NOTES ON ALMOST SPLIT SEQUENCES, I

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1. Introduction

Let k be a field and A an algebra of finite dimension over k. The category of finitely generated left A-modules is denoted mod A, and if $X,Y \in \text{mod } A$, the k-space $\text{Hom}_A(X,Y)$ is denoted (X,Y). Let

$$\mathcal{E}: 0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$$

be a short exact sequence in mod A. Auslander and Reiten [4, p.443] say that $\mathcal{E}$ is <u>almost split</u> if it satisfies the three conditions

<u>A1</u> $\mathcal{E}$ is not split,

<u>A2</u> S and N are both indecomposable, and

A3 if $X \in \text{mod } A$, $h \in (X,S)$ and h is not split epi, then h factors through g .

(Recall that $h \in (X,S)$ is split epi, or is a splitable epimorphism, or is a retraction, if there exists $h' \in (S,X)$ with $hh' = 1_S$.)

M. Auslander and I. Reiten proved in [3, §4] the following theorem, which has initiated an avalanche of new research in the representation theory of algebras.

(1.1) Theorem Given any non-projective, indecomposable $S \in \text{mod } A$, there is an almost split sequence $\mathcal{E}$ ending with S. Moreover $\mathcal{E}$ is determined by S , uniquely up to isomorphism of short exact sequences.

In [1] and [2], Auslander and Reiten develop a theory of finitely presented functors on suitably symmetric ('self-dualizing') categories. In this theory is a computational process which can be regarded as an algorithm for calculating a projective resolution for a functor which is presented in a certain way (see (2.17), below); almost split sequences can be calculated from a special case. The purpose of this paper is to describe a new 'trace formula', which, I believe, makes one step of the Auslander-Reiten process more calculable. Section 2 is an account of the process itself - there is little new here, and I have much use of P. Gabriel's important exposition [9]. The trace formula is proved in section 3, and applied to almost split sequences in section 4. The resulting 'recipe' seems to be easier than that of M.C.R. Butler ([5, p.84]; see also [9, p.17]); it has been used by A.J. Chanter to calculate some components of Auslander-Reiten quivers ([6]). Section 5 is an appendix, on the case where A is symmetric.

Let $X \in \text{mod } A$. We have two left-exact, k-linear functors $(X, \ )$ and $(\ ,X)$ from $\text{mod } A \to \text{mod } k$; these are co- and contra- variant respectively. $(X, \ )$ takes $M \in \text{mod } A$ to (X,M), and it takes a map $u:M\to M'$ in mod A to the k-map $(X,u):(X,M) \to (X,M')$ given by $(X,u)(s) = us$, for $s \in (X,M)$. $(\ ,X)$ takes M to (M,X), and u to $(u,X):(M',X) \to (M,X)$ given by $(u,X)(t) = tu$, for $t \in (M',X)$. If we apply $(X, \ )$ to a short exact sequence $\mathcal{E}$ in mod A, we get the sequence, exact in mod k,

$$(1.2) \qquad 0 \to (X,N) \xrightarrow{(X,f)} (X,E) \xrightarrow{(X,g)} (X,S) \ ;$$

clearly $\text{Im}(X,g)$ is the set of $h \in (X,S)$ which factor through g. Write $\underline{H}(X,S)$ for the set of all $h \in (X,S)$ which are not split epi. Then $\mathcal{E}$ satisfies condition A3 if and only if $\underline{H}(X,S) \leq \text{Im}(X,g)$, for all $X \in \text{mod } A$. It is elementary to check that $\mathcal{E}$ is non-split if and only if $\text{Im}(X,g) \leq \underline{H}(X,S)$, for all X. Hence $\mathcal{E}$ satisfies both A1 and A3, if and only if it satisfies the condition

$$(1.3) \qquad \text{Im}(X,g) = \underline{H}(X,S), \quad \text{for all } X \in \text{mod } A .$$

For any $X, S \in \text{mod } A$, we make the

(1.4) Definition

$$\underline{R}(X,S) = \{f \in (X,S) \mid fg \in \text{rad End}(S), \text{ for all } g \in (S,X)\}.$$

This is a subspace of (X,S), whose importance to us is that if S is indecomposable, then $\underline{R}(X,S) = \underline{H}(X,S)$, for all $X \in \text{mod } A$. The proof is an easy application of Fitting's lemma. So by (1.3), if S is indecomposable, then E satisfies A1 and A3 if and only if it satisfies the condition

(1.5) $\text{Im}(X,g) = \underline{R}(X,S)$, for all $X \in \text{mod } A$.

We shall often use the functor category Fun A (this is Gabriel's term, Auslander and Reiten call it Mod(mod A)), although mainly as a source of convenient notation. We recall some definitions here, for details see [8, chapter 5] and [1, §2]. The objects of Fun A are all k-linear, contravariant functors $F, G, \ldots$ from mod $A \to$ mod k. Morphisms are natural transformations, i.e. a morphism $\alpha: F \to G$ is the same as a family of k-maps $\alpha(X): F(X) \to G(X)$, $X \in \text{mod } A$, which is natural in X. If $F \in \text{Fun } A$, and if for each $X \in \text{mod } A$ there is given a subspace $G(X) \leq F(X)$, in such a way that for each $h: X \to Y$ in mod A, the map $F(h): F(Y) \to F(X)$ takes $G(Y)$ into $G(X)$, then we define $G(h): G(Y) \to G(X)$ to be the restriction of $F(h)$, and we now have an object $G \in \text{Fun } A$, called a subfunctor of F (notation $G \leq F$). One may then define the quotient functor F/G. Fun A is an abelian category. For example, $(\ ,E)$ is an object of Fun A, for every $E \in \text{mod } A$. If $g: E \to S$ is a map in mod A , then the family of k-maps $(X,g):(X,E) \to (X,S)$ is natural in X , and so defines a morphism $(,g):(,E) \to (,S)$ in Fun A . The statement that (1.2) is exact, for all $X \in$ mod A , can be expressed by saying that the sequence

$0 \to (,N) \xrightarrow{(,f)} (,E) \xrightarrow{(,g)} (,S)$ is exact in Fun A .

