}} + \varphi^{\{i,i+1\}}.$$

By an easy observation as in (7.1), we get

$$[S_{\mathbf{k}}([i,j]) \otimes_{B_{w,\rho}} \boldsymbol{X}_w(q)] = [U_{\mathbf{k}}(\{i,j\})^{\rho}] \tag{7.3}$$

for $1 \leq i < j \leq e$.

Hence, in order to prove our claim (7.2) firstly we investigate modules indexed by $[e]$ and $[e,e]$. In section we have already seen that these module have trivial source. By Lemma 6.2.3, $S_{\mathbf{k}}([e]) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q) \cong U_{\mathbf{k}}(\{e\})^\rho$ and $S_{\mathbf{k}}([e,e]) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q) \cong U_{\mathbf{k}}(\{e,e\})^\rho$.

Let $\Lambda^{\checkmark}_{e,2,\rho}$ be the set of partition of $2e+|\rho|$ as follows:

$$\{[i],[i,i] \mid i = 1,\dots,e\}.$$

By the observation (7.3), we have known the composition factors of the Green correspondent of $S_{\mathbf{k}}(\lambda)$ with respect to (G,D,H_2) for $\lambda \in \Lambda_{e,2,\rho} \backslash \Lambda^{\checkmark}_{e,2,\rho}$.

So, we would like to determine the composition factors of the Green correspondent of $S_{\mathbf{k}}(\lambda)$ with respect to (G,D,H_2) for $\lambda \in \Lambda^{\checkmark}_{e,2,\rho}$ by induction on $(\Lambda^{\checkmark}_{e,2,\rho}, >)$.

We claim that

$$D_{\mathbf{k}}(\lambda) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q) \cong V_{\mathbf{k}}(\overline{\lambda})^\rho \tag{7.4}$$

for $\lambda \in \Lambda^{\checkmark}_{e,2,\rho}$.

By Lemma 6.2.3 we have already known the Green correspondents of $D_{\mathbf{k}}([e])$ and $D_{\mathbf{k}}([e,e])$ with repect to (G,D,H_2).

Consider $S_{\mathbf{k}}(\lambda) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q)$. Assume $\lambda = [i]$ for some i. $S_{\mathbf{k}}([i])$ has $D_{\mathbf{k}}([i+1,i+1])$ as a composition factor by Proposition 7.2.2. By the assumption of induction, we get

$$D_{\mathbf{k}}([i+1,i+1]) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q) = V_{\mathbf{k}}(\overline{[i+1,i+1]})^\rho = V_{\mathbf{k}}(\{i+1,i+1\}).$$

So, by (7.1) and decomposition matrices as in Proposition 7.2.4,

$$[S_{\mathbf{k}}([i]) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q)] = [U_{\mathbf{k}}(\overline{[i]})^{\rho}] = [U_{\mathbf{k}}(\{i\})].$$

Similary, we can prove our claim (7.4) for $\lambda = [i, i]$ for some i. So, our claim (7.4) is valid. Hence, our claim (7.2) is also valid for weight 2, and we get

Lemma 7.2.5. *For any $\lambda \in \Lambda_{e,2,\rho}$,*

$$D_{\mathbf{k}}(\lambda) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q) \cong V_{\mathbf{k}}(\overline{\lambda})^{\rho}$$

and

$$S_{\mathbf{k}}(\lambda) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q) \cong U_{\mathbf{k}}(\overline{\lambda})^{\rho}.$$

In particular,

$$[S_{\mathbf{k}}(\lambda) \otimes_{B_{2,\rho}} \boldsymbol{X}_w(q)] = [U_{\mathbf{k}}(\overline{\lambda})^{\rho}].$$

Here, $\overline{\lambda}$ is the e-quotient of λ.

7.3 The unipotent block with e-core ρ and e-weight w

7.3.1 Lemmas

Let n be an integer greater than 1. Let $\mathcal{P}(n)$ be the set of partitions of n. For $\alpha \vdash n-1$, let $\mathcal{P}_{\alpha}(n) := \{\lambda \vdash n \mid LR^{\lambda}_{(1)\alpha} = 1\}$.

Lemma 7.3.1. *If a bijection $\sigma : \mathcal{P}(n) \longrightarrow \mathcal{P}(n)$ satisfies that $\sigma(\mathcal{P}_{\alpha}(n)) = \mathcal{P}_{\alpha}(n)$ for any $\alpha \vdash n-1$, then σ is the identity map.*

Proposition 7.3.2. *There exist a $(\mathbf{k}[G_1], B_{w,\rho})$-bimodule $\boldsymbol{M}$ such that*

$$b_0 R_{G_1}^{G_0} \oplus \boldsymbol{M} \cong b_0 \circ \mathrm{Ind}_{G_1}^{G_0}$$

as $(\mathbf{k}[G_1], B_{w,\rho})$-bimodules and the vertex of each indecomposable direct summand V of $S \otimes_{\mathbf{k}[G_1]} \boldsymbol{M}$ is properly contained in D,i.e.

$$\mathrm{vx}(V) \lneqq_{G_0} D.$$

for any indecomposable $\mathbf{k}[G_1]$ module S with vertex D.

proof.

Let P_1 be a parabolic subgroup of G_0 with Levi decomposition $P_1 = G_1 U_1$. Let e_{U_1} be the idempotent of $\mathbf{k}G_0$ defined by

$$e_{U_1} := \frac{1}{|U_1|} \sum_{u \in U_1} u.$$

Then, $b_0 \circ R_{G_1}^{G_0}$ is isomorphic to $e_{U_1}\mathbf{k}[G_0]b_0 \cong \mathbf{k}[U_1 \backslash G_0]b_0$. Since e_{U_1} is an idempotent of $\mathbf{k}[G_0]$ and G_1 stabilize e_{U_1}, we have the following decomposition:

$$e_{U_1}\mathbf{k}[G_0]b_0 \oplus (1_{G_0} - e_{U_1})\mathbf{k}[G_0]b_0 = \mathbf{k}[G_0]b_0$$

as $(\mathbf{k}[G_1], B_{w,\rho})$-bimodules. Let $\boldsymbol{M}$ be $(1_{G_0} - e_{U_1})\mathbf{k}[G_0]b_0$. Take an indecomposable $\mathbf{k}[G_1]$-module S with vertex D. Since $N_{P_1}(D) \subset G_1$, we can define the Green correspondence f_{P_1} with respect to (P_1, D, G_1). Then, the Green correspondent $f_{P_1}(S)$ of S is $\mathrm{Inf}_{G_1}^{P_1}(S)$. So, we know the following decomposition:

$$\mathrm{Ind}_{G_1}^{P_1} S \cong \mathrm{Inf}_{G_1}^{P_1}(S) \oplus Q_S$$

where Q_s is a relative E-projective module with $E \lneqq_{P_1} D$. Since $b_0 R_{G_1}^{G_0} = b_0 \mathrm{Ind}_{P_1}^{G_0} \circ \mathrm{Inf}_{G_1}^{P_1}$, we complete the proof.

$\square$

Definition 7.3.3. *Let $\mathfrak{F}_{w,\rho}$ be the functor induced by $- \otimes \hat{\boldsymbol{X}}_w(q)^\vee$. Let ρ_- be the e-core obtained by removing the bottom most bead from each runner except 0-th runner of the abucus configuration of ρ. Since both $B_{0,\rho}$ and B_{0,ρ_-} have defect zero, $\mathcal{O}B_0(N_w) \boxtimes \mathcal{O}B_{0,\rho}$ and $\mathcal{O}B_0(N_w) \boxtimes \mathcal{O}B_{0,\rho_-}$ are Morita equivalent. We denote this equivalece from* mod-$\mathcal{O}B_0(N_w) \boxtimes \mathcal{O}B_{0,\rho}$ *to* mod-$\mathcal{O}B_0(N_w) \boxtimes \mathcal{O}B_{0,\rho_-}$ *by $p_{\rho_-,\rho}$. Moreover, one can know that there exists a sequence of e-cores ($\rho_- = \gamma_0 \subset \gamma_1 \subset \cdots \subset \gamma_s = \rho$) such that γ_i and γ_{i+1} form a $[w-1 : k_i]$-pair with $k_i \geq w-1$ for any $0 \leq i \leq s-1$. Hence, $\mathcal{O}B_{w-1,\rho}$ and $\mathcal{O}B_{w-1,\rho_-}$ are Morita equivalent. We denote this equivalence from* mod-$\mathcal{O}B_{w-1,\rho_-}$ *to* mod-$\mathcal{O}B_{w-1,\rho}$ *by* $\mathrm{scp}_{\rho,\rho_-}$. *Let $\mathfrak{F}^\natural_{w-1,\rho}$ be the functor* $\mathrm{scp}_{\rho,\rho_-} \circ \mathfrak{F}_{w-1,\rho_-} \circ p_{\rho_-,\rho}$ *from* mod-$(\mathcal{O}B_0(N_w) \boxtimes \mathcal{O}B_{0,\rho})$ *to* mod-$\mathcal{O}B_{w,\rho}$.

In particular, for $w > 2$, the following diagram commutes and each of arrows is an equivalence:

$$\begin{tikzcd}
\text{mod-}\mathcal{O}B_{w-1,\rho} & & \text{mod-}\mathcal{O}B_{w-1,\rho_-} \arrow[ll, "\mathrm{scp}_{\rho,\rho_-}"'] \\
\text{mod-}(\mathcal{O}B_0(N_{w-1}) \boxtimes \mathcal{O}B_{0,\rho}) \arrow[u, "\mathfrak{F}^\natural_{w-1,\rho_-}"] & & \text{mod-}(\mathcal{O}B_0(N_{w-1}) \boxtimes \mathcal{O}B_{0,\rho_-}) \arrow[ll, "p^{-1}_{\rho_-,\rho}"] \arrow[u, "\mathfrak{F}_{w-1,\rho}"']
\end{tikzcd}$$

Moreover, we may use these functors for module categories of block algebras over $\mathbf{k}$ *and K.*

Lemma 7.3.4. *Let $w > 2$. Asumme that the claim (7.2) is true for $w-1$. Then, the following diagram commutes*:

$$\begin{tikzcd}
\text{mod-}KB_{w,\rho} & & \text{mod-}(K[B_{1,\emptyset} \boxtimes_{\mathbf{k}} B_{w-1,\rho}]) \arrow[ll, "b_0 R^{G_0}_{G_1}"'] \\
\text{mod-}(K[B_0(N_w) \boxtimes_{\mathbf{k}} B_{0,\rho}]) \arrow[u, "\mathfrak{F}_{w,\rho}"] & & \text{mod-}(K[B_{1,\emptyset} \boxtimes_{\mathbf{k}} B_0(N_{w-1}) \boxtimes_{\mathbf{k}} B_{0,\rho}]) \arrow[ll, "\mathrm{Ind}"] \arrow[u, "1 \times \mathfrak{F}^{\natural}_{w-1,\rho}"']
\end{tikzcd}$$

proof.

In order to prove

$$\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}([M]) = b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}^{\natural}_{w-1,\rho})([M])$$

for any simple $K(B_{1,\emptyset} \boxtimes B_0(N_{w-1}) \boxtimes B_{0,\rho})$-module M, we need to show that

$$\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}(\overline{M}) = b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}^{\natural}_{w-1,\rho})(\overline{M})$$

since we deal Green correspondences $\mathfrak{F}_{w,\rho}$ and $\mathfrak{F}^{\natural}_{w-1,\rho}$.

ℓ-Modular reduction $\overline{M}$ of M depends on $\mathcal{O}$, but we may regard $\overline{M}$ as an indecomposable module with vertex D since M has the shape $M = M' \boxtimes S_K(\rho)$ for some simple $GL_e(q) \times N_{w-1}$-module M' and $\ell \nmid \dim_{\mathbf{k}} \overline{M'}$.

Put $S := \overline{M}$. Let A (resp. B) be $B_{1,\emptyset} \boxtimes_{\mathbf{k}} B_0(N_{w-1}) \boxtimes_{\mathbf{k}} B_{0,\rho}$ (resp. $B_{w,\rho}$). Obviously, the following diagram commutes:

$$\begin{tikzcd}
\text{mod-}RG_0 & & \text{mod-}RG_1 \arrow[ll, "\mathrm{Ind}^{G_0}_{G_1}"'] \\
\text{mod-}R[N_w \times GL_{|\rho|}(q)] \arrow[u, "\mathrm{Ind}^{G_0}_{H_w}"] & & \text{mod-}R[GL_e(q) \times N_{w-1} \times GL_{|\rho|}(q)] \arrow[ll, "\mathrm{Ind}"] \arrow[u, "\mathrm{Ind}^{G_1}_{GL_e(q)\times N_{w-1}\times GL_{|\rho|}(q)}"']
\end{tikzcd}$$

for any $R \in \{K, \mathcal{O}, \mathbf{k}\}$. Since $A = B_{1,\emptyset} \boxtimes_{\mathbf{k}} B_0(N_{w-1}) \boxtimes_{\mathbf{k}} B_{0,\rho}$ is a block of $GL_e(q) \times N_{w-1} \times GL_{|\rho|}(q)$ which contains $C_{G_0}(D)$, obviously, the following diagram commutes and its image exists for any object:

$$\begin{tikzcd}
\text{mod-}B_{w,\rho} & & \text{mod-}\mathbf{k}G_1 \arrow[ll, "b_0 \circ \mathrm{Ind}_{G_1}^{G_0}"'] \\
\text{mod-}\mathbf{k}[N_w \times GL_{|\rho|}(q)] \arrow[u, "b_0 \circ \mathrm{Ind}_{H_w}^{G_0}"] & & \text{mod-}A \arrow[ll, "\mathrm{Ind}"] \arrow[u, "\mathrm{Ind}_{GL_e(q)\times N_{w-1}\times GL_{|\rho|}(q)}^{G_1}"']
\end{tikzcd}$$

$$b_0 \circ \mathrm{Ind}_{H_w}^{G_0} \circ \mathrm{Ind} = b_0 \circ \mathrm{Ind}_{G_1}^{G_0} \circ \mathrm{Ind}_{GL_e(q)\times N_{w-1}\times GL_{|\rho|}(q)}^{G_0} \tag{7.5}$$

Let ${}_A\boldsymbol{Z}_B := b_0 \circ \mathrm{Ind}_{H_w}^{G_0} \circ \mathrm{Ind} \circ 1_A$.

Then, by the structure of $N_w \times GL_{|\rho|}(q)$ as a group,

$$\mathrm{Ind} : \text{mod-}A \to \text{mod-}\mathbf{k}[N_w \times GL_{|\rho|}(q)]$$

is nothing but

$$\mathrm{Ind} : \text{mod-}A \to \text{mod-}B_0(N_w) \boxtimes_{\mathbf{k}} B_{0,\rho}.$$

Moreover, for any simple A-module $\mathrm{Ind}(V)$ is semisimple and a $B_0(N_w) \boxtimes_{\mathbf{k}} B_{0,\rho}$-module. In particular, each indecomposable direct summand of $\mathrm{Ind}(S)$ has vertex D.

So, each indecomposable direct summand of $\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}(S)$ must have vertex D. (Clearly, $\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}(S)$ is a direct summand of $S \otimes_A \boldsymbol{Z}$.)

Moreover, there exists an (A, B)-bimodules $\boldsymbol{M}_w$ such that

$$(\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}) \oplus \boldsymbol{M}_w \cong b_0 \circ \mathrm{Ind}_{H_w}^{G_0} \circ \mathrm{Ind}$$

as (A,B)-bimodules. Here, by the property of Green correspondence, $\boldsymbol{M}_w$ has the following property:

For any indecomposable A-module U with vertex D and for any indecomposable direct summand V of $U \otimes_A \boldsymbol{M}_w$, the vertex of V is strictly contained in D, i.e.

$$\mathrm{vx}(V) \lneqq_{G_0} D.$$

On the other hand, we take a $(\mathbf{k}[G_1], B_{w,\rho})$-bimodule $\boldsymbol{M}$ as in Proposition 7.3.2. Let $\boldsymbol{M}^{\natural}_{w-1}$ be an $(A, \mathbf{k}[G_1])$-bimodule such that

$$(1 \times \mathfrak{F}^{\natural}_{w-1}) \oplus \boldsymbol{M}^{\natural}_{w-1} = \mathrm{Ind}^{G_1}_{GL_e(q)\times N_{w-1}\times GL_{|\rho|}(q)}$$

as $(A, \mathbf{k}[G_1])$-bimodules.

