INTRODUCTION TO A-INFINITY ALGEBRAS AND MODULES

BERNHARD KELLER

Dedicated to H. Keller on the occasion of his seventy fifth birthday

ABSTRACT. These are expanded notes of four introductory talks on A_∞-algebras, their modules and their derived categories.

CONTENTS

1. INTRODUCTION

1.1. **These notes.** These are expanded notes of a minicourse of three lectures given at the Euroconference 'Homological Invariants in Representation Theory' in Ioannina, Greece, March 16 to 21, 1999, and of a talk at the Instituto de Matemáticas, UNAM, México, on April 28, 1999. They present basic results on A_∞-algebras, their modules and their derived categories.

1.2. **History.** A_∞-spaces and A_∞-algebras (= sha algebras = strongly homotopy associative algebras) were invented at the beginning of the sixties by J. Stasheff [36] [37] as a tool in the study of 'group-like' topological spaces. In the subsequent two decades, A_∞-structures found applications and developments [24] [3] [1] in homotopy theory; their use remained essentially confined to this subject (cf. however [27], [33]). This changed at the beginning of the nineties when the relevance of A_∞-structures in algebra, geometry and mathematical physics became more and more apparent (cf. e.g. [11], [39]). Of special influence was M. Kontsevich's talk [21] at the International Congress in 1994: Inspired by K. Fukaya's preprint [8] Kontsevich gave a conjectural interpretation of mirror symmetry as the 'shadow' of an equivalence between two triangulated categories associated with A_∞-categories. His conjecture was proved in the case of elliptic curves by A. Polishchuk and E. Zaslow [29].

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1.3. **Scope and sources.** In these notes, we aim at presenting some basic theorems on A_∞-algebras, their modules and their derived categories. The notes may serve as an introduction to a (very) small part of Kontsevich's lectures [22]. Besides these, our main sources were [30], [37], [34], [14], [15], [16]. Some of the results we state are not proved in this form in the literature. Complete proofs will be given in [23].

1.4. **Contents.** In section 2, we motivate the introduction of A_∞-algebras and modules by two basic problems from homological algebra:

1. the reconstruction of a complex from its homology,
2. the reconstruction of the category of iterated selfextensions of a module from its extension algebra.

Then we briefly present the topological origin of A_∞-structures. Section 3 is devoted to A_∞-algebras and their morphisms. The central result is the theorem on the existence of minimal models. In sections 4 and 5, we introduce the derived category of an A_∞-algebra and we present the natural framework for the solution of problem 1. In section 6, we sketch the formalism of standard functors and arrive at the solution of problem 2. Section 7 presents the category of twisted objects, which is of importance because it is 'computable'.

1.5. **Omissions.** The links with Morita theory for derived categories [31] or dg categories [19] have not been made explicit, cf. however [20].

The relevance of A_∞-algebras to boxes and matrix problems was discovered by S. Ovsienko [27] (cf. also [28]). We only give a hint of this important development in example 7.9.

The notions of A_∞-equivalence and of A_∞-enhanced triangulated categories, crucial for [22], will be treated in [23].

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2. Motivation and topological origin

2.1. **Motivation.** Let k be a field. The following two problems will guide us in our exploration of the world of A_∞-algebras and modules.

Problem 1. Let A be an associative k-algebra with 1, let

$$M = (\ldots \to M^p \xrightarrow{d^p} M^{p+1} \to \ldots)$$

be a complex of (unital right) A-modules and let

$$H^*M = \operatorname{ker} d^* / \operatorname{im} d^* \ , \ * \in \mathbf{Z} \ ,$$

be its homology. It is a naive but natural question to ask whether, up to quasi-isomorphism, we can reconstruct the complex M from its homology H^*M. This is of course impossible, except under very restrictive hypotheses, for example when A is an hereditary algebra (i.e. an algebra such that submodules of projective A-modules are projective, see [6]). So we ask *what additional structure is needed if we want to reconstruct M from its homology.* The answer is that *H^*M carries a unique* *(up to isomorphism) A_∞-module structure which encodes exactly the additional information needed for this task.*

Problem 2. Let B be an associative k-algebra with 1, let $M_1, \ldots, M_n$ be B-modules and let $\mathsf{filt}(M_i)$ denote the full subcategory of the category of right B-modules whose objects admit finite filtrations with subquotients among the M_i. In other words, $\mathsf{filt}(M_i)$ is the closure under extensions of the M_i. A natural question to ask is whether $\mathsf{filt}(M_i)$ is determined by the extension algebra

$$\mathsf{Ext}^*_B(M, M), \quad \text{where } M = \bigoplus M_i ,$$

together with its idempotents corresponding to the M_i. Again, the answer is no. This time, it is no even if we assume that B is hereditary. Again, we ask what additional structure on the extension algebra is needed to reconstruct the category of iterated extensions. The answer is that *the extension algebra admits an A_∞-structure which encodes the additional information needed for this task.*

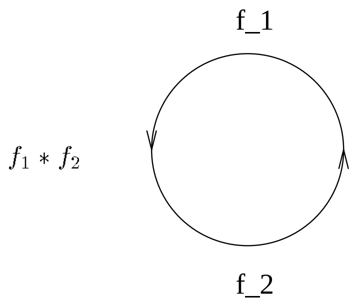
The aim of these lectures is to explain the answers to the two problems and to show how they fit into a general theory.

2.2. **Topological origin.** At the beginning of the sixties, J. Stasheff invented A_∞-spaces and A_∞-algebras [36], [37] as a tool in the study of 'group-like' spaces. Let us consider the basic example:

Let $(X, *)$ be a topological space with a base point $*$ and let ΩX denote the space of based loops in X: a point of ΩX is thus a continuous map $f : S^1 \to X$ taking the base point of the circle to the base point $*$. We have the composition map

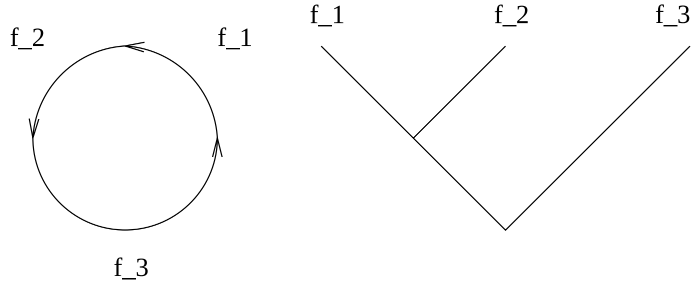
$$m_2 : \Omega X \times \Omega X \to \Omega X$$

sending a pair of loops (f_1, f_2) to the loop $f_1 * f_2 = m_2(f_1, f_2)$ obtained by running through f_1 on the first half of the circle and through f_2 on the second half

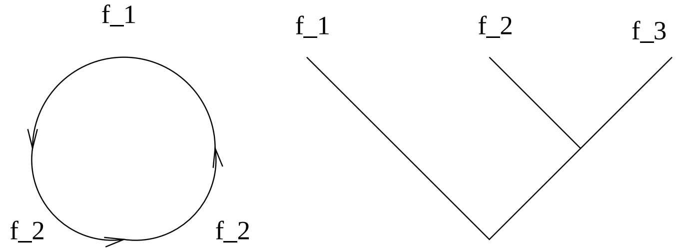


This composition is non associative: for three loops f_1, f_2, f_3, the composition $(f_1*f_2)*f_3$ runs through f_1 on the first *quarter* of the circle whereas the composition $f_1 * (f_2 * f_3)$ runs through f_1 on the first *half* of the circle. We symbolize the two possibilities by the two binary trees with three leaves

$(f_1 * f_2) * f_3$



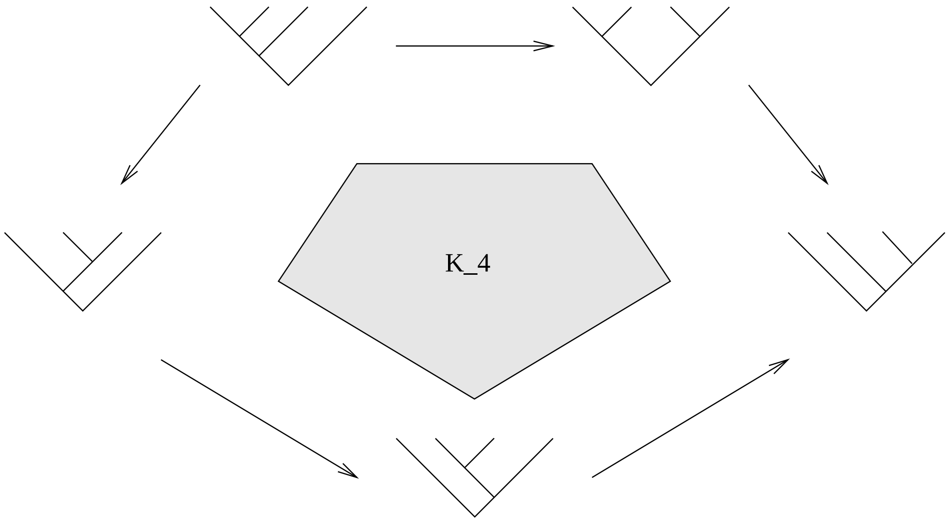
$f_1 * (f_2 * f_3)$



Clearly, there is an homotopy

$$m_3 : [0, 1] \times \Omega X \times \Omega X \to \Omega X.$$

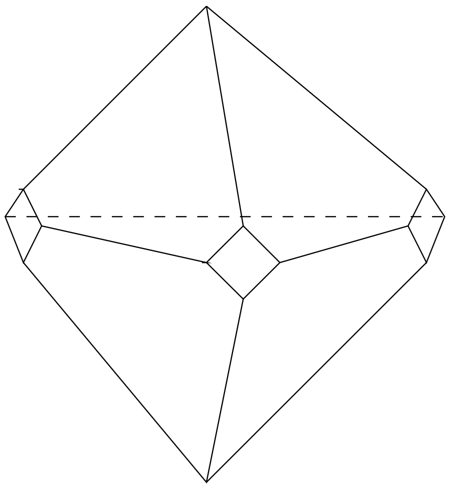
joining the two possibilities of composing three loops. When we want to compose 4 factors, there are 5 possibilities corresponding to the 5 binary trees with 4 leaves. Using m_3, we obtain two paths of homotopies joining the composition $(f_1, f_2, f_3, f_4) \mapsto ((f_1 * f_2) * f_3) * f_4$ to the composition $(f_1, f_2, f_3, f_4) \mapsto f_1 * (f_2 * (f_3 * f_4))$.