Definition (1.4) provides an important subfunctor $\underline{R}(,S)$ of $(,S)$, called the <u>radical</u> of $(,S)$, [2, p.319], [9, p.2]. We have a subfunctor $\text{Im}(,g)$ of $(,S)$, which takes each $X \in$ mod A to $\text{Im}(X,g)$. Condition (1.5) becomes simply the condition $\text{Im}(,g) = \underline{R}(,S)$. We summarize our <u>functorial reformulation</u> of the definition of an almost split sequence as follows.

(1.6) <u>Proposition</u> Let $E: 0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$ be a short exact sequence in mod A , with S indecomposable. Then E satisfies conditions <u>A1</u> and <u>A3</u> if and only if $\text{Im}(,g) = \underline{R}(,S)$. Hence E is almost split, if and only if $\text{Im}(,g) = \underline{R}(,S)$ and N is indecomposable.

<u>Notation</u> If $X \in$ mod A , then $\text{End}(X) = (X,X)$ is the endomorphism algebra of X ; 1_X is the identity map on X ; $\underline{r}X$ is the radical of X . Finitely generated <u>right</u> A-modules are considered as objects of mod A^{op} (see [1, p.278]). Two k-linear, contravariant functors D , d:mod A $\to$ mod A^{op} are in constant use. (1) $DX = \text{Hom}_k(X,k)$, with A acting on the right by $(\phi a)(x) = \phi(ax)$, $\phi \in DX$, $a \in A, x \in X$. See [7, p.410],

where DX is denoted X* . (2) $dX = Hom_A(X,{}_AA) = (X,A)$, with A acting on the right by $(fa)(x) = f(x)a$, $f \in dX$, $a \in A$, $x \in X$. See [7, p.394], [3, p.247] and [9, p.5], where dX is denoted X' , X* and X^t , respectively.

D is exact, and turns projectives into injectives and vice versa; d is only left exact, and turns projectives into projectives. It is useful to notice that $d(Ae) \cong eA$, for any idempotent e of A .

$\underline{N} = Dd$: mod A → mod A is the <u>Nakayama functor</u> [9, p.10]. It is k-linear, covariant and right exact, and turns projectives into injectives.

2. The Auslander-Reiten-Gabriel diagram

In this section we sketch a general procedure, by which one can construct short exact sequences $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ in mod A , in a way which gives an explicit formula for the functor Im(,g) . All the essentials of this method go back to Auslander and Reiten (see [2, §7] in particular); the version given here is based on Gabriel's exposition ([9, pp.5,6]).

Let $M \in$ mod A be given, and also a 2-step projective resolution of M in mod A ,

$$(2.1) \qquad P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0 .$$

It is always possible to choose a resolution (2.1) which is <u>minimal</u>, i.e. for which $\text{Ker } p_i \leq \underline{r}P_i$, $i = 0,1$. However we do not assume this in general.

We apply the right exact functor $\underline{N}$ to (2.1) and get the exact sequence $\underline{N}P_1 \xrightarrow{\underline{N}p_1} \underline{N}P_0 \xrightarrow{\underline{N}p_0} \underline{N}M \to 0$ in mod A , hence the exact sequence

$$(2.2) \qquad 0 \to \underline{A}M \xrightarrow{\text{inc}} \underline{N}P_1 \xrightarrow{\underline{N}p_1} \underline{N}P_0 \xrightarrow{\underline{N}p_0} \underline{N}M \to 0 ,$$

where $\underline{A}M = \operatorname{Ker} \underline{N}p_1$. Gabriel calles $\underline{A}M$ the Auslander-Reiten translate of M ; Auslander and Reiten denote it DTrM .

(2.3) Remark $\underline{A}M$ depends on the resolution (2.1), but is uniquely determined by M up to an injective summand. In fact there is a category equivalence $DTr = \underline{A}:\underline{\text{mod}}\ A \to \overline{\text{mod}}\ A$, where $\underline{\text{mod}}\ A$ [$\overline{\text{mod}}\ A$] denotes the category mod A , taken 'modulo projectives' [injectives]; see [3, pp.246-252] or [9, §2].

If (2.1) is minimal then $\underline{A}M$ has no non-zero injective direct summands. If also M is indecomposable and not projective, then $\underline{A}M$ is indecomposable and not injective ([3, p.265], [9, p.6]).

We are going to describe, for each $X \in \text{mod } A$, the commutative 'ARG diagram' below. Its rows are exact, and all its maps are natural in X . The reader who prefers to see this as a diagram in Fun A , has only to erase the symbol X throughout. From (2.1) we get the exact sequence $0 \to (M,X) \xrightarrow{(p_0,X)} (P_0,X) \xrightarrow{(p_1,X)} (P_1,X)$, and then apply D to get the exact sequence (2.5). There is a k-map $\alpha_Y(X):D(Y,X) \to (X,\underline{N}Y)$, natural for

X,Y in mod A , whose definition we shall recall in section 3 (or see [9, p.5]). This is bijective when Y is projective, hence all the verticals joining (2.5) to (2.6) are isomorphisms. We define b(X) by

$$(2.4) \qquad b(X) = D(p_0,X)\alpha_{P_0}(X)^{-1} ;$$

it is then clear that the upper half of the diagram commutes. We can now see that (2.6) is exact: it is exact at $(X,\underline{A}M)$ and $(X,\underline{N}P_1)$ because (2.2) is exact at $\underline{A}M$ and $\underline{N}P_1$, and it is exact at $(X,\underline{N}P_0)$ and D(M,X) because (2.5) is exact.