By Equation (7.5), we have

$$\boldsymbol{Z} \cong b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}^{\natural}_{w-1}) \oplus \boldsymbol{M}'$$

where $-\otimes \boldsymbol{M}' = b_0 \mathrm{Ind}^{G_0}_{G_1} \circ (-\otimes \boldsymbol{M}^{\natural}_{w-1}) \oplus (-\otimes \boldsymbol{M}) \circ (1 \times \mathfrak{F}^{\natural}_{w-1})$.

Since $\boldsymbol{M}'$ has the same property as $\boldsymbol{M}_w$, we deduce that $\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}(S)$ is a direct summand of $b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}^{\natural}_{w-1,\rho})(S)$.

Hence, we just compare the character of $\mathfrak{F}_{w,\rho} \circ \mathrm{Ind}(S)$ with that of $b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}^{\natural}_{w-1,\rho})(S)$.

By Theorem 4.1.3, Lemma 2.2.2 and Equation (5.5) we have

$$\sum_{\chi \in \mathrm{Irr}(KA)} \mathfrak{F}_{w,\rho} \circ \mathrm{Ind}(\chi) = \sum_{\chi \in \mathrm{Irr}(KA)} b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}^{\natural}_{w-1,\rho})(\chi)$$

Indeed, $\mathfrak{F}_{w,\rho}$ induces an isometry with all positive sign. So is $1 \times \mathfrak{F}^{\natural}_{w-1,\rho}$. Hence, we just compare

$$\sum_{\chi \in \mathrm{Irr}(KB_{w,\rho})} \left(\chi, \sum_{\chi_{t,\boldsymbol{\mu}} \in \mathrm{Irr}(KB_{1,\emptyset} \boxtimes KB_{w-1,\rho})} b_0 R^{G_0}_{G_1}(\chi_{t,\boldsymbol{\mu}}) \right)_{G_0}$$

with

$$\sum_{\chi \in \mathrm{Irr}(KB_0(N_w))} \left(\chi, \sum_{\varphi \in \mathrm{Irr}(KB_0(GL_e(q) \times N_{w-1}))} \mathrm{Ind}(\varphi) \right)_{N_w}.$$

Then, the former scalar products are determined by Equation (5.5), and the latter are also determined by the same branching rule. We complete the proof.

□

Definition 7.3.5. *For $\alpha \vdash n-e$ and $1 \leq i \leq e$ let $\Gamma_{i,\alpha}$ be the set of e-tuple multipartition $\boldsymbol{\mu}$ of w_0 such that $\boldsymbol{\mu}_j = \alpha^{(j)}$ for $j \neq i$ and $\alpha^{(i)} \subseteq \boldsymbol{\mu}_i$ with $|\boldsymbol{\mu}_i| = |\alpha^{(i)}| + 1$.*

Lemma 7.3.6. *Let $n := ew_0 + |\rho|$. Assume that the claim (7.2) is true for $w \leq w_0 - 1$. Then,*

$$\mathfrak{F}_{w_0,\rho}\Big(\sum_{\boldsymbol{\mu} \,\in\, \Gamma_{i,\alpha}} [U_K(\boldsymbol{\mu})^{\rho}]\Big) = \sum_{\overline{\lambda} \,\in\, \Gamma_{i,\alpha}} [S_K(\lambda)]. \tag{7.6}$$

Put $S := [S_K(\nu_i) \boxtimes_K U_K(\overline{\alpha})^{\rho}]$ for some $\nu_i \vdash e$ and $\alpha \vdash n-e$ whose e-core is ρ. Here, $S_K(\nu_i) \boxtimes_K U_K(\overline{\alpha})^{\rho}$ is a $K[GL_e(q) \times H_{w_0-1}]$-module.

Then, by the assumption,

$$b_0 R^{G_0}_{G_1} \circ (1 \times \mathfrak{F}_{w-1})(S) = b_0 R^{G_0}_{G_1}([S_K(\nu) \boxtimes S_K(\alpha)]) = \sum_{\overline{\lambda} \,\in\, \Gamma_{i,\alpha}} [S_K(\lambda)]. \tag{7.7}$$

On the other hand,

$$\mathrm{Ind}(S) = \sum_{\boldsymbol{\mu} \,\in\, \Gamma_{i,\alpha}} [U_K(\boldsymbol{\mu})^\rho]. \tag{7.8}$$

Hence, by Lemma 7.3.4, we get the equation (7.6).

7.3.2 A proof for e-weight w

Take a $\lambda \in \Lambda_{e,w,\rho}$. We claim as in (7.2) that if there exists a partition μ with $[S_K(\lambda)] = \mathfrak{F}_{w_0}([U_K(\overline{\mu})^\rho])$, then $\lambda^{(i)} = \mu^{(i)}$ for any i.

First we consider the case where there exist integers a and b such that $1 \le a < b \le e, \lambda^{(a)} \ne \emptyset$ and $\lambda^{(b)} \ne \emptyset$. So, by the observation (7.1) this mean that $\mu^{(a)} \ne \emptyset$ and $\mu^{(b)} \ne \emptyset$.

Choose two partitions $\alpha, \beta \vdash n - e$ with e-core ρ and e-weight $w_0 - 1$ satisfying the following:

$$\begin{array}{c} \alpha^{(a)} \subseteq \lambda^{(a)}, |\alpha^{(a)}| = |\lambda^{(a)}| - 1 \text{ and } \alpha^{(j)} = \lambda^{(j)} \text{ for } j \ne a; \\ \beta^{(b)} \subseteq \lambda^{(b)}, |\beta^{(b)}| = |\lambda^{(b)}| - 1 \text{ and } \beta^{(j)} = \lambda^{(j)} \text{ for } j \ne b. \end{array} \tag{7.9}$$

Consider modules $S_a := S_K(\nu_a) \boxtimes U_{\mathbf{k}}(\overline{\alpha})^\rho$ and $S_b := S_K(\nu_b) \boxtimes U_K(\overline{\beta})^\rho$ and apply the functor $1 \times \mathfrak{F}^\natural_{w_0-1,\rho}$ to these modules. By Lemma 7.3.4 and the assumption of induction, we also know that

$$\begin{array}{c} \alpha^{(a)} \subseteq \mu^{(a)}, |\alpha^{(a)}| = |\mu^{(a)}| - 1 \text{ and } \alpha^{(j)} = \mu^{(j)} \text{ for } j \ne a; \\ \beta^{(b)} \subseteq \mu^{(b)}, |\beta^{(b)}| = |\mu^{(b)}| - 1 \text{ and } \beta^{(j)} = \mu^{(j)} \text{ for } j \ne b. \end{array} \tag{7.10}$$

By the equations (7.9) and (7.10), we get

Hence, we may assume that $|\lambda^{(i)}| = |\mu^{(i)}|$ and $\lambda^{(j)} = \mu^{(j)} = \emptyset$ for any $j \ne i$.

Definition 7.3.7. *Let $\mathcal{MP}(w,e)$ be the set of e-tuple multipartions of w. For $1 \le i \le e$ let an injection ι_i from $\mathcal{P}(w)$ to $\mathcal{MP}(w,e)$ as follows:*

$$(\iota_i(\gamma))_j = \begin{cases} \gamma & \text{if } j = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

For $1 \le i \le e$, let $\mathcal{MP}(w,e)^i$ be the set of all elements of $\boldsymbol{\mu}$ in $\mathcal{MP}(w,e)$ such that $\boldsymbol{\mu}_j = \emptyset$ for $j \neq i$.

For $1 \le i \le e$ let a projection π_i from $\mathcal{MP}(w,e)^i$ such that to $\mathcal{P}(w)$ as follows:

$$\pi_i(\boldsymbol{\mu}) := \boldsymbol{\mu}_i.$$

Let σ be an automorphism of $\mathcal{MP}(w,e)$ as follows:

$$\sigma(\boldsymbol{\mu}) = \overline{\lambda} \text{ if } \mathfrak{F}_{w,\rho}([U_K(\boldsymbol{\mu})^\rho]) = [S_K(\lambda)].$$

Let $\widetilde{\sigma} := \pi_i \circ \sigma \circ \iota_i$. Then, this is well-defined by (7.1), and becomes an automorphism of $\mathcal{P}(w_0)$. Moreover, by the assumption of induction and Lemma 7.3.6, we get that $\sigma(\Gamma_{i,\alpha}) = \Gamma_{i,\alpha}$ and $\iota_i \circ \pi_i$ is the identity map of $\Gamma_{i,\alpha}$ for any $\alpha^{(i)} \vdash w_0 - 1$. So, $\pi_i(\Gamma_{i,\alpha}) = \mathcal{P}_{\alpha^{(i)}}(w_0)$ and $\Gamma_{i,\alpha} = \iota_i(\mathcal{P}_{\alpha^{(i)}}(w_0))$. Hence, $\widetilde{\sigma}(\mathcal{P}_{\alpha^{(i)}}(w_0)) = \mathcal{P}_{\alpha^{(i)}}(w_0)$. However, by Lemma 7.3.1, $\widetilde{\sigma}$ is the identity map of $\mathcal{P}(w_0)$. Therefore $\lambda^{(i)} = \mu^{(i)}$ for $i = 0, \ldots, e-1$ as in claim (7.2), and we complete the proof of Theorem 7.1.1.

$\square$

Chapter 8

Independence of q

The purpose of this chapter is to investigate the module categories of groups algebra whose groups are defined over various finite fields.

In this chapter, for convenience, we will use rather different notation from section that express some groups and modules. (i.e. $G, H, L_\lambda, M_{\mathbf{k}}(\lambda), S_{\mathbf{k}}(\lambda)$, $D_{\mathbf{k}}(\lambda), X_{\mathbf{k}}(\lambda)$,etc.)

For a power of prime number q, $G(q), N_w(q), H_w(q)$, and $L_\lambda(q)$ denote G, N_w, H_w and L_λ respectively if the defining field of G is $\mathbb{F}_q$. Moreover, in order to distinguish $\mathbf{k}GL_n(q)$-modules and $\mathbf{k}GL_n(q')$-modules lying in unipotent blocks, we attend a subindex to some modules as follows:

$$M_{\mathbf{k},q}(\lambda), S_{\mathbf{k},q}(\lambda), D_{\mathbf{k},q}(\lambda), X_{\mathbf{k},q}(\lambda).$$

Moreover, q and q' always mean some powers of some prime numbers such that $e(q) = e(q') = e$ and $r(q) = r(q')$. Here, $r(q)$ means the maximal integer i such that ℓ^i divides $q^{e(q)} - 1$.

Theorem 8.0.8. *For any $\lambda \in \Lambda_{e,w,\rho}$, the Green correspondent $D_{\mathbf{k},q}(\lambda)\otimes_{kG_0(q)}$*

$\boldsymbol{X}_w(q)$ of $D_{\mathbf{k},q}(\lambda)$ is independent of q in the following sense:

Assume that $e(q) = e(q')$ and $r(q) = r(q')$. Let $\mathcal{M}_{q,q'}$ be the canonical $(\mathbf{k}H_w(q)f_q, \mathbf{k}H_w(q')f_{q'})$-bimodule which induces a Morita equivalence between $\mathbf{k}H_w(q)f_q$ and $\mathbf{k}H_w(q')f_{q'}$, due to A. Marcus. Then $B_{w,\rho}(q)$ and $B_{w,\rho}(q')$ are Morita equivalent by $\boldsymbol{X}_{q,q'} := \boldsymbol{X}_w(q) \otimes_{\mathbf{k}H_w(q)f_q} \mathcal{M}_{q,q'} \otimes_{\mathbf{k}H_w(q')f_{q'}} \boldsymbol{X}_w(q')^\vee$ such that

$$D_{\mathbf{k},q}(\lambda) \otimes_{B_{w,\rho}(q)} \boldsymbol{X}_{q,q'} \cong D_{\mathbf{k},q'}(\lambda). \tag{8.1}$$

Remark

One might consider this special block $B_{w,\rho}$ as a starting point to investigate the other unipotent blocks with e-weight w.

This theorem means that if one wants to classify all the blocks, up to Morita equivalence, arising in the abelian defect unipotent blocks of various finite general linear groups with various defining fields, then one need not move q.

In particular if $B_{w,\rho}(q)$ and $B_{w,\lambda}(q)$ form a $[w : k]$-pair with $k \geq w$, and $e(q) = e(q')$ and $r(q) = r(q')$, then $B_{w,\lambda}(q)$ and $B_{w,\lambda}(q')$ are Morita equivalent. Moreover, this equivalence translates simple objects to simple objects and preserves the natural indices of simple objects, say partitions.

We will discuss unipotent blocks $B_{w,\lambda}$ and $B_{w,\mu}$ which form a $[w:w\text{–}1]$-pair in the next section.

Proof of Theorem 8.0.8

At first, if $e = e(q) = e(q')$ and $r(q) = r(q')$, then $B_0(L_{(e)}(q))$ and $B_0(L_{(e)}(q'))$ are Morita equivalent due to Fong-Srinivasan [17]. (Moreover, this is also Puig equivalent.) Moreover, there exists a $(B_0(L_{(e)}(q)), B_0(L_{(e)}(q')))$-bimodule $\mathcal{M}_{(e)}$ such that

$$\begin{array}{ccc}
\mathcal{M}_{(e)} \bigotimes_{B_0(L_{(e)}(q'))} \mathcal{M}^{\vee}_{(e)} & \cong & B_0(L_{(e)}(q)) \\
\mathcal{M}^{\vee}_{(e)} \bigotimes_{B_0(L_{(e)}(q))} \mathcal{M}_{(e)} & \cong & B_0(L_{(e)}(q')) \\
\text{(this means a} & \text{Morita} & \text{equivalence.)} \\
D_{\mathbf{k},q}(\lambda) \otimes_{B_0(L_{(e)}(q))} \mathcal{M}_{(e)} & \cong & D_{\mathbf{k},q'}(\lambda) \\
 & & \text{for any } \lambda \vdash e.
\end{array}$$

So, the same thing is true for $L_{(e^w)}(q)$ and $L_{(e^w)}(q')$. i.e.

