Cohen-Macaulay Modules over Cohen-Macaulay Algebras

Yuri A. Drozd *

1 Introduction

This survay is devoted to three main subjects concerning tame and wild Cohen-Macaulay algebras (all of Krull dimension 1 and without nilpotent ideals):

1. Families of Cohen-Macaulay modules and "the number of parametres" in such families:
   (a) construction of "almost versal" families with projective bases;
   (b) definition of the "number of parametres";
   (c) semi-continuity of this number in families of algebras.
2. Tame and wild algebras:
   (a) definition via the "universality property";
   (b) definition via the number of parametres in families of modules;
   (c) tame/wild dichotomy and equivalence of definitions;
   (d) tame and wild algebras in families of algebras.
3. Criteria of tameness:

*Supported by Foundation for Fundamental Research of Ukraine and Deutsche Forschungsgemeinschaft.

(a) commutative case;

(b) case of 2×2 matrix rings.

For technical reasons we consider only the "geometrical situation", where all algebras are those over some base filed (usually supposed algebraically closed).

Most of these results have been obtained by G.-M. Greuel and the author. Here the proofs are mainly sketched (and sometimes omitted). For technical details we refer to [8]-[11].

2 Basic Definitions

2.1 Cohen-Macaulay algebras

Throughout this paper we use the following definitions and notations.

Definition 2.1.1 1. *Call a noetherian commutative ring* R Cohen-Macaulay ring *if all its localizations at prime ideals are Cohen-Macaulay local rings.*

2. *For a Cohen-Macaulay ring* R *denote* $\mathcal{CM}(\mathsf{R})$ *the category of* maximal Cohen-Macaulay R-modules M, *i.e. such (finitely generated) Cohen-Macaulay* R*-modules* M, *that* $\dim M_{\mathbf{m}} = \dim \mathsf{R}_{\mathbf{m}}$ *for each maximal ideal* $\mathbf{m} \subset \mathsf{R}$. *Later on we usually omit the word "maximal" in this description.*

3. *Call a ring* Λ Cohen-Macaulay algebra *if it is an algebra over some Cohen-Macaulay ring* R *and belong to* $\mathcal{CM}(\mathsf{R})$ *as* R*-module. To precise, if necessary, we call iit also* Cohen-Macaulay R-algebra.

4. *For a Cohen-Macaulay* R*-algebra* Λ *denote* $\mathcal{CM}(\Lambda)$ *the category of* (maximal) Cohen-Macaulay (left) Λ-modules, *i.e. those, which belong to* $\mathcal{CM}(\mathsf{R})$ *considered as* R*-modules.*

5. *We call a Cohen-Macaulay algebra (in particular, a Cohen-Macaulay ring)* reduced *if it contains no non-zero nilpotent ideals, and* analiticcally reduced *if, for each maximal ideal* $\mathbf{m} \subset \mathsf{R}$, *its* $\mathbf{m}$*-adique completion is also reduced.*

If a Cohen-Macaulay ring R is reduced, its *total ring of fractions* $\mathsf{Q} = \mathsf{Q}((\mathsf{R})$ is a direct product of fields and any Cohen-Macaulay R-algebra Λ is an *order* in the artinian Q-algebra $\mathsf{Q}\Lambda = \mathsf{Q} \otimes_{\mathsf{R}} \Lambda$. Moreover, if Λ is also reduced, then the algebra $\mathsf{Q}\Lambda$ is semi-simple.

Definition 2.1.2 *Let R be a reduced Cohen-Macaulay ring and Λ a Cohen-Macaulay R-algebra.*

1. *For a Cohen-Macaulay Λ-module $\mathcal{M}$ call its* rational length $\tilde{\ell}_\Lambda(M)$ *the length of the $\mathsf{Q}\Lambda$-module $\mathsf{Q}M = \mathsf{Q} \otimes_{\mathsf{R}} M$.*

2. *Call a Cohen-Macaulay Λ-module M* irreducible *if it is of rational length 1, i.e. the $\mathsf{Q}\Lambda$-module $\mathsf{Q}M$ is simple.*

3. *Call* over-ring *of Λ any Cohen-Macaulay R-algebra Γ such that $\Lambda \subseteq \Gamma \subseteq \mathsf{Q}\Lambda$. If Λ has no proper over-rings, it is called* maximal order *(or, if necessary,* maximal R-order.