ARG DIAGRAM

$$\begin{array}{lccccccccc}
(2.5) & & & D(P_1,X) & \xrightarrow{D(p_1,X)} & D(P_0,X) & \xrightarrow{D(p_0,X)} & D(M,X) & \to & 0 \\
 & & & \downarrow{\scriptstyle \alpha_{P_1}(X)} & & \downarrow{\scriptstyle \alpha_{P_0}(X)} & & \downarrow{\scriptstyle id} & & \\
(2.6) & 0 \to & (X,\underline{A}M) & \xrightarrow{(X,inc)} & (X,\underline{N}P_1) & \xrightarrow{(X,\underline{N}p_1)} & (X,\underline{N}P_0) & \xrightarrow{b(X)} & D(M,X) & \to 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle (X,\ell)} & & \uparrow{\scriptstyle (X,\theta)} & & \uparrow{\scriptstyle id} & \\
(2.7) & 0 \to & (X,\underline{A}M) & \xrightarrow{(X,f)} & (X,E(\theta)) & \xrightarrow{(X,g)} & (X,S) & \xrightarrow{a_\theta(X)} & D(M,X) &
\end{array}$$

To construct the sequence (2.7), we introduce a module $S \in$ mod A and a map $\theta \in (S,\underline{N}P_0)$. From these we make an exact sequence $E(\theta)$ by the standard 'pullback over $\underline{N}p_1$ and θ' . Thus $E(\theta) = \{(u,s) \in \underline{N}P_1 \amalg S \mid (\underline{N}p_1)(u) = \theta(s)\}$, a submodule of $\underline{N}P_1 \amalg S$. The maps f,g are given by f(u) = (u,0), g(u,s) = = s . It is easy to check

(2.8) The sequence $E(\theta):0 \to \underline{A}M \xrightarrow{f} E(\theta) \xrightarrow{g} S$ is exact.

The map g is surjective if and only if $\operatorname{Im} \theta \leq \operatorname{Im} \underline{N}p_1 = \operatorname{Ker} \underline{N}p_0$.

Now let $\ell : E(\theta) \to \underline{N}P_1$ be the projection $(u,s) \to u$, and define $a_\theta(X)$ by

$$a_\theta(X) = b(X)(X,\theta) .$$

All the maps in the ARG diagram are now defined. It is easy to check that (2.7) is exact and that the lower half of the diagram commutes. In particular we have

(2.9) $\operatorname{Im}(X,g) = \operatorname{Ker} a_\theta(X)$, for all $X \in \operatorname{mod} A$.

Since $a_\theta(X):(X,S) \to D(M,X)$ is natural in X, it is completely determined by the element $T_\theta = a_\theta(S)(1_S) \in D(M,S)$; this is an application of 'Yoneda's lemma' (see [11, p.61] and [8, p.112]). In fact for any $f \in (X,S)$ one has by naturality the commutative diagram

$$\begin{array}{ccc} (X,S) & \xrightarrow{a_\theta(X)} & D(M,X) \\ {\scriptstyle (f,S)}\uparrow & & \uparrow{\scriptstyle D(M,f)} \\ (S,S) & \xrightarrow[a_\theta(S)]{} & D(M,S) \end{array} ,$$

from which $a_\theta(X)(f) = a_\theta(X)(f,S)(1_S) = D(M,f)a_\theta(S)(1_S) =$ $= D(M,f)(T_\theta)$. This means that $a_\theta(X)(f)$ is the element of $D(M,X)$ given by

(2.10) $a_\theta(X)(f):g \to T_\theta(fg)$, for all $g \in (M,X)$.

We are interested in the kernel of $a_\theta(X)$, and (2.10) shows that it consists of those $f \in (X,S)$ such that the space $f(M,X) = \{fg | g \in (M,X)\}$ lies in Ker T_θ . Now T_θ is a linear form on (M,S) , and (M,S) has natural structure as right End(M)-module. Let us, for any $T \in D(M,S)$, define the <u>right core</u> $\underline{rc}(T)$ of T to be the unique maximal right End(M)-submodule of (M,S) which lies in Ker T . Since $f(M,X)$, for given $f \in (X,S)$, is clearly a right End(M) submodule of (M,S) , it lies in Ker T_θ if and only if it lies in $\underline{rc}(T_\theta)$. We have then, for all $X \in$ mod A ,

$$\text{(2.11)} \qquad \operatorname{Ker} a_\theta(X) = \{f \in (X,S) | fg \in \underline{rc}(T_\theta) \text{ for all } g \in (M,X)\}.$$

This equation prompts the following <u>definition</u>: if $M, S \in$ mod A and if V is any right End(M)-submodule of (M,S) , we define

$$\text{(2.12)} \qquad \underline{z}_V(X,S) = \{f \in (X,S) | fg \in V \text{ for all } g \in (M,X)\} .$$

Then it is elementary to prove the next proposition.

(2.13) <u>Proposition</u> (i) (2.12) defines a subfunctor $\underline{z}_V(\ ,S)$ of $(\ ,S)$. This means, $\underline{z}_V(X,S)$ is a subspace of (X,S) , for all $X \in$ mod A , and if $h:X \to Y$ is any map in mod A , then (h,S) maps $\underline{z}_V(Y,S)$ into $\underline{z}_V(X,S)$.

