space. These are the center (3.3a) and rationality (3.3b) conditions. Throughout this subsection we assume G and $\mathbf{C}$ satisfy these.

DEFINITION 3.19. *HM-Nielsen classes.* Let $r = 2s$. We say the Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ is H(arbater)M(umford) if there exists $\mathbf{g} \in \mathbf{C}$ of this form: $(g_1, g_1^{-1}, \dots, g_s, g_s^{-1})$. Then, $\mathbf{g}$ is an HM representative of the Nielsen class.

Mumford considered curves $\mathcal{C} \to \mathrm{Spec}(\mathbb{Z}_p)$ with nonsingular generic fiber, and totally degenerate special fiber. An analog arises in the proof of Theorem 3.21. Harbater's early deformation approach used this branch cycle condition [**Ha**].

It is easy (and similar to the proofs of the next lemmas) to show there is a $Q \in H_r$ that takes an HM representative as in Def. 3.19 to

$$(g_1, \dots, g_s, g_s^{-1}, \dots, g_1^{-1}) = [g_1, \dots, g_s] \stackrel{\text{def}}{=} [\mathbf{g}]. \tag{3.15}$$

LEMMA 3.20. *Suppose* $\mathbf{g}$ *is from* (3.15) *and* $\pi \in S_s$. *Then, there exists* $Q \in H_r$ *for which*

$$[g_1, \dots, g_s]Q = [g_{(1)\pi}, \dots, g_{(s)\pi}].$$

PROOF. Transpositions generate S_s. It suffices to show this when $\pi = (1\,2)$ using elements $Q \in H_r$ that only affect the first four coordinates. In the notation of (3.1d) take $Q_{1,2} = Q_1^{-1}Q_3$. Then:

$$\begin{aligned}(g_1, g_2, g_2^{-1}, g_1^{-1})Q_{1,2} &= (g_2, g_2^{-1}g_1g_2, g_2^{-1}, g_1^{-1})Q_3 \\ &= (g_2, g_2^{-1}g_1g_2, g_2^{-1}g_1^{-1}g_2, g_2^{-1}).\end{aligned} \tag{3.16}$$

Lemma 3.3 gives $Q' \in H_r$ conjugating (3.16) by g_2 (fixing the coordinates beyond the first four). Conclude by taking $Q = Q_{1,2}Q'$. □

The next result uses a hypothesis that you can remove any two pairs of conjugacy classes C and C′ from $\mathbf{C}$ (of certain type) and what remains should be gcomplete (Def. 3.18). Specifically, you can remove C and C′, inverse to C, and this property holds. We call such a $\mathbf{C}$, *HM-gcomplete* because it goes with the HM Nielsen hypothesis. Examples include $\mathbf{C}$ containing every conjugacy class at least 4 times. Also, if G is p-gcomplete, then $\mathbf{C}$ is HM-gcomplete when every p'-conjugacy class appears at least 4 times.

THEOREM 3.21. *Assume the following conditions on* $\mathbf{C}$.

(3.17a) $\mathbf{C}$ *is HM-gcomplete.*
(3.17b) $\mathrm{Ni}(G, \mathbf{C})$ *is an HM-Nielsen class.*

Then, all HM representatives of $\mathrm{Ni}(G, \mathbf{C})$ *are in one* H_r *orbit* O. *Further, if* (3.3a) *and* (3.3b) *hold,* $\mathcal{H}_O$ *is an absolutely irreducible component of* $\mathcal{H}(G, \mathbf{C})$ *with field of definition* $\mathbb{Q}$.

PROOF. Apply Lemma 3.20 to an HM representative $[g_1, \dots, g_s]$. Suppose g_s' is conjugate to g_s in G: $\alpha g_s \alpha^{-1} = g_s'$. It suffices to find $Q \in H_r$ with

$$[g_1, \dots, g_s]Q = [g_1, \dots, g_{s-1}, g_s'].$$

From (3.17a), among $g_1, \dots, g_{s-1}$ are generators of G. Apply Lemma 3.20 again to assume these are $g_t, \dots, g_{s-1}$ for some t.

Apply Lemma 3.3 to find Q with

$$[g_1, \dots, g_s]Q = [g_1, \dots, g_{t-1}, \alpha g_t \alpha^{-1}, \dots, \alpha g_s \alpha^{-1}]. \tag{3.18}$$

Lemma 3.20 gives $Q \in H_r$ that sends (3.18) to

$$[g_1, \dots, g_{t-1}, \alpha g_s \alpha^{-1}, \alpha g_t \alpha^{-1}, \dots, \alpha g_{s-1} \alpha^{-1}]. \tag{3.19}$$

Apply Lemma 3.3 to conjugate the entries from $\alpha g_t \alpha^{-1}$ to $\alpha g_{s-1} \alpha^{-1}$ by α^{-1}. A final application of Lemma 3.20 produces $[g_1, \dots, g_{s-1}, g_s']$. This concludes the proof that all HM representatives of $\mathrm{Ni}(G, \mathbf{C})$ are in one H_r orbit O. Now, Lemma 3.22 (below) shows $\mathcal{H}_O$ has $\mathbb{Q}$ as field of definition. □

Recall the discriminant locus D_r in $\mathbb{P}^r$. The fat diagonal $\Delta_r \subset (\mathbb{P}^1)^r$ lies above it in the natural map $(\mathbb{P}^1)^r \to \mathbb{P}^r$. A partition of $\{1, \dots, r\}$ is a collection of disjoint sets $\mathcal{I} = \{I_1, \dots, I_u\}$ with these properties. The union of the I_j s is $\{1, \dots, r\}$, and each I_j consists of consecutive integers. Given $\mathcal{I}$, define $\Delta_{\mathcal{I}}$ to be the subset of Δ_r with equations $x_i = x_j$, if $i, j \in I_k$ for some k. Similarly, let $D_{\mathcal{I}}$ be the image of $\Delta_{\mathcal{I}}$. Inequalities $x_i \neq x_j$, if i and j aren't in a common subset of $\mathcal{I}$, define a Zariski open subset $D^0_{\mathcal{I}}$ of $D_{\mathcal{I}}$. We use the following sequence of partitions:

$$\begin{aligned} \mathcal{I}_0 = \{1, \dots, r\}, \mathcal{I}_1 = \{\{1,2\}, 3, \dots, r\}, \dots, \\ \mathcal{I}_s = \{\{1,2\}, \{3,4\}, \dots, \{r-1, r\}\}. \end{aligned} \tag{3.20}$$

Each partition $\mathcal{I}$ also defines a *coalescing* operator $\mathcal{C}_{\mathcal{I}}$ on $\mathrm{Ni}(G, \mathbf{C})$. Given $\mathbf{g} \in \mathrm{Ni}(G, \mathbf{C})$, $\mathcal{C}_{\mathcal{I}}(\mathbf{g})$ is the u-tuple

$$\Big(\prod_{i \in \mathcal{I}_1}{}' g_i, \dots, \prod_{i \in \mathcal{I}_u}{}' g_i\Big).$$

The notation $\prod'$ means take the product of the g_i s in the correct order. Note: The result of coalescing is a u-tuple that may not generate a transitive subgroup of $G \leq S_n$. That is, it gives branch cycle descriptions of covers of $\mathbb{P}^1$ that may not be connected.

For the partition of (3.20) we define a *specialization sequence*. This consists of a sequence of étale covers

$$\psi_0 : V_0 \to \mathbb{P}^r \setminus D_r \text{ and } \psi_{\mathcal{I}_j} : V_{\mathcal{I}_j} \to D^0_{\mathcal{I}_j},\ j = 1, \dots, s-1,$$

with the following property. Let $\bar\psi_{\mathcal{I}_j} : \bar V_{\mathcal{I}_j} \to D_{\mathcal{I}_j}$ be the unique normalization of the completion of ψ to a cover of $D_{\mathcal{I}_j}$. Form this as follows. Let $\bar V_{\mathcal{I}_j}$ be the normalization of $D_{\mathcal{I}_j}$ in the function field of $V_{\mathcal{I}_j}$ [**M**, p. 396-7]. Then,

(3.21a) $\psi_{\mathcal{I}_{j+1}} : V_{\mathcal{I}_{j+1}} \to D^0_{\mathcal{I}_{j+1}}$ factors through the restriction $\bar\psi_{\mathcal{I}_j} : \bar V_{\mathcal{I}_j} \to D_{\mathcal{I}_j}$ over $D^0_{\mathcal{I}_{j+1}}$, $j = 1, \dots, s-2$.

(3.21b) An analog to (3.21a) for relating ψ_0 to $\psi_{\mathcal{I}_1}$.

We say this is a specialization sequence starting at V_0.

Hypotheses (3.3a) and (3.3b) produce a total family $\Phi : \mathcal{T} \to \mathcal{H}(G, \mathbf{C}) \times \mathbb{P}^1$ over $\mathbb{Q}$ with the following property. For $\boldsymbol{p} \in \mathcal{H}$, the fiber $\mathcal{T}_{\boldsymbol{p}} = \Phi^{-1}(\boldsymbol{p} \times \mathbb{P}^1)$ covers $\mathbb{P}^1$ by projection on the second factor. This cover should be in the equivalence class of covers $\boldsymbol{p}$ represents. In particular, each such cover has a fixed degree n. Suppose you have a specialization sequence. Lemma 3.22 requires extra hypotheses to define the pullback over a specialization sequence.

First, consider pullback over V_0. As before, U_r is $\mathbb{P}^r \setminus D_r$. Form the fiber product $V_0 \times_{U_r} \mathcal{H} = \mathcal{H}_{V_0}$. Then, form the fiber product (with obvious maps):

$$\mathcal{T}_{V_0} = \mathcal{T} \times_{\mathcal{H} \times \mathbb{P}^1} (\mathcal{H}_{V_0} \times \mathbb{P}^1). \tag{3.22}$$

Thus, $\mathcal{T}_{V_0}$ is a family of degree n covers by projection onto $\mathcal{H}_{V_0} \times \mathbb{P}^1$. All covers in the family are in the Nielsen class $\mathrm{Ni}(G, \mathbf{C})$. As above, complete and normalize $\mathcal{H}_{V_0}$ to produce $\bar{\mathcal{H}}_{V_0} \to \mathbb{P}^r$. Also, complete and normalize $\mathcal{T}_{V_0}$ over $\bar{\mathcal{H}}_{V_0} \times \mathbb{P}^1$ to get a cover $\bar{\mathcal{T}}_{V_0} \to \bar{\mathcal{H}}_{V_0} \times \mathbb{P}^1$. Finally, let $\mathcal{H}_1$ (resp., $\mathcal{T}_1$) be the normalization of the restriction of $\bar{\mathcal{H}}_{V_0}$ (resp., $\bar{\mathcal{T}}_{V_0}$) over $D^0_{\mathcal{I}_1}$ (resp., $D^0_{\mathcal{I}_1} \times \mathbb{P}^1$). Here is the hypothesis we need.