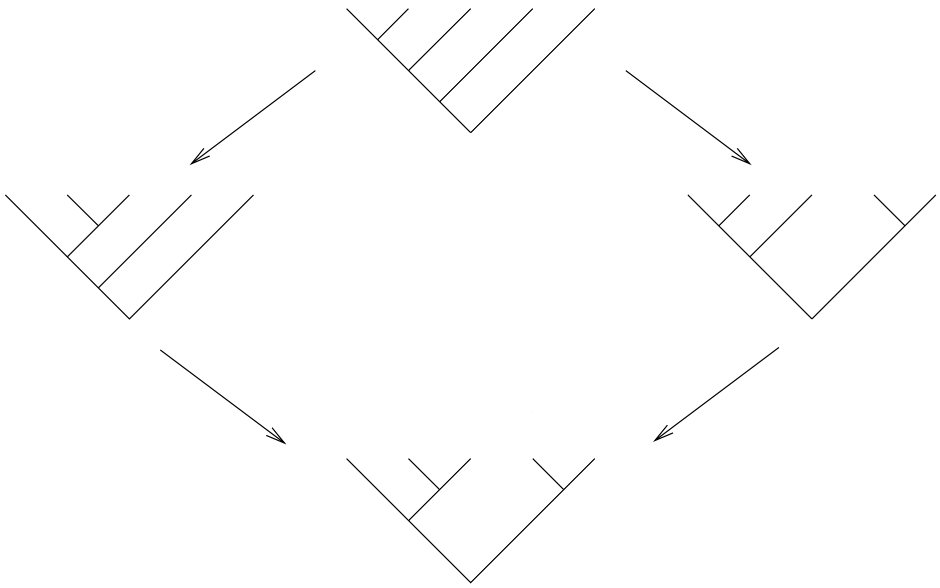
These two paths are clearly homotopic. Let us denote an homotopy by

$$m_4 : K_4 \times (\Omega X)^4 \to \Omega X$$

where K_4 denotes the pentagon bounded by the two paths. When we want to compose 5 factors, there are 14 possibilities corresponding to the 14 binary trees with 5 leaves. Using m_4 and m_3, we obtain paths linking the compositions and faces linking the paths. The resulting sphere is the boundary of the polytope K_5 depicted [39] below.



The pentagonal faces are copies of K_4. A rectangular face appears for example in the following situation



More generally, Stasheff [36] defined polytopes K_n of dimension $n-2$ for all $n \geq 2$ (we have $K_2 = *$, $K_3 = [0,1]$) and he defined an A_∞-*space* to be a topological space Y endowed with maps $m_n : K_n \times Y^n \to Y$, $n \geq 2$, satisfying suitable compatibility conditions and admitting a 'strict unit'. The loop space ΩX is the prime example of such a space Y. Conversely [1, 2.2], a topological space that admits the structure of an A_∞-space and whose connected components form a group is homotopy equivalent to a loop space.

If Y is an A_∞-space, the singular chain complex of Y is the paradigmatic example of an A_∞-algebra [37].

3. A_∞-algebras and their morphisms

3.1. **Definition of an A_∞-algebra.** Let k be a field. An *A_∞-algebra over k* (also called a 'strongly homotopy associative algebra' or an 'sha algebra') is a $\mathbf{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbf{Z}} A^p$$

endowed with graded maps (=homogeneous k-linear maps)

$$m_n : A^{\otimes n} \to A \; , \; n \geq 1 \; ,$$

of degree $2-n$ satisfying the following relations

- We have $m_1\, m_1 = 0$, i.e. (A, m_1) is a differential complex.
- We have

$$m_1\, m_2 = m_2\,(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$

  as maps $A^{\otimes 2} \to A$. Here $\mathbf{1}$ denotes the identity map of the space A. So m_1 is a (graded) derivation with respect to the multiplication m_2.
- We have

$$\begin{aligned} &m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) \\ &\qquad = m_1 m_3 + m_3\,(m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1) \end{aligned}$$

  as maps $A^{\otimes 3} \to A$. Note that the left hand side is the associator for m_2 and that the right hand side may be viewed as the boundary of m_3 in the morphism complex $\mathsf{Hom}^\bullet{}_k\,(A^{\otimes 3}, A)$ (the definition of the morphism complex is recalled below). This implies that m_2 is associative up to homotopy.
- More generally, for $n \geq 1$, we have

$$\sum (-1)^{i+jk}\, m_l\,(\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k}) = 0$$

  where the sum runs over all decompositions $n = i + j + k$ and we put $l = i + 1 + k$.

We have adopted the sign conventions of Getzler-Jones [11]. Note that when these formulas are applied to elements, additional signs appear because of the Koszul sign rule:

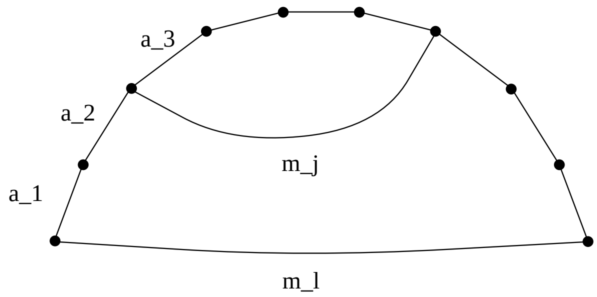
$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$$

where f, g are graded maps, x,y homogeneous elements and the vertical bars denote the degree. For example, we have

$$(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)(x \otimes y) = m_1(x) \otimes y + (-1)^{|x|} x \otimes m_1(y)$$

so that $m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1$ is the usual differential on the tensor product. The defining identities are pictorially represented [22] by

$\sum \pm$



$= 0.$

We note three immediate consequences of the definition:

(1) In general, an A_∞-algebra A is not associative but its homology $H^*A = H^*(A, m_1)$ is an associative graded algebra for the multiplication induced by m_2.
(2) If $A^p = 0$ for all $p \neq 0$, then $A = A^0$ is an ordinary associative algebra. Indeed, since m_n is of degree $2 - n$, all m_n other than m_2 have to vanish.
(3) If m_n vanishes for all $n \geq 3$, then A is an associative dg (=differential $\mathbf{Z}$-graded) algebra and conversely each dg algebra yields an A_∞-algebra with $m_n = 0$ for all $n \geq 3$.

3.2. **Minimal models.** Let us recall the dg algebra which is implicit in problem 2: Let B be an ordinary associative k-algebra with 1 and and M a B-module. Choose an injective resolution

$$0 \to M \to I^0 \to I^1 \to \dots$$

of M and denote by I the complex

$$0 \to I^0 \to I^1 \to \dots .$$

Let $\mathsf{Hom}^\bullet{}_B(I, I)$ be the morphism complex: recall that its nth component is the group of graded B-linear maps $f : I \to I$ of degree n (no compatibility with d_I) and its differential is defined by $d(f) = d_I \circ f - (-1)^n f \circ d_I$. Then $A = \mathsf{Hom}^\bullet{}_B(I, I)$ endowed with its differential becomes a dg algebra for the natural composition of graded maps and its homology algebra is the extension algebra of M:

$$H^*A = \mathsf{Ext}^*_B(M, M).$$

We claim that this has a canonical (up to equivalence) structure of A_∞-algebra with $m_1 = 0$ and m_2 the usual multiplication. Indeed, more generally, we have the

Theorem (Kadeishvili [14], see also [13], [34], [12], [25])**.** *Let A be an A_∞-algebra. Then the homology H^*A has an A_∞-algebra structure such that*

1) *we have $m_1 = 0$ and m_2 is induced by m_2^A and*
2) *there is a quasi-isomorphism of A_∞-algebras $H^*A \to A$ lifting the identity of H^*A.*

The notion of morphism and quasi-isomorphism of A_∞-algebras will be defined below (3.3). For the moment, let us introduce some terminology: In the situation of the theorem, the A_∞-algebra H^*A is called a *minimal model* for A. The A_∞-algebra A is called *formal* if the minimal model can be chosen to be an ordinary associative graded algebra, i.e. such that $m_n = 0$ for all $n \geq 3$.

One can also show that each A_∞-algebra A admits an 'anti-minimal' model, i.e. there is a dg algebra C and a quasi-isomorphism of A_∞-algebras $A \to C$. Thus, passing from dg algebras to A_∞-algebras does not yield new quasi-isomorphism classes. What it does yield is a *new description* of these classes by minimal models.