(ii) $\underline{z}_V(M,S) = V$. Hence if V,V' are right End(M)-submodules of (M,S) , then the functors $\underline{z}_V(\ ,S)$ and $\underline{z}_{V'}(\ ,S)$ are equal, if and only if $V = V'$.

Combining (2.9), (2.11) and (2.12) we have the main result from the ARG diagram, as follows.

(2.14) Theorem Let $M,S \in \text{mod } A$, and let (2.1) be any 2-step projective resolution of M. Let θ be any element of $(S,\underline{NP}_0)$.

Define the exact sequence $E(\theta):0 \to \underline{A}M \xrightarrow{f} E(\theta) \xrightarrow{g} S$ by pullback, as in (2.8). Define $T_\theta = a_\theta(S)(1_S) \in D(M,S)$.

Then we have the formula

$$\text{(2.15)} \qquad \text{Im}(\ ,g) = \underline{z}_V(\ ,S) \ , \quad \text{where} \quad V = \underline{rc}(T_\theta) \ .$$

The ARG diagram displays an algorithm to solve the following problem: given a subfunctor of $(\ ,S)$ of the form $\underline{z}_V(\ ,S)$, where $M \in \text{mod } A$, and $V = \underline{rc}(T)$ for some $T \in D(M,S)$, to find an exact sequence $E:0 \to N \xrightarrow{f} E \xrightarrow{g} S$ in mod A such that $\text{Im}(\ ,g) = \underline{z}_V(\ ,S)$. For by (2.15) we get a solution to this problem by taking $E = E(\theta)$, where θ is any element of $(S,\underline{NP}_0)$ such that $T_\theta = T$. Such a θ always exists, since we have

$$\text{(2.16)} \qquad T_\theta = a_\theta(S)(1_S) = b(S)(S,\theta)(1_S) = b(S)(\theta) \ ,$$

and the map $b(S):(S,\underline{NP}_0) \to D(M,S)$ is surjective (put $X = S$ in (2.6)). In section 3 we shall give a formula for T_θ which is more explicit than (2.16). And in section 4 we shall see that the problem of finding an almost split sequence ending with S is a special case of the problem just described. We end the present section with some general comments.

<u>Finitely presented functors.</u> To say that the sequence $\mathcal{E}$ above has the property $\mathrm{Im}(\ ,g) = z_V(\ ,S)$, is the same as to say that the following sequence in Fun A is exact

$$(2.17) \qquad 0 \to (\ ,N) \xrightarrow{(\ ,f)} (\ ,E) \xrightarrow{(\ ,g)} (\ ,S) \xrightarrow{\text{nat}} F \to 0 ,$$

where $F = (\ ,S)/z_V(\ ,S)$. This implies that the object $F \in$ Fun A is <u>finitely presented</u> and that (2.17) is a <u>projective resolution</u> of F (see [1, §4]). Conversely, Auslander and Reiten have shown that for any finitely presented $F \in$ Fun A , there exist $S,M \in$ mod A and a morphism $a:(\ ,S) \to D(M,\ )$ in Fun A such that Im a = F (see [2, p.319]. The fact that mod A is a 'dualizing k-variety' is proved in [2, Props. 2.5, 2.6].) The 'ARG algorithm' gives a resolution (2.17) for any F defined in this way. Namely let $T = a(S)(1_S)$, and choose $\theta \in (S,\underline{N}P_0)$ such that $T_\theta = T$. Then $a = a_\theta$ (see (2.10)), so if $\mathcal{E} = \mathcal{E}(\theta)$, (2.17) is a resolution of the kind required.

<u>Surjectivity of the map</u> g . We take $\mathcal{E} = \mathcal{E}(\theta)$ and go back to (2.8): g is surjective if and only if $\mathrm{Im}\ \theta \leq \mathrm{Im}\ \underline{N}p_1$. In that case we may regard θ as an element of $(S,\mathrm{Im}\ \underline{N}p_1)$, and identify $\mathcal{E}(\theta)$ with the short exact sequence obtained by pull-back from the short exact sequence (2.18) below. Notice that (2.18) is an injective presentation of $\underline{A}M$.

(2.18)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{A}M & \xrightarrow{inc} & \underline{N}P_1 & \xrightarrow{\underline{N}p_1} & \text{Im } \underline{N}p_1 & \to & 0 \\ & & id\uparrow & & \ell\uparrow & & \theta\uparrow & & \\ E(\theta):0 & \longrightarrow & \underline{A}M & \xrightarrow[f]{} & E(\theta) & \xrightarrow[g]{} & S & \longrightarrow & 0 . \end{array}$$

(2.19) Proposition The map g is surjective if and only if either (i) $\underline{z}_V(A,S) = (A,S)$, where $V = \underline{rc}(T_\theta)$, and A stands for the left regular A-module ${}_A A$, or

(ii) $T_\theta(P(M,S)) = 0$, where $P(M,S)$ is the space of all maps $h \in (M,S)$ which factor through some projective module in mod A.

Proof (i) By (2.15) and the (elementary) fact that a map $g:E \to S$ in mod A is surjective if and only if $(A,g):(A,E) \to (A,S)$ is surjective.

(ii) $\underline{z}_V(A,S) = (A,S)$ holds if and only if $f(M,A) \leq \underline{rc}(T_\theta)$ for all $f \in (A,S)$, i.e. if and only if $T_\theta((A,S)(M,A)) = 0$. But it is easy to prove that $(A,S)(M,A) = P(M,S)$ (see [3, p.245]).

From this we may deduce the following remarkable identity of Auslander and Reiten ([2, Props, 7.2, 7.3]; see also [9, p.13]).