4. *Denote* $\operatorname{crit}(\Lambda)$ *the set of all such prime ideals $\mathbf{p} \subset \mathsf{R}$ that $\Lambda_{\mathbf{p}}$ is not a maximal order (the* critical set *of Λ).*

Remind a criterion of boundness [7].

Proposition 2.1.3 *The following conditions are equivalent for a Cohen-Macaulay* [1] *R-algebra Λ:*

1. *Λ is bound.*

2. *The over-rings of Λ satisfy the maximality condition.*

3. *For each prime ideal $\mathbf{p} \subset \mathsf{R}$ of height 1 the $\mathsf{R}_{\mathbf{p}}$-algebra $\Lambda_{\mathbf{p}}$ is analytically reduced and the critical set* $\operatorname{crit}(\Lambda)$ *contains only finitely many prime ideals of height 1.*

We need also the following well-known relations between "global" and "local" Cohen-Macaulay modules and algebras as well as between "non-complete" and "complete" situation. (cf. [3],[20]).

[1] Really this coondition holds for a wider class of algebras than Cohen-Macaulay ones, but we do not need such generality.

Proposition 2.1.4 *Let* R *be a reduced Cohen-Macaulay ring of Krull dimension* 1 *or* 2*. Suppose given a Cohen-Macaulay* R*-module* M *and for each prime ideal* $\mathbf{p} \subset \mathsf{R}$ *of height* 1 *given a Cohen-Macaulay* $\mathsf{R}_{\mathbf{p}}$*-module* $N(\mathbf{p})$ *such that* $\mathsf{Q}N(\mathbf{p}) = \mathsf{Q}M_{\mathbf{p}}$ *for all* $\mathbf{p}$ *and* $N(\mathbf{p}) = M_{\mathbf{p}}$ *for* almost all $\mathbf{p}$ *(i.e. for all but a finite set). Then there exists a Cohen-Macaulay* R*-module* N *such that* $N_{\mathbf{p}} = N(\mathbf{p})$ *for all* $\mathbf{p}$ *(of height* 1*).*

Of course, if each of these given modules is really a Cohen-Macaulay algebra (or a module over such algebra), then the resulting module N is also Cohen-Macaulay algebra (or module over it).

Proposition 2.1.5 *Let* R *be a local analitically reduced Cohen-Macaulay ring,* Λ *be an analitically reduced Cohen-Macaulay* R*-algebra. Denote by* ˆ *the completions with respect to the unique maximal ideal of* R*. Then:*

1. *If* M *and* N *are such* Λ*-modules that* $\hat{M} \simeq \hat{N}$*, then* $M \simeq N$*.*

2. *If* N' *is a Cohen-Macaulay* $\hat{\Lambda}$*-module such that* $\mathsf{Q}N' \simeq \mathsf{Q}\hat{M}$ *for some Cohen-Macaulay* Λ*-module* M*, then there exists a Cohen-Macaulay* Λ*-module* N *such that* $N' \simeq \hat{N}$*.*

3. *If* $\hat{N}$ *is isomorphic to a direct summand of* $\hat{M}$*, then also* N *is isomorphic to a direct summand of* M*.*

Remind also the notions related to *ranks* of Cohen-Macaulay modules.