There exists a $(B_0(L_{(e^w)}(q)), B_0(L_{(e^w)}(q')))$-bimodule $\mathcal{M}_{(e^w)}$ such that

$$\begin{array}{ccc}
\mathcal{M}_{(e^w)} \bigotimes_{B_0(L_{(e^w)}(q'))} \mathcal{M}^{\vee}_{(e^w)} & \cong & B_0(L_{(e^w)}(q)) \\
\mathcal{M}^{\vee}_{(e^w)} \bigotimes_{B_0(L_{(e^w)}(q))} \mathcal{M}_{(e^w)} & \cong & B_0(L_{(e^w)}(q')) \\
(\boxtimes_{i=1}^{w} D_{\mathbf{k},q}(\lambda(i))) \otimes_{B_0(L_{(e^w)}(q))} \mathcal{M}_{(e^w)} & \cong & \boxtimes_{i=1}^{w} D_{\mathbf{k},q'}(\lambda(i)) \\
 & & \text{for any } \lambda(i) \vdash e \text{ for } i = 1, \ldots, w.
\end{array}$$

By [43, § 3], $\mathcal{M}_{(e^w)} \otimes_{B_0(L_{(e^w)}(q'))} B_0(N_w(q'))$ and $B_0(N_w(q)) \otimes_{B_0(L_{(e^w)}(q'))} \mathcal{M}_{(e^w)}$ are isomorphic $\mathfrak{S}_w$-graded $(B_0(N_w(q)), B_0(N_w(q')))$-bimodule, and denote it by $\mathcal{M}_{w,q,q'}$. Then the following holds:

$$\begin{array}{ccccc}
\mathcal{M}_{w,q,q'} & \bigotimes_{B_0(N_w(q'))} & \mathcal{M}^{\vee}_{w,q,q'} & \cong & B_0(N_w(q)) \\
\mathcal{M}^{\vee}_{w,q,q'} & \bigotimes_{B_0(N_w(q))} & \mathcal{M}_{w,q,q'} & \cong & B_0(N_w(q')) \\
V_{k,q}(\boldsymbol{\lambda}) & \otimes_{B_0(N_w(q))} & \mathcal{M}_{w,q,q'} & \cong & V_{k,q'}(\boldsymbol{\lambda}) \\
 & & & & \text{for any } \boldsymbol{\lambda} \in \mathcal{MP}(w, e).
\end{array} \tag{8.2}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\text{mod-}B_0(N_w(q)) \arrow[r, "- \otimes \mathcal{M}_{w,q,q'}"] & \text{mod-}B_0(N_w(q')) \\
\text{mod-}B_0(L_{(e^w)}(q)) \arrow[u, "\mathrm{Ind}"] \arrow[r, "- \otimes \mathcal{M}_{(e^w)}"'] & \text{mod-}B_0(L_{(e^w)}(q')) \arrow[u, "\mathrm{Ind}"']
\end{tikzcd} \tag{8.3}$$

Hence, since both $B_{0,\rho}(q)$ and $B_{0,\rho}(q')$ have defect zero, we also know that there exist exact $(\mathbf{k}H_w(q)f_q, \mathbf{k}H_w(q')f_{q'})$-bimodule $\mathcal{M}^{\rho}_{w,q,q'}$ and its dual $\mathcal{M}^{\rho\ \vee}_{w,q,q'}$ such that

$$\begin{array}{cccccc} \mathcal{M}^{\rho}_{w,q,q'} & \bigotimes_{\mathbf{k}H_w(q')f_{q'}} & \mathcal{M}^{\rho\ \vee}_{w,q,q'} & \cong & \mathbf{k}H_w(q)f_q & \\ \mathcal{M}^{\rho\ \vee}_{w,q,q'} & \bigotimes_{\mathbf{k}H_w(q)f_q} & \mathcal{M}_{w,q,q'} & \cong & \mathbf{k}H_w(q')f_{q'} & \\ V_{k,q}(\boldsymbol{\lambda})^{\rho} & \otimes_{\mathbf{k}H_w(q)f_q} & \mathcal{M}^{\rho}_{w,q,q'} & \cong & V_{k,q'}(\boldsymbol{\lambda})^{\rho} & \\ & & & \text{for} & \text{any } \boldsymbol{\lambda} \in \mathcal{MP}(w,e). & \end{array} \tag{8.4}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd} \text{mod-}H_w(q)f_q \arrow[r, "- \otimes \mathcal{M}^{\rho}_{w,q,q'}"] & \text{mod-}H_w(q')f_{q'} \\ \text{mod-}(B_0(L_{(e^w)}(q)) \boxtimes_{\mathbf{k}} B_{0,\rho}(q)) \arrow[u, "\text{Ind}"] \arrow[r, "- \otimes \mathcal{M}_{(e^w,\rho)}"'] & \text{mod-}(B_0(L_{(e^w)}(q')) \boxtimes_{\mathbf{k}} B_{0,\rho}(q')) \arrow[u, "\text{Ind}"'] \end{tikzcd} \tag{8.5}$$

Combining these observations and Theorem 7.1.1, we complete the proof.

□

Chapter 9

$[w:w{-}1]$-pair

In this chapter we would like to discuss an equivalence between $B_{w,\tau}$ and $B_{w,\upsilon}$ where τ and υ form a $[w:w-1]$-pair. Actually, one can easily know that this equivalence is a splendid equivalence once one knows Scopes' article [56]. But we present this result here for one application of Theorem 7.1.1. We would like to just mimic the argument in [56] and [8].

9.1 Harish-Chandra induction and restriction for characters

Suppose that τ and υ are e-cores such that $\tau \supset \upsilon$ and τ and υ form a $[w:w-1]$-pair. Let G be $GL_{ew+|\tau|}(q)$ and $\overline{G}$ be $GL_{ew+|\upsilon|}(q)$. Let $\overline{G} \times G'$ be a Levi subgroup of G. $\{D_{\mathbf{k}}(\nu) \mid \nu \in \Lambda_{e,w,\tau}\}$ (resp. $\{D_{\mathbf{k}}(\nu) \mid \nu \in \Lambda_{e,w,\upsilon}\}$) is a complete set of the simple $B_{w,\tau}$-modules (resp. $B_{w,\upsilon}$-modules). Let $f_{w,\tau}$ (resp. $f_{w,\upsilon}$) be the unipotent block idempotent of $B_{w,\tau}$ (resp. $B_{w,\upsilon}$). Let i be a non-negative integer from 0 to $e-1$ such that the abacus configuration with n beads of υ is obtained from the abacus configuration with n-beads of τ by sliding the bottom $w-1$ beads of the i-th runner into the $(i-1 \mod e)$-th runner.

Definition 9.1.1. *For $j = 0, \ldots, w$ let β^j be an element of $\Lambda_{e,w,\tau}$ whose e-quotient is such that $(\beta^j)^{(i-1)} = (j)$ and $(\beta^j)^{(i)} = (1^{w-j})$.*

For $j = 0, \cdots, w$, let β_-^j be an element of $\Lambda_{e,w,v}$ whose e-quotient is such that $(\beta_-^j)^{(i-1)} = (1^j)$ and $(\beta_-^j)^{(i)} = (w-j)$.

For $\lambda \in \Lambda_{e,w,\tau}$ other than $\beta^0, \beta^1, \ldots, \beta^{w-1}, \beta^w$, let $\Phi(\lambda)$ equal λ_- where ${f_{w,v}}^* R^G_{L_{(n-w+1,1^{w-1})}}([S_{\mathbf{k}}(\lambda)]) \cong (w-1)![S_{\mathbf{k}}(\lambda_-)]$.

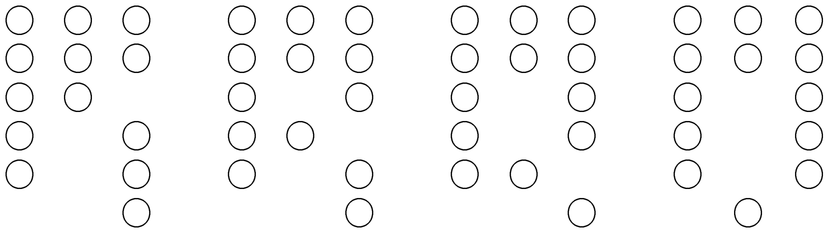
Let $\Lambda^{\checkmark}_{e,w,\tau}$ be the set of elements in $\Lambda_{e,w,\tau}$ other than $\beta^0, \beta^1, \ldots, \beta^{w-1}, \beta^w$.

Let $\Lambda^{\checkmark}_{e,w,v}$ be the set of elements in $\Lambda_{e,w,v}$ other than $\beta^0_-, \beta^1_-, \ldots, \beta^{w-1}_-, \beta^w_-$.

Clearly we can know from the abacus configuration that $\beta^0 > \beta^1 > \cdots > \beta^{w-1} > \beta^w$, $\beta^0_- > \beta^1_- > \cdots > \beta^{w-1}_- > \beta^w_-$ and β^0 is e-regular.

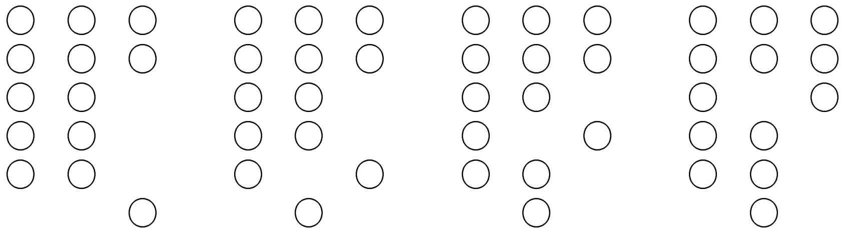
Example

We would like to deal a quite small example. Consider the case $e = 3, w = 3$, $GL_{11}(q)$ and $GL_{13}(q)$. Then we find two blocks $B_{3,(1^2)}$ of $GL_{11}(q)$ and $B_{3,(2,1^2)}$ of $GL_{13}(q)$. We use 13 beads since it is enough to display all the partitions of $n \leq 13$ on our abacus.



β^0 β^1 β^2 β^3_-

Here, $\beta^0 = (5, 3, 2^2, 1), \beta^1 = (5, 3, 2, 1^3), \beta^2 = (5, 2^3, 1^2)$ and $\beta^3 = (4, 3, 2^2, 1^2)$.



β^0_- β^1_- β^2_- β^3_-

Here, $\beta^0_- = (5, 2^2, 1^2), \beta^1_- = (4, 3, 2, 1^2), \beta^2_- = (4, 2^3, 1)$ and $\beta^3_- = (4, 2^2, 1^3)$.

Easily we get the next two lemmas once one see Scopes' article [56].

Lemma 9.1.2. (*Compare with* [*56*, *p.*208 *Lemma* 4.2.])

1. *$D_{\mathbf{k}}(\beta^0)$ occurs exactly once as a composition factor in each of $S_{\mathbf{k}}(\beta^0)$, $S_{\mathbf{k}}(\beta^1)$, ..., $S_{\mathbf{k}}(\beta^{w-1})$ and $S_{\mathbf{k}}(\beta^w)$. For any $\lambda \in \Lambda^{\checkmark}_{e,w,\tau}$, $S_{\mathbf{k}}(\lambda)$ has no composition factor isomorphic to $D_{\mathbf{k}}(\beta^0)$.*

2. *The diagnal Cartan entry c_{β^0,β^0} corresponding to $D_{\mathbf{k}}(\beta^0)$ is equal to $w + 1$.*

3. *For $j = 1, \dots, w-1$, $D_{\mathbf{k}}(\beta^j)$ occurs as a composition factor of $S_{\mathbf{k}}(\beta^{j+1})$.*

4. *$D_{\mathbf{k}}(\beta^0_-)$ occurs exactly once as a composition factor in each of $S_{\mathbf{k}}(\beta^0_-)$, $S_{\mathbf{k}}(\beta^1_-)$, ... , $S_{\mathbf{k}}(\beta^{w-1}_-)$ and $S_{\mathbf{k}}(\beta^w_-)$. For any $\mu \in \Lambda^{\checkmark}_{e,w,v}$, $S_{\mathbf{k}}(\mu)$ has no composition factor isomorphic to $D_{\mathbf{k}}(\beta^0_-)$.*

5. *The diagnal Cartan entry $c_{\beta^0_-,\beta^0_-}$ corresponding to $D_{\mathbf{k}}(\beta^0_-)$ is equal to $w+1$.*

6. *For $j = 1, \ldots, w-1$, $D_{\mathbf{k}}(\beta^j_-)$ occurs as a composition factor of $S_{\mathbf{k}}(\beta^{j+1}_-)$.*

Lemma 9.1.3. (*Compare with* [*56*, *p.*206 *Lemma* 3.3])

1. *For $\lambda \in \Lambda^{\checkmark}_{e,w,\tau}$, there exists a unique $\lambda_- \in \Lambda^{\checkmark}_{e,w,v}$ such that*

$$f_{w,v}{}^*R^G_{L_{(n-w+1,1^{w-1})}}([S_{\mathbf{k}}(\lambda)]) \cong (w-1)![S_{\mathbf{k}}(\lambda_-)]$$
$$f_{w,\tau}R^G_{L_{(n-w+1,1^{w-1})}}([S_{\mathbf{k}}(\lambda_-)]) \cong (w-1)![S_{\mathbf{k}}(\lambda)].$$

2. *The corresponding results for $\beta^0, \beta^1, \ldots, \beta^{w-1}$ and β^w are*

$$\begin{aligned} f_{w,\tau}R^G_{L_{(n-w+1,1^{w-1})}}([S_{\mathbf{k}}(\beta^m_-)]) &\cong (w-1)!(\textstyle\bigoplus_{j=0,\neq w-m}^{w}[S_{\mathbf{k}}(\beta^j)])), \\ f_{w,v}{}^*R^{GL_n(q)}_{L_{(n-w+1,1^{w-1})}}([S_{\mathbf{k}}(\beta^m)]), &\cong (w-1)!(\textstyle\bigoplus_{j=0,\neq w-m}^{w}[S_{\mathbf{k}}(\beta^j_-)]). \end{aligned}$$

3. *For $\beta^0, \beta^1, \ldots, \beta^{w-1}$ and β^w, we have*

$$\begin{aligned} f_{w,v}{}^*R^{GL_n(q)}_{L_{(n-w+1,1^{w-1})}}([D_{\mathbf{k}}(\beta^0)]) &\cong (w-1)! \ ([D_{\mathbf{k}}(\beta^0_-)] \oplus (\textstyle\bigoplus_{j=1}^{w-1}[S_{\mathbf{k}}(\beta^j_-)])) \\ f_{w,\tau}R^{GL_n(q)}_{L_{(n-w+1,1^{w-1})}}([D_{\mathbf{k}}(\beta^0_-)]) &\cong (w-1)! \ ([D_{\mathbf{k}}(\beta^0)] \oplus (\textstyle\bigoplus_{j=1}^{w-1}[S_{\mathbf{k}}(\beta^j)])), \\ f_{w,v}{}^*R^{GL_n(q)}_{L_{(n-w+1,1^{w-1})}}([D_{\mathbf{k}}(\beta^m)]) &\cong (w-1)! \ [D_{\mathbf{k}}(\beta^m_-)], \\ f_{w,\tau}R^{GL_n(q)}_{L_{(n-w+1,1^{w-1})}}([D_{\mathbf{k}}(\beta^m_-)]) &\cong (w-1)! \ [D_{\mathbf{k}}(\beta^m)]. \end{aligned}$$

Example

Consider the case where $e = 3, w = 3, \tau = (1^2)$ and $\nu = (2, 1^2)$. Recall that $\beta^0_- = (5, 2^2, 1^2), \beta^1 = (4, 3, 2, 1^2)$, $\beta^2_- = (4, 2^3, 1)$, $\beta^3_- = (4, 2^2, 1^3)$, $\beta^0 =$