(2.20) Theorem There is a k-isomorphism, natural in $S,M \in$ mod A, $\text{Ext}^1_A(S,\underline{A}M) \to D(\underline{M,S})$. Here $(\underline{M,S}) = (M,S)/P(M,S)$.

Proof From (2.18), we can identify $\text{Ext}^1_A(S,\underline{A}M)$ with the quotient space $(S,\mathrm{Im}\ \underline{N}p_1)/\mathrm{Im}(S,\underline{N}p_1)$. The map $b(S):(S,\underline{N}P_0) \to D(M,S)$ has kernel $\mathrm{Im}(S,\underline{N}p_1)$. By (2.16), (2.19,ii) the counter-image under $b(S)$ of the space $D(\underline{M,S})$ (which we identify with the set of those $T \in D(M,S)$ which vanish on $P(M,S)$) , is the set of those $\theta \in (S,\underline{N}P_0)$ such that the map g in $E(\theta)$ is surjective, i.e. ((2.8)), it is $(S,\mathrm{Im}\ \underline{N}p_1)$. Therefore $b(S)$ induces the required isomorphism.

3. A 'trace formula' for T_θ

Keeping the notation of the last section, we have for each $\theta \in (S,\underline{N}P_0)$ that $T_\theta = b(S)(\theta)$, hence by (2.4)

$$(3.1) \qquad T_\theta = D(p_0,S)\alpha_{P_0}(S)^{-1}(\theta) .$$

The part of this which is difficult to calculate is $\alpha_{P_0}(S)^{-1}(\theta)$. So we begin by giving a procedure for calculating the map $\alpha_{P_0}(X)^{-1}:(X,\underline{N}P_0) \to D(P_0,X)$, for an arbitrary $X \in \mathrm{mod}\ A$. It is worth noticing that we never require a map $\alpha_Y(X)$, either in setting up the ARC diagram, or in calculations of the type we have in mind, unless Y is projective.

Since P_0 is projective, we can find a (left) A-isomorphism $\kappa:\coprod_{\nu=1}^{n} Ae_\nu \to P_0$, where $e_1,..,e_n$ are idempotents of A , not necessarily distinct. For example, we could write P_0 as direct sum of indecomposable submodules $P_{0,\nu}$, and then use the fact that each $P_{0,\nu} \cong Ae_\nu$, for some primitive idempotent

e_ν of A. But, in general, we do not assume that the e_ν are primitive - for example, if P_0 were a free A-module, it might be more convenient to take all the $e_\nu = 1$. In any case, we have $P_0 = Ay_1 \oplus \ldots \oplus Ay_n$, where for each $\nu = 1,\ldots,n$ we define $y_\nu = \kappa(0,\ldots,0,e_\nu,0,\ldots,0) \in P_0$. It is clear that $e_\nu y_\nu = y_\nu$, for each ν.

There is a right A-isomorphism $\kappa' : \coprod_\nu e_\nu A \to dP_0 = (P_0,A)$, most easily described by saying that we define the elements $z_\mu = \kappa'(0,\ldots,0,e_\mu,0,\ldots,0) \in dP_0$ in such a way that $z_\mu(y_\nu) = \delta_{\mu\nu}e_\mu$, for $\mu,\nu = 1,\ldots,n$. Explicitly, $z_\mu(a_1y_1+\ldots+a_ny_n) = a_\mu e_\mu$, for any $a_1,\ldots,a_n \in A$. We have then $dP_0 = z_1A \oplus \ldots \oplus z_nA$, and $z_\mu e_\mu = z_\mu$, for each μ. The sets $\{y_\nu\}$, $\{z_\nu\}$ are 'dual bases' of P_0, dP_0 in the sense described, for example, in [10, p.152].

Now take any $X \in \text{mod } A$. To each $\sigma \in (P_0,X)$ we assign its <u>vector</u> $v_X(\sigma) = (s_1,\ldots,s_n)$, where $s_\nu = \sigma(y_\nu)$, for each ν. Evidently $s_\nu \in e_\nu X$, and we find easily that

$$(3.2) \qquad v_X : (P_0,X) \to \coprod_\nu e_\nu X$$

is a k-isomorphism.

In a similar way, we assign to each $\rho \in (X,\underline{N}P_0)$ its vector $v'_X(\rho) = (r_1,\ldots,r_n)$, where for each ν, r_ν is the element of DX given by $r_\nu(x) = \rho(x)(z_\nu)$, $x \in X$. (Notice that $\underline{N}P_0 = D(dP_0)$, so that $\rho(x)$ is a linear map $dP_0 \to k$.) We check that $r_\nu \in (DX)e_\nu$ - DX being a right A-module as usual - and that

$$(3.3)\qquad v'_X : (X, \underline{N}P_0) \to \coprod_{\nu} (DX)e_\nu$$

is a k-isomorphism.

It is useful to record the inverses of v_X, v'_X. If $v_X(\sigma) = (s_1,\dots,s_n)$ and $v'_X(\rho) = (r_1,\dots,r_n)$, then $\sigma \in (P_0,X)$ and $\rho \in (X,\underline{N}P_0)$ are given by

$$(3.4)\qquad \sigma(\Sigma a_\nu y_\nu) = \Sigma a_\nu s_\nu \text{ , for all } a_1,\dots,a_n \in A \text{ , and}$$

$$(3.5)\qquad \text{For each } x \in X \text{ , } \rho(x)(\Sigma z_\nu a_\nu) = \Sigma r_\nu(a_\nu x), \text{ for all } a_1,\dots,a_n \in A \text{ .}$$

<u>Definition</u> Let $\langle\ ,\ \rangle : (X,\underline{N}P_0) \times (P_0,X) \to k$ be the k-bilinear form given by the formula

$$(3.6)\qquad \langle\rho,\sigma\rangle = \sum_\nu r_\nu(s_\nu) \text{ ,}$$

where $(r_1,\dots,r_n)$ and $(s_1,\dots,s_n)$ are the vectors of $\rho \in (X,\underline{N}P_0)$ and $\sigma \in (P_0,X)$, respectively.