(3.23a) $\mathcal{T}_1 \to \mathcal{H}_1 \times \mathbb{P}^1$ is a family of degree n covers.

Continue inductively for the analog of property (3.23a).

(3.23b) $\mathcal{T}_j \to \mathcal{H}_j \times \mathbb{P}^1$ is a family of degree n covers; the specialization of the family for $j-1$ using $V_{\mathcal{I}_{j-1}}$, $j = 1, \dots, s-1$.

Some covers in the family may not be irreducible. For example, Lemma 3.22 produces examples were certain covers will be n copies of $\mathbb{P}^1$. Zariski's connectedness theorem say all fibers of a connected family are connected. So, for this example you would think we mean n copies pasted together over s points. That last normalization, however, before declaring the definition of $\mathcal{H}_1$ could disconnect components.

We can restrict a specialization sequence to $\mathcal{H}_O$ for any orbit O of H_r on $\mathrm{Ni}(G, \mathbf{C})$. The proof of Lemma 3.22 needs a technical point. We can restrict construction (3.23) over $\mathbb{A}^r$, the subset of $\mathbb{P}^r$ in the image of $\mathbb{A}^r \subset (\mathbb{P}^1)^r$ where $x_i \neq \infty$, $i = 1, \dots, r$.

LEMMA 3.22. *Assume the hypotheses of Theorem 3.21. Let O be any orbit of H_r acting on $\mathrm{Ni}(G, \mathbf{C})$. Consider the image $\mathcal{C}_{\mathcal{I}_s}(O)$ of O under the last of the coalescing operators from sequence (3.20). Then, $\mathcal{C}_{\mathcal{I}_s}(O)$ contains $(1, 1, \dots, 1)$ if and only if O contains HM representatives. Further, suppose we have a specialization sequence (over $\mathbb{A}^r$) having property (3.23). Restrict the sequence to $\mathcal{H}_O$. The family induced over $V_{\mathcal{I}_s, O}$ consists of covers with branch cycles in $\mathcal{C}_{\mathcal{I}_s}(O)$. Finally, there is a specialization sequence satisfying (3.23). In particular, $\mathcal{H}_O$ has $\mathbb{Q}$ as field of definition.*

PROOF. The statement on $\mathcal{C}_{\mathcal{I}_s}(O)$ follows from the definitions once we know there is only one orbit containing HM representatives (Theorem 3.21). Existence and properties of a specialization sequence over $\mathbb{A}^r$ is implicit in [**Fr5**, §4]. In fact, this proves its existence without the center hypothesis. (This is necessary, because after coalescing, we lose that hypothesis.) The last statement is similar to the end of the proof of Galois Invariance Theorem 3.16. Any $\tau \in G_{\mathbb{Q}}$ acts on the whole induced family over a specialization sequence. Suppose $\mathcal{H}_O$ and $\mathcal{H}_{O'}$ are conjugate. Then, τ extends to take the sequence (3.23) applied to $\mathcal{H}_O$ to the sequence applied to $\mathcal{H}_{O'}$. In particular, $\mathcal{C}_{\mathcal{I}_s}(O')$ contains $(1,\dots,1)$. This contradicts the first statement in Theorem 3.21. $\square$

COROLLARY 3.23. *Let p be a prime dividing $|G|$. Assume $\mathbf{C}$ are p' conjugacy classes G satisfying* (3.17) *of Theorem* 3.21. *Assume, also, the Center Hypothesis (Def.* 3.5*) for (G,p). Then, each level $\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ of the modular tower for $(G,\mathbf{C})$ has a (nonempty) absolutely irreducible component over $\mathbb{Q}$.*

PROOF. We show (3.17a) and (3.17b) hold for $\mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$. First (3.17a). Let $\mathbf{C}'$ be $\mathbf{C}$ after removing two conjugacy classes that must pass the HM-gcomplete test. Let $\mathbf{g} \in \mathbf{C}'$. By assumption, $\langle \mathbf{g} \rangle$ generates G. Now let $\tilde{\mathbf{g}} \in {}^n_p\tilde{\mathbf{C}}'$ lie over $\mathbf{g}$. As ${}^n_p\tilde{G} \to G$ is a frattini cover, $\langle \tilde{\mathbf{g}} \rangle$ generates ${}^n_p\tilde{G}$. This concludes the proof (3.17a) holds.

That leaves showing $\mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ is nonempty and contains an HM representative. As in Ex. 3.13, the lift of an HM representative to ${}^n_p\tilde{G}$ is automatically an HM representative of $\mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$. $\square$

APPENDIX. DISTINGUISHED PROBLEMS

We use the notation of Nielsen classes $\mathrm{Ni}(G,\mathbf{C})$ freely from §III.A.

§APP.A. Modular representation problems. Recall the Center Hypothesis 3.5 on the characteristic quotients ${}^n_p\tilde{G}$ of the universal p-frattini cover ${}_p\tilde{G}$ of G. It applies to Th. 3.4 to produce rationality results as in Cor. 3.23. Unless $\mathbf{C}$ is a rational union of conjugacy classes, the Hurwitz space $\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ has *no* component with field of definition $\mathbb{Q}$ (see end of §III.A). This comes from the most mundane use of the Branch Cycle Argument applying the action of the cyclotomic character (Part A of the proof of Theorem 3.16). Thus, for finding $\mathbb{Q}$ components, the hypothesis $\mathbf{C}$ is a rational union of conjugacy classes is both necessary and convenient. The converse, however, that the moduli space $\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ has $\mathbb{Q}$ as field of definition, requires ${}^n_p\tilde{G}$ to have no center. (See Moduli Space Condition (C.8).) While a finer analysis of relaxing this hypothesis is welcome, we should know when it holds. Lemma 3.6 says it holds when G is perfect, including when G is a nonabelian simple group.

PROBLEM A.1. *For what groups (and primes p dividing their orders) does the Center Hypothesis hold?*

LEMMA A.2. *If $\mathbf{1}_{{}^n_p\tilde{G}}$ appears at the left of a Loewy display for $\ker_n/\ker_{n+1}$, then $\mathbf{1}_{{}^k_p\tilde{G}}$ appeasr at the left of a Loewy display for $\ker_k/\ker_{k+1}$ for $k > n$.*

PROOF. Let $P_{\mathbf{1},k}$ be the projective indecomposable for the trivial representation of ${}^k_p\tilde{G}$. Also, P_k is the minimal projective module for ${}^k_p\tilde{G}$ that maps surjectively to the kernel of $P_{\mathbf{1},k} \to \mathbf{1}$. By hypothesis, $\mathbf{1}$ appears at the far left of the Loewy display of P_n. From the argument of Lemma 3.6, it appears at the far right of the Loewy display of P_n. Lemma 2.6 gives a surjective map $P_{\mathbf{1},k} \to P_{\mathbf{1},n}$ of ${}_p^k\tilde{G}$ modules for $k \ge n$. Thus, the far right of the Loewy display for the kernel of $P_{\mathbf{1},k} \to \mathbf{1}$ contains the far right of the Loewy display for $P_{\mathbf{1},n}$. Conclude: $\mathbf{1}$ appears in the far right of the Loewy display of P_k. So, it appears in the far left of this Loewy display. Conclude this lemma from Lemma 2.3. □

Lemma A.2 says a nontrivial center to ${}_p^n\tilde{G}$ will persist to a nontrivial center of ${}_p^k\tilde{G}$, $k \ge n$. Still, there may be a way around this. Assume n is the first value of k for which ${}_p^k\tilde{G}$ has a nontrivial center. Let V_n be the largest submodule of $\ker_{n-1} / \ker_n$ on which ${}_p^n\tilde{G}$ acts trivially. Then, ${}_p^n\tilde{G} \to G$ factors through a natural frattini cover $H = {}_p^n\tilde{G}/V_n \to G$. If $\mathbf{1} \to \mathbf{1}$ does not appear at the left of the Loewy display of $\ker_{n-1} / \ker_n$, then H has no center. A further hypothesis would allow replacing ${}_p^n\tilde{G}$ with H in the modular tower construction.

PROBLEM A.3. *In the notation above, for what groups is $\dim_{\mathbb{F}_p}(V_n)$ bounded as a function of n? When this holds, when can we assure ${}_p^n\tilde{G}/V_n$ has no center?*

Self-normalizing stabilizer subgroups arise in interpreting the moduli spaces $\mathcal{H}^{\mathrm{ab}}$ (Addendum to Thm. 3.4). §III.C has a natural case for considering extending permutation representations on the characteristic quotients of ${}_p^n\tilde{G}$. This has $G \le S_k$ and the stabilizer $G(1)$ in the representation has order prime to p. Then, all characteristic quotients have one conjugacy class of copies of $G(1)$. This is the analog of Problem A.1 for that question.

PROBLEM A.4. *Consider (G, T, p) with G a nonabelian simple group, T a primitive permutation representation of G, and p a prime not dividing $|G(1)|$. For which (G, T, p) does the conclusion of Self-Normalizing Problem 3.8 hold?*

Remark 2.10 notes that $G = A_5$ and $p = 5$ is the other extreme of Prop. 2.9. Here the module $\ker_0 / \ker_1$ is equal to the module induced from the universal 5-frattini module for A_4. Example 3.9 checks the case $n = 1$ in this example. We haven't yet checked if there is a positive conclusion to Problem A.4 for all n in this example.

