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Integral Homology of PGL_2 over Elliptic Curves

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The Friedlander–Milnor Conjecture [**1**] asserts that if G is a reductive algebraic group over an algebraically closed field k, then the comparison map

$$H^{\bullet}_{\text{ét}}(BG_k, \mathbb{Z}/p) \longrightarrow H^{\bullet}(BG, \mathbb{Z}/p)$$

is an isomorphism for all primes p not equal to the characteristic of k. Gabber's rigidity theorem [**2**] implies that this map is indeed an isomorphism for the stable general linear group GL (this is due to Suslin [**6**] for $k = \mathbb{C}$ and to Jardine [**3**] for arbitrary k). Similarly, a proof of an unstable version of rigidity would lead to a proof of the unstable Friedlander–Milnor Conjecture.

In this note we consider unstable rigidity for the group PGL_2 over an elliptic curve E. We assume that E is defined by the equation $F(x,y) = 0$, where

$$F(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6,$$

and the a_i lie in an infinite field k. Denote by $\overline{E}$ the projective curve $E \cup \{\infty\}$. Denote by A the coordinate ring of the affine curve E. If $l \in k$ and $F(l,y) = 0$ has no rational solutions, denote by $k(\omega)$ the quadratic extension of k inside the algebraic closure $\overline{k}$ for which $F(l,\omega) = 0$. Our main result is the following.

Theorem. *For all $i \geq 1$,*

$$\begin{aligned} H_i(PGL_2(A), \mathbb{Z}) \quad &= \quad \bigoplus_{\substack{p \in \overline{E} \\ 2p = 0}} H_i(PGL_2(k), \mathbb{Z}) \oplus \bigoplus_{\substack{p \in \overline{E}, 2p \neq 0 \\ p \sim -p}} H_i(k^{\times}, \mathbb{Z}) \\ &\quad \oplus \bigoplus_{\substack{l \in k, F(l,y) = 0 \\ \text{has no rat. sol.}}} H_i(k(\omega)^{\times}/k^{\times}, \mathbb{Z}). \end{aligned}$$

Here, $p \sim -p$ means that we identify the factors corresponding to the points p and $-p$.

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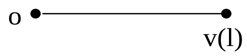

FIGURE 1. $F(l, y) = 0$ has no rational solutions

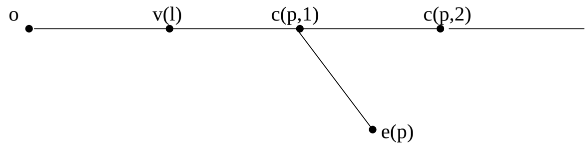

FIGURE 2. $F(l, y) = 0$ has a unique rational solution

This is proved by examining the action of $PGL_2(A)$ on the Bruhat–Tits tree $\mathcal{X}$ associated to a two-dimensional vector space over the function field of E (see, *e.g.,* Serre's book [**5**]). The quotient graph $PGL_2(A)\backslash\mathcal{X}$ is a tree (this is due to S. Takahashi [**8**]) and the various simplex stabilizers are easily described.

As a corollary, we have the following rigidity result.

COROLLARY. *Suppose the field k is algebraically closed and let x, y be distinct points on $\overline{E}$. Then the specialization homomorphisms*

$$s_x, s_y : H_\bullet(PGL_2(A), \mathbb{Z}) \longrightarrow H_\bullet(PGL_2(k), \mathbb{Z})$$

coincide.

1. The Quotient and the Stabilizers

In [**8**], Takahashi described a fundamental domain for the $PGL_2(A)$-action on $\mathcal{X}$; denote this subtree by $\mathcal{D}$. There is a distinguished vertex $o \in \mathcal{D}$. For each l in $k \cup \{\infty\}$ there is a vertex $v(l)$ adjacent to o. The rest of $\mathcal{D}$ may be described as follows. Let $\mathcal{D}(l)$ denote the subtree of $\mathcal{D} - \{o\}$ which contains $v(l)$. The tree $\mathcal{D}$ is the union of o and the various $\mathcal{D}(l)$ (which are disjoint). The trees $\mathcal{D}(l)$ are as follows.