The space $(DX)e_\nu$ may and shall be identified with $D(e_\nu X)$ (each $f \in (DX)e_\nu$ vanishes on $(1-e_\nu)X$, and is identified with its restriction to $e_\nu X$). Therefore $\langle\ ,\ \rangle$ is non-singular, for the right side of (3.6) is just the direct sum of the natural pairings $D(e_\nu X) \times e_\nu X \to k$. Notice that $(X,\underline{N}P_0)$ and (P_0,X) have the same dimension, as is clear by comparing (3.2) and (3.3).

We may use $\langle\ ,\ \rangle$ to define a k-isomorphism $\zeta_{P_0}(X):(X,\underline{N}P_0) \to D(P_0,X)$ by the rule

(3.7) $$\zeta_{P_0}(X)(\rho)(\sigma) = \langle\rho,\sigma\rangle, \text{ for all } \rho \in (X,\underline{N}P_0),\ \sigma \in (P_0,X).$$

This discussion culminates in the following proposition.

(3.8) Proposition For all $X \in \text{mod } A$, $\zeta_{P_0}(X) = \alpha_{P_0}(X)^{-1}$.

Proof It is time to define $\alpha_{P_0}(X):D(P_0,X) \to (X,\underline{N}P_0)$. If $f \in dP_0$, $x \in X$, let $\beta_{f,x} \in (P_0,X)$ be given by $\beta_{f,x}(y) = f(y)x$ for all $y \in P_0$. Let Φ be an element of $D(P_0,X)$. Then we define the element $\alpha_{P_0}(X)(\Phi) = \rho$ of $(X,\underline{N}P_0)$ by

(3.9) $$\rho(x)(f) = \Phi(\beta_{f,x}) , \text{ for all } x \in X , f \in dP_0 .$$

(See [9, p.5]. For the purposes of this paper we may take (3.9) as definition of $\alpha_{P_0}(X)$. To construct the ARG diagram, we need to know that it is natural in both P_0 and X , which is easy. That it is an isomorphism for any projective P_0 , can be deduced from the proof of the present proposition.)

Our ambition is to prove that $\zeta_{P_0}(X)\alpha_{P_0}(X)$ is the identity map on $D(P_0,X)$ - this will prove (3.8), since $\zeta_{P_0}(X)$ is an isomorphism between finite-dimensional k-spaces. Let $\Phi \in D(P_0,X)$, and let $(r_1,\ldots,r_n)$ be the vector of

$\rho = \alpha_{P_0}(X)(\Phi)$. Using (3.9) we find $r_\nu(x) = \rho(x)(z_\nu) = \Phi(\beta_{z_\nu,x})$, for all ν , and all $x \in X$. Now take any $\sigma \in (P_0,X)$ and let $(s_1,\dots,s_n)$ be its vector. We have $s_\nu = \sigma(y_\nu)$ for all ν ; so by (3.6), (3.7), $\zeta_{P_0}(X)(\alpha_{P_0}(X)(\Phi))$ takes σ to $\zeta_{P_0}(X)(\rho)(\sigma) = \langle\rho,\sigma\rangle = \Sigma r_\nu(\sigma(y_\nu)) = \Sigma\Phi(\beta_{z_\nu,\sigma(y_\nu)}) = \Phi(\Sigma\beta_{z_\nu,\sigma(y_\nu)})$. But $\Sigma\beta_{z_\nu,\sigma(y_\nu)} = \sigma$, because it takes each $y \in P_0$ to $\Sigma z_\nu(y)\sigma(y_\nu) = \sigma(\Sigma z_\nu(y)y_\nu) = \sigma(y)$ – the last equality from the fact that $\{y_\nu\}$, $\{z_\nu\}$ are dual bases of P_0 , dP_0 . This proves that $\zeta_{P_0}(X)(\alpha_{P_0}(X)(\Phi)) = \Phi$, which proves (3.8).

(3.10) Corollary The form $\langle\ ,\ \rangle$ given in (3.6) is independent of the choice of bases $\{y_\nu\}$, $\{z_\nu\}$ of P_0, dP_0 . For $\alpha_{P_0}(X)$ is independent of this choice; now use (3.8), (3.7).

Formula With the notation above, we have for all $\theta \in (S,\underline{N}P_0)$ the following formula for the element $T_\theta \in D(M,S)$,

$$(3.11) \qquad T_\theta(h) = \langle\theta,hp_0\rangle = \sum_{\nu=1}^{n} t_\nu(h(c_\nu)) \ , \quad \text{all } h \in (M,S) \ ,$$

where $v_S'(\theta) = (t_1,\dots,t_n) \in \coprod(DS)e_\nu$ is the vector of θ , and $c_1,\dots,c_n \in M$ are given by $c_\nu = p_0(y_\nu)$, $\nu = 1,\dots,n$.

Proof of (3.11) By (3.1) and (3.9), $T_\theta = D(p_0,S)\zeta_{P_0}(S)(\theta)$. Hence for all $h \in (M,S)$, $T_\theta(h) = \zeta_{P_0}(S)(\theta)(hp_0)$, which equals $\langle\theta,hp_0\rangle$ by (3.7). Since $hp_0 \in (P_0,S)$ has vector $v_S(hp_0) = (hp_0(y_1),\dots,hp_0(y_n)) = (h(c_1),\dots,h(c_n))$, the second equality in (3.11) follows from (3.6).