$(5,3,2^2,1)$, $\beta^1 = (5,3,2,1^3)$, $\beta^2 = (5,2^3,1^2)$ and $\beta^3 = (4,3,2^2,1^2)$ in the previous example. Then, for $\lambda \in \Lambda_{3,3,(1^2)}$ $f_{3,(2,1^2)} R^{GL_{13}(q)}_{L_{(11,1^2)}} S_{\mathbf{k}}(\lambda)$ is as follows:

$$
\begin{array}{lcl}
S_{\mathbf{k}}(10,1) & \to & 2\,(S_{\mathbf{k}}(11,1^2)) \\
S_{\mathbf{k}}(9,2) & \to & 2\,(S_{\mathbf{k}}(9,3,1)) \\
S_{\mathbf{k}}(7,4) & \to & 2\,(S_{\mathbf{k}}(8,4,1)) \\
S_{\mathbf{k}}(7,2^2) & \to & 2\,(S_{\mathbf{k}}(8,3,2)) \\
S_{\mathbf{k}}(7,1^4) & \to & 2\,(S_{\mathbf{k}}(8,1^5)) \\
S_{\mathbf{k}}(6,5) & \to & 2\,(S_{\mathbf{k}}(6^2,1)) \\
S_{\mathbf{k}}(6,2^2,1) & \to & 2\,(S_{\mathbf{k}}(6,3,2^2)) \\
S_{\mathbf{k}}(6,2,1^3) & \to & 2\,(S_{\mathbf{k}}(6,3,1^4)) \\
S_{\mathbf{k}}(5,2^2,1^2) & \to & 2\,(S_{\mathbf{k}}(5,3,2^2,1) \oplus S_{\mathbf{k}}(5,3,2,1^3) \oplus S_{\mathbf{k}}(5,2^3,1^2)) \\
S_{\mathbf{k}}(4^2,3) & \to & 2\,(S_{\mathbf{k}}(5,4^2)) \\
S_{\mathbf{k}}(4^2,2,1) & \to & 2\,(S_{\mathbf{k}}(5,4,2^2)) \\
S_{\mathbf{k}}(4^2,1^3) & \to & 2\,(S_{\mathbf{k}}(5,4,1^4)) \\
S_{\mathbf{k}}(4,3,2,1^2) & \to & 2\,(S_{\mathbf{k}}(5,3,2^2,1) \oplus S_{\mathbf{k}}(5,3,2,1^3) \oplus S_{\mathbf{k}}(4,3,2^2,1^2)) \\
S_{\mathbf{k}}(4,2^3,1) & \to & 2\,(S_{\mathbf{k}}(5,3,2^2,1) \oplus S_{\mathbf{k}}(5,2^3,1^2) \oplus S_{\mathbf{k}}(4,3,2^2,1^2)) \\
S_{\mathbf{k}}(4,2^2,1^3) & \to & 2\,(S_{\mathbf{k}}(5,3,2,1^3) \oplus S_{\mathbf{k}}(5,2^3,1^2) \oplus S_{\mathbf{k}}(4,3,2^2,1^2)) \\
S_{\mathbf{k}}(4,1^7) & \to & 2\,(S_{\mathbf{k}}(5,1^8)) \\
S_{\mathbf{k}}(3^3,1^2) & \to & 2\,(S_{\mathbf{k}}(3^3,2,1^2)) \\
S_{\mathbf{k}}(3,2^4) & \to & 2\,(S_{\mathbf{k}}(3^2,2^3,1)) \\
S_{\mathbf{k}}(3,2^2,1^4) & \to & 2\,(S_{\mathbf{k}}(3^2,2^2,1^3)) \\
S_{\mathbf{k}}(3,2,1^6) & \to & 2\,(S_{\mathbf{k}}(3^2,1^7)) \\
S_{\mathbf{k}}(2^3,1^5) & \to & 2\,(S_{\mathbf{k}}(2^4,1^5)) \\
S_{\mathbf{k}}(1^{11}) & \to & 2\,(S_{\mathbf{k}}(2,1^{11}))
\end{array}
$$

Definition 9.1.4. *Let $\Phi(\beta^j) = \beta^j_-$ for $j \in [0,\dots,w]$.*

Lemma 9.1.5. *Let $P(D_{\mathbf{k}}(\beta^0))$ be the projective cover of $D_{\mathbf{k}}(\beta^0)$ as a $B_{w,\tau}$-module and let $P(D_{\mathbf{k}}(\Phi(\beta^0)))$ be the projective cover of $D_{\mathbf{k}}(\Phi(\beta^0))$ as a $B_{w,\nu}$-module. Then,*

$$\mathbf{ch}\left(K \otimes_{\mathcal{O}} P_{\mathcal{O}}(D_{\mathbf{k}}(\beta^0))\right) = \sum_{j=0}^{w} \chi_{\beta^j}$$

and

$$\mathbf{ch}\left(K \otimes_{\mathcal{O}} P_{\mathcal{O}}(D_{\mathbf{k}}(\Phi(\beta^0)))\right) = \sum_{j=0}^{w} \chi_{\Phi(\beta^j)}$$

So, Lemma 9.1.3 means the following:

Lemma 9.1.6. *1. For any* $\lambda_- \in \Lambda^{\surd}_{e,w,v}$,

$$f_{w,\tau} R^{GL_n(q)}_{L_{(n-w+1,w-1)}}([S_K(\lambda_-) \boxtimes_K S_K(w-1)]) = [S_K(\lambda)]$$

2. For any $\beta^m,\ m \in [0,\dots,w]$

$$f_{w,\tau} R^{GL_n(q)}_{L_{(n-w+1,w-1)}}([S_K(\beta^m) \boxtimes_K S_K(w-1)]) \cong \bigoplus_{j=0,\neq w-m}^{w} [S_K(\beta^j_-)]$$

3. For any $\lambda_- \in \Lambda_{e,w,v} \backslash \left\{\beta^0_-\right\}$,

$$f_{w,\tau} R^{GL_n(q)}_{L_{(n-w+1,w-1)}}(D_{\mathbf{k}}(\lambda_-) \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(w-1)) \cong D_{\mathbf{k}}(\lambda)$$

4. For any $\lambda \in \Lambda_{e,w,\tau} \backslash \{\beta^0\}$,

$$f_{w,v}\, {}^*R^{GL_n(q)}_{L_{(n-w+1,w-1)}}(D_{\mathbf{k}}(\lambda)) \cong D_{\mathbf{k}}(\lambda_-) \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(w-1).$$

5.

$$f_{w,\tau} R^{GL_n(q)}_{L_{(n-w+1,w-1)}}([D_{\mathbf{k}}(\beta^0_-) \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(w-1)]) = [D_{\mathbf{k}}(\beta^0)] \oplus \bigoplus_{j=1}^{w-1} [S_{\mathbf{k}}(\beta^j)].$$

6.

$$f_{w,v}\, {}^*R^{GL_n(q)}_{L_{(n-w+1,w-1)}}([D_{\mathbf{k}}(\beta^0)])$$
$$= [D_{\mathbf{k}}(\beta^0_-) \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(w-1)] \oplus \bigoplus_{j=1}^{w-1} [S_{\mathbf{k}}(\beta^j_-) \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(w-1)].$$

Proposition 9.1.7. (*Compare with* [*56*, *p.*207 *Lemma* 3.5.])

Φ preserves the lexicographic ordering of partitions between $\Lambda^{\surd}_{e,w,\tau}$ and $\Lambda^{\surd}_{e,w,v}$.

Φ preserves e-singularity and regularity between $\Lambda^{\surd}_{e,w,\tau}$ and $\Lambda^{\surd}_{e,w,v}$.

9.2 Derived equivalences

Since $(K, \mathcal{O}, \mathbf{k})$ is an ℓ-modular system we can lift idempodents $f_{w,\tau}$ and $f_{w,v}$ to idempotents $\widehat{f}_{w,\tau}$ and $\widehat{f}_{w,v}$ in $\mathcal{O}G$ and $\mathcal{O}\overline{G}$. Let $\mathcal{O}B_{w,\tau}$ and $\mathcal{O}B_{w,v}$ be the corresponding blocks.

We would like to introduce the argument in [55, Section 4]. The functors G_3 and G_3'

$$\text{mod-}\mathbf{k}[\overline{G} \times G'] \underset{\mathsf{G}_3{}'}{\overset{\mathsf{G}_3}{\rightleftarrows}} \text{mod-}\mathbf{k}[\overline{G}]$$

are defined as follows. Consider the module categories of $\mathbf{k}[\overline{G} \times G']$ and of $\mathbf{k}\overline{G}$ as the categories of $\left(\mathbf{k}G'^{\circ}, \mathbf{k}\overline{G}\right)$- and $(\mathbf{k}, \mathbf{k}\overline{G})$-bimodules, respectively. (i.e. $\text{mod-}\mathbf{k}[\overline{G} \times G'] \cong \mathbf{k}[G'^{\circ}]\text{-mod-}\mathbf{k}[\overline{G}]$ and $\text{mod-}\mathbf{k}[\overline{G}] \cong \mathbf{k}\text{-mod-}\mathbf{k}[\overline{G}]$.)

Since $S_{\mathbf{k}}(w-1)$ is the trivial right $\mathbf{k}G'$-module, this module is simple and self-dual. We may regard it as $(\mathbf{k}, \mathbf{k}[G'])$-bimodule, and denote it by $\mathbf{k}[w-1]$.

Given a $(\mathbf{k}G'^{\circ}, \mathbf{k}\overline{G})$-bimodule M define

$$\mathsf{G}_3(M) = \mathbf{k}[w-1] \otimes_{\mathbf{k}G'^{\circ}} M.$$

Given a $(\mathbf{k}G'^{,\circ}, \mathbf{k}\overline{G})$-bimodule homomorphism $h : M \to M'$ define

$$\mathsf{G}_3(h) = 1 \otimes h.$$

Given a $(\mathbf{k}, \mathbf{k}\overline{G})$-bimodule N define

$$\mathsf{G}_3'(N) = \mathrm{Hom}_{\mathbf{k}}({}_{\mathbf{k}}\mathbf{k}[w-1]_{\mathbf{k}G'^{\circ}}, {}_{\mathbf{k}}N_{\mathbf{k}\overline{G}}).$$

Given a $(\mathbf{k}, \mathbf{k}\overline{G})$-bimodule homomorphism $h' : N \to N'$ define $\mathsf{G}_3'(h')$ by composition with h'. By [55, Lemma 4.1] and [34, p.137], the functor G_3 is left adjoint to G_3' and the functor G_3' is exact.

In particular, if $w \leq e$ then G_3' induces a Morita equivalence between $B_{w,v}$ and $B_{w,v} \otimes_{\mathbf{k}} B_{0,(w-1)}$.

Just mimicking the argument in J. Chuang's paper [8, pp.117-119], we get the following:

Proposition 9.2.1. *Suppose that $w < \ell$. Let M be a $(B_{w,\tau}, B_{w,v})$ -bimodule which induces $f_{w,v} \circ {}^*R^{GL_n(q)}_{L_{(n-w+1,w-1)}} \circ \mathsf{G}_3' \circ f_{w,\tau}$.*

Let $\delta' : P' \to M$ be a projective cover of M. Then

$$P' \cong \bigoplus_{\lambda \in \Lambda_{e,w,\tau}} P(D_{\mathbf{k}}(\lambda))^{\vee} \boxtimes_{\mathbf{k}} P(D_{\mathbf{k}}(\Phi(\lambda)))$$

Here, $P(N)$ denotes the projective cover of a module N. Let P be a direct summand of P' isomorphic to $P(D_{\mathbf{k}}(\alpha))^{\vee} \boxtimes_{\mathbf{k}} P(D_{\mathbf{k}}(\Phi(\alpha)))$, and let δ be the restriction of δ' to P. Then, the following complex of $(B_{w,\tau}, B_{w,v})$-bimodules with M in degree 0

$$\boldsymbol{X} = (0 \to P \to M \to 0)$$

give a splendid Rickard equivalence between $B_{w,\tau}$ and $B_{w,v}$. Namely, $B_{w,\tau}$ and $B_{w,v}$ are splendid Richard equivalent

Remark

We can also prove this proposition as follows:

At first show that $B_{w,\tau}$ and $B_{w,\upsilon}$ are stable equivalent of Morita type by [5, p.24] ,[53, § 4] and checking that for any proper non-trivial subgroup Q of defect group of $B_{w,\tau}$ and $B_{w,\upsilon}$ blocks $\mathrm{Br}_{\Delta(Q)}(B_{w,\tau})$ of $\mathbf{k}C_{GL_{ew+|\tau|}(q)}(Q)$ and $\mathrm{Br}_{\Delta(Q)}(B_{w,\upsilon})$ of $\mathbf{k}C_{GL_{ew+|\upsilon|}(q)}(Q)$ form a $[w-1:k]$-pair with $k \geq w-1$ and are Morita equivalent by an ℓ-parmutation bimodule due to [34]. Here, Br is the Brauer functor.

Next choose an except index set as $\{\beta^0\}$ (I_0 in [48]) , check Lemma 9.1.2 and 9.1.6 , and apply Okuyama's lemma [48, Lemma 2.1] to $B_{w,\tau}$ and $B_{w,\upsilon}$.

By Proposition 9.2.1, T. Jost's result[34], and Theorem 8.0.8 we get

Corollary 9.2.2. *If there exists a sequence of e-cores*

$$(\rho = \tau^0, \tau^1, \tau^2, \cdots, \tau^s = \upsilon)$$

for some intger s such that Λ_{e,w,τ^j} and $\Lambda_{e,w,\tau^{j+1}}$ form a $[w : k_j]$-pair with $k_j \geq w-1$, then Broué's conjecture holds for $B_{w,\upsilon}$.

Moreover, if $e(q) = e(q')$ and $r(q) = r(q')$ then $B_{w,\upsilon}(q)$ and $B_{w,\upsilon}(q')$ are Morita equivalent. And this Morita equivalence preserves the natural indices of simple modules of $B_{w,\upsilon}(q)$ and $B_{w,\upsilon}(q')$. More presicely, $D_{\mathbf{k},q}(\lambda)$ (resp. $S_{\mathbf{k},q}(\lambda), X_{\mathbf{k},q}(\lambda)$) corresponds to $D_{\mathbf{k},q'}(\lambda)$ (resp. $S_{\mathbf{k},q'}(\lambda), X_{\mathbf{k},q'}(\lambda)$) for $\lambda \in \Lambda_{e,w,\upsilon}$ by this Morita equivalence.

Chapter 10

Decomposition matrices for $B_{w,\rho}$

In this chapter, we construct the decomposition matrix of $B_{w,\rho}$ where $\ell > w$ and ρ being a CKR e-core relative to w (see the Definition 2.2.1). Recall the indices of simple $B_{w,\rho}$-modules which are described in Chapter 6.

Let $(-)^\natural : \mathcal{MP}(e,w) \to \mathcal{MP}(e,w)$ be a bijection defined by

$$(\boldsymbol{\lambda}^\natural)_i = \begin{cases} \boldsymbol{\lambda}_i & \text{if } i+e : \text{even} \\ \boldsymbol{\lambda}'_i & \text{if } i+e : \text{odd} \end{cases}$$

for $\boldsymbol{\lambda} \in \mathcal{MP}(e,w)$.

So, for $\boldsymbol{\lambda} \vdash \boldsymbol{n}$, $U_R(\boldsymbol{\lambda}^\natural)$ (resp. $V_{\mathbf{k}}(\boldsymbol{\lambda}^\natural)$) is isomorphic to $\mathrm{Ind}_{L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}}^{GL_e(q) \wr \mathfrak{S}_w} \left(S_R(\boldsymbol{n}) \otimes S_R^{\boldsymbol{\lambda}}\right)$ (resp. $\mathrm{Ind}_{L_{(e^w)} \rtimes \mathfrak{S}_{\boldsymbol{n}}}^{GL_e(q) \wr \mathfrak{S}_w} \left(D_R(\boldsymbol{n}) \otimes S_R^{\boldsymbol{\lambda}}\right)$).