Even if p does divide $|G(1)|$, there may be groups ${}_n^p\tilde{G}(1)$ extending $G(1)$ in each of the characteristic quotients ${}_p^n\tilde{G}$. We must, however, assure the extensions of $G(1)$ give a faithful coset representation for each ${}_p^n\tilde{G}$. This doesn't always happen. Again take $G = A_5$ and $p = 2$. From Prop. 2.9, the only subgroup of ${}_p^n\tilde{G}$ mapping surjectively to $G(1) = A_4$ is the full pullback of A_4 in ${}_p^n\tilde{G}$. This can't give a faithful representation of ${}_p^n\tilde{G}$. Is this outcome characteristic of being on the extreme end of the Prop. 2.9 conclusion?

§III.E discusses groups G for which $|G|$ has prime divisors p that ramify exceptionally. Lemma 3.17 says all p' conjugacy classes lie in a proper normal subgroup N of G. Is there a natural converse?

Problem A.5. *Suppose N is a proper normal subgroup of G and N contains all conjugacy classes of elements of order r^t with $r \neq p$ a prime. When does this imply N contains a normal p' complement?*

Kantor used the classification of finite simple groups to show the following (in the notation of Def. 3.18) [**FKS**].

(A.1) (3.14) holds if $\mathbf{C}$ contains each prime-power order conjugacy class of G.

Corollary 3.23 applies to a group that is p-gcomplete. Lemma 3.17 suggests (A.1) could help classify groups with no p' compliment that aren't p-gcomplete.

§**APP.B. Curves tesselating a general Hurwitz space.** As in §III.F, let D_r be the discriminant in $\mathbb{P}^r$. The fat diagonal $\Delta_r \subset (\mathbb{P}^1)^r$ lies above it in the symmetry map $\Theta_r : (\mathbb{P}^1)^r \to \mathbb{P}^r$. [**Fr5**] and [**DFr2**] refer to the fundamental group of $(\mathbb{P}^1)^r \setminus \Delta_r$ as the *straight Hurwitz* monodromy group SH_r. It is the kernel of the natural map $H_r \to S_r$ with the effect $Q_i \mapsto (i\,i{+}1)$, $i = 1, \dots, r{-}1$ (§III.A). Given a Hurwitz space cover $\mathcal{H} \to \mathbb{P}^r \setminus D_r$, form the fiber product

$$\mathcal{H}' = \mathcal{H} \times_{\mathbb{P}^r} (\mathbb{P}^1)^r \setminus \Delta_r.$$

Suppose $\mathcal{H}$ comes from H_r acting on some absolute Nielsen classes $\mathrm{Ni}(\mathbf{C})^{\mathrm{ab}}$ (see (3.1d)). Components of $\mathcal{H}'$ correspond to orbits of SH_r on the same absolute Nielsen classes.

Consider Def. 3.1a for Nielsen classes. An $\omega \in S_r$ acts on $\mathbf{C}$ by permutating the entries of $\mathbf{C}$. Form the group of symmetries of $\mathbf{C}$: $I_{\mathbf{C}} = \{\omega \in S_r \mid (\mathbf{C})\omega = \mathbf{C}\}$. Note: SH_r preserves the order of the conjugacy classes in a Nielsen class $\mathrm{Ni}(\mathbf{C})$. Thus, the pullback of each component of $\mathcal{H}$ to $\mathcal{H}'$ breaks into at least $(S_r : I_{\mathbf{C}})$ components. Each ordering $(\mathbf{C})\omega$ of $\mathbf{C}$ produces a union $\mathcal{H}'_\omega$ of components on $\mathcal{H}'$ lying above the given component of $\mathcal{H}$. To assess the components of $\mathcal{H}'_\omega$ consider the action of SH_r on

$$\mathrm{SNi}((\mathbf{C})\omega) = \{\mathbf{g} \in G^r \mid \langle \mathbf{g} \rangle = G,\ g_1 \cdots g_r = 1,\ g_i \in \mathrm{C}_{(i)\omega}, i = 1, \dots, r\}.$$

Let $N_{S_n}(\mathbf{C})'$ be the subgroup of $N_{S_n}(\mathbf{C})$ that stabilizes $\mathrm{SNi}(G, (\mathbf{C})\omega)$. Orbits, modulo the action of $N_{S_n}(\mathbf{C})'$ corresponds exactly to (absolutely irreducible) components of $\mathcal{H}'_\omega$. Similar statements hold for representations of H_r on $\mathrm{Ni}(\mathbf{C})$ corresponding to components of $\mathcal{H}^{\mathrm{in}}$.

Rational Conjugacy Class Observation B.1. The proof of Lemma 3.20 gives a stronger result: HM-representatives in $\mathrm{SNi}(\mathbf{C})$ fall in one SH_r orbit. Thus, there is a version of Theorem 3.21 referring to SNi. One point, however, complicates this result and the corresponding corollaries. Assume $\mathbf{C}$ is a rational union of conjugacy classes with a particular conjugacy class, say, C_1, not rational. Suppose $\mathcal{H}_1$ is an absolutely irreducible $\mathbb{Q}$ component of $\mathcal{H}$. Then, no absolutely irreducible component of $\mathcal{H}'$ lying over $\mathcal{H}_1$ will have field of definition $\mathbb{Q}$. So, if our search is for $\mathbb{Q}$ components of $\mathcal{H}'$, *all* conjugacy classes must be rational. This condition holds in many important examples (modular curves and Ex. 3.13

included). Still, this illustrates that adding an ordering to the branch points, defies the generality of our results. □

Suppose $\mathcal{H}' \to (\mathbb{P}^1)^r \setminus \Delta_r$ has field of definition F. Let $(\mathbb{P}^1)^{r-1}$ be the *last* $r-1$ copies of $\mathbb{P}^1$ in the $(\mathbb{P}^1)^r$. Choose any $(x'_2, \dots, x'_r) = \boldsymbol{x}' \in (\mathbb{P}^1)^{r-1} \setminus \Delta_{r-1}$. Regard the fiber of $\mathcal{H}'$ over $\boldsymbol{x}'$ as a curve $\mathcal{H}'_{\boldsymbol{x}'}$ over $\mathbb{P}^1_{x_1} \times \boldsymbol{x}' \cong \mathbb{P}^1$. The copy of $\mathbb{P}^1$ is the first coordinate of $(\mathbb{P}^1)^r$. Here, $\mathcal{H}'_{\boldsymbol{x}'}$ is missing points over the values of x_1 equal to $(x'_2, \dots, x'_r)$. Complete this curve according to Riemann's Existence Theorem to get a ramified cover $\bar{\mathcal{H}}'_{\boldsymbol{x}'} \to \mathbb{P}^1_{x_1} \times \boldsymbol{x}'$. If $\boldsymbol{x}'$ has coordinates in F, this cover has field of definition F. Express a description of the branch cycles of this curve using the classical generators $Q_1, \dots, Q_{r-1}$ (for example, [**Fr7**]) as

$$\text{(B.1)} \qquad \begin{aligned} a_{1\,2} = Q_1^{-2}, a_{1\,3} = &Q_1 Q_2^{-2} Q_1^{-1}, \dots, \\ &a_{1\,r-1} = Q_1 \dots Q_{r-2} Q_{r-1}^{-2} Q_{r-2}^{-1} \dots Q_1^{-1}. \end{aligned}$$

Notice the product 1 condition ((i) of §I.F) translates to condition (3.1c) for $Q_1, \dots, Q_{r-1}$. The $a_{1\,j}$ s, however, may not generate a transitive group in their action on a given SH_r orbit. For example, we haven't checked the following.

PROBLEM B.2. *Do all HM-representatives of* SNi *fall in one orbit under the action of the $a_{1\,j}$ s, $j = 2, \dots, r-1$ (Th. 3.21)?*

One approach to formulating generalizations of Theorem 1.1 is to fix a point $\boldsymbol{x}' \in (\mathbb{P}^1)^{r-1} \setminus \Delta_{r-1}$. Assume a situation with a positive conclusion for Problem B.2 and all conjugacy classes in $\mathbf{C}$ are rational. For each n, let $\mathcal{H}_n$ be the level n Hurwitz space of the modular tower. Then, Corollary 3.23 produces an absolutely irreducible $\mathbb{Q}$ component $\mathcal{H}^*_n$ of the pullback $\mathcal{H}'_n$ of $\mathcal{H}_n$. Choose these compatibly so the natural map $\mathcal{H}'_n \to \mathcal{H}'_{n-1}$ maps $\mathcal{H}^*_n$ to $\mathcal{H}^*_{n-1}$. Finally, restrict these maps to the fibers over any $\mathbb{Q}$ point $\boldsymbol{x}' \in (\mathbb{P}^1)^{r-1} \setminus \Delta_{r-1}$ to get a sequence of curves $\mathcal{H}^*_{n,\boldsymbol{x}'}$ similar to the modular curve tower of §I.A.

PROBLEM B.3. *Is there a result analogous to Theorem 1.1 for the sequence of curves $\mathcal{H}^*_{n,\boldsymbol{x}'}$, $n = 1, 2, \dots$?*

Recall: A variety W is *unirational* (over a field F) if there is an integer n and a birational map $\mathbb{P}^n \to W$ (over F) onto a Zariski open subset of W. Consider the case of many repetitions of any given conjugacy classes (as in the discussion around condition (3.9)). Then, we've greatly weakened the possible conclusions for the original problem by pulling back to $(\mathbb{P}^1)^r$. For example, $\mathcal{H}^*_{n,\boldsymbol{x}'}$ supports the first cohomology of $\mathcal{H}'$. (This is because the first cohomology group of a compactification of $\mathcal{H}'$ is a birational invariant, and this space is birational to $\mathcal{H}^*_{n,\boldsymbol{x}'} \times (\mathbb{P}^1)^{r-1}$.) Thus, as [**Fr7**] examples show, even if r is small (like 4), $\mathcal{H}^*_0$ may be far from unirational while its image in $\mathcal{H}$ is. As in Observation B.1, an ordering of the branch points destroys important questions in the area. Part IV of the sequel will refine these questions to deal directly with $\mathcal{H}$.

§APP.C. Complex conjugation on Hurwitz space orbits. Our diophantine starting point is real points and points over finite fields on a modular tower. We define our terms with real points. Comments C.5 below model, for complex conjugation, applying the Drinfeld-Ihara relations for the Grothendieck-Teichmüller group. The goal is to determine fields of definition of components of a modular tower. §APP.D outlines an approach to finding projective sequences of real points on a modular tower. Finally, §APP.E discusses analogous ideas over finite fields.