If, in the above example, we choose B to be a finite-dimensional algebra and M to be the sum of the simple B-modules, then it is easy to show that the dg algebra A is formal if B is hereditary or Koszul. The smallest examples where A is not formal is the algebra B defined by the quiver with relations (cf. [32], [9], [2])

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet\ , \ \gamma\,\beta\,\alpha = 0.$$

Here the extension algebra H^*A is given by the quiver with relations

$$\begin{tikzcd} \circ & \circ \arrow[l, "a"'] & \circ \arrow[l, "b"'] & \circ \arrow[l, "c"'] \arrow[lll, bend left, "e"] \end{tikzcd}\ , \ ab = 0\ , \ bc = 0\ ,$$

where the arrows a, b, c are of degree 1 and the arrow e is of degree 2. We have $m_n = 0$ for $n \neq 2, 3$, $m_3(a, b, c) = e$ and all other products vanish (the method for computing this is given at the end of section 3.3). This example illustrates the fact that the higher products of the A_∞-algebra H^*A lift the Massey products [37] whenever these are defined.

It was shown in [7] that the dg algebra of smooth real forms on a compact Kähler manifold is formal. In particular, this holds for smooth, complex projective manifolds.

3.3. **Morphisms of A_∞-algebras.** A *morphism of A_∞-algebras* $f : A \to B$ is a family

$$f_n : A^{\otimes n} \to B$$

of graded maps of degree $1 - n$ such that

- we have $f_1 m_1 = m_1 f_1$, i.e. f_1 is a morphism of complexes,
- we have

  $$f_1 m_2 = m_2\, f_1 \otimes f_1 + m_1 f_2 + f_2\,(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)\,,$$

  which means that f_1 commutes with the multiplication m_2 up to homotopy,
- more generally, for $n \geq 1$, we have

  $$\sum (-1)^{i+jk}\, f_l\,(\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k}) = \sum (-1)^s\, m_r\,(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r})$$

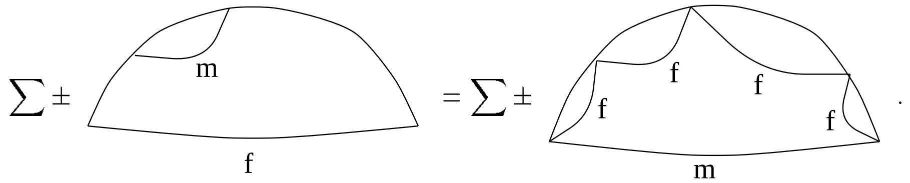
  where the first sum runs over all decompositions $n = i + j + k$, we put $l = i+1+k$, and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \cdots i_r$; the sign on the right hand side is given by

  $$s = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1).$$

The morphism f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism. It is *strict* if $f_i = 0$ for all $i \neq 1$. The *identity morphism* is the strict morphism f such that f_1 is the identity of A. The *composition* of two morphisms $f : B \to C$ and $g : A \to B$ is given by

$$(f \circ g)_n = \sum (-1)^s f_r\ \circ\ (g_{i_1} \otimes \cdots \otimes g_{i_r})$$

where the sum and the sign are as in the defining identity. Here is a (slightly stylized) pictorial representation of the defining identity for a morphism of A_∞-algebras



The minimal model alluded to in (3.2) is unique up to non unique (and non strict!) isomorphism of A_∞-algebras [30].

As an illustration, suppose that A is a dg algebra and let us construct the multiplication m_3 on H^*A and, at the same time, the first three terms of a quasi-isomorphism $f : H^*A \to A$ lifting the identity of H^*A. For this, we view H^*A as a complex with zero differential and we choose a morphism of complexes $f_1 : H^*A \to A$ inducing the identity in homology. Then f_1 commutes with the multiplication up to boundaries, i.e. there exists a graded map $f_2 : H^*A \otimes H^*A \to A$ degree -1 such that

$$f_1\, m_2 = m_2\,(f_1 \otimes f_1) + m_1\, f_2.$$

Now we look for f_3 and m_3 such that

$$\begin{aligned} & f_1 \circ m_3 + f_2 \circ (m_2 \otimes \mathbf{1} - \mathbf{1} \otimes m_2) \\ & \quad + f_3 \circ (m_1 \otimes \mathbf{1}^{\otimes 2} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1}^{\otimes 2} \otimes m_1) \\ = \; & m_3 \circ (f_1 \otimes f_1 \otimes f_1) + m_2 \circ (f_1 \otimes f_2 - f_2 \otimes f_1) + m_1 f_3. \end{aligned}$$

In our situation, this simplifies into

$$f_1 \circ m_3 = m_2 \circ (f_1 \otimes f_2 - f_2 \otimes f_1) + m_1 f_3.$$

Now it is easy to check that the map $m_2\,(f_1 \otimes f_2 - f_2 \otimes f_1)$ has its image in the cycles of A. So there is indeed a map $f_3 : (H^*A)^{\otimes 3} \to A$ of degree -2 and a map $m_3 : (H^*A)^{\otimes 3} \to H^*A$ of degree -1 such that the identity holds.

3.4. **Augmented A_∞-algebras.** If an A_∞-algebra A has a unital homology algebra H^*A, one can often simplify the computation of the minimal model for A by passing to the category of augmented A_∞-algebras. Here are the relevant definitions:

A *strictly unital A_∞-algebra* is an A_∞-algebra A endowed with an element 1_A of degree 0 such that we have $m_1(1_A) = 0$, $m_2(1_A, a) = a = m_2(a, 1_A)$ for all $a \in A$ and such that, for all $i > 2$, and all $a_1, \ldots, a_i$ in A, the product $m_i(a_1, \ldots, a_i)$ vanishes if one of the a_j equals 1_A. If A and B are strictly unital A_∞-algebras, a *morphism* of A_∞-algebras $f : A \to B$ is *strictly unital* if we have $f_1(1_A) = 1_B$ and if for all $i > 1$ and all $a_1, \ldots, a_i$, the element $f_i(a_1, \ldots, a_i)$ vanishes if one of the a_j equals 1. Each strictly unital A_∞-algebra is canonically endowed with a strict (hence strictly unital) morphism $\eta : k \to A$ mapping 1_k to 1_A. It is *augmented* if it is moreover endowed with a strictly unital morphism $\varepsilon : A \to k$ such that $\varepsilon \circ \eta = \mathbf{1}_k$. A *morphism of augmented A_∞-algebras* is a strictly unital morphism $f : A \to B$ such that $\varepsilon_B \circ f = \varepsilon_A$. It is then easy to check that the functor $A \mapsto \ker \varepsilon$ is an equivalence from the category of augmented A_∞-algebras to the category of

A_∞-algebras. The quasi-inverse functor sends an A_∞-algebra B to $k \oplus B$ endowed with the unique strictly unital structure extending the structure of B.

Let A be an augmented A_∞-algebra and let $\overline{A} = \ker \varepsilon$. Clearly we obtain a minimal model for A by taking $k \oplus \overline{A}'$, where $\overline{A}'$ is a minimal model for $\overline{A}$ in the category of A_∞-algebras. The computation of the minimal model for $\overline{A}$ is simpler than that of A, because we do not have to deal with the 'loop' 1.

The construction of minimal models and the above discussion extend to the case where the ground ring k is no longer a field but a semi-simple ring R, for example a product of fields. We then consider A_∞-algebras in the monoidal category of R-R-bimodules. Suppose for example that A is a unital dg k-algebra endowed with a decomposition

$$1_A = e_1 + \ldots + e_N$$

of 1_A into idempotents $e_i = e_i^2$ such that the complexes $e_j A e_i$ are acyclic for $i < j$ and that $e_i A e_i$ is quasi-isomorphic to $k\, e_i$. Clearly A is isomorphic to the matrix algebra

$$\bigoplus_{i,j=1}^{N} e_j A e_i$$

and this algebra is quasi-isomorphic to the subalgebra of upper triangular matrices

$$B = ke_1 \oplus \ldots \oplus ke_N \oplus \bigoplus_{i<j} e_j A e_i .$$

Clearly B is an augmented algebra over $R = k \times \ldots \times k$ (N factors). So we obtain a minimal model for A as the sum of R with a minimal model for

$$\overline{B} = \bigoplus_{i<j} e_j A e_i .$$

viewed as an A_∞-algebra over R. Let $\overline{B}'$ be this minimal model. Then we have a decomposition

$$\overline{B}' = \bigoplus_{i<j} H^*(e_j A e_i) = \bigoplus_{i<j} e_j H^*(A) e_i$$

of R-R-bimodules and the tensor power

$$\overline{B}' \otimes_R \overline{B}' \otimes_R \ldots \otimes_R \overline{B}' \quad (n \text{ factors})$$

is isomorphic to the sum of the

$$e_{i_{n+1}} H^*(A) e_{i_n} \otimes_k e_{i_n} H^*(A) e_{i_{n-1}} \otimes \ldots \otimes_k e_{i_2} H^*(A) e_{i_1}$$

taken over all sequences $i_1 < i_2 < \ldots < i_{n+1}$. These spaces vanish for $n \geq N$ so that in the minimal model we automatically have $m_n = 0$ for $n \geq N$. For example, if $N = 3$, i.e. if the unit of A decomposes into $1 = e_1 + e_2 + e_3$, then $\overline{B}'$ is automatically formal. Hence $\overline{B}$ and A are also formal in this case.

The following lemma presents another case where the passage to the category of augmented A_∞-algebras is easy. A *subalgebra* of an A_∞-algebra A is a graded subspace B such that m_i maps $B^{\otimes i}$ to B for all $i \geq 1$.