For $i = 1, \dots, e$ and an integer $\boldsymbol{n}_i$, we know the following decomposition:

$$\mathrm{Ind}_{L_{(e^{\boldsymbol{n}_i})}}^{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{\boldsymbol{n}_i}} \left(S_{\mathbf{k}}(\nu_i)^{\boxtimes \boldsymbol{n}_i}\right) = \bigoplus_{\lambda \vdash \boldsymbol{n}_i} \left(S_{\mathbf{k}}(\nu_i)^{\boxtimes \boldsymbol{n}_i}\right) \otimes_{\mathbf{k}} (\dim_{\mathbf{k}} S_{\mathbf{k}}^{\lambda}) \cdot S_{\mathbf{k}}^{\lambda}.$$

For $i = 1, \dots, e$ and a partition λ, let $U_{\mathbf{k}}(i,\lambda)$ be $\left(S_{\mathbf{k}}(\nu_i)^{\boxtimes \boldsymbol{n}_i}\right) \otimes_{\mathbf{k}} S_{\mathbf{k}}^{\lambda}$ and let

$V_{\mathbf{k}}(i,\lambda)$ be $\left(D_{\mathbf{k}}(\nu_i)^{\boxtimes \boldsymbol{n}_i}\right) \otimes_{\mathbf{k}} S_{\mathbf{k}}^{\lambda}$ and $V_{\mathbf{k}}^{\natural}(i,\lambda)$ be

$$\begin{cases} V_{\mathbf{k}}(i,\lambda) & \text{if } i+e \text{ is even,} \\ V_{\mathbf{k}}(i,\lambda') & \text{if } i+e \text{ is odd.} \end{cases}$$

First, we look at $U_{\mathbf{k}}(i,(\boldsymbol{n}_i))$ for $i=1,\ldots,e$. Note that $\mathfrak{S}_{\boldsymbol{n}_i}$ acts $\left(S_{\mathbf{k}}(\nu_i)^{\boxtimes \boldsymbol{n}_i}\right)$ as permutations of each bases $\{v_1 \otimes v_2 \otimes \cdots \otimes v_{\boldsymbol{n}_i} \in (S_{\mathbf{k}}(\nu_i)^{\otimes \boldsymbol{n}_i})\}$. So, for $j=0,\ldots,\boldsymbol{n}_i$ the $j+1$-th Loewy layer $\mathcal{L}^j(U_{\mathbf{k}}(i,(\boldsymbol{n}_i)))$ is

$$\begin{aligned} & \operatorname{Ind}_{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{(\boldsymbol{n}_i-j,j)}}^{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{\boldsymbol{n}_i}} \left(\left(D_{\mathbf{k}}(\nu_i)^{\boxtimes \boldsymbol{n}_i-j} \otimes_{\mathbf{k}} S_{\mathbf{k}}^{(\boldsymbol{n}_i-j)}\right)\right. \\ & \qquad\qquad\qquad\qquad \left.\boxtimes_{\mathbf{k}} \left(D_{\mathbf{k}}(\nu_{i+1})^{\boxtimes j} \otimes_{\mathbf{k}} S_{\mathbf{k}}^{(j)}\right)\right) \\ \cong\ & \operatorname{Ind}_{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{(\boldsymbol{n}_i-j,j)}}^{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{\boldsymbol{n}_i}} \left(V_{\mathbf{k}}(i,(\boldsymbol{n}_i-j)) \boxtimes_{\mathbf{k}} V_{\mathbf{k}}(i+1,(j))\right). \end{aligned}$$

In particular, $\mathcal{L}^j(U_{\mathbf{k}}(i,(\boldsymbol{n}_i)))$ is simple for any $0 \le j \le \boldsymbol{n}_i$, and $U_{\mathbf{k}}(i,(\boldsymbol{n}_i))$ is uniserial and has Loewy length $\boldsymbol{n}_i+1$. Let $l_{\mathbf{k}}(\lambda)$ be the Loewy length of $S_{\mathbf{k}}(\lambda)$.

For $\alpha \vdash \boldsymbol{n}_i - j$ and $\beta \vdash j$, let $V_{\mathbf{k}}(i,[\alpha,\beta])$ be

$$\operatorname{Ind}_{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{(\boldsymbol{n}_i-j,j)}}^{L_{(e^{\boldsymbol{n}_i})} \rtimes \mathfrak{S}_{\boldsymbol{n}_i}} \left(V_{\mathbf{k}}(i,\alpha) \boxtimes V_{\mathbf{k}}(i+1,\beta)\right).$$

For $\lambda \vdash \boldsymbol{n}_i$, tensoring $S_{\mathbf{k}}^{\lambda}$ to $U_{\mathbf{k}}(i,(\boldsymbol{n}_i))$, we get

$$\mathcal{L}^j(U_{\mathbf{k}}(i,\lambda)) \cong \bigoplus_{\alpha \vdash \boldsymbol{n}_i - j} \bigoplus_{\beta \vdash j} LR^{\lambda}_{\alpha,\beta} V_{\mathbf{k}}(i,[\alpha,\beta]).$$

For $k \le w$ and $\boldsymbol{n} \in \Lambda(e,w)$, let $\Lambda(e,k,\boldsymbol{n})$ be the set of all the elements $\mu \in \Lambda(e,k)$ satisfying $\mu_e = 0$ and $\mu_i \le \boldsymbol{n}_i$ for any i.

For $\boldsymbol{n}$ and $\gamma \in \Lambda(e,k,\boldsymbol{n})$, let $\boldsymbol{n}^{\gamma}$ be a composition of w into e parts defined by

$$(\boldsymbol{n}^{\gamma})_i := \begin{cases} \boldsymbol{n}_i - \gamma_1 & \text{if } i=1 \\ \boldsymbol{n}_i + \gamma_{i-1} - \gamma_i & \text{if } i=2,\ldots,e. \end{cases} \tag{10.1}$$

$$\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\lambda})) \cong \bigoplus_{\gamma\in\Lambda(e,k,\boldsymbol{n})} \mathrm{Ind}_{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}}^{L_{(e^w)}\rtimes\mathfrak{S}_w} \left(\boxtimes_{i=1}^{e} \mathcal{L}^{\gamma_i}(U_{\mathbf{k}}(i,\boldsymbol{\lambda}_i)) \right).$$

Hence, for $\boldsymbol{\lambda} \vdash \boldsymbol{n}$ with $|\boldsymbol{n}| = w$, we get

$$\mathrm{Ind}_{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}}^{L_{(e^w)}\rtimes\mathfrak{S}_w} \left(\boxtimes_{i=1}^{e} \mathcal{L}^{\gamma_i}(U_{\mathbf{k}}(i,\boldsymbol{\lambda}_i)) \right)$$

$$\cong \mathrm{Ind}_{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}}^{L_{(e^w)}\rtimes\mathfrak{S}_w} \boxtimes_{i=1}^{e} \left(\bigoplus_{\alpha^i\vdash\boldsymbol{n}_i-\gamma_i} \bigoplus_{\beta^i\vdash\gamma_i} LR_{\alpha^i,\beta^i}^{\boldsymbol{\lambda}_i} V_{\mathbf{k}}(i,[\alpha^i,\beta^i]) \right)$$

$$\cong \mathrm{Ind}_{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}}}^{L_{(e^w)}\rtimes\mathfrak{S}_w} \boxtimes_{i=1}^{e} \left(\bigoplus_{\alpha^i\vdash\boldsymbol{n}_i-\gamma_i} \bigoplus_{\beta^i\vdash\gamma_i} LR_{\alpha^i,\beta^i}^{\boldsymbol{\lambda}_i} \left(\mathrm{Ind}_{L_{(e^{\boldsymbol{n}_i})}\rtimes\mathfrak{S}_{(\boldsymbol{n}_i-\gamma_i,\gamma_i)}}^{L_{(e^{\boldsymbol{n}_i})}\rtimes\mathfrak{S}_{\boldsymbol{n}_i}} V_{\mathbf{k}}(i,\alpha^i) \boxtimes V_{\mathbf{k}}(i+1,\beta^i) \right) \right)$$

$$\cong \mathrm{Ind}_{L_{(e^w)}\rtimes\mathfrak{S}_{\boldsymbol{n}^\gamma}}^{L_{(e^w)}\rtimes\mathfrak{S}_w} \boxtimes_{i=1}^{e} \left(\bigoplus_{\alpha^i\vdash\boldsymbol{n}_i-\gamma_i} \bigoplus_{\beta^i\vdash\gamma_i} LR_{\alpha^i,\beta^i}^{\boldsymbol{\lambda}_i} \left(\bigoplus_{\boldsymbol{\mu}_i\vdash(\boldsymbol{n}^\gamma)_i} LR_{\beta^{i-1},\alpha^i}^{\boldsymbol{\mu}_i} V_{\mathbf{k}}^{\natural}(i,\boldsymbol{\mu}_i) \right) \right)$$

$$\cong \bigoplus_{\alpha\in\mathcal{MP}(\boldsymbol{n},\gamma)} \bigoplus_{\beta\vdash\gamma} \left(\prod_{j=1}^{e} LR_{\alpha^j,\beta^j}^{\boldsymbol{\lambda}_j} \right) \cdot \left(\bigoplus_{\boldsymbol{\mu}\in\mathcal{MP}(e,w)} (\prod_{j=1}^{e} LR_{\beta^{j-1},\alpha^j}^{\boldsymbol{\mu}_j}) \cdot V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural}) \right) \tag{10.2}$$

Here, for $\boldsymbol{n} \in \Lambda(e,w)$ and $\gamma \in \Lambda(e,k,\boldsymbol{n})$, $\mathcal{MP}(\boldsymbol{n},\gamma)$ denotes the set of multipartitions defined by $\{\alpha = (\alpha^1,\dots,\alpha^e) \mid \alpha^i \vdash \boldsymbol{n}_i - \gamma_i\}$. Conventionally, we set $LR_{\emptyset,\emptyset}^{\emptyset} = 1$ and $\beta^0 = \emptyset$.

Then, the equation (10.2) is modified as follows:

$$\cong \bigoplus_{\boldsymbol{\mu}\in\mathcal{MP}(e,w)} \bigoplus_{\alpha\in\mathcal{MP}(\boldsymbol{n},\gamma)} \bigoplus_{\beta\vdash\gamma} \left(\prod_{j=1}^{e} LR^{\boldsymbol{\lambda}_j}_{\alpha^j,\beta^j} \cdot LR^{\boldsymbol{\mu}_j}_{\beta^{j-1},\alpha^j} \right) \cdot V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural})$$

$$\cong \bigoplus_{\boldsymbol{\mu}\in\mathcal{MP}(e,w)} \left(\sum_{\alpha\in\mathcal{MP}(\boldsymbol{n},\gamma)} \sum_{\beta\vdash\gamma} \prod_{j=1}^{e} LR^{\boldsymbol{\lambda}_j}_{\alpha^j,\beta^j} \cdot LR^{\boldsymbol{\mu}_j}_{\beta^{j-1},\alpha^j} \right) \cdot V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural})$$

Therefore,

$$\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\lambda})) \cong \bigoplus_{\boldsymbol{\mu}\in\mathcal{MP}(e,w)} \bigoplus_{\gamma\in\Lambda(e,k,\boldsymbol{n})} \left(\sum_{\alpha\in\mathcal{MP}(\boldsymbol{n},\gamma)} \sum_{\beta\vdash\gamma} \prod_{i=1}^{e} LR^{\boldsymbol{\lambda}_i}_{\alpha^i,\beta^i} \cdot LR^{\boldsymbol{\mu}_i}_{\beta^{i-1},\alpha^i} \right) \cdot V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural}). \tag{10.3}$$

In particular, considering the case $LR^{\boldsymbol{\lambda}_i}_{\alpha^i,\beta^i} \neq 0$ and $LR^{\boldsymbol{\mu}_i}_{\beta^{i-1}\alpha^i} \neq 0$ in the right hand side of (10.3) for given $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we have

$$[\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\lambda})) : V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural})] = \sum_{\gamma\in\Lambda(e,k,\boldsymbol{n})} \sum_{\substack{\alpha\in\mathcal{MP}(\boldsymbol{n},\gamma),\beta\vdash\gamma \\ |\alpha^i|+|\beta^i|=|\boldsymbol{\lambda}_i| \text{ if } i\neq e, \\ |\beta^{i-1}|+|\alpha^i|=|\boldsymbol{\mu}_i|, \\ \beta^0=\emptyset,}} \prod_{i=1}^{e} LR^{\boldsymbol{\lambda}_i}_{\alpha^i,\beta^i} \cdot LR^{\boldsymbol{\mu}_i}_{\beta^{i-1},\alpha^i}. \tag{10.4}$$

We want to refine the above formula (10.4) in order to find non-zero parts. Looking at a relation between the size of $\boldsymbol{\lambda}_i$ and the size of $\boldsymbol{\mu}_i$ in (10.1) and the second summation in the formula (10.4), we know that β must satisfy

$$\beta^0 = \emptyset, |\beta^{i-1}| = |\boldsymbol{\mu}_i| - |\boldsymbol{\lambda}_i| + |\beta^i|.$$

So,

$$|\beta^i| = \sum_{j=1}^{i}(|\boldsymbol{\lambda}_j| - |\boldsymbol{\mu}_j|), |\alpha^i| = \sum_{j=i}^{e} |\boldsymbol{\lambda}_j| - \sum_{j=i-1}^{e} |\boldsymbol{\mu}_j| \geq 0.$$

Definition-Proposition 10.0.3. *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{MP}(e,w)$, let $\partial(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be an e-tuple list of integers defined by*

$$\partial(\boldsymbol{\lambda}, \boldsymbol{\mu})_i := \sum_{j=1}^{i}(|\boldsymbol{\lambda}_j| - |\boldsymbol{\mu}_j|)$$

and let $\wp(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be an e-tuple list of integers defined by

$$\wp(\boldsymbol{\lambda}, \boldsymbol{\mu})_i := \sum_{j=1}^{i} |\boldsymbol{\mu}_j| - \sum_{j=1}^{i-1} |\boldsymbol{\lambda}_j|.$$

By the equation (10.4), if $d_{\lambda,\mu} > 0$, then $\partial(\overline{\lambda}, \overline{\mu})_i \geq 0$ and $\wp(\overline{\lambda}, \overline{\mu})_i \geq 0$ for any $1 \leq i \leq e$. We write $\partial(\boldsymbol{\lambda}, \boldsymbol{\mu}) \geq 0$ (resp. $\wp(\boldsymbol{\lambda}, \boldsymbol{\mu}) \geq 0$) if $\partial(\boldsymbol{\lambda}, \boldsymbol{\mu})_i \geq 0$ (resp. $\wp(\boldsymbol{\lambda}, \boldsymbol{\mu})_i \geq 0$) for any $1 \leq i \leq e$.

Therefore, by (10.4) we get

$$[\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\lambda})) : V_{\mathbf{k}}(\boldsymbol{\mu}^\natural)] = \sum_{\substack{\alpha \vdash \wp(\boldsymbol{\lambda},\boldsymbol{\mu}) \geq 0, \beta \vdash \partial(\boldsymbol{\lambda},\boldsymbol{\mu}) \geq 0, \\ |\partial(\boldsymbol{\lambda},\boldsymbol{\mu})| = k}} \prod_{i=1}^{e} LR^{\boldsymbol{\lambda}_i}_{\alpha^i,\beta^i} \cdot LR^{\boldsymbol{\mu}_i}_{\beta^{i-1},\alpha^i}.$$

Hence, $l_{\mathbf{k}}(\lambda) = w + 1 - |\lambda^{(e-1)}|$ if $\lambda \in \Lambda_{e,w,\rho}$.