Let $\{\mathcal{H}_n = \mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}}),\ n = 0, 1, \dots\}$ be a modular tower attached to the characteristic quotients of ${}_p\tilde{G}$. Certain circumstances require treating the action of H_r on inner classes differently than on absolute classes (§III.A). For the former, use the notation $\mathcal{H}_n^{\mathrm{in}}$ and for the latter $\mathcal{H}_n^{\mathrm{ab}}$. In some contexts we drop this notation, implying one or the other of these hold. Consider finding components of $\mathcal{H}_n$ and points on $\mathcal{H}_n$ defined over $\mathbb{R}$. Thus, we may relax our standard assumption that $\mathbf{C}$ is a rational union of conjugacy classes. If $\mathbf{C}^{-1} = \mathbf{C}$ ($a = -1$ in Def. 3.2) then [**FrV**, Prop. 3] shows (under the assumption G has no center) $\mathcal{H}_0 = \mathcal{H}(G, \mathbf{C})$ has field of definition $\mathbb{R}$. The argument that shows ${}_n\tilde{\mathbf{C}}^{-1} = {}_n\tilde{\mathbf{C}}$ for all n appears in Ex. 3.13. Putting these conclusions together gives the following criterion for all levels of a modular tower to have field of definition $\mathbb{R}$.

LEMMA C.1. *Assume $\mathbf{C}$ is a collection of p' conjugacy classes of G, and $\mathbf{C}^{-1} = \mathbf{C}$. Also, suppose the center hypothesis holds for (G, p) as in Def. 3.5. Then, all levels $\mathcal{H}_n$ of the modular tower have field of definition $\mathbb{R}$.*

[**DFr2**, Th. 4.4] explicitly describes real points on a Hurwitz space $\mathcal{H}(G, \mathbf{C})^{\mathrm{ab}}$ or $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$. This, however, uses the assumptions of no center. [**DFr**, Comment 3 of §3.5] gives exact criteria without the center hypothesis, for a cover to have field of definition *or* field of moduli in $\mathbb{R}$.

PROBLEM C.2. *Generalize* [**DFr2**, Th. 4.4] *even if G has a center.*

We come to the main discussion of this subsection. Let W be any absolutely irreducible variety over a field F. The arithmetic fundamental group $\hat{\pi}_1^{ar}(W)$ is the profinite limit of the Galois groups of étale Galois covers $V \to W$. Let M be the maximal extension of $F(W)$ étale over W in the algebra closure $\overline{F(W)}$. Then, $G(M/F(W))$ is isomorphic to $\hat{\pi}_1^{ar}(W)$. Similarly, $\hat{\pi}_1^{ge}$ is isomorphic to $G(M/\bar{F}(W))$. Now suppose F has characteristic 0.

Take $\hat{\pi}_1$ to be the profinite completion of $\pi_1(W)$ over all subgroups of finite index in $\pi_1(W)$. From the Grauert-Remmert generalization of Riemann's Existence Theorem [**GR**]: The natural homomorphism $\hat{\pi}_1(W) \to \hat{\pi}_1^{ge}$ is surjective. A profinite group is *residually finite* if the intersection of all subgroups of finite index is $\{1\}$. If, however, $\pi_1(W)$ isn't residually finite, then the map is not injective. Serre [**Se4**, p. 60] seems to say it isn't known if fundamental groups of open subsets of algebraic varieties over $\mathbb{C}$ are residually finite. He, however, outlines a proof they are when $\pi_1(W)$ embeds in a linear group. See [**I3**] for this result when $W = \mathbb{P}^r \setminus D_r$. Use the notation of (3.1) for the images in $\hat{\pi}_1^{ge}(\mathbb{P}^r \setminus D_r)$ of the generators Q_i, $i = 1, \dots, r{-}1$.

Real components-points problem.

(C.1a) *Give a procedure to identify the existence of a projective system of (absolutely irreducible) components $\mathcal{H}'_n$ of $\mathcal{H}_n$ with field of definition $\mathbb{R}$?*

(C.1b) *Give a procedure to identify the existence of a projective system of points $\{\boldsymbol{p}_n \in \mathcal{H}_n\}_{n=0,1,\dots}$ in $\mathbb{R}$?*

This section translates (C.1a) to a question on the universal frattini cover of G. §App.D does the same for (C.1b), by introducing the *universal Artin-Schreier* cover of a group with chosen involution conjugacy classes. The meaning of (C.1a) should be clear. By projective system of points we mean the analog of the situation of §I.A: the natural map (following Def. 3.10) takes $\boldsymbol{p}_n \in \mathcal{H}_n$ to $\boldsymbol{p}_{n-1}$, $n = 1, 2, \dots$. Start with an action of complex conjugation on paths representing homotopy classes in $\mathbb{P}^r \setminus D_r$ from a basepoint $\boldsymbol{x}_0$. Then, this choice produces paths on $\mathbb{P}^1 \setminus \boldsymbol{x}_0$ to decide what complex conjugation does to the covers representing points of $\mathcal{H}_n$. For there to be a projective system of real points over a specific $\boldsymbol{x}_0$, we must have $\boldsymbol{x}_0 \in \mathbb{P}^r \setminus D_r$. This implies $\boldsymbol{x}_0$ is the image from $(\mathbb{P}^1)^r \setminus \Delta_r$ of an ordered r-tuple $\boldsymbol{x}'$ closed under complex conjugation. Thus, we may assume

(C.2) $\boldsymbol{x}' = (x'_1, \dots, x'_r)$ with $x'_1 < x'_2 < \cdots < x'_s$ in $\mathbb{P}^1(\mathbb{R})$ and $x'_{s+j}, x'_{s+r-j+1}$ complex conjugate pairs, $j = 1, \dots, (r-s)/2$.

The relation $x'_1 < x'_2 < \cdots < x'_s$ means these fall on the real circle in the given order on the Riemann sphere ($r \geq 3$). For the sequel include the relation $x'_s < x'_1$, meaning that going counterclockwise around $\mathbb{R} \cup \{\infty\}$, x'_1 follows x'_s.

The case $s = r$ is most significant: other cases revert to this following computations of [**DFr2**]. Therefore, restrict to this case:

(C.3) $\boldsymbol{x}'$ is a set of real points ordered on $\mathbb{P}^1(\mathbb{R})$ according to their subscripts.

Let B_i be a clockwise circle on $\mathbb{P}^1$ with a marked diameter on the real axis having x'_i and x'_{i+1} as endpoints. (One of these has x'_r and x'_1 at the endpoints of the directed diameter.) Parametrize the top of B_i with $t \mapsto B_i^+(t)$ on $[0,1]$, so $B_i^+(0) = x'_i$ and $B_i^+(1) = x'_{i+1}$. Similarly, parametrize the bottom of B_i with $t \mapsto B_i^-(t)$ on $[0,1]$ so $B_i^-(0) = x'_{i+1}$ and $B_i^-(1) = x'_i$. Consider the path

$$\text{(C.4a)} \qquad t \mapsto (x'_1, \dots, x'_{i-1}, B_i^+(t), B_i^-(t), x'_{i+2}, \dots, x'_r),\ t \in [0,1].$$

The image of (C.4a) in $\mathbb{P}^r \setminus D_r$ represents the braid group generator Q_i^{-1} (from (3.1)). The inverse of path (C.4a) is

$$\text{(C.4b)} \qquad t \mapsto (x'_1, \dots, x'_{i-1}, \bar{B}_i^-(t), \bar{B}_i^+(t), x'_{i+2}, \dots, x'_r),\ t \in [0,1].$$

where the notation $\bar{B}$ means we applied complex conjugation to the coordinate.

Complex conjugation acts on all paths based at $\boldsymbol{x}_0$. So, it induces an operator κ on homotopy classes of paths and thus on $\hat{\pi}_1^{ar}$ (see Proof–Comments below).

Also, κ acts on functions expanded around $\boldsymbol{x}_0$ by acting on their coefficients. This latter action gives a section of $\hat{\pi}_1^{ar} = \hat{\pi}_1^{ar}(\mathbb{P}^r \setminus D_r, \boldsymbol{x}_0) \to G(\mathbb{C}/\mathbb{R})$. Use κ for the corresponding element of $\hat{\pi}_1^{ar}$.

Complex Conjugation Proposition C.2. *Complex conjugation of the image of* (C.4a) *in* $\mathbb{P}^r \setminus D_r$ *is the image of* (C.4b). *Conjugation by* κ *gives an automorphism of* $\hat{\pi}_1^{ge}(\mathbb{P}^r \setminus D_r, \boldsymbol{x}_0)$ *mapping* Q_i *to* Q_i^{-1}.

Exchange $r+1$ *for* r *and consider the induced effect of* κ *on the elements*

$$\text{(C.5a)}\quad a_{1\,2} = Q_1^{-2}, a_{1\,3} = Q_1 Q_2^{-2} Q_1^{-1}, \ldots, a_{1\,r} = Q_1 \cdots Q_{r-1} Q_r^{-2} Q_{r-1}^{-1} \cdots Q_1^{-1}.$$

Then, κ *maps this set in order to*

$$\text{(C.5b)}\qquad a_{1\,2}^{-1}, a_{1\,2} a_{1\,3}^{-1} a_{1\,2}^{-1}, \ldots, a_{1\,2} \cdots a_{1\,r-1} a_{1\,r}^{-1} a_{1\,r-1}^{-1} \cdots a_{1\,2}^{-1}.$$

Proof–Comments. All unramified covers of $\mathbb{P}^r \setminus D_r$ have field of definition $\bar{\mathbb{Q}}$: consequence of the *general* Riemann's existence theorem [**GR**]. For such a cover $Y \to \mathbb{P}^r \setminus D_r$, α in the ordinary fundamental group acts through analytic continuation as a permutation of the fiber on Y above $\boldsymbol{x}_0$. Running through the fibers $Y_{\boldsymbol{x}_0}$ of covers, α produces compatible actions on all these fibers in the following sense. If $\psi : Y' \to \mathbb{P}^r \setminus D_r$ factors through Y then the action of α commutes with ψ. Thus, interpret $\hat{\pi}_1^{ge}(\mathbb{P}^r \setminus D_r, \boldsymbol{x}_0)$ as *all* compatible maps on the system of fibers of covers over $\boldsymbol{x}_0$ (as in [**FrV**, §1.2] or [**I**, §2]). Equivalently, α gives an analytic continuation on the germs of algebraic functions on Y, after expansion around points of the fiber over $\boldsymbol{x}_0$.