Lemma. *Let A be a strictly unital A_∞-algebra (for example a unital dg algebra). Suppose that $H^i A = 0$ for $i < 0$ and that $H^0 A = k$. Then there is a subalgebra $B \subset A$ whose inclusion is a quasi-isomorphism and which is augmented.*

3.5. **The bar construction.** Following [37] and [15] we will restate the definitions of A_∞-algebras and their morphisms in a more efficient way. Let V be a $\mathbf{Z}$-graded vector space and let

$$\overline{T}V = V \oplus V^{\otimes 2} \oplus \cdots$$

be the reduced tensor algebra (in the category of graded spaces). We make $\overline{T}V$ into a graded coalgebra via the comultiplication

$$\Delta : \overline{T}V \to \overline{T}V \otimes \overline{T}V$$

defined by

$$\Delta(v_1, \ldots, v_n) = \sum_{1 \le i \le n} (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n) \, .$$

Note that $\Delta(v) = 0$ and $\Delta(v_1, v_2) = v_1 \otimes v_2$. It is then easy to see that each graded map $b : \overline{T}V \to V$ of degree 1 uniquely lifts to a graded coderivation $b : \overline{T}V \to \overline{T}V$ of degree 1, where *coderivation* means that we have

$$\Delta b = b \otimes \mathbf{1} + \mathbf{1} \otimes b.$$

(To see that this is plausible, one should remember that algebra derivations $\overline{T}V \to \overline{T}V$ are in bijection with maps $V \to \overline{T}V$). Thus we have a bijection between coderivations $b : \overline{T}V \to \overline{T}V$ of degree 1 and families of maps $b_n : V^{\otimes n} \to V$, $n \ge 1$, of degree 1.

Now let A be another $\mathbf{Z}$-graded space and let SA be its shift by one: $(SA)^p = A^{p+1}$. Define a bijection between the families of maps $b_n : (SA)^{\otimes n} \to SA$ and maps $m_n : A^{\otimes n} \to A$, $n \ge 1$, by the commutative square

$$\begin{array}{ccc} (SA)^{\otimes n} & \xrightarrow{b_n} & SA \\ s^{\otimes n} \uparrow & & \uparrow s \\ A^{\otimes n} & \xrightarrow{m_n} & A \end{array} \, ,$$

where $s : A \to SA$ is the canonical map $a \mapsto a$ of degree -1. Note that m_n is of degree $-n + 1 - (-1) = 2 - n$. Then we have the following easy

Lemma. *The following are equivalent*

(i) *The maps $m_n : A^{\otimes n} \to A$ yield an A_∞-algebra structure on A.*
(ii) *The coderivation $b : \overline{T}SA \to \overline{T}SA$ satisfies $b^2 = 0$.*
(iii) *For each $n \ge 1$, we have*

$$\sum b_l \, (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes k}) = 0 \, ,$$

where the sum runs over all decompositions $n = i+j+k$ and we put $l = i+1+k$.

Note that no signs appear in the last formula. The lemma yields a bijection between A_∞-algebra structures on A and coalgebra differentials on $\overline{T}SA$. It is also easy to check that if A and B are A_∞-algebras, then morphisms of A_∞-algebras $f : A \to B$ are in bijection with morphisms of dg coalgebras $\overline{T}SA \to \overline{T}SB$.

3.6. **Homotopy equivalences and quasi-isomorphisms.** Let A and B be A_∞-algebras and f, g morphisms from A to B. Let F, G denote the corresponding morphisms of dg coalgebras $\overline{T}SA \to \overline{T}SB$. One defines f and g to be *homotopic* if there is a homogeneous map $H : \overline{T}SA \to \overline{T}SB$ of degree -1 such that

$$\Delta H = F \otimes H + H \otimes G \quad \text{and} \quad F - G = b \circ H + H \circ b.$$

It is easy to translate this into the existence of a family of maps $h_n : A^{\otimes n} \to B$ satisfying suitable conditions.

Theorem (Prouté [30], cf. also [16]). (a) *Homotopy is an equivalence relation on the set of morphisms of A_∞-algebras $A \to B$.*

(b) *A morphism of A_∞-algebras $s : A \to B$ is a quasi-isomorphism iff it is an homotopy equivalence (i.e. it becomes an isomorphism in the category obtained by dividing the category of A_∞-algebras by the homotopy relation).*

4. A_∞-modules and the derived category

4.1. **Reminder on the derived category.** We refer to [18, Ch. I] for a concise presentation of derived categories and to [42] and [10] for more leisurely introductions. We refer to [35] for foundational material on resolutions of unbounded complexes.

Let A be an associative k-algebra with 1 and denote by $\mathsf{Mod}\, A$ the category of unital right A-modules. The derived category $\mathcal{D}A$ is classically constructed [40] [41] in three steps:

1st step: We define the *category $\mathcal{C}A$* to be the category of complexes

$$\ldots \to M^p \to M^{p+1} \to \ldots$$

with morphisms of complexes as morphisms.

2nd step: We define the *homotopy category $\mathcal{H}A$*. Its objects are the same as those of $\mathcal{C}A$. The group of morphisms $M \to N$ in $\mathcal{H}A$ is the group of morphisms of complexes divided by the subgroup of null homotopic morphisms.

3rd step: We define the *derived category $\mathcal{D}A$* to be the localization of the homotopy category $\mathcal{H}A$ with respect to all quasi-isomorphisms. Thus the objects of $\mathcal{D}A$ are the same as those of $\mathcal{H}A$ and its morphisms are obtained from morphisms in the homotopy category by formally inverting all quasi-isomorphisms.

To get an idea of what the morphisms in the derived category are it is helpful to consider the following special case: Let M and N be two A-modules. Identify M with the complex

$$\ldots \to 0 \to M \to 0 \to \ldots$$

concentrated in degree 0. For any complex K and any $n \in \mathbf{Z}$, denote by $K[n]$ the complex K shifted by n degrees to the left: $K[n]^p = K^{n+p}$ and $d_{K[n]} = (-1)^n d_K$. Then there is a canonical quasi-isomorphism

$$\mathsf{Hom}_{\mathcal{D}A}(M, N[n]) = \mathsf{Ext}^n_A(M, N)$$

valid for any $n \in \mathbf{Z}$ if we convene that the right hand side vanishes for negative n.

4.2. **Generalization to A_∞-algebras.** Let A be an A_∞-algebra. We generalize the above step by step.

1st step: An *A_∞-module over A* is a $\mathbf{Z}$-graded space M endowed with maps

$$m_n^M : M \otimes A^{\otimes n-1} \to M \ , \ n \geq 1 \ ,$$

of degree $2-n$ such that an identity of the same form as the one in the definition (3.1) of an A_∞-algebra holds. However, the term

$$m_l\,(\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k})$$

has to be interpreted as

$$m_l^M\,(\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k})$$

if $i > 0$ and as

$$m_l^M\,(m_j^M \otimes \mathbf{1}^{\otimes k})$$

if $i = 0$. A *morphism of A_∞-modules* $f : L \to M$ is a sequence of graded morphisms

$$f_n : L \otimes A^{\otimes n-1} \to M \tag{4.1}$$

of degree $1-n$ such that for each $n \geq 1$, we have

$$\sum (-1)^{i+jk}\, f_l \circ (\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k}) = \sum (-1)^{(i+1)j}\, m_l \circ (f_i \otimes \mathbf{1}^{\otimes j})\ , \tag{4.2}$$

where the left hand sum is taken over all decompositions $n = i+j+k$, $i, k \geq 0$, $j \geq 1$ and we put $l = i+1+k$; and the right hand sum is taken over all decompositions $n = i+j$, $i \geq 0$, $j \geq 1$ and we put $l = i+1$. The morphism f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism. The *identical morphism* $f : M \to M$ is given by $f_1 = \mathbf{1}_M$, $f_i = 0$, for all $i > 0$. The *composition* fg is given by

$$(fg)_n = \sum (-1)^{(i-1)j} f_l(g_i \otimes \mathbf{1}^{\otimes j})$$

where the sum runs over all decompositions $n = i+j$ and we put $l = 1+j$.

In analogy with (3.5), one can easily check that A_∞-structures on a graded space M are in bijection with comodule differentials on $SM \otimes TSA$, where TSA is the coaugmented tensor coalgebra on SA :

$$TSA = k \oplus \overline{T}SA.$$

The comultiplication on TSA is given by

$$\Delta(x) = 1 \otimes x + \Delta_{\overline{T}SA}(x) + x \otimes 1.$$

The differential of TSA is induced by b on $\overline{T}SA$ and maps k to zero. The comodule structure on $SM \otimes TSA$ is induced from the coalgebra structure on TSA. Morphisms of A_∞-modules then correspond bijectively to morphisms of dg comodules.

We define $\mathcal{C}_\infty A$ to be the *category of A_∞-modules.* If A is an ordinary associative algebra, we have a faithful functor

$$\mathcal{C}A \to \mathcal{C}_\infty A$$

but $\mathcal{C}_\infty A$ has more objects and more morphisms than $\mathcal{C}A$. A typical example of an A_∞-module over A is obtained as follows: Suppose that A is the endomorphism algebra of a module M' over an ordinary algebra B and let M be a projective resolution of M' over B. Then it is well-known that each element of A lifts to an endomorphism of the complex M and this endomorphism is unique up to homotopy. The resulting 'action up to homotopy' of A on M may be shown to arise from a structure of A_∞-module.

2nd step: By definition, a morphism $f : L \to M$ of A_∞-modules is *nullhomotopic* if there is a family of graded maps

$$h_n : L \otimes A^{\otimes n-1} \to M\ ,\ n \geq 1\ ,$$

homogeneous of degree $1-n$ such that

$$\begin{aligned} f_n &= \sum(-1)^{ij}\, m_{1+j} \circ (h_i \otimes \mathbf{1}^{\otimes j}) \\ &\quad + \sum(-1)^{i+jk}\, h_l \circ (\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k})\,, \end{aligned}$$

where the first sum runs over all decompositions $n = i+j$, $i \geq 1$, $j \geq 0$ and the second sum over all decompositions $n = i+j+k$, $i,k \geq 0$, $j \geq 1$ and we put $l = i+1+k$.