Definition 2.1.6 *Let* Λ *be a reduced Cohen-Macaulay algebra. Consider all pairwise non-isomorphic simple* $\mathsf{Q}\Lambda$*-modules:* $W_1, W_2, \dots, W_s$*.*

1. *For any Cohen-Macaulay* Λ*-module* M *let* $\mathsf{Q}M \simeq \oplus_{i=1}^{s} r_i W_i$*. Denote* $\mathbf{r}(M) = (r_1, r_2, \dots, r_s)$ *and call it* vector-rank *of* M*.*

2. *For any vector* $\mathbf{r} \in \mathbf{N}^s$ *denote* $\mathcal{CM}(\mathbf{r}, \Lambda)$ *the set of all Cohen-Macaulay* Λ*-modules of vector-rank* $\mathbf{r}$*.*

For a vector $\mathbf{r} = (r_1, r_2, \dots, r_s)$ denote $|\mathbf{r}| = \sum_{i=1}^{s} r_i$. Then, of course, $|\mathbf{r}(M)| = \tilde{\ell}_{\Lambda}(M)$.

2.2 Dense subrings

Definition 2.2.1 *Let A be a subring of some ring B. Call A* dense in B *if any simple B-module U is also simple as A-$\mathrm{End}_A U$-bimodule.*

The following technical lemma was the reason for this notion.

Lemma 2.2.2 *Let D be a skewfield, $B = \mathrm{Mat}\,(n\,,\,D)$ (the ring of $n \times n$ matrices with entries in D) and $A \subseteq B$ a dense subring. Consider the set $W = \mathrm{Mat}\,(m \times n\,,\,D)$ of $m \times n$ matrices with entries in D as B-module (in the natural way) and suppose that $V \subseteq W$ is such A-submodule in W that $BV = W$. Let $m = qn + r$ with $0 \leq r < n$. Then there exists a B-automorphism σ of W such that $\sigma(V)$ contains the matrices:*

$$E_1 = (\,E \quad 0 \quad \ldots \quad 0 \quad 0'\,)\,,\; E_2 = (\,0 \quad E \quad \ldots \quad 0 \quad 0'\,)\,,\; \ldots\,,$$
$$E_q = (\,0 \quad 0 \quad \ldots \quad E \quad 0'\,)$$

and a matrix of the form $(\,Y_1 \quad Y_2 \quad \ldots \quad Y_q \quad Y'\,)$, where

E denotes the $n \times n$ unit matrix,

0 denotes the $n \times n$ zero matrix,

$0'$ denotes the $n \times r$ zero matrix,

Y_i are some $n \times n$ matrices ($i = 1, 2, \ldots, q$),

Y' is an $n \times r$ matrix of rank r.

(Of course, if $r = 0$, then $0'$ and Y' are empty, so the assertion means that V contains a B-basis of W).

For the proof (consisting of some straightforward calculations) v. [10]. Later on, we use mainly the following corollary of this lemma.

Corollary 2.2.3 *Let $A \subseteq B$ be a dense subring. Suppose that $\overline{B} = B/\mathrm{rad}\,B$ is an artinian ring and $\overline{A} = A/(A \cap \mathrm{rad}\,B)$ contains an infinite central subfield F of $\overline{B}$ (e.g., this is the case if B is an artinian F-algebra and A its subalgebra). Let $V \subseteq nB$ be a B-submodule such that $BV = nB$. Then V contains a B-basis of nB.*

Proof. Of course, we may suppose that $B = \overline{B}$ and $A = \overline{A}$, so $B = \prod_{i=1}^{s} B_i$ with B_i simple artinian. Put $A_i = \mathrm{pr}_i(A)$ and $V_i = \mathrm{pr}_i(V)$, where pr_i is the projection of B onto B_i. Then A_i is dense in B_i and $B_iV_i = nB_i$, hence, each V_i contains a B_i-basis $\{ b_{ij} \mid j = 1, 2, \dots, n \}$ of nB_i. Choose $e_{ij} \in V_i$ such that $\mathrm{pr}_i(e_{ij}) = b_{ij}$ and consider, for each n-tuple $(\lambda_1, \lambda_2, \dots, \lambda_n) \in nF$, the elements $e_j(\lambda_1, \lambda_2, \dots, \lambda_n) = \sum_{i=1}^{s} \lambda_i e_{ij} \in V$. The sets $T_i = \{ (\lambda_1, \lambda_2, \dots, \lambda_n) \mid \mathrm{pr}_i e_j(\lambda_1, \lambda_2, \dots, \lambda_n) \text{ form a basis of } n_iB_i \}$ are Zariski open in nF and non-empty. As F is infinite, their intersection is also non-empty **q.e.d.**