(3.12) <u>Remarks</u> (i) (3.11) may be called a 'trace formula', from its similarity to the formula $\mathrm{Tr}(h) = \Sigma t_\nu(h(c_\nu))$ for the ordinary trace of an endomorphism h of a k-space U ($\{c_\nu\}$, $\{t_\nu\}$ being dual k-bases of U, DU respectively).

(ii) (3.5) shows that θ is expressed in terms of $t_1,\dots,t_n$ by $\theta(s)(\Sigma z_\nu a_\nu) = \Sigma t_\nu(a_\nu s)$, for $s \in S$, $a_1,\dots,a_n \in A$.

(iii) The following is sometimes useful:

$$(3.13) \qquad \mathrm{Ker}\,\theta = \bigcap_{\nu=1}^{n} \underline{c}(t_\nu) ,$$

where for any $t \in DS$, the 'core' $\underline{c}(t)$ of t is the largest submodule of S which is contained in $\mathrm{Ker}\,t$. To prove (3.13), notice that for each $s \in S$, $\theta(s) \in DdP_0$ is a linear form on $dP_0 = z_1A \oplus \dots \oplus z_nA$. Hence, and using (ii), $\theta(s) = 0$ if and only if $\theta(s)(z_\nu A) = t_\nu(As) = 0$, i.e. if and only if $s \in \underline{c}(t_\nu)$, for all ν.

4. <u>Existence and construction of almost split sequences</u>

(4.1) <u>Proposition</u> Take any indecomposable $S \in \mathrm{mod}\,A$, put $M = S$, choose a resolution (2.1) of S, choose $\theta \in (S,\underline{NP}_0)$ and make the ARG diagram as in section 2. Then the sequence $E(\theta): 0 \to \underline{AS} \xrightarrow{f} E \xrightarrow{g} S$ satisfies $\mathrm{Im}(\ ,g) = \underline{R}(\ ,S)$ if and only if the element $T_\theta = T \in D(S,S)$ satisfies

$$(4.2) \qquad T \neq 0 , \quad T(\mathrm{rad}\,\mathrm{End}(S)) = 0 .$$

Proof By (2.15), $\mathrm{Im}(\ ,g) = \underline{z}_V(\ ,S)$, where $V = \underline{rc}(T_\theta)$. By definitions (1.4), (2.12), $\underline{R}(\ ,S) = \underline{z}_J(\ ,S)$, where $J = \mathrm{rad}\ \mathrm{End}(S)$. Then (2.13,ii) shows that $\mathrm{Im}(\ ,g) = \underline{R}(\ ,S)$ if and only if $\underline{rc}(T_\theta) = J$. But this is equivalent to $T_\theta \neq 0$, $T_\theta(J) = 0$, since J is the unique maximal right ideal of $(S,S) = \mathrm{End}(S)$.

(4.3) Corollary If $S \in \mathrm{mod}\ A$ is indecomposable and not projective, and if the resolution (2.1) is minimal, then $E(\theta)$ is an almost split sequence if and only if $T_\theta = T$ satisfies (4.2).

Proof In the circumstances given, $\underline{A}S$ is indecomposable (see (2.3)). Also $\mathrm{Im}(\ ,g) = \underline{R}(\ ,S)$ implies $\mathrm{Im}(A,g) = \underline{R}(A,S)$, and $\underline{R}(A,S) = \underline{H}(A,S)$ (see section 1), and $\underline{H}(A,S) = (A,S)$ because there is no split epi $h:A \to S$. Therefore g is surjective by (2.19,i). The corollary now follows from (1.6).

(4.4) Corollary (= 'Existence' part of Auslander-Reiten's theorem (1.1).) If $S \in \mathrm{mod}\ A$ is indecomposable and non-projective, then there is an almost split sequence ending with S .

Proof It is clear that there is some $T \in (S,S)$ satisfying (4.2). By (2.16) there is some $\theta \in (S,\underline{NP}_0)$ such that T_θ equals this T . Then (4.3) finishes the proof of (4.4).

If we combine (4.3) with the trace formula (3.11), we get a 'recipe' for constructing almost split sequences. (Cf. the recipe of Gabriel [9, p.17], which is based on Butler's [5, p.84]. I believe that (4.5) is easier and more flexible than this; it can be regarded as a refinement of a method of Auslander which is quoted in [5, p.85].)

(4.5) <u>Recipe</u> Given S and (2.1) as in (4.3), we choose generators $y_1, \dots, y_n$ of P_0 as in section 3. Clearly $S = Ac_1 + \dots + Ac_n$, where $c_\nu = p_0(y_\nu)$, $\nu = 1, \dots, n$. Let $J = \text{rad End}(S)$. Choose a A-submodule Y of S such that $JS \leq Y < S$. This can surely be done, since JS is an A-submodule of S, and $JS < S$ because J is nilpotent. We choose the numbering of the y_ν, as we clearly may, so that $c_1 \notin Y$. Then $c_1 \notin e_1 Y$, and so there is some t_1 in $(DS)e_1$ such that $t_1(Y) = 0$, $t_1(c_1) \neq 0$. Now let $\theta \in (S, \underline{N}P_0)$ be defined by requiring that its vector $v_S'(\theta)$ be equal to $(t_1, 0, \dots, 0)$. By (3.11), $T_\theta(h) = t_1(h(c_1))$, for all $h \in \text{End}(S)$. In particular $T_\theta(1_S) = t_1(c_1) \neq 0$, and $T_\theta(J) = t_1(Jc_1) \leq t_1(Y) = = 0$. So $T = T_\theta$ satisfies (4.2), hence $E(\theta)$ is almost split by (4.3).