The *homotopy category* $\mathcal{H}_\infty A$ has the same objects as $\mathcal{C}_\infty A$ and morphisms from L to M are morphisms of A_∞-modules modulo nullhomotopic morphisms.

3rd step: To define the derived category, we should formally invert all quasi-isomorphisms. This turns out not to be necessary:

Theorem. *Each quasi-isomorphism of A_∞-modules is an homotopy equivalence.*

We therefore define the *derived category* $\mathcal{D}_\infty A$ to be equal to the homotopy category $\mathcal{H}_\infty A$. If k is only assumed to be a commutative ring, the assertion of the theorem is no longer true in general and the derived category is different from the homotopy category.

4.3. **Solution of problem 1.**

Theorem. *Let A be an A_∞-algebra.*

a) *If A is an ordinary associative algebra with 1, the canonical functor*

$$\mathcal{D}A \to \mathcal{D}_\infty A$$

is an equivalence onto the full subcategory of homologically unital A_∞-modules, i.e. A_∞-modules M such that $1 \in A$ acts by the identity on H^*M.

b) *For each A_∞-module M, the graded space H^*M admits a canonical A_∞-module structure such that $m_1 = 0$ and M is isomorphic to H^*M in $\mathcal{D}_\infty A$.*

Part a) is proved in [20, section 7]. The A_∞-module H^*M of b) is called the *minimal model* of M. Its existence is shown in [12]. It is unique up to (non unique, non strict) isomorphism of A_∞-modules.

The theorem explains the answer to problem 1: Indeed, if M is a complex over an algebra A as in (2.1), then we may view it as an A_∞-module, or, in other words, consider its image in $\mathcal{D}_\infty A$. By b), the image is isomorphic to H^*M endowed with its canonical A_∞-structure $m_1 = 0, m_2, m_3, \ldots$. The full faithfulness of the functor $\mathcal{D}A \to \mathcal{D}_\infty A$ then shows that $(H^*M, m_1 = 0, m_2, \ldots)$ determines the complex M up to isomorphism in the derived category $\mathcal{D}A$.

In fact the theorem tells us more: If we also use the essential surjectivity of the functor $\mathcal{D}A \to \mathcal{D}_\infty A$ onto its image, we see that if V is a graded vector space, then we have a bijection between complexes of A-modules M with $H^*M \xrightarrow{\sim} V$ (as graded vector spaces) up to isomorphism in $\mathcal{D}A$, and homologically unital A_∞-structures $m_1 = 0, m_2, m_3, m_4, \ldots$ on V up to isomorphism of such structures in $\mathcal{C}_\infty\mathcal{A}$. Note that the latter set is naturally the set of orbits under the action of a group, namely the group of families f_i, $i \geq 1$, as in (4.1) acting on the set of homologically unital module structures by a conjugation action deduced from (4.2). If the total dimensions of V and A are finite, we even obtain an algebraic group acting on an algebraic variety.

5. Triangulated structure

5.1. **Towards problem 2.** Consider problem 2 of (2.1) for the special case $n = 1$: We then have an ordinary associative algebra B with 1 and a B-module M and we wish to reconstruct the category $\mathsf{filt}\, M \subset \mathsf{Mod}\, B$ of iterated extensions of M by itself from the extension algebra

$$E = \mathsf{Ext}^*_B\,(M, M)$$

with its canonical A_∞-structure. The basic idea is to realize $\mathsf{filt}\, M$ as a subcategory of the derived category of E. In fact, this subcategory will simply be the subcategory of iterated extensions of the free E-module of rank one by itself. Now the derived category is not abelian in general so that a priori it makes no sense to speak of extensions. However, like the derived category of an ordinary associative algebra, the derived category of an A_∞-algebra does admit a triangulated structure and this suffices for our purposes.

5.2. **Triangulated structure on $\mathcal{D}_\infty A$.** Recall that a *triangulated category* is an additive category $\mathcal{T}$ endowed with an additive endofunctor $S : \mathcal{T} \to \mathcal{T}$ called its *suspension* (or *shift*) and with a class of sequences of the form

$$X \to Y \to Z \to SX \quad (*)$$

called *triangles* (or *distinguished triangles*). These have to satisfy a number of axioms (see the references at the beginning of 4.1). If Y occurs in a triangle $(*)$, it is called an *extension* of Z by X.

For example, if A is an ordinary associative algebra with 1, the derived category $\mathcal{D}A$ is triangulated: The suspension of a complex K is defined by

$$(SK)^p = K^{p+1}\ ,\ d_{SK} = -d_K.$$

Each short exact sequence of complexes

$$0 \to K \to L \to M \to 0$$

gives rise to a canonical triangle

$$K \to L \to M \to SK$$

and (up to isomorphism) all triangles are obtained in this way. This generalizes to A_∞-algebras as follows

Proposition. *Let A be an A_∞-algebra. Then the derived category $\mathcal{D}_\infty A$ admits a triangulated structure with suspension functor S defined by*

$$(SM)^p = M^{p+1}\ ,\ m_n^{SM} = (-1)^n\, m_n^M$$

and such that each short exact sequence

$$0 \to K \xrightarrow{i} L \xrightarrow{p} M \to 0$$

of A_∞-modules with strict *morphisms i and p gives rise to a canonical triangle. Up to isomorphism in $\mathcal{D}_\infty A$, all triangles are obtained in this way.*

6. Standard functors

6.1. **More on problem 2.** If $\mathcal{T}$ is a triangulated category and M an object of $\mathcal{T}$, we denote by $\mathsf{filt}\, M = \mathsf{filt}_{\mathcal{T}}\, M$ the full subcategory of $\mathcal{T}$ whose objects are iterated extensions of M by itself. We denote by $\mathsf{tria}\, M$ the closure of M under suspension, desuspension and forming extensions. Thus $\mathsf{tria}\, M$ is the smallest triangulated subcategory of $\mathcal{T}$ containing M and $\mathsf{filt}\, M$ is a full subcategory of $\mathsf{tria}\, M$.

If $\mathcal{S}$ and $\mathcal{T}$ are triangulated categories a *triangle functor* $\mathcal{S} \to \mathcal{T}$ is an additive functor $F : \mathcal{S} \to \mathcal{T}$ together with an isomorphism of functors $\varphi : FS \to SF$ such that for each triangle (X, u, Y, v, Zw) of $\mathcal{S}$, the sequence

$$FX \stackrel{Fu}{\to} FY \stackrel{Fv}{\to} FZ \xrightarrow{(\varphi X)(Fw)} SFX$$

is a triangle of $\mathcal{T}$. If M is an object of $\mathcal{T}$ and F a triangle functor then clearly $F\,\mathsf{tria}\, M \subset \mathsf{tria}\, FM$ and similarly for $\mathsf{filt}\, M$.

In the situation of (5.1), consider the subcategory $\mathsf{filt}\, A \subset \mathcal{D}_\infty A$ obtained by taking the closure under extensions of the free A-module of rank one (also denoted by A). We would like to show that the subcategory $\mathsf{filt}\, A$ is equivalent to $\mathsf{filt}_{\mathsf{Mod}\, B}\, M$. For this, we first observe that the inclusion

$$\mathsf{Mod}\, B \to \mathcal{D}B$$

induces an equivalence

$$\mathsf{filt}_{\mathsf{Mod}\, B}\, M \xrightarrow{\sim} \mathsf{filt}_{\mathcal{D}B}\, M.$$

It will thus suffice to construct a triangle functor $\mathcal{D}B \to \mathcal{D}_\infty A$ inducing an equivalence from $\mathsf{filt}\, M$ onto $\mathsf{filt}\, A$. The following diagram summarizes the setup:

$$\begin{array}{ccccc} \mathsf{Mod}\, B & \to & \mathcal{D}B & \to & \mathcal{D}_\infty A \\ \uparrow & & \uparrow & & \uparrow \\ \mathsf{filt}\, M & \xrightarrow{\sim} & \mathsf{filt}_{\mathcal{D}B}\, M & \xrightarrow{\sim} & \mathsf{filt}\, A \end{array}$$

The triangle functor $\mathcal{D}B \to \mathcal{D}_\infty A$ is constructed as a composition of standard functors.

6.2. **Restriction.** Let $f : A \to B$ be a morphism of A_∞-algebras and let M be a module over B. Define

$$m_n^A : M \otimes A^{\otimes n-1} \to M \;,\; n \geq 1 \;,$$

by

$$m_n^A = \sum (-1)^s \, m_{r+1}(\mathbf{1} \otimes f_{i_1} \otimes \cdots \otimes f_{i_r})$$

where the sum runs over all $1 \leq r \leq n-1$ and all decompositions $n-1 = i_1 + \cdots + i_r$ and the sign is like in (3.3). Then it is not hard to check that the m_n^A define an A_∞-module f^*M over A. The assignment $M \to f^*M$ becomes a functor

$$\mathcal{C}_\infty B \to \mathcal{C}_\infty A$$

and induces a triangle functor

$$\mathcal{D}_\infty B \to \mathcal{D}_\infty A$$

which will still be denoted by f^*.