The main examples (for our purpose) of dense subrings are obtained in the following way.

Definition 2.2.4 *1. Let D be a skewfield, $B = \mathrm{Mat}\,(n\,,\,D)$, $U = nD$ (considered as B-module) and $\mathcal{F} = \{ U = U_0 \supset U_1 \supset \dots \supset U_s = 0 \}$ be a flag of D-subspaces in U. Put*

$$\mathcal{E}(\mathcal{F}) = \{ a \in B \mid aU_i \subseteq U_i \text{ for all } i \}$$

and call it a flag subalgebra *in B.*

2. If $B = \prod_k B_k$ with all B_k simple artinian, call flag subalgebra *in B any subalgebra of the form $\mathcal{E} = \prod_k \mathcal{E}_k$, where each $\mathcal{E}_k$ is a flag subalgebra in B_k.*

The following quite obvious "existence lemma" enables us to construct dense subrings.

Lemma 2.2.5 *Let B be a semi-simple artinian ring, $\mathcal{E}$ a flag subalgebra in B and $A \subseteq \mathcal{E}$ a subring. Then there exists a flag subalgebra $\mathcal{E}'$ such that $A \subseteq \mathcal{E}' \subseteq \mathcal{E}$ and A is dense in $\mathcal{E}'$.*

Flag algebras are closely related to *hereditary orders*, i.e. hereditary Cohen-Macaulay algebras of Krull dimension 1. Namely, one has the following description of the hereditary orders in local case (cf. [20]).

Proposition 2.2.6 *Let R be a reduced local Cohen-Macaulay ring of Krull dimension* 1*, Ω a maximal R-order. Then:*

1. *Ω is hereditary.*

2. *If $\mathcal{E}$ is any flag subalgebra in $\Omega/\mathrm{rad}\,\Omega$, then its pre-image in Ω is also hereditary.*

3. *Any hereditary R-order can be obtained in this way.*

Taking into account Proposition 2.1.4, we get the following corollary.

Corollary 2.2.7 *Let Λ be a bound Cohen-Macaulay algebra of Krull dimension 1 and Ω be its hereditary (e.g. maximal) over-ring. Then there exists a hereditary over-ring Ω' such that $\Lambda \subseteq \Omega' \subseteq \Omega$ and Λ is dense in Ω'.*

We need also the following simple result concerning extensions of the base field.

Proposition 2.2.8 *Let B be an algebra over a separably closed field F such that $B/\mathrm{rad}\,B$ is finite-dimensional, A be a dense subalgebra of B and K a separably generated extension of F. Then $A \otimes_F K$ is also dense in $B \otimes_F K$.*

3 Families of Modules

3.1 Sandwiched families

From now on let R be an analitically reduced Cohen-Macaulay ring of Krull dimension 1, which is an algebra over a field k such that all residue fields $\mathsf{R}/\mathbf{m}$, $\mathbf{m}$ being maximal ideals, are finite-dimensional over k. Moreover, in this section we suppose the ring R to be *local*. Fix an analitically reduced Cohen-Macaulay R-algebra Λ. For an algebraic k-scheme X with structure sheaf $\mathcal{O}_x = \mathcal{O}$, denote $\Lambda_X = \Lambda \otimes_{\mathsf{k}} \mathcal{O}$ and $\Lambda(x) = \Lambda \otimes_{\mathsf{k}} \mathsf{k}(x)$, x being a point of X and $\mathsf{k}(x)$ its residue field.