Consider $\sigma \in G_{\mathbb{Q}}$ (resp., $\sigma = \kappa$) and $\boldsymbol{x}_0$ with coordinates in $\mathbb{Q}$ (resp., $\mathbb{R}$). Then, σ takes $\alpha \in \hat{\pi}_1^{ge}$ to the element $\sigma(\alpha)$ which has the effect

$$\text{(C.6)}\qquad \boldsymbol{y} \in Y_{\boldsymbol{x}_0} \mapsto \sigma \circ \alpha \circ \sigma^{-1}(\boldsymbol{y}).$$

Calculate this when $\alpha = Q_i$, $\boldsymbol{x}_0$ is the image of $\boldsymbol{x}' \in (\mathbb{P}^1(\mathbb{R}))^r \setminus \Delta_r$ and $\sigma = \kappa$ as above. Direct application of κ to the representing path $\hat{Q}_i$ for Q_i gives

$$\text{(C.7)}\qquad \kappa \circ Q_i \circ \kappa^{-1}(\boldsymbol{y}) = \kappa(\hat{Q}_i)(\kappa \circ \kappa^{-1}(\boldsymbol{y})) = \hat{Q}_i^{-1}(\boldsymbol{y}).$$

Check κ on (C.5a) by substituting Q_i^{-1} for Q_i. □

Now we interpret if components of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}} = \mathcal{H}$, *as moduli spaces*, have field of definition in $\mathbb{R}$. Let $\mathcal{H}^*$ be an absolutely irreducible component of $\mathcal{H}$. Consider $\sigma \in G_{\mathbb{Q}}$. Suppose the cover $\Psi : \mathcal{H}^* \to \mathbb{P}^r \setminus D_r$ is a moduli space defined over the fixed field of σ. From [**Fr5**, Th. 5.1] or [**FrV**, Th. 1], this requires more than the field of definition of Ψ is in the fixed field of σ. For $\boldsymbol{p} \in \mathcal{H}^*(\bar{\mathbb{Q}})$, let $\psi : X \to \mathbb{P}^1$ represent $\boldsymbol{p}$. Then, the following holds.

(C.8a) Extension of σ to the cover ψ produces a cover $\psi^\sigma : X^\sigma \to \mathbb{P}^1$ (corresponding to $\boldsymbol{p}^\sigma$) with $\boldsymbol{p}^\sigma \in \mathcal{H}^*(\bar{\mathbb{Q}})$.

For example, it is possible the cover $\Psi : \mathcal{H}^* \to \mathbb{P}^r \setminus D_r$ has a model with field of definition $\mathbb{Q}$ even if $\mathbf{C}$ isn't a rational union of conjugacy classes (Def. 3.2). This condition, however, is necessary for a model of $\Psi : \mathcal{H}^* \to \mathbb{P}^r \setminus D_r$ representing covers in the Nielsen class to have field of definition $\mathbb{Q}$.

Now turn (C.8a) into a combinatorial criterion. Assume (for simplicity) the branch points $\boldsymbol{x} = (x_1, \dots, x_r)$ of ψ are in the fixed field of σ. Let $(g_1, \dots, g_r)$ be a branch cycle description of ψ with respect to a specific set $(\bar{\Sigma}_1, \dots, \bar{\Sigma}_r)$ of homotopy classes of paths on $\mathbb{P}^1 \setminus \{\boldsymbol{x}\}$. Thus, $(g_1, \dots, g_r)$ is in an H_r orbit O in $\mathrm{Ni}(G, \mathbf{C})$ (as in §III.A). Using this data, here is the meaning of (C.8a).

(C.8b) The branch cycle description $(g'_1, \dots, g'_r)$ for $\psi^\sigma : X^\sigma \to \mathbb{P}^1$ relative to the same paths $(\bar{\Sigma}_1, \dots, \bar{\Sigma}_r)$ is also in O.

Similar statements apply for $\sigma \in G(\bar{F}/F)$ where F is any field; choose $\boldsymbol{x}$ with coordinates in F. This includes $F = \mathbb{R}$ or an archimedian completion of $\mathbb{Q}$. Also, we may replace $\mathrm{Ni}(G, \mathbf{C})$ by $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{ab}}$ corresponding to an appropriate permutation representation of G. Finally, let complex conjugation, κ, act on $\mathrm{Ni}(G, \mathbf{C})$ through the following formula:

$$\begin{aligned}\mathbf{g} = (g_1, \dots, g_r) \in &\mathrm{Ni}(G, \mathbf{C}) \mapsto \kappa(\mathbf{g}) = \\ &(g_1^{-1}, g_1 g_2^{-1} g_1^{-1}, \dots, g_1 \cdots g_{r-1} g_r^{-1} g_{r-1}^{-1} \cdots g_1^{-1}).\end{aligned} \tag{C.9}$$

For H a subgroup of G_F, let $F^{(H)}$ be the fixed field of H in $\bar{F}$.

Theorem C.3. *Assume G has no center. Choose an H_r orbit O on* $\mathrm{Ni}(G, \mathbf{C})$. *Denote the elements $\sigma \in G_{\mathbb{Q}}$ for which* (C.8b) *holds by H_O. Then, the moduli space component $\mathcal{H}^*$ of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ corresponding to O has the fixed field $\mathbb{Q}^{(H_O)}$ as its minimal field of definition.*

The analogous result holds if $\mathcal{H}^*$ is an absolutely irreducible component of $\mathcal{H}^{\mathrm{ab}}$ and if (3.5a) *or* (3.5b) *holds for the corresponding permutation representation $G \le S_n$. That is, $\boldsymbol{p} \in \mathcal{H}^*(\mathbb{Q}^{(H)})$ over $\boldsymbol{x}$ produces a regular field extension $L/\mathbb{Q}^{(H)}(x)$ of degree n with branch cycle description (relative to $(\bar{\Sigma}_1, \dots, \bar{\Sigma}_r)$) in the orbit O.*

Replace $\mathbb{Q}$ with $\mathbb{R}$ and $G_{\mathbb{Q}}$ by $G_{\mathbb{R}}$. Then, the results above apply. More explicitly, the following tests for these conditions hold.

(C.10a) (C.8b) *is true if and only if for some* $\mathbf{g} \in O$, $\kappa(\mathbf{g}) \in O$. *Similarly for the $\mathcal{H}^{\mathrm{ab}}$ version: Replace O by an H_r orbit in* $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{ab}}$.

(C.10b) *Points $\boldsymbol{p} \in \mathcal{H}^{\mathrm{in}}(\mathbb{R})$ over $\boldsymbol{x}_0$ correspond exactly to* $\mathbf{g} \in \mathrm{Ni}(G, \mathbf{C})$ *with* $\kappa(\mathbf{g}) = \gamma \mathbf{g} \gamma^{-1}$ *for some involution $\gamma \in G$.*

(C.10c) *If* (3.5a) *or* (3.5b) *holds, points $\boldsymbol{p} \in \mathcal{H}^{\mathrm{ab}}(\mathbb{R})$ over $\boldsymbol{x}_0$ correspond exactly to* $\mathbf{g} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{ab}}$ *with* $\kappa(\mathbf{g}) = \gamma \mathbf{g} \gamma^{-1}$ *for some involution $\gamma \in N_{S_n}(G)$.*

Proof. We've given references for everything excluding (C.10). Suppose $\psi : X \to \mathbb{P}^1$ has branch points $\boldsymbol{x}'$ (over $\boldsymbol{x}_0$ as above) and $\mathbf{g}$ as a description of branch cycles. [**DFr2**, Fig. 2] gives a set of paths $\boldsymbol{\Sigma} = (\bar{\Sigma}_1, \dots, \bar{\Sigma}_r)$on $\mathbb{P}^1 \setminus \{\boldsymbol{x}'\}$ with $\kappa(\mathbf{g})$ a description of branch cycles of $\psi^\kappa : X^\kappa \to \mathbb{P}^1$ relative to $\boldsymbol{\Sigma}$. Thus, there is a precise condition for complex conjugation to map points of $\mathcal{H}^*$ into $\mathcal{H}^*$. It is that κ maps branch cycle descriptions of covers corresponding to the

H_r orbit $O \subset \mathrm{Ni}(G, \mathbf{C})$ into O. From (C.8a) and (C.8b), this proves (C.10a). Replace these statements by $O \subset \mathrm{Ni}(G, \mathbf{C})^{\mathrm{ab}}$ for the $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{ab}}$ analog. Also, the κ operator commutes with the action of H_r, so we need only check this condition on one element of O.

Statement (C10.b) restates (***) of [**DFr**, §3.5, Comment 3]. Similarly, (C10c) restates [**DFr**, §3.5, Comment 2]. Both statements are addendum to [**DFr**, Th. 1.1]. For the continuing discussion below, we use the equivalent statements from replacing $\mathbf{g}$ by

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{r-1}) = (g_2 \cdots g_r, \dots, g_{r-1}g_r, g_r). \tag{C.11a}$$

[**DFr**, (2.3)] says γ of (C.10b) or (C.10c) conjugates generators $\boldsymbol{\alpha}$ of G into their inverses:

$$\gamma\boldsymbol{\alpha}\gamma^{-1} = (\alpha_1^{-1}, \dots, \alpha_{r-1}^{-1}). \qquad \square \tag{C.11b}$$

Note: Application of [**DFr**, §3.5, Comment 3], to get (C.10b) doesn't require G to have a trivial center. Also, suppose the Center Hypothesis holds and $\{\mathcal{H}_n = \mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})^{\mathrm{in}},\ n = 0, 1, \dots\}$ is a modular tower. Let $\{O_n,\ n = 0, 1, \dots\}$ be a compatible system of H_r orbits on the respective Nielsen classes $\mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$. It is convenient to choose these through a compatible system of representatives ${}^n_p\tilde{\mathbf{g}} \in \mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$. That is, ${}^n_p\tilde{\mathbf{g}} \mapsto {}^{n-1}_p\tilde{\mathbf{g}}$ through the natural map of §III.C. Here is the combinatorial check for $\mathcal{H}_{O_n}$ giving a projective system of (absolutely irreducible) Hurwitz space components over $\mathbb{R}$.