Proposition. *The functor $f^* : \mathcal{D}_\infty A \to \mathcal{D}_\infty B$ is an equivalence if f is a quasi-isomorphism.*

6.3. **Hom and Tensor.** Let B be an A_∞-algebra and X and M two B-modules. For $n \in \mathbf{Z}$, let $\overset{\infty}{\mathsf{Hom}}{}^n_B(X, M)$ be the space of graded comodule maps $f : SX \otimes TSB \to SM \otimes TSB$ of degree n and define

$$d(f) = b \circ f - (-1)^n f \circ b.$$

Then $\overset{\infty}{\mathsf{Hom}}{}^\bullet_B(X, M)$ is a complex which is bifunctorial in X and M. If B is an ordinary algebra and X, M are ordinary modules, then $\overset{\infty}{\mathsf{Hom}}{}^\bullet_B(X, M)$ computes $\mathsf{Ext}^*_B(X, M)$.

Now suppose that A is another A_∞-algebra and that the B-module structure on X comes from an A-B-bimodule structure, i.e. X is endowed with graded maps

$$m_{k,l} : A^{\otimes k-1} \otimes X \otimes B^{\otimes l-1} \to X \ , \ k, l \geq 1 \ ,$$

of degree $3 - k - l$ such that we have the associativity identity of (3.1), where the terms $\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k}$ have to be interpreted according to the nature of their arguments. Then $\overset{\infty}{\mathsf{Hom}}{}^\bullet_B(X, M)$ becomes a (right) A_∞-module over A and we have the Hom-functor

$$\overset{\infty}{\mathsf{Hom}}{}^\bullet_B(X, ?) : \mathcal{C}_\infty B \to \mathcal{C}_\infty A$$

Proposition. *The Hom-functor admits a left adjoint*

$$? \overset{\infty}{\otimes}_A X : \mathcal{C}_\infty A \to \mathcal{C}_\infty B.$$

We have an induced pair of adjoint triangle functors

$$\begin{array}{ccc} & \mathcal{D}_\infty A & \\ ? \overset{\infty}{\otimes}_A X & \downarrow\uparrow & \overset{\infty}{\mathsf{Hom}}{}^\bullet_B(X, ?) \\ & \mathcal{D}_\infty B & \end{array} .$$

It is not hard to make $L \overset{\infty}{\otimes}_A X$ explicit. Its underlying graded space is $L \otimes_k TSA \otimes_k X$. If A is an ordinary associative algebra with 1 and L, X are ordinary modules, then $L \overset{\infty}{\otimes}_A X$ is a complex computing $\mathsf{Tor}^A_*(L, X)$.

6.4. **Application to problem 2.** Let B be an A_∞-algebra and M a B-module. Then

$$A = \overset{\infty}{\mathsf{Hom}}{}^\bullet_B(M, M)$$

is a dg algebra and in particular an A_∞-algebra. The module M becomes an A-B-bimodule.

Proposition. *The functor*

$$F = \overset{\infty}{\mathsf{Hom}}{}^\bullet_B(M, ?) : \mathcal{D}_\infty B \to \mathcal{D}_\infty A$$

induces a triangle equivalence

$$\mathsf{tria}\, M \xrightarrow{\sim} \mathsf{tria}\, A$$

taking M to A and thus an equivalence $\mathsf{filt}\, M \xrightarrow{\sim} \mathsf{filt}\, A$.

Now consider the situation of (5.1). In this case, A is a dg algebra whose homology is the extension algebra of M. Let us simply write

$$E = \mathsf{Ext}^*_B(M, M)$$

for the extension algebra endowed with its canonical structure of A_∞-algebra. Then we know by Theorem 3.2 that we have a quasi-isomorphism $E \to A$. This yields the restriction functor

$$\mathsf{res} : \mathcal{D}_\infty A \to \mathcal{D}_\infty E$$

which is an equivalence by Proposition 6.2. Consider the square

$$\begin{array}{ccc} \mathcal{D}_\infty B & \xrightarrow{\overset{\infty}{\mathsf{Hom}}{}^{\bullet}_B(M,?)} & \mathcal{D}_\infty A \\ \mathsf{can} \uparrow & & \downarrow \mathsf{res} \\ \mathcal{D} B & \longrightarrow & \mathcal{D}_\infty E \end{array}$$

The left vertical arrow induces an equivalence onto its image and maps M to M. By the above proposition, the top horizontal arrow induces an equivalence $\mathsf{filt}\, M \xrightarrow{\sim} \mathsf{filt}\, A$. The right vertical arrow is an equivalence mapping the free module A to the free module E and hence induces an equivalence $\mathsf{filt}\, A \xrightarrow{\sim} \mathsf{filt}\, E$. It follows that the bottom arrow induces an equivalence $\mathsf{filt}_{\mathcal{D}B}\, M \xrightarrow{\sim} \mathsf{filt}\, E$. Finally, we know that $\mathsf{filt}_{\mathcal{D}B}\, M$ is equivalent to $\mathsf{filt}\, M \subset \mathsf{Mod}\, B$ so that we do obtain the required equivalence between $\mathsf{filt}\, E \subset \mathcal{D}_\infty E$ and $\mathsf{filt}\, M \subset \mathsf{Mod}\, B$.

6.5. **Solution of the original problem 2.** In order to treat problem 2 as it is stated in (2.1) with n modules $M_1, \ldots, M_n$, let e_i be the idempotent of E corresponding to the direct factor M_i of $M = \bigoplus M_i$. There is a canonical isomorphism

$$E \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}_\infty E}(E, E)$$

and thus e_i yields an idempotent endomorphism of E in the derived category. Now the derived category is a triangulated category with infinite sums so that by a well-known trick [4], the idempotent e_i splits in $\mathcal{D}_\infty E$. Let X_i denote its image. We have

$$E \xrightarrow{\sim} \bigoplus X_i$$

in $\mathcal{D}_\infty E$. The functor constructed in 6.4 sends X_i to M_i and thus induces an equivalence $\mathsf{filt}\, X_i \xrightarrow{\sim} \mathsf{filt}\, M_i$.

7. Twisted objects

7.1. **Motivation.** Let A be an associative algebra with 1 and $\mathsf{tria}\, A$ the triangulated subcategory of $\mathcal{D}A$ generated by the free A-module of rank one (as in 6.1). Up to isomorphism, the objects of $\mathsf{tria}\, A$ are the bounded complexes of finitely generated free A-modules and the morphisms between two such complexes are in bijection with the homotopy classes of morphisms of complexes. In other words, $\mathsf{tria}\, A$ is *triangle equivalent to the bounded homotopy category of finitely generated free A-modules.* The advantage of this description of $\mathsf{tria}\, A$ is that morphisms and objects are readily computable. The aim of this section is to give an analogous description of $\mathsf{tria}\, A$ for any A_∞-algebra A with a strict unit (cf. below). This description will be of the form

$$\mathsf{tria}\, A \xrightarrow{\sim} H^0\, \mathsf{tw}\, A$$

where $\mathsf{tw}\, A$ is the A_∞*-category* of twisted objects over A. An A_∞-category is an 'A_∞-algebra with several objects' analogous to a 'ring with several objects' [26]. The above equivalence illustrates the general fact that working with several objects is natural and useful. From the outset, we will therefore replace the A_∞-algebra A by an A_∞-category with strict identities $\mathcal{A}$. We will then have more generally

$$\mathsf{tria}\,\mathcal{A} \xrightarrow{\sim} H^0\,\mathsf{tw}\,\mathcal{A}.$$

The material in this section is adapted from [21], [22]. The special case of dg categories goes back to Bondal-Kapranov's article [5].

7.2. A_∞**-categories.** Let k be a field. An A_∞*-category* $\mathcal{A}$ is the datum of

- a class of objects $\mathsf{obj}(\mathcal{A})$,
- for all $A, B \in \mathcal{A}$ (we write $A \in \mathcal{A}$ instead of $A \in \mathsf{obj}(\mathcal{A})$), a $\mathbf{Z}$-graded vector space $\mathsf{Hom}_{\mathcal{A}}(A, B)$, often *denoted by* (A, B),
- for all $n \geq 1$ and all $A_0, \ldots, A_n \in \mathcal{A}$, a graded map

  $$m_n : (A_{n-1}, A_n) \otimes (A_{n-2}, A_{n-1}) \otimes \cdots \otimes (A_0, A_1) \to (A_0, A_n)$$

  of degree $2 - n$

such that for each $n \geq 1$ and all $A_0, \ldots, A_n \in \mathcal{A}$, we have the identity

$$\sum (-1)^{i+jk}\, m_l\, (\mathbf{1}^{\otimes i} \otimes m_j \otimes \mathbf{1}^{\otimes k}) = 0$$

of maps

$$(A_{n-1}, A_n) \otimes (A_{n-2}, A_{n-1}) \otimes \cdots \otimes (A_0, A_1) \to (A_0, A_n)\ ,$$

where the sum runs over all decompositions $n = i + j + k$ and we put $l = i + 1 + k$.

For example, the datum of an A_∞-category $\mathcal{A}$ with one object $*$ is equivalent to the datum of the endomorphism A_∞-algebra $\mathsf{Hom}_{\mathcal{A}}(*, *)$. Another class of examples is provided by dg categories : these are exactly the A_∞-categories whose composition maps m_n vanish for all $n \geq 3$.

Note that for two reasons, an A_∞-category is not a category in general : firstly, the composition m_2 may not be associative; secondly, there may not be identity morphisms.

Let $\mathcal{A}$ be an A_∞-category and $A \in \mathcal{A}$. A morphism $e_A \in \mathsf{Hom}^0_{\mathcal{A}}(A, A)$ is a *strict identity* if we have

$$m_2(f, e_A) = f\ ,\ m_2(e_A, g) = g$$

whenever these make sense and

$$m_n(\ldots, e_A, \ldots) = 0$$

for all $n \neq 2$. In particular, e_A is then a cycle of the complex $(\mathsf{Hom}_{\mathcal{A}}(A, A), m_1)$. Clearly, if e_A exists, it is unique. The A_∞-category $\mathcal{A}$ *has strict identities* if there is a strict identity e_A for each object $A \in \mathcal{A}$.