Definition 3.1.1 *Let X be a k-scheme and $\mathcal{M}$ a coherent sheaf of Λ_X-modules. Call $\mathcal{M}$* family *of Cohen-Macaulay Λ-modules on X (or with the base X) if the following conditions hold:*

1. *$\mathcal{M}$ is torsion-free over R.*

2. *$\mathcal{M}$ is flat over $\mathcal{O}$.*

3. *For each non-zero-divisor* $a \in \mathsf{R}$ *the sheaf* $\mathcal{M}/a\mathcal{M}$ *is also flat over* $\mathcal{O}$ *.*

It is easy to see that, under conditions *1* and *2,* the condition *3* is equivalent to the following one:

3′. For each point $x \in X$ *, the* $\Lambda(x)$*-module* $\mathcal{M}(x) = \mathcal{M} \otimes_{\mathcal{O}} \mathsf{k}(x)$ *is Cohen-Macaulay one.*

Call the dimension $\operatorname{dim} X$ *dimension* of this family.

Among the families of Cohen-Macaulay modules, the following class will be of special interest for us.

Definition 3.1.2 *Let* Γ *be an over-ring of* Λ *. Call a coherent sheaf* $\mathcal{M}$ *of* Λ_X*-modules modules* sandwiched family (with respect to Γ and with the base X), *if:*

1. $n\Lambda_X \subseteq \mathcal{M} \subseteq n\Gamma_X$ *for some* n *(called* the rank *of* $\mathcal{M}$ *).*
2. *The* $\mathcal{O}_X$*-sheaf* $n\Gamma_X/\mathcal{M}$ *is locally free of some constant rank* d *(called* the codimension *of* $\mathcal{M}$ *).*

One can easily see that each sandwiched family is really a family of Λ-modules with the base X . The following rather simple fact shows the role of sandwiched families.

Proposition 3.1.3 *Suppose* k *to be separably closed and all fields* $\mathsf{k}(x)$ *for* $x \in X$ *separably generated over* k *(e.g.* k *algebraically closed). Let* Γ *be an over-ring of* Λ *such that* Λ *is dense in* Γ *and* $\mathcal{M}$ *a family of Cohen-Macaulay* Λ*-modules on* X *such that* $\Gamma\mathcal{M}$ *is flat over* Γ_X *. Then there exist an open dense subscheme* $Y \subseteq X$ *and a projective* Γ*-module* P *such that* $\mathcal{M}' = \mathcal{M} \oplus (P \otimes_{\mathsf{k}} \mathcal{O}_X)$ *is isomorphic to a sandwiched family with respect to* Γ *. Moreovere, one can choose* P *in such a way that the rank of* $\mathcal{M}'$ *were not greater than* $\max\left\{ \tilde{\ell}_\Lambda \mathcal{M}(g) \right\}$*, where* g *runs through the* minimal *points of* X *(i.e. the general points of its irreducible components).*

Proof. Of course, we may suppose X irreducible with the generic point g . The $\Gamma(g)$-module $\Gamma\mathcal{M}(g)$ is finitely generated and flat, hence, projective [3]. Then one can easily choose a projective Γ-module P such that $\Gamma\mathcal{M}(g) \oplus$

$P(g) \simeq n\Gamma(g)$ for some $n \le \tilde{\ell}_\Lambda \mathcal{M}(g)$. Replacing $\mathcal{M}$ by $\mathcal{M} \oplus P_X$, we may now suppose that $\Gamma\mathcal{M}(g) = n\Gamma(g)$. Due to Proposition 2.2.8 and Corollary 2.2.3, we may suppose that $\mathcal{M}(g)$ contains a basis of $\Gamma\mathcal{M}(g)$. Then the same is true for each point of an open subscheme $Y \subseteq X$. Hence, the restriction of $\mathcal{M}$ on Y is isomorphic to such family $\mathcal{N}$ that $n\Lambda_Y \subseteq \mathcal{N} \subseteq n\Gamma_Y$. Shrinking Y, we may also suppose that $n\Gamma_Y/\mathcal{N}$ is locally free over $\mathcal{O}_Y$ **q.e.d.**