(1) Suppose $F(x, y) = 0$ has no rational solution with $x = l$. Then $\mathcal{D}(l)$ consists only of $v(l)$ (see Figure 1).

(2) Suppose $l = \infty$ or $F(x, y) = 0$ has a unique rational solution with $x = l$. Let p be the point at infinity of E or the rational point corresponding to the solution. Note that p is a point of order 2. Then $\mathcal{D}(l)$ consists of an infinite path $c(p, 1), c(p, 2), \ldots$ and an extra vertex $e(p)$ (see Figure 2).

(3) Suppose $F(x, y) = 0$ has two different solutions such that $x = l$. Let p, q be the corresponding points on E. Then $\mathcal{D}(l)$ consists of two infinite paths $c(p, 1), c(p, 2), \ldots$ and $c(q, 1), c(q, 2), \ldots$ (see Figure 3).

The infinite path $c(p, 1), c(p, 2), \ldots$ is called a *cusp.* Note that there is a one-to-one correspondence between cusps and the rational points of $\overline{E}$.

Since $\mathcal{X}$ is contractible, we have a spectral sequence with E^1–term

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset \mathcal{D}} H_q(\Gamma_\sigma, \mathbb{Z}) \Longrightarrow H_{p+q}(PGL_2(A), \mathbb{Z})$$

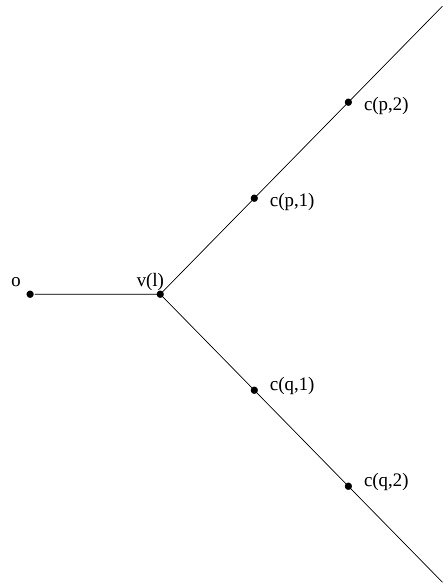

FIGURE 3. $F(l,y)=0$ has two distinct solutions

where Γ_σ is the stabilizer of the p–simplex σ in $PGL_2(A)$.

The various stabilizers of the $GL_2(A)$ action were described in [**8**]. Denote these by $\tilde{\Gamma}_\sigma$.

PROPOSITION 1.1 (cf. [**8**],Theorem 5). *The stabilizers $\tilde{\Gamma}_\sigma$ are (up to isomorphism)*

$$\begin{aligned}
\tilde{\Gamma}_o &\cong k^\times \\
\tilde{\Gamma}_{v(l)} &\cong \begin{cases} k(\omega)^\times & \textit{in case } (1) \\ k^\times \times k & \textit{in case } (2) \\ k^\times \times k^\times & \textit{in case } (3) \end{cases} \\
\tilde{\Gamma}_{e(p)} &\cong GL_2(k) \\
\tilde{\Gamma}_{c(p,n)} &\cong (k^n \times_\theta k^\times) \times k^\times
\end{aligned}$$

where $k^n \times_\theta k^\times$ is the semidirect product of k^n and $k^\times$ and θ is the automorphism of k^n given by coordinate-wise multiplication by elements of $k^\times$. Furthermore, the stabilizer of an edge is the intersection of the stabilizers of its vertices.

The groups $\tilde{\Gamma}_{c(p,n)}$ are of the form

$$\left\{ \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} : p, s \in k^\times, q \in k^n \right\}.$$

Denote the diagonal subgroups of these $\tilde{\Gamma}$ by $\tilde{L}$. By Theorem 1.11 of [**4**], these stabilizers satisfy

$$H_\bullet(\tilde{\Gamma}, \mathbb{Z}) \cong H_\bullet(\tilde{L}, \mathbb{Z}),$$

the isomorphism being induced by the inclusion $\tilde{L} \to \tilde{\Gamma}$.