7.3. A_∞**-functors.** Let $\mathcal{A}, \mathcal{B}$ be two A_∞-categories. An A_∞*-functor* $F : \mathcal{A} \to \mathcal{B}$ is the datum of

- a map $F : \mathsf{obj}(\mathcal{A}) \to \mathsf{obj}(\mathcal{B})$,
- for all $n \geq 1$ and all $A_0, \ldots, A_n \in \mathcal{A}$ a graded map

  $$F_n : (A_{n-1}, A_n) \otimes (A_{n-2}, A_{n-1}) \otimes \cdots \otimes (A_0, A_1) \to \mathsf{Hom}_{\mathcal{B}}(FA_0, FA_n)$$

  of degree $1 - n$

such that conditions analogous to those of (3.3) hold. The A_∞-functor F is *strict* if we have $F_n = 0$ for all $n \geq 2$. This implies that F_1 'commutes with all compositions'.

7.4. **Modules.** We leave it as an exercise to the reader to generalize (4.2) to A_∞-categories. In particular, if $\mathcal{A}$ is an A_∞-category, he may

- define the category $\mathcal{C}_\infty\mathcal{A}$ of $\mathcal{A}$-modules
- define the derived category $\mathcal{D}_\infty\mathcal{A}$
- define a structure of dg category
$$\overset{\infty}{\mathsf{Hom}}\,(L, M)\ ,\ L, M \in \mathcal{C}_\infty\mathcal{A}$$
on $\mathcal{C}_\infty\mathcal{A}$
- define the Yoneda functor
$$Y : \mathcal{A} \to \mathcal{C}_\infty\mathcal{A}\ ,\ A \mapsto \overset{\infty}{\mathsf{Hom}}_{\mathcal{A}}(?, A).$$
Here we view $\mathcal{C}_\infty\mathcal{A}$ as a dg category (and hence an A_∞-category) and the Yoneda functor is to be an A_∞-functor.

7.5. **Factorization of the Yoneda functor.** Let $\mathcal{A}$ be an A_∞-category and let $\mathsf{tria}\,\mathcal{A}$ denote the triangulated subcategory of $\mathcal{D}_\infty\mathcal{A}$ generated by the image of the Yoneda functor.

Theorem. *Suppose that $\mathcal{A}$ has strict identities. Then there is an A_∞-category $\mathsf{tw}\,\mathcal{A}$ and a factorization*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Y_1"] \arrow[dr, "Y"'] & \mathsf{tw}\,\mathcal{A} \arrow[d, "Y_2"] \\ & \mathcal{C}_\infty\mathcal{A} \end{tikzcd}$$

of the Yoneda functor Y such that Y_1 is a strict fully faithful A_∞-functor and Y_2 is an A_∞-functor inducing an equivalence

$$H^0\,\mathsf{tw}\,\mathcal{A} \xrightarrow{\sim} \mathsf{tria}\,\mathcal{A} \subset \mathcal{D}_\infty\mathcal{A}.$$

The theorem may be enhanced so as to *characterize the category* $\mathsf{tw}\,\mathcal{A}$ *of twisted objects by a universal property.*

7.6. **Explicit description of $\mathsf{tw}\,\mathcal{A}$.** Let $\mathcal{A}$ be an A_∞-category with strict identities. We will explicitly construct the category of twisted objects $\mathsf{tw}\,\mathcal{A}$ in two steps. First, we construct the A_∞-category $\mathbf{Z}\mathcal{A}$, which is the 'closure under shifts' of $\mathcal{A}$: the objects of $\mathbf{Z}\mathcal{A}$ are the pairs (A, n) of an object $A \in \mathcal{A}$ and an integer $n \in \mathbf{Z}$. The morphism space from (A, n) to (A', n') is in bijection with the graded space

$$\mathsf{Hom}_{\mathcal{A}}(A, A')[n' - n].$$

We will identify $\mathcal{A}$ with the full sub-A_∞-category of $\mathbf{Z}\mathcal{A}$ on the objects $(A, 0)$. The compositions of $\mathbf{Z}\mathcal{A}$ are obtained by shifting the compositions of $\mathcal{A}$. The reader is invited to write down an explicit formula with correct signs.

In a second step, we construct $\mathsf{tw}\,\mathcal{A}$ as the 'closure under extensions' of $\mathbf{Z}\mathcal{A}$: Objects of $\mathsf{tw}\,\mathcal{A}$ are pairs (B, δ), where B is a sequence $(A_1, \ldots, A_r)$ of objects of $\mathbf{Z}\mathcal{A}$, and $\delta = (\delta_{ij})$ a matrix of morphisms of degree 1

$$\delta_{ij} \in \mathsf{Hom}^1_{\mathbf{Z}\mathcal{A}}\,(A_j, A_i)$$

such that we have $\delta_{ij} = 0$ for all $i \geq j$ and

$$\sum_{t=1}^{\infty} (-1)^{\frac{t(t-1)}{2}} m_t\,(\delta, \delta, \ldots, \delta) = 0.$$

Here we denote by m_n the extension of $m_n^{\mathbf{Z}\mathcal{A}}$ to matrices. Note that since δ is strictly upper triangular, the sum contains only finitely many non zero terms. The strange sign in the formula disappears in the b_n-language (3.5).

Alternatively, we can view an object (B, δ) as the datum of an object B of the additive hull of $\mathbf{Z}\mathcal{A}$ endowed with a finite split filtration and with an endomorphism δ of degree 1 which strictly decreases the filtration.

The space of morphisms from (B, δ) to (B', δ') is defined as

$$\bigoplus_{i,j} \mathsf{Hom}_{\mathbf{Z}\mathcal{A}}(A_i, A'_j).$$

Let $n \geq 1$ and let $(B_0, \delta_0), \ldots, (B_n, \delta_n)$ be objects of $\mathsf{tw}\,\mathcal{A}$. We will define the composition map (where the δ_i are suppressed from the notation)

$$m_n : (B_{n-1}, B_n) \otimes \cdots \otimes (B_0, B_1) \to (B_0, B_n)$$

In the following formula, we write δ for all δ_i

$$m_n^{\mathsf{tw}\,\mathcal{A}} = \sum_{t=0}^{\infty} \sum \pm m_{n+t}^{\mathbf{Z}\mathcal{A}} \circ (\mathbf{1}^{\otimes i_1} \otimes \delta^{\otimes j_1} \otimes \mathbf{1}^{\otimes i_2} \otimes \delta^{\otimes j_2} \otimes \ldots \otimes \mathbf{1}^{\otimes i_r} \otimes \delta^{\otimes j_r}).$$

Here the terms of the second sum are in bijection with the non-commutative monomials

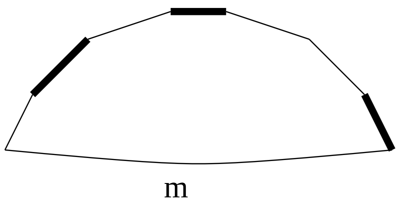
$$X^{i_1} Y^{j_1} \ldots X^{i_r} Y^{j_r}$$

of degree n in X and degree t in Y and the sign is given by the identity

$$(SX)^{i_1} (SY)^{j_1} \ldots (SX)^{i_r} (SY)^{j_r} = \pm S^{n+t} X^{i_1} Y^{j_1} \ldots X^{i_r} Y^{j_r}$$

in the algebra $k < X, Y, S > /(SX - XS, SY + YS)$. The pictorial version of the formula is

$$m_n = \sum \pm$$



.

Here the fat edges symbolize tensor powers of δ's to be inserted in all possible places and in any number. The concatenation of the thin edges represents a given tensor $f_n \otimes f_{n-1} \otimes \cdots \otimes f_1$.

It is worthwile to make the formula explicit for m_1. If f is a morphism of even degree from (B, δ) to (B', δ'), we have

$$\begin{aligned} m_1^{\mathsf{tw}\,\mathcal{A}}(f) = m_1(f) - m_2(\delta', f) + m_2(f, \delta) \\ - m_3(\delta', \delta', f) + m_3(\delta', f, \delta) - m_3(f, \delta, \delta) \\ + m_4(\delta', \delta', \delta', f) - m_4(\delta', \delta', f, \delta) + m_4(\delta', f, \delta, \delta) - m_4(f, \delta, \delta, \delta) + \ldots \end{aligned}$$

If f is of odd degree, we have

$$\begin{aligned} m_1^{\mathsf{tw}\,\mathcal{A}}(f) = m_1(f) - m_2(\delta', f) - m_2(f, \delta) \\ - m_3(\delta', \delta', f) - m_3(\delta', f, \delta) - m_3(f, \delta, \delta) \\ + m_4(\delta', \delta', \delta', f) + m_4(\delta', \delta', f, \delta) + m_4(\delta', f, \delta, \delta) + m_4(f, \delta, \delta, \delta) + \ldots \end{aligned}$$

In the sequel, we identify $\mathcal{A}$ with the subcategory of the objects $(A,0)$ of $\mathbf{Z}\mathcal{A}$. With this convention, the functor Y_1 of (7.5) sends an object $A \in \mathcal{A}$ to $(A, \delta = 0)$. The functor Y_2 sends $B = (A_1, \dots, A_n)$ to $YA_1 \oplus \dots \oplus YA_n$ (where Y denotes the canonical extension of the Yoneda functor to $\mathbf{Z}\mathcal{A}$) endowed with the multiplications

$$m_n = \sum_{t=0}^{\infty} (-1)^{\frac{t(t-1)}{2}} m_{n+t} \circ (\delta^{\otimes n+t} \otimes \mathbf{1}^{\otimes n-1}) \,,\ n \geq 1 \,,$$

where m_{n+t} on the right hand side denotes the multiplications of the direct sum of the YA_i.