For $\ell > w$, we define $rad_{\lambda,\mu}(v) \in \mathbb{N}[v]$ as follows:

$$rad_{\lambda,\mu}(v) = \sum_{k=0}^{l_{\mathbf{k}}(\lambda)-1} [\operatorname{Rad}^k(U_{\mathbf{k}}(\overline{\lambda}))/\operatorname{Rad}^{k+1}(U_{\mathbf{k}}(\overline{\lambda})) : V_{\mathbf{k}}(\overline{\mu})]v^k$$

$$= \sum_{k=0}^{w-|\lambda^{(e-1)}|} \left[\sum_{\substack{\alpha \vdash \wp(\overline{\lambda},\overline{\mu}^\natural) \geq 0, \beta \vdash \partial(\overline{\lambda},\overline{\mu}^\natural) \geq 0, \\ |\partial(\overline{\lambda},\overline{\mu}^\natural)| = k}} \prod_{i=1}^{e} LR^{(\overline{\lambda})_i}_{\alpha^i,\beta^i} \cdot LR^{(\overline{\mu}^\natural)_i}_{\beta^{i-1},\alpha^i} \right] v^k. \qquad (10.5)$$

Note that $\partial(\boldsymbol{\mu}', \boldsymbol{\lambda}')_{e-i} = \sum_{j=i+1}^{e}(|\boldsymbol{\mu}_j| - |\boldsymbol{\lambda}_j|)$ for $1 \leq i \leq e-1$. So, $\partial(\boldsymbol{\lambda}, \boldsymbol{\mu})_i - \partial(\boldsymbol{\mu}', \boldsymbol{\lambda}')_{e-i} = \sum_{j=1}^{e}(|\boldsymbol{\lambda}_j| - |\boldsymbol{\mu}_j|) = 0$. Therefore, $\partial(\boldsymbol{\lambda}, \boldsymbol{\mu})_i = \partial(\boldsymbol{\mu}', \boldsymbol{\lambda}')_{e-i}$,

$\wp(\boldsymbol{\lambda},\boldsymbol{\mu})_i=\wp(\boldsymbol{\mu}',\boldsymbol{\lambda}')_{e-i+1}$ and $|\partial(\boldsymbol{\lambda},\boldsymbol{\mu})|=|\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')|$. We would like to compare $[\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\mu}')):V_{\mathbf{k}}((\boldsymbol{\lambda}')^{\natural})]$ with $[\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\lambda})):V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural})]$.

$$
\begin{aligned}
[\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\mu}')):V_{\mathbf{k}}((\boldsymbol{\lambda}')^{\natural})] &= \sum_{\substack{\alpha\vdash\wp(\boldsymbol{\mu}',\boldsymbol{\lambda}')\geq 0,\beta\vdash\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')\geq 0,\\ |\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')|=k}} \prod_{i=1}^{e} LR^{(\boldsymbol{\mu}')_i}_{\alpha^i,\beta^i}\cdot LR^{(\boldsymbol{\lambda}')_i}_{\beta^{i-1},\alpha^i}\\
&= \sum_{\substack{\alpha\vdash\wp(\boldsymbol{\mu}',\boldsymbol{\lambda}')\geq 0,\beta\vdash\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')\geq 0,\\ |\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')|=k}} \prod_{i=1}^{e} LR^{(\boldsymbol{\mu}_{e-i+1})'}_{\alpha^i,\beta^i}\cdot LR^{(\boldsymbol{\lambda}_{e-i+1})'}_{\beta^{i-1},\alpha^i}\\
&= \sum_{\substack{\alpha\vdash\wp(\boldsymbol{\mu}',\boldsymbol{\lambda}')\geq 0,\beta\vdash\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')\geq 0,\\ |\partial(\boldsymbol{\mu}',\boldsymbol{\lambda}')|=k}} \prod_{i=1}^{e} LR^{(\boldsymbol{\mu}_i)'}_{\alpha^{e-i+1},\beta^{e-i+1}}\cdot LR^{(\boldsymbol{\lambda}_i)'}_{\beta^{e-i},\alpha^{e-i+1}}\\
&= \sum_{\substack{\alpha\vdash\wp(\boldsymbol{\lambda},\boldsymbol{\mu})\geq 0,\beta\vdash\partial(\boldsymbol{\lambda},\boldsymbol{\mu})\geq 0,\\ |\partial(\boldsymbol{\lambda},\boldsymbol{\mu})|=k}} \prod_{i=1}^{e} LR^{(\boldsymbol{\mu}_i)'}_{\alpha^i,\beta^{i-1}}\cdot LR^{(\boldsymbol{\lambda}_i)'}_{\beta^i,\alpha^i}\\
&= \sum_{\substack{\alpha\vdash\wp(\boldsymbol{\lambda},\boldsymbol{\mu})\geq 0,\beta\vdash\partial(\boldsymbol{\lambda},\boldsymbol{\mu})\geq 0,\\ |\partial(\boldsymbol{\lambda},\boldsymbol{\mu})|=k}} \prod_{i=1}^{e} LR^{\boldsymbol{\mu}_i}_{\alpha^i,\beta^{i-1}}\cdot LR^{\boldsymbol{\lambda}_i}_{\beta^i,\alpha^i}\\
&= [\mathcal{L}^k(U_{\mathbf{k}}(\boldsymbol{\lambda})):V_{\mathbf{k}}(\boldsymbol{\mu}^{\natural})].
\end{aligned}
$$

Hence, by (10.5) and Theorem 7.1.1, we get the following theorem:

Theorem 10.0.4. *Suppose that $\ell > w$. Then, $rad_{\lambda,\mu}(1)=d_{\lambda,\mu}$ and $rad_{\lambda,\mu}(v)=rad_{\mu',\lambda'}(v)$ for any $\lambda,\mu\in\Lambda_{e,w,\rho}$. The k-th coefficient of $rad_{\lambda,\mu}(v)$ equals to* *the multiplicity of* $D_{\mathbf{k}}(\mu)$ *in* $\mathcal{L}^k(S_{\mathbf{k}}(\lambda))$ *as a composition factor. Here,*

$$rad_{\lambda,\mu}(v) = \sum_{k=0}^{w-|\lambda^{(e-1)}|} \left[\sum_{\substack{\alpha \vdash \wp(\overline{\lambda},\overline{\mu}^{\natural}) \geq 0, \beta \vdash \partial(\overline{\lambda},\overline{\mu}^{\natural}) \geq 0, \\ |\partial(\overline{\lambda},\overline{\mu}^{\natural})| = k}} \prod_{i=1}^{e} LR_{\alpha^i,\beta^i}^{(\overline{\lambda})_i} \cdot LR_{\beta^{i-1},\alpha^i}^{(\overline{\mu}^{\natural})_i} \right] v^k \in \mathbb{N}[v].$$

Remark.

By the construction of the Loewy series of $S_{\mathbf{k}}(\lambda)$ for $\lambda \in \Lambda_{e,w,\rho}$, we also know $\mathcal{L}^i(S_{\mathbf{k}}(\lambda)) \cong \mathrm{Soc}^{l_{\mathbf{k}}(\lambda)-i}(S_{\mathbf{k}}(\lambda))/\,\mathrm{Soc}^{l_{\mathbf{k}}(\lambda)-i+1}(S_{\mathbf{k}}(\lambda))$.

Chapter 11

Conjecture

11.1 The theory of Lascoux-Leclerc-Thibon

Let v be an indeterminate over $\mathbb{Q}$. Let $U_v(\widehat{\mathfrak{sl}_e})$ be the quantized enveloping algebra over $\mathbb{Q}(v)$ corresponding to the Dynkin diagram $A^{(1)}_{e-1}$.

The so-called "Fock space"

$$\mathcal{F}_v = \bigoplus_{\lambda} \mathbb{Q}(v)|\lambda\rangle$$

is the $\mathbb{Q}(v)$-vector space with basis $|\lambda\rangle$ indexed by the set of all partitions. In [38] Lascoux, Leclerc and Thibon introduced an algorithm to compute the *canonical basis* of the basis representation $M_v(\Lambda_0)$ and conjectured that it also compute the decomposition matrix of the Iwahori-Hecke algebras of type A at root of unity over $\mathbb{C}$. This is so called the LLT conjeture.

The LLT conjecture is now a theorem (see, for example, [45, Chap. 6] and the references of Chapter 6).

In [40], Leclerc and Thibon define a canonical basis of the v-deformed Fock space representation $\mathcal{F}_v$ of the affine Lie algebra $\widehat{\mathfrak{gl}}_e$. They conjectured that the entries of the transition matrix between these basis and $\{|\lambda\rangle\}_\lambda$ are also crystallized decomposition numbers of Dipper-James Schur algebras for ζ specialized at a primitive e-th root of unity.

This LT conjecture is now a theorem over $\mathbb{Q}(\zeta)$ where ζ is a primitive e-th root of unity over $\mathbb{C}$, due to Varagnolo and Vasserot [59].

Leclerc and Thibon showd that

Theorem 11.1.1 (Leclerc-Thibon). *There exist bases $\{G(\lambda)\}$ and $\{G^-(\lambda)\}$ of $\mathcal{F}_v$ characterized by :*

1. $\overline{G(\lambda)} = G(\lambda)$, $\overline{G^-(\lambda)} = G^-(\lambda)$

2. $G(\lambda) \equiv |\lambda\rangle \mod vL, \quad G^-(\lambda) \equiv |\lambda\rangle \mod v^{-1}L^-.$

Here, L (resp. L^-) denotes the $\mathbb{Z}[v]$(resp. $\mathbb{Z}[v^{-1}]$)-lattice in $\mathcal{F}_v$ with basis $\{|\lambda\rangle\}$.

Let

$$G(\mu) = \sum_\lambda d_{\lambda,\mu}(v)|\lambda\rangle, \quad G^-(\lambda) = \sum_\lambda e_{\lambda,\mu}(v)|\mu\rangle.$$

and $\mathbf{D}_{m,e,0}(v) = [d_{\lambda,\mu}(v)]_{\lambda,\mu\vdash m}$.

Theorem 11.1.2 (Leclerc-Thibon, Varagnolo-Vasserot).
The matrix $\mathbf{D}_{m,e,0}(1)$ is equal to the decomposition matrix $\mathbf{D}_{m,\zeta,0}$.

Remark

If both ζ and ζ' are primitive e-th roots of unity in $\mathbb{C}$, then $\mathbf{D}_{m,\zeta,0} = \mathbf{D}_{m,\zeta',0}$. So, we may write $\mathbf{D}_{m,e,0}$ instead of $\mathbf{D}_{m,\zeta,0}$.

11.2 Our hope

Theorem 11.2.1 (Geck). [19] *There exists a square lower unitriangular matrix* $\mathbf{A}$ *such that each entry of* $\mathbf{A}$ *is non-negative and* $\mathbf{D}_{m,\overline{q},\ell} = \mathbf{D}_{m,e,0} \cdot \mathbf{A}$.

By the above theorem and Theorem 10.0.4, we deduce

Corollary 11.2.2. *If* $rad_{\lambda,\mu}(v) = 0$, *then* $d_{\lambda,\mu}(v) = 0$ *for any* $\lambda, \mu \in \Lambda_{e,w,\rho}$..

Not only do we want to show that James' conjecture is true, but we want to know an explicit formula for $d_{\lambda,\mu}(v)$ which are now known to be certain parabolic Kazhdan-Lusztig polynomials.

According to James conjecture, Rouquier conjecture and Leclerc-Thibon conjecture [45, 6.33 (see also 6.27)] on Jantzen filtrations over $\mathbb{C}$, we hope the following:

Conjecture 11.2.3. $d_{\lambda,\mu}(v) = rad_{\lambda,\mu}(v)$ *for any* $\lambda, \mu \in \Lambda_{e,w,\rho}$.

Remark.

By using Veigneau's package ACE [60], the author checked that the above conjecture is ture where $e = 2, w \leq 4$ or $e = 3, w \leq 2$. By using Mathas' package SPECHT [44], the author checked that the above conjecture is true for $d_{\lambda,\mu}(v)$ where μ is e-regular and $e = 3, w = 3$.

Moreover, these matrices will be appended to the Appendix of this article.

Chapter 12

Some observations

We write Scott(G, H) for the *Scott* module for H in G (see [46, Chap. 4, Sect. 8]). Let G be $GL_n(q)$. For an ℓ-subgroup Q of G, we write $[Q]$ be the conjugacy class of Q in G. For a composition μ of n let L_λ be the standard Levi subgroup corresponding to μ. Let $\mathcal{L}_G := \{L_\lambda \mid \lambda \vdash n\}$. Let $\mathfrak{Q}_n$ be

$$\{[Q] \mid Q \in \mathrm{Syl}_\ell(L), L \in \mathcal{L}_G\}.$$

Let $\Lambda_n(e, \ell)$ be the set of partitions λ of n such that $\lambda_i = e\ell^j, 1$ or 0 for some $j \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots, n$.

Let $\mathcal{L}_{G,(e,\ell)}$ be

$$\{L_\lambda \in \mathcal{L}_G \mid \lambda \in \Lambda_n(e, \ell)\}.$$

The following proposition is easily proved by [61].

Proposition 12.0.4. *$\mathfrak{Q}_n = \left\{[Q] \mid Q \in \mathrm{Syl}_\ell(L), L \in \mathcal{L}_{G,(e,\ell)}\right\}$. In particular, $|\,\mathfrak{Q}_n\,| = |\,\mathcal{L}_{G,(e,\ell)}\,|$.*

So, we may denote Q by Q_λ if $Q \in \mathrm{Syl}_\ell(L_\lambda)$ for some $\lambda \in \Lambda_n(e, \ell)$.

Following [30, p.236 (3.4)] we summarize a result of R. Dipper and G.D. James we need in this subsection (see also [15, p.37 Theorem 4.7.]).

Theorem 12.0.5 (Dipper-James). *1. For each $\lambda \vdash n$, $\mathrm{Ind}_{P_\lambda}^G \mathbf{k}_{P_\lambda}$ is isomorphic to a direct sum of Young modules.*

2. $R_{L_\lambda}^G(k_{L_\lambda}) \cong \mathrm{Ind}_{P_\lambda}^G \mathbf{k}_{P_\lambda} \cong (\bigoplus_{\mu \rhd \lambda} b_{\lambda,\mu} X_{\mathbf{k}}(\mu)) \oplus X_{\mathbf{k}}(\lambda)$. Here $b_{\lambda,\mu}$ denote the multiplicity of $X_{\mathbf{k}}(\mu)$ in $\mathrm{Ind}_{P_\lambda}^G(\mathbf{k}_{P_\lambda})$ as indecomposable direct summands.

3. The multiplicity of $S_K(\lambda)$ as a composition factor of $K \otimes_{\mathcal{O}} X_{\mathcal{O}}(\mu)$ is $d_{\lambda',\mu'}$.

4. The three lower unitriangluar matrices $(K_{\lambda,\mu}), (b_{\lambda,\mu})$ and $(d_{\lambda,\mu})$ have the following relation:

$$K_{\lambda,\mu} = \sum_{\nu \vdash n} b_{\lambda,\nu} d_{\mu',\nu'}.$$

Definition 12.0.6. *For $\lambda \in \Lambda_n(e,\ell)$, let $\Upsilon_\lambda(e,\ell)$ be the set of partitons μ of n such that $Q \in [Q_\lambda]$ for any $Q \in \mathrm{Syl}_\ell(L_\mu)$. For $\lambda \in \Lambda_n(e,\ell)$, let $\mathbf{m}_{e,\ell}(\lambda)$ be the maximal element of $\Upsilon_n(e,\ell)$ with respect to the lexicographic order on partitions.*

Lemma 12.0.7. *For any $\lambda \in \Lambda_n(e,\ell)$,*

$$X_{\mathbf{k}}(\mathbf{m}_{e,\ell}(\lambda)) \cong \mathrm{Scott}(G, Q_\lambda) \text{ and } d_{(1^n),(\mathbf{m}_{e,\ell}(\lambda))'} = 1.$$

proof.

By definition of the Scott module, there exists a $\mu \in \Upsilon_\lambda(e,\ell)$ such that $X_{\mathbf{k}}(\mu) \cong \mathrm{Scott}(G, Q_\lambda)$. Suppose that $\mu < \mathbf{m}_{e,\ell}(\lambda)$. Since $Q \in [Q_\lambda]$ for any $Q \in \mathrm{Syl}_\ell(L_{\mathbf{m}_{e,\ell}(\lambda)})$, $\mathrm{Scott}(G, Q_\lambda)$ is an indecomposable direct summand of $x_{\mathbf{m}_{e,\ell}(\lambda)}M$. However, $\mathrm{Scott}(G, Q_\lambda)$ is isomorphic to $X_{\mathbf{k}}(\mu)$. This contradicts Theorem 12.0.5 (2). Hence, $X_{\mathbf{k}}(\mathbf{m}_{e,\ell}(\lambda)) \cong \mathrm{Scott}(G, Q_\lambda)$.

Moreover, $(\eta_{\mathbf{m}_{e,\ell}(\lambda)}, \chi_{(n)}) = 1$. So, by Theorem 12.0.5 (3), $d_{(1^n),\mathbf{m}_{e,\ell}(\lambda)'} = d_{(n)',\mathbf{m}_{e,\ell}(\lambda)'} = 1$ as required.

□

Corollary 12.0.8. *Fix $\lambda \in \Lambda_n(e,\ell)$. For any $\mu \in \Upsilon_\lambda(e,\ell)$, $\mathbf{m}_{e,\ell}(\lambda) \trianglerighteq \mu$ and $b_{\mu,\mathbf{m}_{e,\ell}(\lambda)} = 1$.*

Note that as in the proof of Lemma 12.0.7 if $d_{(1^n),\mu} > 2$ then we have $(\eta_{\mu'}, \chi_{(n)}) > 2$. However, this contradict $X_{\mathbf{k}}(\mu')$ is a permutation module. Hence, $d_{(1^n),\mu}$ must be smaller or equal to 1. So the following proposition is easily proved.