COROLLARY 1.2. *The stabilizers Γ_σ satisfy*

$$\begin{aligned} H_\bullet(\Gamma_o, \mathbb{Z}) &\cong H_\bullet(\{1\}, \mathbb{Z}) \\ H_\bullet(\Gamma_{v(l)}, \mathbb{Z}) &\cong \begin{cases} H_\bullet(k(\omega)^\times/k^\times, \mathbb{Z}) & \textit{in case } (1) \\ H_\bullet(k, \mathbb{Z}) & \textit{in case } (2) \\ H_\bullet(k^\times, \mathbb{Z}) & \textit{in case } (3) \end{cases} \\ H_\bullet(\Gamma_{c(p,n)}, \mathbb{Z}) &\cong H_\bullet(k^\times, \mathbb{Z}) \\ H_\bullet(\Gamma_{e(p)}, \mathbb{Z}) &\cong H_\bullet(PGL_2(k), \mathbb{Z}). \end{aligned}$$

2. The Main Theorem

Note that our spectral sequence consists of two columns and that each row $E^1_{\bullet,q}$ is the chain complex $C_\bullet(\mathcal{D}, \mathcal{H}_q)$, where $\mathcal{H}_q$ is the coefficient system $\sigma \mapsto H_q(\Gamma_\sigma)$. Fix a positive integer $q \geq 1$. Note that $H_q(\Gamma_o, \mathbb{Z}) = 0$ and that $H_q(\Gamma_e, \mathbb{Z}) = 0$ for any edge incident with o. It follows that the chain complex $C_\bullet(\mathcal{D}, \mathcal{H}_q)$ is a direct sum of chain complexes

$$C_\bullet(\mathcal{D}, \mathcal{H}_q) = \bigoplus_{l \in k \cup \{\infty\}} C_\bullet(\mathcal{D}(l), \mathcal{H}_q).$$

PROPOSITION 2.1. *Suppose $F(x,y) = 0$ has no rational solutions with $x = l$. Then*

$$H_i(\mathcal{D}(l), \mathcal{H}_q) = \begin{cases} H_q(k(\omega)^\times/k^\times, \mathbb{Z}) & i = 0 \\ 0 & i > 0. \end{cases}$$

PROOF. This is clear since $C_\bullet(\mathcal{D}(l), \mathcal{H}_q)$ consists only of the single group $H_q(k(\omega)^\times/k^\times, \mathbb{Z})$ sitting in degree zero. □

PROPOSITION 2.2. *Suppose $l = \infty$ or $F(x,y) = 0$ has a unique rational solution with $x = l$. Then*

$$H_i(\mathcal{D}(l), \mathcal{H}_q) = \begin{cases} H_q(PGL_2(k), \mathbb{Z}) & i = 0 \\ 0 & i > 0. \end{cases}$$

PROOF. The stabilizer of $v(l)$ and of the edge joining $v(l)$ to $c(p,1)$ is isomorphic to k. The map $\Gamma_{\langle v(l), c(p,1)\rangle} \to \Gamma_{v(l)}$ is an isomorphism on homology and the map $\Gamma_{\langle v(l), c(p,1)\rangle} \to \Gamma_{c(p,1)}$ induces the zero map on homology. It follows that $H_\bullet(\mathcal{D}(l), \mathcal{H}_q) \cong H_\bullet(\mathcal{D}(l)', \mathcal{H}_q)$ where $\mathcal{D}(l)'$ is the tree obtained by deleting $v(l)$ and the edge joining it to $c(p,1)$. One checks easily that the map

$$C_1(\mathcal{D}(l), \mathcal{H}_q) \longrightarrow C_0(\mathcal{D}(l), \mathcal{H}_q)$$

is injective with cokernel $H_q(PGL_2(k), \mathbb{Z})$ (or equivalently, check that the relative homology groups $H_\bullet(\mathcal{D}(l), e(p); \mathcal{H}_q)$ vanish). □