(C.12a) There is $Q_n \in H_r$ with $\kappa({}^n_p\tilde{\mathbf{g}}) = ({}^n_p\tilde{\mathbf{g}})Q_n$, $n = 0, 1, \dots$.

Here is the corresponding statement with $\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})^{\mathrm{ab}}$ replacing $\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})^{\mathrm{in}}$.

(C.12b) There is $Q_n \in H_r$, $\gamma_n \in N_{S_{k(n)}}({}^n_p\tilde{G})$, with $\kappa({}^n_p\tilde{\mathbf{g}}) = \gamma_n({}^n_p\tilde{\mathbf{g}})Q_n\gamma_n^{-1}$, $n = 0, 1, \dots$.

PROBLEM C.4. *Suppose $(G, p, \mathbf{C})$ has G satisfying the center hypothesis, and $\mathbf{C}$ is a set of p' conjugacy classes. Give a procedure for checking existence of ${}^n_p\tilde{\mathbf{g}}$, $n = 0, 1, \dots$ satisfying* (C.12a) *(or* (C.12b)*).*

COMMENTS C.5. *Relating* Th. C.3 *to the Ihara-Matsumoto section.* With the center hypothesis in force, apply Th. C.3 to the modular tower $\mathcal{H}_n = \mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$, $n = 0, 1, \dots$ (inner or absolute version). Consider the total family $\mathcal{T}_n \to \mathcal{H}_n \times \mathbb{P}^1$ (as in (3.22)) attached to the moduli space $\mathcal{H}_n$. The (ramified) cover $\mathbb{P}^r \times \mathbb{P}^1$ restricts over the open set $\mathbb{P}^{r+1} \setminus D_{r+1}$ to give the (unramified) cover $U_{r+1} \to \mathbb{P}^{r+1} \setminus D_{r+1}$. Restrict $\mathcal{T}_n$ over U_{r+1} to give unramified covers

$$\mathcal{T}_n^0 \xrightarrow{\Phi_n} (\mathcal{H}_n \times \mathbb{P}^1)^0 \to U_{r+1}. \tag{C.13}$$

Here $(\mathcal{H}_n \times \mathbb{P}^1)^0$ is a natural affine subvariety of $\mathcal{H}_n \times \mathbb{P}^1$ from removing the loci $(\boldsymbol{p}, \boldsymbol{x})$ where $\boldsymbol{x}$ is the branch point set of a cover representing $\boldsymbol{p} \in \mathcal{H}_n$.

Take $\boldsymbol{x}_0$, the image of $\boldsymbol{x}' \in (\mathbb{P}^1(\mathbb{R}))^{r+1} \setminus \Delta_{r+1}$ as in (C.2) with $s = 0$ and $r+1$ replacing r. Restrict covers over the locus $(\boldsymbol{x}_0, x)$ (x runs over $\mathbb{P}^1 \setminus \{x'_2, \dots, x'_{r+1}\}$).

Then, (C.5a) gives words in $Q_1, \dots, Q_r$ expressing convenient paths (based at x'_1) for generators of $\pi_1^* = \pi_1(\mathbb{P}^1 \setminus \{x'_2, \dots, x'_{r+1}\})$. Let β be the homomorphism $\pi_1^* \to S_n$ sending the r-tuple of (C.5a) to the coordinates of $\mathbf{g} \in \mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$. This extends $\pi_1(\mathbb{P}^r \setminus D_r)$ acting on $\mathrm{Ni}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$. Thus, β maps r-tuple (C.5b) to $\kappa(\mathbf{g})$. This expresses the compatibility of the choice of paths in [**DFr2**, Fig. 2] to those of (C.5a).

These topological considerations defining κ don't depend on $\boldsymbol{x}'$ if we stay in the simply connected set given by (C.2) (with $s = 0$ and $r+1$ replacing r). Consider, however, a *general* $\sigma \in G_{\mathbb{Q}}$. The fixed field $\bar{\mathbb{Q}}^{(\sigma)}$ of σ supports no ordering that allows an analog of a simply connected set like (C.2). (Here's one way to see that. For a measure one subset of $G_{\mathbb{Q}}$, $\bar{\mathbb{Q}}^{(\sigma)}$ is a PAC field [**FrJ**, Th. 16.47]. Then, the Frey-Prestel Theorem shows any Henselian closure of a characteristic 0 PAC field is algebraically closed [**FrJ**, Th. 10.14].) So, it behooves us to find a way to preserve the result for complex conjugation that gives a valuable extension to all $\sigma \in G_{\mathbb{Q}}$. A limit $\boldsymbol{x}' \to (x'_1, \dots, x'_1)$ in the set (C.2) preserves the construction above. Here $(x'_1, \dots, x'_1)$ lies in the deepest part of the fat diagonal. We can extend local algebraic functions on an étale cover to a neighborhood of a chosen lift of the simply connected space of (C.2). Thus, we may consider their Puiseux expansions around this point. These expansions will have fractional exponents in local parameters for the point $(x'_1, \dots, x'_1)$, so there will be several choices for the expansion. [**I**] uses data for a direction from $x'_i \mapsto x'_j$ to normalize these functions to be positive in their values along this direction.

The point $(x'_1, \dots, x'_1)$ alone, doesn't support this extra information. Thus [**IM**] extends the Puiseux expansions to a point $\boldsymbol{x}''$ on a partial blow-up of the point $(x'_1, \dots, x'_1)$. An encoding of the direction from the i-th coordinate to the $i+1$-th coordinate, $i = 1, \dots, r+1$, appears algebraically in the ring structure of this point. This data supports a proof of the Ihara-Drinfield relations.

For a general $\sigma \in G_{\mathbb{Q}}$, the Drinfeld-Ihara action on $\mathbb{P}^{r+1} \setminus D_{r+1}$ maps the r-tuple $(Q_1, \dots, Q_r)$ to

$$A_\sigma(Q_1, \dots, Q_r) \stackrel{\mathrm{def}}{=} (f_{1,\sigma} Q_1^{\psi(\sigma)} f_{1,\sigma}^{-1}, \dots, f_{r,\sigma} Q_r^{\psi(\sigma)} f_{r,\sigma}^{-1}).$$

Here, ψ is the cyclotomic character (discussion before Th. 3.16). Further, suppose we fix $\lambda = \psi(\sigma)$. Then, A_σ depends only on a single profinite word f. That is $f_{i,\sigma} = f(x_i, y_i)$, $i = 1, \dots, r$. Here, (x_i, y_i) are elements of H_r, independent of σ and $f = f_\sigma$ lies in the commutator subgroup of the free group $\hat{F}_2$ on two generators. [**IM**, Appendix] lists explicit properties of f_σ. The conjecture on these relations is that this list exactly characterizes the words f that arise as an f_σ.

Thus, fixing λ, test Ihara-Drinfeld by running through all such allowable fs giving transformations A_f (notationally replacing A_σ). To detect fields of definition of components of the spaces $\mathcal{H}_n$, apply A_f to the elements of (C.5a). Reexpress the result as an r-tuple of words in the r-tuple of (C.5a). This produces an analog κ_f of the κ operator of expression (C.5b) and (C.9). Finally, for all levels of the modular tower $\mathcal{H}_n$, $n = 0, 1, \ldots$, apply κ_f to check (C.8b). For example, checking this action against Galois Invariance Th. 3.16 gives information on the $\widehat{\mathcal{G}T}$ conjectures. Is this really possible to carry out? Part IV looks at examples. □

PROBLEM C.6. *Use the computations of* [**DFr**] *and* [**DFr2**] *to give an analog to* Th. C.3 *for any real point* $\boldsymbol{x}_0 \in \mathbb{P}^r \setminus D_r$. *That is, instead of* (C.3), *let* $\boldsymbol{x}_0$ *be the image of* $\boldsymbol{x}' \in (\mathbb{P}^1)^r$ *with some coordinates real and others in complex conjugate pairs as in* (C.2).

§**APP.D** $\mathbb{R}$ **points on modular towers.** For perspective, reconsider Problem D_2 of §I.B. This is the most important diophantine problem applicable to modular towers at this stage. Assume $G \le \hat{G} \le S_n$ are transitive subgroups with G normal in $\hat{G}$ and G has no center. As in (3.9), suppose $\mathbf{C}$, a rational union of conjugacy classes of G, has its entries from the distinct conjugacy classes $\mathrm{C}'_1, \ldots, \mathrm{C}'_t$. Analogous to Problem B of §I.D, here are necessary conditions for a $(G, \hat{G}, \mathbf{C})$-realization over $\mathbb{Q}$.

(D.1a) No element of $\hat{G} \setminus G$ centralizes G [**Fr5**, Prop. 2].
(D.1b) $\hat{G} \le N_{S_n}(\mathbf{C})$ (as in Def. 3.1a).

PROBLEM D.1. *Assume* (D.1) *and* $N > 0$. *Is there* $\mathbf{C}$ *with* C'_i *appearing at least* N *times in* $\mathbf{C}$, $i = 1, \ldots, t$, *where* $\mathcal{H}^{\mathrm{in}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ *(resp.,* $\mathcal{H}^{\mathrm{ab}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$*) has* $\mathbb{Q}$ *points? Continue these assumptions. Are there* $\mathbb{Q}$ *points of* $\mathcal{H}^{\mathrm{ab}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ *where* $\hat{G}/G$ *is the decomposition group in the natural cover*

$$\mathcal{H}^{\mathrm{in}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}}) \to \mathcal{H}^{\mathrm{ab}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})?$$

A positive answer to Problem D.1 gives $(G, \hat{G}, \mathbf{C})$-realizations under the hypotheses above (application of Theorem 3.4 following [**FrV**]). (In particular, a $\mathbb{Q}$ point on $\mathcal{H}^{\mathrm{in}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ gives a regular realization of G over $\mathbb{Q}$.) For example, the $(D_p, \mathbb{A}_p, \mathbf{C}_{p+1})$-realization Lemma of §I.D produces $\mathbb{Q}$ points on $\mathcal{H}(D_p, \mathbf{C}_{p+1})^{\mathrm{ab}}$ for each odd prime p. It shouldn't be hard to improve this to give $(D_{p^n}, \mathbb{A}_{p^n}, \mathbf{C}_{p^n+1})$-realizations. Since D_p is a quotient of D_{p^n}, taking n large automatically produces $(D_p, \mathbb{A}_p, \mathbf{C}_r)$-realizations with r arbitrarily large. This is for the case of fixed D_p, and involutions as conjugacy classes. No one has yet found $\mathbb{Q}$ points on $\mathcal{H}(D_p, \mathbf{C}_r)^{\mathrm{in}}$ (Problem A of §I.D) for each prime p (r depends on p). This is the analog for hyperelliptic curves (of genus $(r-2)/2$) for $\mathbb{Q}$ non-cusp points on $X_1(p)$. §App.E continues these problems over $\mathbb{Q}$ as they apply to modular towers.