Proposition 12.0.9. *If $d_{(1^n),\mu} > 0$, then $d_{(1^n),\mu} = 1$ and there exists $\lambda \in \Lambda_n(e,\ell)$ such that $\mu' = \mathbf{m}_{e,\ell}(\lambda)$ and $Q \in [Q_\lambda]$ for any $Q \in \mathrm{Syl}_\ell(L_{\mu'})$.*

We would like to know what is $\mathbf{m}_{e,\ell}(\lambda)$ for $\lambda \in \Lambda_n(e,\ell)$.

12.1 Notation

Definition 12.1.1. *For a non-negative integer a and a positive integer b, $\lceil a/b \rceil$ denotes the integral part of a/b.*

Definition-Proposition 12.1.2. *For non-negative integers a, b and c with $1 < a$ and $0 \leq c < a$, let $\tau[a,b,c,b]$ be the partition of $a \cdot b + c$ defined by*

$$\tau[a,b,c,b]_i := \begin{cases} a-1 & \text{for } i = 1, \ldots, b + \lceil (b+c)/(a-1) \rceil, \\ \mathrm{Mod}(a \cdot b + c, a-1) & \text{for } i = b + \lceil (b+c)/(a-1) \rceil + 1. \end{cases}$$

$$\tau[a,0,c,0] := (c).$$

Then, the a-core of $\tau[a,b,c,b]$ is (c) and the a-weight of $\tau[a,b,c,b]$ is b.

For a non-negative $d \leq b$, let $\tau[a,b,c,b-d]$ be the partition $\tau[a,b-d,c,b-d] + (a \cdot d)$ of $a \cdot b + c$.

Then, the a-core of $\tau[a,b,c,d]$ is (c) and the a-weight of $\tau[a,b,c,d]$ is b.

Remark.

The above definition is well-defined. By the definition of $\tau[a,b,c,b]$, the length of $\tau[a,b,c,b]'$ is equal to $a-1$ for any b. In particular, $\tau[a,b,c,b]'$ is a-regular for any b.

Proposition 12.1.3.

$$\tau[e_\circ, w, r, w] \lhd \tau[e_\circ, w, r, w-1] \lhd \cdots \lhd \tau[e_\circ, w, r, 0]$$

and

$$\tau[e_\circ, w, r, w]' \rhd \tau[e_\circ, w, r, w-1]' \rhd \cdots \rhd \tau[e_\circ, w, r, 0]'$$

12.2 Steinberg and Scott modules

In this subsection we consider the composition factors of ℓ-modular reduction of the Steinberg character of $GL_{e_\circ w}(q)$.

12.2.1 James' construction

Definition 12.2.1. *For a non-negative integer b, let $\mathfrak{l}_\ell(b)$ be the least non-negative integer i such that $b < \ell^i$.*

In order to investigate the socle of $S_{\mathbf{k}}(\tau[e_\circ, w, r, i])$, we use the following submodule theorem due to G.D. James:

Theorem 12.2.2 (G.D. James). [27, p.127 19.6 *Theorem*]
Let $\mu = (\mu_1, \mu_2, \ldots, \mu_h)$ be a partition of n, with $\mu_h > 0$. Then the trivial $\mathbf{k}GL_n(q)$-module $D_{\mathbf{k}}((n))$ is a submodule of $S_{\mathbf{k}}(\mu)$ if and only if

1. *e divides $(\mu_d + 1)$, and*

2. $$\left\lceil \frac{\mu_d + 1}{e} \right\rceil \equiv 0 \textit{ modulo } \ell^{\mathfrak{l}_\ell(\lceil \frac{\mu_{d+1}}{e} \rceil)}$$

for each d with $1 \leq d \leq h-1$.

Proposition 12.2.3. *For w and i with $0 \leq i \leq w$, the trivial $\mathbf{k}GL_{e_\circ w+r}(q)$-module $D_{\mathbf{k}}((e_\circ w + r))$ is a submodule of $S_{\mathbf{k}}(\tau[e_\circ, w, r, i])$. In particular, $D_{\mathbf{k}}((e_\circ w))$ is a submodule of the Young module $X_{\mathbf{k}}(\tau[e_\circ, w, r, i])$.*

proof.

Fix $0 \leq i \leq w$ and put $\tau = (\tau[e_\circ, w, r, w-i])$. And, write $\tau' = (\tau'_1, \tau'_2, \ldots, \tau'_h)$ with $\tau'_h > 0$. Then, by the definition of $\tau[e_\circ, w, w-i]$ and Definition 4.0.2, we know that

$$\tau'_1 + 1 = e_\circ(i+1), \tau'_d + 1 = e_\circ \tag{12.1}$$

for any d with $1 < d \leq h-1$, and $\tau'_h \leq e_\circ - 1$. So, by Definition 4.0.2, we also know that

$$\left\lceil \frac{\tau_d + 1}{e} \right\rceil \equiv 0 \text{ modulo } \ell^{\mathfrak{l}_\ell(\lceil \frac{\tau_d+1}{e} \rceil)} \tag{12.2}$$

for any d with $1 \leq d \leq h-1$. Therefore, by Theorem 12.2.2, equations (12.1) and (12.2), we get claimed proposition.

$\square$

Lemma 12.2.4. *$d_{(1^{e_\circ w+r}),\tau[e_\circ,w,r,i]'} = 1$ for any $0 \leq i \leq w$. In particular, these decomposition numbers only depend on $e_\circ$, but not on the characteristic ℓ of $\mathbf{k}$.*

proof.

By Proposition 12.2.3, we know that $X_{\mathbf{k}}(\tau[e_\circ, w, r, i])$ is a Scott module since $X(\tau[e_\circ, w, r, i])$ is a permutation module. Hence $(\eta_{\tau[e_\circ,w,r,i]'}, \chi_{(e_\circ w+r)}) = 1$. The composition factors (multiplicities counted) of $X(\lambda)$ are the same as those of $\bigoplus_{\mu\vdash n} d_{\mu',\lambda'} S_{\mathbf{k}}(\mu)$. Hence, $d_{(1^{e_\circ w}),\tau[e_\circ,w,r,i]'} = 1$.

$\square$

Since $K_{\lambda,(n)} = 1$ for any $\lambda \vdash n$, by Theorem 12.0.5 (4), we get

Corollary 12.2.5. *If $D_{\mathbf{k}}(\lambda)$ and $D_{\mathbf{k}}(\mu)$ are composition factors of $S_{\mathbf{k}}(1^n)$, then $b_{\lambda,\mu} = 0$. In particular, $b_{\tau[e_\circ,w,r,i],\tau[e_\circ,w,r,j]} = 0$ unless $i = j$.*

By 12.0.4, we also get

Corollary 12.2.6. *If $\ell > w$, then $d_{(1^{e_\circ w+r}),\mu} = 0$ unless $\mu = \tau[e_\circ, w, r, i]$ for some $i = 0, \ldots, w$.*

Corollary 12.2.7. *Suppose that $\ell > w$. If the trivial module $S_{\mathbf{k}}(e_\circ w + r)$ is a submodule of $X_{\mathbf{k}}(\mu)$, then $S_{\mathbf{k}}(e_\circ w + r)$ is a submodule of $S_{\mathbf{k}}(\mu)$.*

Remark

This dose not hold in general.

Theorem 12.2.8 (James-Mathas). *[32] Let $d^0_{\lambda,\mu}$ be the decomposition number $[S^\lambda_{\mathbb{Q},-1} : D^\mu_{\mathbb{Q},-1}]$ of $\mathcal{H}_{\mathbb{Q},-1}(\mathfrak{S}_n)$. Then,*

1. *If n is even, then*

$$d^0_{\lambda,(n)} = \begin{cases} 1 & \text{if } \lambda = (i, 1^{n-i}) \text{ for some } i \in [1, \ldots, n], \\ 0 & \text{otherwise.} \end{cases}$$

2. *If n is odd, then*

$$d^0_{\lambda,(n)} = \begin{cases} 1 & \text{if } \lambda = (i, 1^{n-i}) \text{ for some odd } i \in [1, \ldots, n], \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 12.2.9. *Suppose that $\ell > w$ and $e_\circ = e = 2$.*

1. *If n is even, then*

$$d_{\lambda,(n)} = \begin{cases} 1 & \text{if } \lambda = (i, 1^{n-i}) \text{ for some } i \in [1, \ldots, n], \\ 0 & \text{otherwise.} \end{cases}$$

2. *If n is odd, then*

$$d_{\lambda,(n)} = \begin{cases} 1 & \text{if } \lambda = (i, 1^{n-i}) \text{ for some odd } i \in [1, \ldots, n], \\ 0 & \text{otherwise.} \end{cases}$$

proof

Just using Theorem 11.2.1, its proof, Theorem 12.2.8, Theorem 12.0.5.1, Proposition 12.2.3, and induction on n, we can get this result.

$\square$

Question 12.2.10. *Suppose that $\ell > 2$, $e = 2$ and n is even. By James' kernel intersection theorem, there exists a sequence of non-trivial homomorphisms*

$$0 \to S_{\mathbf{k}}(n) \to \cdots \to S_{\mathbf{k}}(n-i+1, 1^{i-1}) \to S_{\mathbf{k}}(n-i, 1^i) \to \cdots \to S_{\mathbf{k}}(1^n).$$

Then, is this sequence a complex? Moreover, applying the Hecke functor H to this sequence, we get a new sequence as follows:

$$0 \to S_{\mathbf{k},\overline{q}}^{(n)} \to S_{\mathbf{k},\overline{q}}^{(n-1,1)} \to \cdots \to S_{\mathbf{k},\overline{q}}^{(n-i+1,1^{i-1})} \to S_{\mathbf{k},\overline{q}}^{(n-i,1^i)} \to \cdots \to S_{\mathbf{k},\overline{q}}^{(1^n)}.$$

Then, is this sequence a complex? More strongly, when is this a long exact sequence? How about $v = -1$ in characteristic zero case?

As a special case which R. Dipper and G.D. James deal with in [15, Sect. 3], we know

Theorem 12.2.11 (Dipper-James). *Let*

$$y_{(n)} := \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} T_w.$$

Let $S_{\mathcal{O}}(1^n)$ be $\sqrt{\ } y_{(n)} M_{\mathcal{O}}(1^n)$. Then, $S_R(1^n) \cong R \otimes_{\mathcal{O}} S_{\mathcal{O}}(1^n)$. Let $Q_{\mathcal{O}}(1^n)$ be the projective cover of $S_{\mathcal{O}}(1^n)$, and let $\alpha_{(1^n)}$ be an epimorphism from $Q_{\mathcal{O}}(1^n)$

onto $S_{\mathcal{O}}(1^n)$. Then, $K \otimes_{\mathcal{O}} (\ker\alpha_{(1^n)})$ has no factor of the form $S_K(\mu)$. In particular, we can get $S_{\mathcal{O}}(1^n)$ uniquely as a unipotent quotient of the projective cover of $S_{\mathcal{O}}(1^n)$.

Remark.

A property the top of $S_{\mathbf{k}}(1^n)$ is simple and is φ_{st} is quite strong. Here, φ_{st} is the ℓ-modular irreducible Steinberg character of $GL_n(q)$. G.D. James obtain this module from "principal series module", a permutation module induced from a standard Borel subgroup of $GL_n(q)$, by cutting it by the idempodent of Gelfand-Graev character of $GL_n(q)$ (see Definition 4.2.6). In general case $\mathbf{G}^F$ is a finite reductive group, we could not get such a property. For example, in the case $\mathbf{G}^F = Sp_4(q), \ell \geq 3, \ell \mid q+1$ [49], we could not get the above property. The ℓ-modular reduction of some unipotent cuspidal characters might be contained in the top of the one which is obatined by cutting the "principal series" module by the idempotent of Gelfand-Graev representation and has the same composition factors as the ℓ-modular reduction of Steinberg character $\mathrm{St}_{\mathbf{G}^F}$.

12.2.2 Gruber and Hiss' construction

We assume that the centre of $\mathbf{G}$ is connected. In this subsection we investigate the local submodules of the ℓ-regular quotient of the Gelfand-Graev lattice of G affording the Steinberg character.

The Gelfand-Graev character of G has a multiplicity-free property (see, Carter's book [7, Chapter 8]). Its modular representation theoritical interpretation is found in [26].

Definition 12.2.12. *For a group with split BN-pair M we denote the set of standard Levi subgroups by $\mathcal{L}_M$. For $L \in \mathcal{L}_M$, $\mathcal{L}_{M \geq L}$ denote the set of standard Levi subgroups L' of $\mathcal{L}_M$ such that L' contains L. For $L \in \mathcal{L}_M$, we denote the functor of Harish-Chandra induction and restriction by R_L^M and ${}^*R_L^M$ respectively. For an $R \in \{K, \mathbf{k}\}$ and a simple cuspidal $R[L]$-module X, we denote the set of isomorphism classes of simple submodules of $R_L^M(X)$ by $\mathcal{S}(M, L, X)_R$.*

Since we concentrate upon the unipotent blocks, we only deal with an ℓ-modular reduction of the Steinberg $K[GL_n(q)]$-module. So, we would like to write weaker theorems in Gruber-Hiss[25, p.61-63] and Gruber[24, Sect. 5]. One of their points of view is a correspondence between the decomposition numbers for $\overline{\mathrm{St}_G}$ and Harish-Chandra series in G.

Definition 12.2.13. *For $M \in \mathcal{L}_G$ let*

$$\mathcal{L}_M^{c,1} := \{L \in \mathcal{L}_M \mid \mathcal{E}_\ell(L, 1) \text{ contains a cuspidal Brauer character.}\}$$

For $M \in \mathcal{L}_G^{c,1}$, we denote the unique unipotent cuspidal $\mathbf{k}[M]$-module by $D_{M,1}$. For $L \in \mathcal{L}_G$ let Γ_L be the projective $\mathcal{O}[L]$-lattice affording the Gelfand-Graev representation of L over $\mathcal{O}$. We denote by $Y_{L,1}$ the indecomposable direct summand of Γ_L lying in $\mathcal{E}_\ell(L, 1)$, i.e. a projective indecomposable module obtained from Γ_L by the unipotent regular block idempodent of L. We denote *by $X_{L,1}$ the unipotent quotient of $Y_{L,1}$. It is known that $X_{L,1}$ is an $\mathcal{O}L$-lattice for the Steinberg character of L.*

Theorem 12.2.14 (Gruber-Hiss). *Let $M \in \mathcal{L}_G^{c,1}$. Then $X_{G,1}$ contains a unique composition factor from the series $\mathcal{S}(G, M, D_M)_{\mathbf{k}}$*

Theorem 12.2.15 (Dipper-James). *[29], [11], If $GL_n(q)$ has a cuspidal unipotent Brauer character φ, then $\varphi = \varphi_{\mathrm{St}}$. Moreover, φ_{St} is cuspidal if and only if $n = 1$ or $n = e\ell^j$ for some $j \geq 0$.*

Note that by [22, Theorem 7.6], it is also known that $\mathcal{L}_G^{c,1} = \mathcal{L}_{G,(e,\ell)}$.

Theorem 12.2.16 (Gruber). *[24, Theorem5.9] Let $M \in \mathcal{L}_G^{c,1}$.*

1. *There exists a unique non-trivial homomorphism*

$$\phi_M : R_M^G X_{M,1} \to X_{G,1}$$

   and its image contains $D_{G,M,1}$ as its unique top composition factor.