7.7. **Description of filtered modules by twisted objects.** Let B be an associative k-algebra with 1, let $M_1, \dots, M_n$ be B-modules and let $\mathsf{filt}(M_i)$ denote the full subcategory of the category of right B-modules whose objects admit finite filtrations with subquotients among the M_i. Let $\mathcal{A}$ be the A_∞-category with objects $1, \dots, n$ and with morphism spaces

$$\mathcal{A}\,(i,j) = \mathsf{Ext}^*_B\,(M_i, M_j).$$

The composition maps of $\mathcal{A}$ are defined as follows: The differential m_1 vanishes, the composition m_2 is the Yoneda composition, and the m_i, $i > 3$, are the canonical higher compositions constructed as in (3.2, 3.3).

Let $\mathsf{filt}\,\mathcal{A}$ denote the full subcategory of $H^0\,\mathsf{tw}\,\mathcal{A}$ whose objects are the (B, δ) such that B is a filtered direct sum of (non shifted) copies of objects of $\mathcal{A}$ (we use the alternative description of $\mathsf{tw}\,\mathcal{A}$ and identify $\mathcal{A}$ with a subcategory of $\mathbf{Z}\mathcal{A}$).

Theorem. *There is an equivalence of categories*

$$\mathsf{filt}\,\mathcal{A} \xrightarrow{\sim} \mathsf{filt}(M_i).$$

Although the category $\mathsf{filt}\,\mathcal{A} \subset H^0\,\mathsf{tw}\,\mathcal{A}$ looks complicated, it is often quite easy to describe in practice. We illustrate this in two examples below.

The theorem is a consequence of (7.5) and (6.5). Indeed, by its construction, the category $\mathcal{A}$ is the A_∞-Ext-category of the B-modules $M_1, \dots, M_n$ in the sense of problem 2 of (2.1). So by (6.4) and (6.5), there is a triangle functor

$$F : \mathcal{D}_\infty \mathcal{A} \to \mathcal{D}B$$

inducing an equivalence

$$\mathsf{tria}(Y1, \dots, Yn) \xrightarrow{\sim} \mathsf{tria}(M_1, \dots, M_n).$$

By (7.5), the Yoneda functor induces an equivalence

$$H^0\,\mathsf{tw}\,\mathcal{A} \to \mathsf{tria}(Y1, \dots, Yn).$$

Thus we get an equivalence

$$H^0\,\mathsf{tw}\,\mathcal{A} \to \mathsf{tria}(M_1, \dots, M_n)$$

which takes the object i to the corresponding module M_i and therefore induces an equivalence

$$\mathsf{filt}\,\mathcal{A} \xrightarrow{\sim} \mathsf{filt}(M_i).$$

7.8. **Example: Reconstruction from simple modules.** Let B be a finite-dimensional algebra. Theorem 7.7 allows us in particular to reconstruct the category $\mathsf{mod}\, B$ of finite-dimensional B-modules from the A_∞-Ext-category of the simple B-modules.

We illustrate this by an example. Let B be the algebra given by the quiver with relations

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} t\ ,\ \gamma\,\beta\,\alpha = 0$$

and let $M_1 = S_x, \dots, M_4 = S_t$ be the four simple B-modules. The A_∞-category $\mathcal{A}$ of (7.7) is then given by the quiver with relations

$$\begin{tikzcd} \overset{x}{\circ} & \overset{y}{\circ} \arrow[l, "a"'] & \overset{z}{\circ} \arrow[l, "b"'] & \overset{t}{\circ} \arrow[l, "c"'] \arrow[lll, bend left, "e"] \end{tikzcd}$$

where the arrows a, b, c are of degree 1, the arrow e is of degree 2, and all compositions other than $m_3(a, b, c) = e$ are defined to be zero (this example has already appeared in 3.2). We would like to determine the subcategory $\mathsf{filt}\,\mathcal{A}$ of $H^0\,\mathsf{tw}\,\mathcal{A}$ whose objects are the (A, δ) where A is a filtered sum of objects of $\mathcal{A}$ (we use the alternative description and identify $\mathcal{A}$ with a subcategory of $\mathbf{Z}\mathcal{A}$). By an obvious abuse of notation, we have

$$A = V_x \otimes x \oplus \dots \oplus V_t \otimes t$$

where $V_x, \dots, V_t$ are finite-dimensional vector spaces and the datum of

$$\delta \in \mathsf{Hom}^1_{\mathbf{Z}\mathcal{A}}\,(A, A)$$

corresponds to the datum of three linear maps V_a, V_b, V_c :

$$V_x \xleftarrow{V_a} V_y \xleftarrow{V_b} V_z \xleftarrow{V_c} V_t.$$

The condition

$$m_1(\delta) + m_2(\delta, \delta) - m_3(\delta, \delta, \delta) - \dots = 0$$

translates into $V_a V_b V_c = 0$. By definition, morphisms from (A, δ) to (A', δ') are in bijection with

$$H^0\,\mathsf{Hom}_{\mathsf{tw}\,\mathcal{A}}\,((A, \delta), (A', \delta')).$$

Now $\mathsf{Hom}^{-1}_{\mathcal{A}}$ and hence

$$\mathsf{Hom}^{-1}_{\mathsf{tw}\,\mathcal{A}}\,((A, \delta), (A', \delta'))$$

vanishes so that the homology classes in degree zero are exactly the zero cycles. Now it is easy to see that the elements f of $\mathsf{Hom}^0_{\mathsf{tw}\,\mathcal{A}}\,((A, \delta), (A', \delta'))$ correspond exactly to the quadruples of linear maps

$$f_x : V_x \to V'_x\ ,\ \dots\ ,\ f_t : V_t \to V'_t$$

and the condition

$$m_1(f) - m_2(\delta', f) + m_2(f, \delta) - m_3(\dots) + \dots = 0$$

translates into

$$f_x V_a = V'_a f_y\ ,\ \dots\ ,\ f_z V_c = V'_c f_t.$$

Hence, as expected, the category $\mathsf{filt}\,\mathcal{A}$ is equivalent to the category of contravariant representations of the quiver with relations

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} t\ ,\ \gamma\,\beta\,\alpha = 0\ ,$$

i.e. to the category $\mathsf{mod}\,B$ of finite-dimensional right modules over the algebra B. This confirms theorem 7.7 in this example.

7.9. **Example: Classifying filtered modules.** This example arose from an email exchange with T. Brüstle. Consider the following quiver Q of type D_4

$$\begin{tikzcd} & & 2 \\ 3 \arrow[r] & 1 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

Denote by S_i its simple representation associated with the point i, $1 \le i \le 4$. Let $\Delta_1 = S_1$, $\Delta_2 = S_2$, $\Delta_4 = S_4$ and let Δ_3 be the following indecomposable representation

$$\begin{tikzcd} & & k \\ k \arrow[r] & k \arrow[ur] \arrow[dr] & \\ & & 0 \end{tikzcd} .$$

We are interested in the category $\mathsf{filt}(\Delta_1, \ldots, \Delta_4)$ of representations of Q admitting a filtration with subquotients among the Δ_i. The A_∞-Ext-category $\mathcal{A}$ of the Δ_i is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, "a", "|" marking] \arrow[rrr, bend right, "c"', "|" marking] & 2 \arrow[r, "f"] & 3 \arrow[r, "b", "|" marking] & 4 \end{tikzcd}$$

where a, b, c are of degree 1, f is of degree 0 and all multiplications other than $m_3(b, f, a) = c$ are defined to be zero. As in example 7.8, the subcategory $\mathsf{filt}\,\mathcal{A}$ of $H^0\,\mathsf{tw}\,\mathcal{A}$ can be described as follows: Its objects are the finite-dimensional representations V of the quiver

$$2 \xleftarrow{a} 1 \xrightarrow{c} 4 \xleftarrow{b} 3$$

of type A_4. Its morphisms $f : V \to V'$ are the quintuples of linear maps $f_i : V_i \to V_i'$ and $\varphi : V_2 \to V_3'$

$$\begin{tikzcd}
V_2 \arrow[d, "f_2"'] \arrow[drrr, "\varphi"] & V_1 \arrow[l, "V_a"'] \arrow[r, "V_c"] \arrow[d, "f_1"'] & V_4 \arrow[d, "f_4"] & V_3 \arrow[l, "V_b"] \arrow[d, "f_3"] \\
V_2' & V_1' \arrow[l, "V_a'"] \arrow[r, "V_c'"'] & V_4' & V_3' \arrow[l, "V_b'"]
\end{tikzcd}$$

satisfying

$$0 = V_a' f_1 - f_2 V_a \ , \ 0 = V_b' f_3 - f_4 V_b$$
$$0 = V_c' f_1 - f_4 V_c + V_b' \varphi V_a \quad (*)$$

Thanks to theorem 7.7, we have an equivalence

$$\mathsf{filt}\,\mathcal{A} \xrightarrow{\sim} \mathsf{filt}(\Delta_1, \ldots, \Delta_4).$$

The term $V_b' \varphi V_a$ in $(*)$ is essential: If it is omitted, the problem is that of classifying the representations of the quiver of type A_4 above. Then we find 10 isoclasses of indecomposables instead of the 9 isoclasses present on the right hand side.

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UFR de Mathématiques, UMR 7586 du CNRS, Case 7012, Université Paris 7, 2, place Jussieu, 75251 Paris Cedex 05, France

E-mail address: `keller@math.jussieu.fr`
`www.math.jussieu.fr/~keller`