This method has a useful bonus. By (3.13) we see that $\text{Ker}\,\theta = \underline{c}(t_1) \geq Y$. So if we choose Y to be a maximal submodule of S (and of course we may do this), then $\text{Ker}\,\theta = Y$. It is some advantage to have $\text{Ker}\,\theta$ maximal in S, since this makes the pullback $E(\theta)$ 'as near as possible' to a split sequence.

(4.6) Remarks (i) If S is indecomposable and not projective, and if $\theta \in (S, \underline{N}P_0)$ is such that $T_\theta = T$ satisfies (4.2), then the proof of (4.3) shows that g is surjective, i.e. $\operatorname{Im} \theta \leq \operatorname{Im} \underline{N}p_1$ holds automatically.

(ii) Auslander and Reiten give (at least) two proofs of the 'uniqueness' part of theorem (1.1): (a) using the case $M = S$ of their identity (2.20), see [3, Prop. 4.3] or [9, §2.4], and (b) using the fact that an almost split sequence E provides a minimal projective resolution (in Fun A) of the simple functor $(,S)/\underline{R}(,S)$. See [9, §1.4].

(iii) (Auslander-Reiten) An almost split sequence E automatically satisfies the 'dual' condition to A3, namely A3' : if $X \in \text{mod } A$ and if $h \in (N,X)$ is not split mono, then h factors through f . See [3, §4]. For a direct proof, see [9, Prop.1.5].

5. Appendix: case where A is symmetric

An algebra A is symmetric if there exists a linear function $j: A \to k$, such that the bilinear form $\{ , \}: A \times A \to k$ given by $\{a,b\} = j(ab)$, for $a,b \in A$, is symmetric and non-singular (see for example [7, p.440]). In this case we have for each $X \in \text{mod } A$ a map $u_X: dX \to DX$, given by $u_X(f) = jf$, for all $f \in dX$. This is seen to be a natural isomorphism of right A-modules. So we get a natural isomorphism $w_X: X \to \underline{N}X = DdX$ given by

(5.1) $\quad w_X(x)(f) = jf(x)$, for all $x \in X$, $f \in dX$.

Therefore we can eliminate the Nakayama functor from our calculations, in case A is symmetric. Since this includes the important case where $A = kG$ is a finite group algebra, it may be worth giving some details. Let (2.1) be a projective resolution for $M \in \text{mod } A$. It is usual to write $\Omega M = \text{Ker } p_0$, $\Omega^2 M = \text{Ker } p_1$ (these depend on (2.1), but are determined by M up to projective summands, cf. (2.3)). The exact sequence

$$(5.2) \qquad 0 \to \Omega^2 M \xrightarrow{inc} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

is isomorphic to (2.2), as we see by applying the maps w_{P_1}, w_{P_1}, w_{P_0}, w_M to the appropriate terms of (5.2). Take any $\theta' \in (S,P_0)$, and define

$$(5.3) \qquad \theta = w_{P_0}\theta' \in (S,\underline{N}P_0) .$$

Let $F(\theta'): 0 \to \Omega^2 M \xrightarrow{f'} F(\theta') \xrightarrow{g'} S$ be the exact sequence obtained from $0 \to \Omega^2 M \to P_1 \to P_0$ by pullback over p_1 and θ'. Thus $F(\theta') = \{(v,s) \in P_1 \amalg S \mid p_1(v) = \theta'(s)\}$, $f'(y) = (y,0)$, $g'(v,s) = s$, for $y \in \text{Ker } p_1 = \Omega^2 M$ and $(v,s) \in F(\theta')$. It is clear that $F(\theta') \cong E(\theta)$, and that $\text{Im}(\ ,g) = \text{Im}(\ ,g')$. By (2.15), $\text{Im}(\ ,g') = \underline{z}_V(\ ,S)$, where $V = \underline{rc}(T_\theta)$. Moreover g' is surjective if and only if g is surjective, and in this case we may regard θ' as element of $(S,\Omega M)$ and identify $F(\theta')$ with the short exact sequence obtained by pullback from the short exact sequence (5.4) below.

(5.4)
$$\begin{array}{ccccccccc} & 0 & \longrightarrow & \Omega^2 M & \xrightarrow{inc} & P_1 & \xrightarrow{p_1} & \Omega M & \to 0 \\ & & & id \uparrow & & \ell' \uparrow & & \theta' \uparrow & \\ F(\theta'): & 0 & \longrightarrow & \Omega^2 M & \xrightarrow[f']{} & F(\theta') & \xrightarrow[g']{} & S & \to 0 \end{array} .$$

In particular, $F(\theta')$ is almost split if and only if $E(\theta)$ is. The recipe (4.5) can be used to find almost split sequences $F(\theta')$; all we need is a formula to calculate $\theta' = w_{P_0}^{-1}\theta$ directly from the vector $v_S'(\theta) = (t_1, \ldots, t_n)$ of θ . Using (3.12,ii) and (5.1) we find the following: $\theta'(s) = t_1'(s)y_1 + \ldots + t_n'(s)y_n$ for all $s \in S$, where for each ν , $t_\nu'(s)$ is the element of A defined by the equations

(5.5) $\qquad \{t_\nu'(s), a\} = t_\nu(as)$, for all $a \in A$.

Here $\{\ ,\ \}$ is the non-singular bilinear form defined above. In the case $A = kG$, G a finite group, one takes j to be the linear form on A such that $j(g) = 1$ or 0 , according as $g \in G$ is 1 or not. Then (5.5) simplifies to give a direct formula for $t_\nu(s)$, viz

(5.6) $$t_\nu'(s) = \sum_{g \in G} t_\nu(g^{-1}s)g .$$

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