PROPOSITION 2.3. *Suppose $F(x,y)=0$ has two distinct solutions with $x=l$. Then*

$$H_i(\mathcal{D}(l),\mathcal{H}_q)=\begin{cases}H_q(k^\times,\mathbb{Z}) & i=0\\ 0 & i>0.\end{cases}$$

PROOF. In this case, $\mathcal{D}(l)$ is a tree and the stabilizer of each vertex and each edge is $k^\times$. The maps $\Gamma_e\to\Gamma_v$ induce isomorphisms on homology for each edge and vertex. It follows that $C_\bullet(\mathcal{D}(l),\mathcal{H}_q)$ is a chain complex with *constant* coefficients. Since $\mathcal{D}(l)$ is contractible, the result follows. □

THEOREM 2.4. *For all $i\geq 1$,*

$$\begin{aligned}H_i(PGL_2(A),\mathbb{Z}) \;\cong\; &\bigoplus_{\substack{p\in\overline{E}\\ 2p=0}} H_i(PGL_2(k),\mathbb{Z})\oplus\bigoplus_{\substack{p\in\overline{E},2p\neq 0\\ p\sim q}} H_i(k^\times,\mathbb{Z})\\ &\oplus\bigoplus_{\substack{l\in k,F(l,y)=0\\ \text{has no solutions}}} H_i(k(\omega)^\times/k^\times,\mathbb{Z}).\end{aligned}$$

PROOF. Note that since $\mathcal{D}$ is contractible, $E^2_{0,0}=\mathbb{Z}$ and $E^2_{1,0}=0$. The preceding propositions show that

$$H_0(\mathcal{D},\mathcal{H}_q)=\bigoplus_{l\in k\cup\{\infty\}} H_0(\mathcal{D}(l),\mathcal{H}_q)$$

and $H_1(\mathcal{D},\mathcal{H}_q)=0$. It remains to identify the various direct summands with points of $\overline{E}$.

Those l for which $F(l,y)=0$ has a unique solution (or $l=\infty$) correspond to points of order 2 in $\overline{E}$. Those l for which $F(l,y)=0$ has two distinct solutions correspond to pairs of points on $\overline{E}$. For such a pair p,q, the groups $H_i(k^\times,\mathbb{Z})$ arising from the cusps associated to p,q are identified together since they are both adjacent to $v(l)$. The claimed direct sum decomposition follows. □

3. Rigidity

Suppose now that the field k is algebraically closed. Note that the projective curve $\overline{E}$ is isomorphic to the group $\mathrm{Pic}^0\overline{E}$ of degree zero line bundles on $\overline{E}$. Moreover, after a suitable linear change of coordinates, the points p,q corresponding to the two solutions of $F(l,y)=0$ may be assumed to satisfy $q=-p$.

COROLLARY 3.1. *If k is algebraically closed, then for all $i\geq 1$,*

$$H_i(PGL_2(A),\mathbb{Z}) \;\cong\; \bigoplus_{\substack{L\in\mathrm{Pic}^0\overline{E}\\ 2L=0}} H_i(PGL_2(k),\mathbb{Z})\oplus\bigoplus_{\substack{L\in\mathrm{Pic}^0\overline{E},2L\neq 0\\ L\sim -L}} H_i(k^\times,\mathbb{Z}).$$

PROOF. This is obvious once one notes that the factors $H_i(k(\omega)^\times/k^\times,\mathbb{Z})$ do not enter the picture when k is algebraically closed. □

As a consequence, we have the following rigidity result.

COROLLARY 3.2. *Suppose k is algebraically closed and let x, y be distinct points on $\overline{E}$. Then the corresponding specialization homomorphisms*

$$s_x, s_y : H_i(PGL_2(A), \mathbb{Z}) \longrightarrow H_i(PGL_2(k), \mathbb{Z})$$

coincide for all $i \geq 0$.

PROOF. This follows from the direct sum decomposition of Corollary 3.1. Since the groups which appear as summands do not involve rational functions on $\overline{E}$, the homomorphisms s_x and s_y must agree on each summand. □

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