Turn now to existence of (a projective system of) real points on a modular tower. Assume G satisfies the Center Hypothesis. Restrict, as in §App.C, to points over $\boldsymbol{x}_0 \in \mathbb{P}^r \setminus D_r$ with $\boldsymbol{x}_0$ the image of an r-tuple with all coordinates real. Apply (C.10b) and (C.10c) to rephrase existence of real points on modular towers. For the absolute version, assume we have transitive representations ${}^n_p\tilde{G} \le S_{k(n)}$. The following joins modular towers with [**DFr**, Th. 1.1]. The involutions $\boldsymbol{\beta}$ in the next lemma arise from (C.11b) by taking $\beta_0 = \gamma$ and $\beta_i = \gamma\alpha_i$, $i = 1, \ldots, r{-}1$.

Real Projective System Lemma D.2. *There exists a projective system* $\{\boldsymbol{p}_n \in \mathcal{H}_n({}_p^n\tilde{G}, {}_p^n\tilde{\mathbf{C}})^{\mathrm{in}}(\mathbb{R})\}_{n=0,1,\ldots}$ *(over* $\boldsymbol{x}_0$*) if and only if there exists a projective system* ${}_p^n\mathbf{g} \in \mathrm{Ni}({}_p^n\tilde{G}, {}_p^n\tilde{\mathbf{C}})$ *with*

(D.2a) $\kappa({}_p^n\mathbf{g}) = \gamma_n {}_p^n\mathbf{g}\gamma_n^{-1}$ for some involution $\gamma_n \in {}_p^n\tilde{G}$, $n = 0, 1, \ldots$.

This is equivalent to a projective system of r*-tuples of involutions*

(D.2b) $\boldsymbol{\beta}_n \in ({}_p^n\tilde{G})^r$ with $\boldsymbol{\beta}_n$ generating ${}_p^n\tilde{G}$, $n = 0, 1, \ldots$.

Similarly, there exists a projective system $\{\boldsymbol{p}_n \in \mathcal{H}_n({}_p^n\tilde{G}, {}_p^n\tilde{\mathbf{C}})^{\mathrm{ab}}(\mathbb{R})\}_{n=0,1,\ldots}$ *if and only if there exists a projective system* ${}_p^n\mathbf{g} \in \mathrm{Ni}({}_p^n\tilde{G}, {}_p^n\tilde{\mathbf{C}})^{\mathrm{ab}}$ *with*

(D.2c) $\kappa({}_p^n\mathbf{g}) = \gamma_n {}_p^n\mathbf{g}\gamma_n^{-1}$ for some involution $\gamma_n \in N_{S_{k(n)}}({}_p^n\tilde{G})$, $n = 0, 1, \ldots$.

This is equivalent to a projective system of $r{-}1$*-tuples* $\boldsymbol{\alpha}_n \in ({}_p^n\tilde{G})^{r-1}$ *and an automorphism* h_n *of* ${}_p^n\tilde{G}$ *with these properties.*

(D.2d)
$$\boldsymbol{\alpha}_n = (\alpha_{1,n}, \ldots, \alpha_{r-1,n}) \text{ generates } {}_p^n\tilde{G}, \text{ and}$$
$$h(\alpha_i) = \alpha_i^{-1},\ i = 1, \ldots, r{-}1,\ n = 0, 1, \ldots.$$

Suppose p *is odd and* $\boldsymbol{\beta}_0 \in (G)^{r-1}$ *exists satisfying* (D.2b) *(involutions exist at level 0). Then, there is a complete projective system satisfying* (D.2b)*. If, however,* $p = 2$*, then no such system as in* (D.2b) *exists.*

The Center Hypothesis assures a γ_n in (D.2b) is an involution. Otherwise its square would centralize ${}_p^n\tilde{G}$ contrary to statement (3.4d) of Th. 3.4. The version of this in [**DFr**, Th. 1.1] considers the case the Center Hypothesis doesn't hold. There are two natural questions for considering Lemma D.2 further.

Lifting Problem D.3. *Assume* G *satisfies the Center Hypothesis.*

(D.3a) *Suppose a specific* r*-tuple* $\boldsymbol{\beta}$ *of involutions generate* G*. Characterize quotients* H *of* ${}_2\tilde{G}$ *that have* r *involutions that map respectively to* $\boldsymbol{\beta}$*.*

Similarly, suppose $(G^* : G) = 2$ *and there are involutions* $\boldsymbol{\beta}^*$ *from* $G^* \setminus G$ *generating* G^**. Characterize quotients* H *of* $\tilde{G}$ *with the following property.*

(D.3b) H *is an index 2 normal subgroup of a group* H^* *covering* G^* *where* H^* *has generating involutions* $\boldsymbol{\beta}_H^*$ *that lift* $\boldsymbol{\beta}^*$*.*

For (D.3a), $\boldsymbol{\beta}$ can't lift all the way to ${}_2\tilde{G}$. For this, [**DFr**, Lemma 1.3] uses that $\tilde{G}$ is projective (§II.A). It thus contains no involutions (or any element of finite order). We don't know when there is a *maximal* quotient H of ${}_2\tilde{G}$ fulfilling the conclusion of (D.3a). Involutions generate S_n for any integer n. So, answering (D.3a) for S_n and some set of generating 2-cycles is a valuable test.

Now consider (D.3b). Here we don't even know when $H = \tilde{G}$ provides a positive answer to this question. The hypothesis that G has no center excludes the case G is a p-group. In that case, however, its universal frattini cover is a free pro-p-group. Thus, we can lift $\boldsymbol{\alpha}$ and γ in (C.11.b) to $\tilde{\boldsymbol{\alpha}} \in (\tilde{G})^{r-1}$ and $\tilde{\gamma}$, an automorphism of $\tilde{G}$, for which $\tilde{\gamma}\tilde{\alpha}_i\tilde{\gamma}^{-1} = \tilde{\alpha}_i^{-1}$, $i = 1, \ldots, r-1$. The key to (D.3b) is the universal *real* frattini cover $R(\tilde{H}^*) \xrightarrow{\psi} H^*$ of H^* (relative to $\boldsymbol{\beta}_H$). We explain this briefly as it will be the subject of [**Fr9**].

Definition D.4. *Real frattini covers.* Let $(\mathrm{C}_1, \ldots, \mathrm{C}_r) = \mathbf{C}$ be r conjugacy classes of involutions of G. Denote the intersection of all maximal subgroups of G meeting each conjugacy class of $\mathbf{C}$ by $\mathrm{RF}(G) = \mathrm{RF}(G, \mathbf{C})$. This is the *real frattini* subgroup of G with respect to $\mathbf{C}$. Suppose $\phi : K \to G$ is a cover of groups with r conjugacy classes of involutions $(\mathrm{C}_1^K, \ldots, \mathrm{C}_r^K)$ mapping respectively to $(\mathrm{C}_1, \ldots, \mathrm{C}_r)$. Then, ϕ is a real frattini cover if its kernel is in $\mathrm{RF}(K, \mathbf{C}^K)$.

[**HJ**] discusses *real projective* groups. Using these, the analog of the §II.A discussion for the universal frattini cover gives this result. There is a *universal real frattini* cover $R(\tilde{G})$ of G: It is universal for all such real frattini covers (relative to $\mathbf{C}$). [**H**] motivates calling this the universal *Artin-Schreier* cover of G (relative to $\mathbf{C}$). Unlike the frattini cover case, the kernel ker of the natural cover $R(\tilde{G}) \xrightarrow{\psi} G$ may not be nilpotent. Arguments, however, similar to those in [**H**] show the following.

(D.4a) ker has a unique profinite 2-sylow, P_2, normal in $R(\tilde{G})$.
(D.4b) $\ker / P_2$ is a nilpotent subgroup of $R(\tilde{G})/P_2$.
(D.4c) If G_1 is a subgroup of G that meets none of the conjugacy classes of $\mathbf{C}$, then $\psi^{-1}(G_1)$ is a projective subgroup of $R(\tilde{G})$.

Apply this to the hypotheses from (D.3b). Conclude the pullback of $G \le G^*$ in $R(\tilde{G}^*)$ is a projective group. Call this $\tilde{G}^*$. By assumption there are generators $\tilde{\boldsymbol{\alpha}}^*$ of $\tilde{G}^*$ and an involution $\tilde{\gamma}^*$ of $R(\tilde{G}^*)$ satisfying (C.11b). As $\tilde{G}^* \to G$ is a cover by a projective group, this factors through the universal frattini cover $\tilde{G} \to G$. Inspecting the action of γ^* on the kernel of the surjective induced map $\tilde{G}^* \to \tilde{G}$ gives an effective approach to (D.3b). This motivates considering universal group covers beyond the universal frattini cover. Finally, the next example shows the universal real frattini cover of a finite group may not be a frattini cover.

Example D.5. *The universal real frattini cover of* $G = \mathbb{Z}/3 \times \mathbb{Z}/2$. There is one class of involutions, consisting of $(0,1) = \tau$. The universal frattini cover $\tilde{G}$ of G is $\mathbb{Z}_3 \times \mathbb{Z}_2$. The universal real frattini cover $R(\tilde{G})$ of G (relative to $\{\tau\}$) has generators an involution and an element of infinite order generating a group isomorphic to $\mathbb{Z}_3$. Call these τ^* and α^*, lying above τ and $(1,0) = \alpha$, respectively. As previously, denote the kernel of $R(\tilde{G}) \xrightarrow{\psi} G$ by ker. Let $\mathcal{D}$ be the group freely generated by an involution τ' and an element α' of infinite order. Form the profinite completion $\tilde{\mathcal{D}}$ of $\mathcal{D}$ using all subgroups of finite index. Inside this group, α' generates a subgroup isomorphic to $\hat{\mathbb{Z}} = \mathbb{Z}_3 \times \prod_{p \neq 3} \mathbb{Z}_p$.