2. *$\phi_M(R_M^G(X_{M,1}))$ contains $D_{G,M,1}$ as a composition factor for $M' \in \mathcal{L}_G^{c,1}$ if and only if $M'^x \leq M$ for some $x \in N$. In particular, every submodule of $X_{G,1}$ containing $D_{G,M,1}$ also contains $D_{G,M',1}$ if $M' \leq M$.*

3. *For any proper sumodule V of $X_{G,1}$ there exist $M \in \mathcal{L}_G^{c,1}$ and ϕ_M such that $V \cong \phi(R_M^G X_{M,1})$.*

Theorem 12.2.17 (Dipper-James). *[12, Theorem 5.1], [30, (3.10)] Let $\lambda \vdash n$ and suppose that α and β are partitions such that $\mu' = \alpha' + e_\circ\beta'$ and α is*

$e_\circ$-regular. Let s be an ℓ-element of degree $e_\circ$ over $\mathbb{F}_q$. Then

$$D_{\mathbf{k}}(1,\mu) \cong R^{GL_n(q)}_{L_{(|\alpha|,e_\circ|\beta|)}}\left(D_{\mathbf{k}}(1,\alpha) \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(s,\beta)\right).$$

Theorem 12.2.18. *Let $G = GL_{e_\circ w+r}(q)$ with $0 \le r < e_\circ$. Suppose that $\ell > w$. Then $X_{G,1}$ is uniserial and has Loewy length $w+1$. Moreover, its Loewy layer is as follows:*

$$S_{\mathbf{k}}(1^{e_\circ w+r}) \cong X_{G,1} \cong \begin{bmatrix} \tau[e_\circ, w, r, 0]' \\ \tau[e_\circ, w, r, 1]' \\ \tau[e_\circ, w, r, 2]' \\ \vdots \\ \tau[e_\circ, w, r, w]' \end{bmatrix}$$

proof

By 12.1.3 and definition of $\tau[e_\circ, w, r, i]$, we have already known that

$$\tau[e_\circ, w, r, i] = \tau[e_\circ, w-i, r, w-i] + e_\circ(i)$$

and $\tau[e_\circ, w-i, r, w-i]$ is $e_\circ$-regular. Hence, by 12.2.17 we know that there exists an ℓ-element s_i of degree $e_\circ$ over $\mathbb{F}_q$ such that

$$D_{\mathbf{k}}(1,\tau[e_\circ, w, r, i]') \cong R^G_{L_{(e_\circ(w-i)+r, e_\circ i)}}(D_{\mathbf{k}}(1,\tau[e_\circ, w-i, r, w-i]') \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(s_i,(1^i))).$$

Therefore, we can know that

$$D_{\mathbf{k}}(1,\tau[e_\circ, w, r, i]') \in \mathcal{S}(G, L_{(e_\circ(w-i)+r, e_\circ i)}, D_{\mathbf{k}}(1,(1))^{\boxtimes e_\circ(w-i)+r} \boxtimes_{\mathbf{k}} D_{\mathbf{k}}(s_i,(1))^{\boxtimes i}).$$

By 12.2.16, we get its Loewy structure.

□

Let

$$rad_{(1^{e_\circ w+r}),\mu}(v) := \sum_{i=0}^{w} [\mathrm{Rad}^i(S_{\mathbf{k}}(1^{e_\circ w+r}))/\,\mathrm{Rad}^{i+1}(S_{\mathbf{k}}(1^{e_\circ w+r})) : D_{\mathbf{k}}(\mu)]v^i$$

Conjecture 12.2.19. *If $\ell > w$, then $rad_{(1^{e_\circ w+r}),\mu}(v) = d_{(1^{e_\circ w}),\mu}(v)$ for any $\mu \vdash e_\circ w + r$. More precisely, $d_{(1^{e_\circ w+r}),\mu}(v) = 0$ except $\mu = \tau[e_\circ, w, r, i]'$ for some $i = 0, 1, \ldots, w$, in which case $d_{(1^{e_\circ w+r}),\tau[e_\circ,w,r,i]'}(v) = v^i$.*

Remark

It is known that $d_{(1^{e_\circ w}),\tau[e_\circ,w,r,w]}(v) = v^w$ and $d_{(1^{e_\circ w}),\tau[e_\circ,w,r,0]}(v) = 1$ (see, [38, Corollary 7.7.] and [45, 6.43 Example (iii)]).

Corollary 12.2.20. *Let $h+1$ be the cardinarity of $\Lambda_n(e, \ell)$. Then, the Loewy length of $S_{\mathbf{k}}(1^{e_\circ w+r})$ is $h + 1$. The Young module $X_{\mathbf{k}}(1^{e_\circ w+r})$ is a Harish Chandra module with simple top $D_{\mathbf{k}}(\tau[e_\circ, w, r, w]')$ and has at least Loewy length $2h + 1$. In particular, the projective cover of $D_{\mathbf{k}}(\tau[e_\circ, w, r, w]')$ has at least Loewy length $2h + 1 \geq 2w + 1$. If $\ell \neq e_\circ$, then the projective cover of $D_{\mathbf{k}}(1^{e_\circ w+r})$ has at least Loewy length $2h + 1 \geq 2w + 1$.*

proof.

Note that $\tau[e_\circ, w, r, w]'$ is $e_\circ$-regular. The Young module $X_{\mathbf{k}}(1^{e_\circ w+r})$ is also a Harish-Chandra module with respect to $(G, T, \mathbf{k}_T)$. So, the top of $X_{\mathbf{k}}(1^{e_\circ w+r})$ and the socle of $X_{\mathbf{k}}(1^{e_\circ w+r})$ is simple and isomorphic to $D_{\mathbf{k}}(\tau[e_\circ, w, r, w]')$. Moreover, the multiplicity of $D_{\mathbf{k}}(\tau[e_\circ, w, r, w])$ in $X_{\mathbf{k}}(1^{e_\circ w+r})$ as a composition factor is 1. By, for example, [36, (1.1) Lemma], we get that $X_{\mathbf{k}}(1^{ew+r})$ has at least Loewy length $2h+1$. So, the projective cover of $D_{\mathbf{k}}(\tau[e_\circ, w, r, w])$ has at least Loewy length $2h+1$. Suppose that $e_\circ \neq \ell$. Then the multiplicity of $D_{\mathbf{k}}(\tau[e_\circ, w, r, w]')$ in the projective cover of $D_{\mathbf{k}}(1^{e_\circ w+r})$ as a compositon factor is 1. Similary, we get that the projective cover of $D_{\mathbf{k}}(1^{e_\circ w+r})$ has at least Loewy length $2h+1 \geq 2w+1$

$\square$

Chapter 13

Appendix

13.1 v-Decomposition matrices

The crystallized decomposition matrix for $\Lambda_{e=2,w=2,\rho=(1)}$.

$$
\begin{array}{ll|ccccc}
[\emptyset,[2]] & (5) & 1 & & & & \\
[\emptyset,[1,1]] & (3,2) & & 1 & & & \\
[[1],[1]] & (3,1^2) & v & v & 1 & & \\
[[2],\emptyset] & (2^2,1) & & v^2 & v & 1 & \\
[[1,1],\emptyset] & (1^5) & v^2 & & v & & 1
\end{array}
$$

The crystallized decomposition matrix for $\Lambda_{e=2,w=3,\rho=(2,1)}$.

$$
\begin{array}{ll|cccccccccc}
[\emptyset,[3]] & (8,1) & 1 & & & & & & & & & \\
[\emptyset,[2,1]] & (6,3) & & 1 & & & & & & & & \\
[\emptyset,[1,1,1]] & (4,3,2) & & & 1 & & & & & & & \\
[[1],[2]] & (6,1^3) & v & v & & 1 & & & & & & \\
[[1],[1,1]] & (4,3,1^2) & & v & v & & 1 & & & & & \\
[[2],[1]] & (4,2^2,1) & & v^2 & v^2 & v & v & 1 & & & & \\
[[1,1],[1]] & (4,1^5) & v^2 & v^2 & & v & v & & 1 & & & \\
[[3],\emptyset] & (3^2,2,1) & & & v^3 & & v^2 & v & & 1 & & \\
[[2,1],\emptyset] & (2^3,1^3) & & v^3 & & v^2 & v^2 & v & v & & 1 & \\
[[1,1,1],\emptyset] & (2,1^7) & v^3 & & & v^2 & & & v & & & 1
\end{array}
$$

Remark This matrix is arranged symmetrically. Moreover, this matrix is also available in [39, 8.2], [45, p.153].

The crystallized decomposition matrix for $\Lambda_{e=2,w=4,\rho=(3,2,1)}$.

$S_K(\lambda)/D_{\mathbf{k}}(\mu)$	$(3,2,1^9)$	$(3,2^3,1^5)$	$(3,2^5,1)$	$(3^3,2,1^3)$	$(4^2,3,2,1)$	$(5,2,1^7)$	$(5,2^3,1^3)$	$(5,3^2,2,1)$	$(5,4,1^5)$	$(5,4,2^2,1)$	$(5,4,3,1^2)$	$(5,4,3,2)$	$(7,2,1^5)$	$(7,2^3,1)$	$(7,4,1^3)$	$(7,4,3)$	$(7,6,1)$	$(9,2,1^3)$	$(9,4,1)$	$(11,2,1)$
regularity	S	S	S	S	S	S	S	S	S	S	S	R	S	S	S	R	R	S	R	R
$(11,2,1)$ $[\emptyset,[4]]$	v^4					v^3							v^2					v		1
$(9,4,1)$ $[\emptyset,[3,1]]$		v^4				v^3	v^3		v^2				v^2	v^2	v			v	1	
$(9,2,1^3)$ $[[1],[3]]$	v^3	v^3				v^2	v^2						v	v				1		
$(7,6,1)$ $[\emptyset,[2,2]]$			v^4				v^3			v^2			v^2		v		1			
$(7,4,3)$ $[\emptyset,[2,1,1]]$				v^4			v^3	v^3	v^2	v^2	v			v^2	v	1				
$(7,4,1^3)$ $[[1],[2,1]]$		v^3	v^3	v^3		v^2	$2v^2$	v^2	v	v			v	v	1					
$(7,2^3,1)$ $[[2],[2]]$		v^2		v^2			v	v						1						
$(7,2,1^5)$ $[[1,1],[2]]$	v^2	v^2	v^2			v	v						1							
$(5,4,3,2)$ $[\emptyset,[1,1,1,1]]$					v^4			v^3		v^2	v	1								
$(5,4,3,1^2)$ $[[1],[1,1,1]]$				v^3	v^3		v^2	v^2	v	v	1									
$(5,4,2^2,1)$ $[[1,1],[2]]$			v^2	v^2	v^2		v	v		1										
$(5,4,1^5)$ $[[1,1],[1,1]]$		v^2		v^2		v	v		1											
$(5,3^2,2,1)$ $[[3],[1]]$				v	v			1												
$(5,2^3,1^3)$ $[[2,1],[1]]$		v	v	v			1													
$(5,2,1^7)$ $[[1,1,1],[1]]$	v	v				1														
$(4^2,3,2,1)$ $[[4],\emptyset]$					1															
$(3^3,2,1^3)$ $[[3,1],\emptyset]$				1																
$(3,2^5,1)$ $[[2,2],\emptyset]$			1																	
$(3,2^3,1^5)$ $[[2,1,1],\emptyset]$		1																		
$(3,2,1^9)$ $[[1,1,1,1],\emptyset]$	1																			

Remark. Note that there are two selfconjugate partitions $(5,4,2^2,1)$ and $(7,2,1^5)$ and we can not arrange the matrix symmetrically on a plane.

This matrix is calculated by “ACE” in [60].

The crystallized decomposition matrix for $\Lambda_{e=3,w=2,\rho=(3,1^2)}$**.**

$[\emptyset,\emptyset,[2]]$	$(9,1^2)$	1								
$[\emptyset,\emptyset,[1,1]]$	$(6,4,1)$		1							
$[\emptyset,[1],[1]]$	$(6,3,2)$	v	v	1						
$[\emptyset,[2],\emptyset]$	$(5,4,2)$		v^2	v	1					
$[[1],\emptyset,[1]]$	$(6,1^5)$			v		1				
$[\emptyset,[1,1],\emptyset]$	$(3^2,2^2,1)$	v^2		v			1			
$[[1],[1],\emptyset]$	$(3^2,2,1^3)$			v^2	v	v	v	1		
$[[2],\emptyset,\emptyset]$	$(3,2^3,1^2)$						v^2	v	1	
$[[1,1],\emptyset,\emptyset]$	$(3,1^8)$				v^2			v		1

Remark This matrix is arranged symmetrically.

The crystallized decomposition matrix for $\Lambda_{e=3,w=3,\rho=(6,4,2^2,1^2)}$

It is not checked that Conjecture 11.2.3 is true or not for the next matrix. At least the Conjecture 11.2.3 is true for $d_{\lambda,\mu}(v)$ where 3-regular partitions $\mu \in \Lambda_{3,3,(6,4,2^2,1^2)}$ by SPECHT [44].

$S_K(\lambda)/D_{\mathbf{k}}(\mu)$	$(15,4,2^2,1^2)$	$(12,7,2^2,1^2)$	$(12,4^2,3,1^2)$	$(12,4,2^2,1^5)$	$(9,7,5,2,1^2)$	$(9,7,4,3,1^2)$	$(9,7,2^2,1^5)$	$(9,6,5,3,1^2)$	$(9,4^2,3^2,2)$	$(9,4^2,3,1^5)$	$(9,4,2^5,1^2)$	$(9,4,2^2,1^8)$	$(8,7,5,3,1^2)$	$(6^2,5,3^2,2)$	$(6^2,5,3,1^5)$	$(6,4^2,3^2,2^2,1)$	$(6,4^2,3^2,2,1^3)$	$(6,4^2,3,2^3,1^2)$	$(6,4^2,3,1^8)$	$(6,4,3^3,2^2,1^2)$	$(6,4,2^5,1^5)$	$(6,4,2^2,1^{11})$
regularity	R	R	R	S	R	R	S	R	R	S	S	S	R	R	S	R	S	S	S	S	S	S
$(15,4,2^2,1^2)$ $[\emptyset,\emptyset,[3]]$	1		v						v^2							v^3						
$(12,7,2^2,1^2)$ $[\emptyset,\emptyset,[2,1]]$		1	v			v		v^2	v^2					v^3								
$(12,4^2,3,1^2)$ $[\emptyset,[1],[2]]$			1	v				v	v	v^2				v^2		v^2	v^3					
$(12,4,2^2,1^5)$ $[[1],\emptyset,[2]]$				1						v							v^2					
$(9,7,5,2,1^2)$ $[\emptyset,\emptyset,[1,1,1]]$					1	v		v^2					v^3									
$(9,7,4,3,1^2)$ $[\emptyset,[1],[1,1]]$						1	v	v	v	v^2			v^2	v^2	v^3							
$(9,7,2^2,1^5)$ $[[1],\emptyset,[1,1]]$							1			v					v^2							
$(9,6,5,3,1^2)$ $[\emptyset,[2],[1]]$								1		v		v^2	v	v	v^2		v^2		v^3			
$(9,4^2,3^2,2)$ $[\emptyset,[1,1],[1]]$									1	v	v^2			v	v^2	v	v^2	v^3				
$(9,4^2,3,1^5)$ $[[1],[1],[1]]$										1	v	v			v		v	v^2	v^2			
$(9,4,2^5,1^2)$ $[[2],\emptyset,[1]]$											1							v				
$(9,4,2^2,1^8)$ $[[1,1],\emptyset,[1]]$												1							v			
$(8,7,5,3,1^2)$ $[\emptyset,[3],\emptyset]$													1		v				v^2			v^3
$(6^2,5,3^2,2)$ $[\emptyset,[2,1],\emptyset]$														1	v		v	v^2	v^2		v^3	
$(6^2,5,3,1^5)$ $[[1],[2],\emptyset]$															1			v	v		v^2	v^2
$(6,4^2,3^2,2^2,1)$ $[\emptyset,[1,1,1],\emptyset]$																1	v	v^2		v^3		
$(6,4^2,3^2,2,1^3)$ $[[1],[1,1],\emptyset]$																	1	v	v	v^2	v^2	
$(6,4^2,3,2^3,1^2)$ $[[2],[1],\emptyset]$																		1		v	v	
$(6,4^2,3,1^8)$ $[[1,1],[1],\emptyset]$																			1		v	v
$(6,4,3^3,2^2,1^2)$ $[[3],\emptyset,\emptyset]$																				1		
$(6,4,2^5,1^5)$ $[[2,1],\emptyset,\emptyset]$																					1	
$(6,4,2^2,1^{11})$ $[[1,1,1],\emptyset,\emptyset]$																						1

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