Then, the universal real frattini cover of G is isomorphic to the closed subgroup of $\tilde{\mathcal{D}}$ generated by the image of α' in $\mathbb{Z}_3$ and τ'. This is clearly a nonabelian group. So, it isn't a quotient of the universal frattini cover of G. R. Guralnick gives the following explicit example quotient K of $R(\tilde{G})$. It has exactly one conjugacy class of involutions mapping to the involution of G.

Let S be the group freely generated by x_i, $1 \le i \le 3$, subject to these relations: $x_i^2 = 1$, $[x_i, [x_j, x_k]] = 1$. That is, the commutator subgroup is central. Therefore, $C = [S, S] = Z(S)$ is an elementary abelian of order 8 with basis $y_{i\,j} = [x_i, x_j]$, $1 \le i < j \le 3$. The map z of S which permutes the x_1, x_2, x_3 cyclically preserves all relations. So, it defines an automorphism of S. Write C as $C_1 \times C_2$ with these properties.

(D.5a) C_1 is a 4-group and C_2 has order 2.

(D.5b) $\langle S, z\rangle$ centralizes C_2 and z acts nontrivially on C_1.

Denote S/C_1 by M and let K be $M \times^s \langle z\rangle$. Thus K maps to G by sending z to α and x_1 to τ. Automatically, x_i maps to τ for each i. Note: M has commutator subgroup N of order 2. Let w generate this. We list elements that map to τ: x_i, $x_i w$, $x_1x_2x_3$ and $x_1x_2x_3w$. The last two have order 4 (their square is w), so there are 6 involutions mapping to τ. The x_i s are all conjugate as are the $x_i w$ s. On the other hand, $x_j x_i x_j = x_i w$ for $i \ne j$, so all six involutions are conjugate. There is a unique class of involutions mapping to τ. □

Here is a harder example. Let $G = A_n$ and $G^* = S_n$. Take $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{n-2})$ equal to $((1\,2\,3), (1\,2\,4), \dots, (1\,2\,n))$. Then, $\boldsymbol{\alpha}$ generates A_n and $\gamma = (1\,2) \in G^*$ conjugates the α_i s to their inverses. That is, we have the condition of (C.11b).

Problem D.6. *For $n \ge 5$, let $\tilde{A}_n$ be the universal frattini cover of A_n. Is there a lift of $\boldsymbol{\alpha}$ to $\tilde{\boldsymbol{\alpha}} \in (\tilde{A}_n)^{n-2}$ and an automorphism h of $\tilde{A}_n$ with $h(\tilde{\alpha}_i) = \tilde{\alpha}_i^{-1}$, $i = 1, \dots, n-2$?*

We don't know the answer to Prob. D.5 even when $n = 5$. Nor, to compare with Prop. 2.9, do we know the universal real frattini cover of S_5.

§APP.E. Groups as regular extensions over finite fields. This brief subsection reminds of the most significant diophantine problems. This includes how Thm. 3.21 applies to explicitly bound exceptional primes for the regular realization of a given group over $\mathbb{F}_p(x)$. The essential hypothesis uses a a gcomplete collection of rational conjugacy classes from G (§III.E).

Ex. 3.13 produces examples of modular towers obstructed at the first level. Counter to this, Cor. 3.23 gives hypothesis (3.17), on p' conjugacy classes $\mathbf{C}$, which assures some H_r orbit of $\mathrm{Ni}(G, \mathbf{C})$ is unobstructed at every level. The center hypothesis is irrelevant for this question. In particular, if G is p-gcomplete, there are many choices of $\mathbf{C}$ for which $\mathrm{Ni}({}_p^n\tilde{G}, {}_p^n\tilde{\mathbf{C}})$ is nonempty at every level n. Characterizing those $(G, \mathbf{C})$ unobstructed at every level is a significant problem requiring careful formulation.

Problem E.1. *Suppose G satisfies the center hypothesis, $p \mid |G|$, and $\mathbf{C}$ are p' conjugacy classes of G. Assume $\mathrm{Ni}(G, \mathbf{C})$ is unobstructed at every level. Also,*

$\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ has a $\mathbb{Q}$ absolutely irreducible component at every level. Does there exist an integer n for which $\mathcal{H}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ has no $\mathbb{Q}$ points?

PROBLEM E.2. *Suppose given G (a transitive subgroup of S_k), p, n. Let $\mathbf{C}'$ be a fixed union of p' conjugacy classes. These are the only conjugacy classes we allow to appear in $\mathbf{C}$. Assume each element of $\mathbf{C}'$ appears at least a times in $\mathbf{C}$. Is there an a (dependent on G, n and $\mathbf{C}'$) for which $\mathcal{H}^{\mathrm{ab}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$ is $\mathbb{C}$-unirational? There is a similar question for $\mathcal{H}^{\mathrm{in}}({}^n_p\tilde{G}, {}^n_p\tilde{\mathbf{C}})$.*

The remainder of this subsection applies Th. 3.21 to bound exceptional primes for groups as Galois groups of regular extensions of $\mathbb{F}_p(x)$. The proof is an outline because we apply **[Fu]** for a statement of good reduction of Hurwitz spaces modulo primes not dividing $|G|$. In fact, Fulton didn't have the Nielsen class concept and proved this only for simple branching. A later paper will show this hypothesis in the following form.

REDUCTION HYPOTHESIS E.3. *Assume G has no center and p is a prime not dividing $|G|$. Let $G \le S_n$ be an embedding satisfying* (3.5a) *or* (3.5b). *Let $\mathbf{C}$ be a rational union of conjugacy classes of G. Then, the natural map $\Psi : \mathcal{H}(G, \mathbf{C})^{\mathrm{in}} \to \mathcal{H}(G, \mathbf{C})^{\mathrm{ab}}$ (before Lemma 3.3) reduces modulo p to produce a cover of smooth affine varieties over $\mathbb{F}_p$. Denote this $\Psi_p : \mathcal{H}(G, \mathbf{C})^{\mathrm{in}}_p \to \mathcal{H}(G, \mathbf{C})^{\mathrm{ab}}_p$. Each G realization over $\mathbb{F}_p$ corresponds to a point of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}_p(\mathbb{F}_p)$. Further, absolutely irreducible components of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ (resp., $\mathcal{H}(G, \mathbf{C})^{\mathrm{ab}}$) correspond one-one to absolutely absolutely irreducible components of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}_p$ (resp., $\mathcal{H}(G, \mathbf{C})^{\mathrm{ab}}_p$).*

PROPOSITION E.4. *Suppose G has no center and $\mathbf{C}$ satisfies the hypotheses of (3.17). Especially, $\mathrm{Ni}(G, \mathbf{C})$ is an HM-Nielsen class for which $\mathbf{C}$ is HM-gcomplete and a rational union of conjugacy classes. Then, there is a constant $A = A(\mathrm{Ni}(G, \mathbf{C}))$ for which the following holds. For any p exceeding A, $G = G(\hat{L}/\mathbb{F}(x))$ with $\hat{L}$ a regular extension of $\mathbb{F}_p(x)$.*

More generally, for any finite group G, we may explicitly compute a finite covering group G' with no center for which there is an r' and $\mathbf{C}'$ satisfying (3.17). A regular realization of G' over $\mathbb{F}_p(x)$ automatically produces a regular realization of G over $\mathbb{F}_p(x)$. Thus, the statement above holds for p exceeding $A(\mathrm{Ni}(G', \mathbf{C}'))$.

PROOF. Apply Hypoth. E.3 in the case G is in S_n as its regular representation. For p not dividing the order of G, we have only to verify there is a $\mathbb{F}_p$ rational point on $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}_p$. From Cor. 3.21, this variety has a $\mathbb{F}_p$ absolutely irreducible component $\mathcal{H}(G, \mathbf{C})'_p$.

The cover $\mathcal{H}(G, \mathbf{C})'_p \to \mathbb{P}^r \setminus D_r$ is a finite morphism of affine varieties. Restrict over the affine subset $\mathbb{A}^r \setminus D_r$ from the embedding $\mathbb{A}^r \to \mathbb{P}^r$. Thus, $\mathcal{H}(G, \mathbf{C})'_p$ is birational to the zero set of a hypersurface V_p defined by $f(\boldsymbol{x}, y)_p = 0$. Further, the degree of f_p in y equals the degree of the cover $\mathcal{H}(G, \mathbf{C})'_p \to \mathbb{P}^r \setminus D_r$. Bound d by the cardinality of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$. Any non-singular $\mathbb{F}_p$ point of V_p corresponds to a $\mathbb{F}_p$ point of $\mathcal{H}(G, \mathbf{C})'_p$. Apply the Lang-Weil estimates to determine a A'

for which V_p has a nonsingular $\mathbb{F}_p$ point when p exceeds A'. [**FrHJ**, Prop. 3.3] gives the best estimates we know. This is in d, and r.

Apply this to a general group G as follows. Use the procedure of §III.B (before the Center Hypothesis 3.5) to explicitly cover G' by a group without a center. Comments before Th. 3.21 (similar to the constructions of [**FrV**, §3] bound the value the number r' of conjugacy classes you need to produce an HM-Nielsen class. $\square$

Apply Reduction Hypothesis E.3 to a modular tower. In particular, consider the tower attached to $(G, p, \mathbf{C})$ satisfying the hypotheses of Corollary 3.23. This produces a corresponding modular tower $\{\mathcal{H}({}_p^n\tilde{G}, {}_p^n\tilde{\mathbf{C}})_r,\ n = 0, 1, \dots\}$ in characteristic r. Here r is *any* prime not dividing the order of G. That is, this tower has all properties we expect of a modular tower in characteristic 0. From Cor. 3.23, this will also have a projective sequence of absolutely irreducible components over $\mathbb{F}_r$. This is suitable for versions of Problem B.3 that deal directly with all points on Hurwitz space components over finite fields.

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