Remark 3.1.4 *If the over-ring* Γ *is hereditary, the condition* "$\Gamma\mathcal{M}$ is flat over Γ_X" *becomes superfluous: it is always satisfied on an open dense subscheme of* X *(namely, that, on which the sheaf* $\Gamma\mathcal{M}/\mathcal{M}$ *is flat over* $\mathcal{O}$ *).*

One can easily construct some universal sandwiched families in the following way. Let Γ be an over-ring of Λ. Put $\Phi = \Gamma/\Lambda$ and consider the Grassmanian $\mathsf{Gr} = \mathsf{Gr}(n\Phi, d)$, i.e. the variety of subspaces of codimension d in $n\Phi$ [18]. The Λ-submodules form in this Grassmanian a closed subvariety $\mathsf{B} = \mathsf{B}(n, d; \Lambda, \Gamma)$. Denote $\mathcal{L}$ the restriction on B of the canonical locally free sheaf on Gr and $\mathcal{F} = \mathcal{F}(n, d; \Lambda, \Gamma)$ the kernel of the natural epimorphism $n\Gamma_{\mathsf{B}} \to \mathcal{L}$. The universal property of the Grassmanian implies immediately the following universal property of B and $\mathcal{F}$.

Proposition 3.1.5 $\mathcal{F}$ *is a sandwiched family of* Λ*-modules with respect to* Γ *and for each sandwiched family* $\mathcal{M}$ *with a base* X *of* Λ*-modules with respect to* Γ *of rank* n *and codimension* d *there exists a unique morphism* $\varphi : X \to \mathsf{B}$ *such that* $\mathcal{M} = \varphi^*(\mathcal{F})$.

Taking into account Proposition 3.1.3, we get also the following "veresality property" of the families $\mathcal{F}(n, d; \Lambda, \Gamma)$.

Corollary 3.1.6 *Under the conditions of Proposition 3.1.3 there exist:*

- *a descending chain of closed subschemes* $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m = \emptyset$;
- *a set of projective* Γ*-modules* $\{ P_i \mid i = 1, 2, \ldots, m \}$;
- *a set of morphisms* $\{ \varphi_i : Y_i \to \mathsf{B}(n_i, d_i; \Lambda, \Gamma) \mid i = 1, 2, \ldots, m \}$, *where* $Y_i = X_{i-1} \setminus X_i$,

such that $\mathcal{M}_{Y_i} \oplus (P_i \otimes_{\mathsf{k}} \mathcal{O}_{Y_i}) \simeq \varphi_i^* \mathcal{F}(n_i, d_i; \Lambda, \Gamma)$ *for all* $i = 1, 2, \dots, m$ *and some values* n_i, d_i.

Remark 3.1.7 *1. One can easily give an upper bound for necessary values of* n_i. *For instance, we can always suppose, that* $n_i \leq \tilde{\ell}_\Lambda(\hat{\mathcal{M}}(x))$, *where* x *is any closed point and* $\hat{}$, *as before, denotes the completion.*

2. According to Corollary 2.2.7, one can always choose a hereditary overring Γ *such that* Λ *is dense in* Γ. *Then, as remarked above, Corollary 3.1.6 can be applied to* any *family of Cohen-Macaulay* Λ*-modules. This shows that the families* $\mathcal{F}(n, d; \Lambda, \Gamma)$ *are really, in some sense, "versal".*

3.2 Number of parametres

Fix now an over-ring Γ of the Cophen-Macaulay algebra Λ and put $\mathsf{B} = \mathsf{B}(n, d; \Lambda, \Gamma)$, $\mathcal{F} = \mathcal{F}(n, d; \Lambda, \Gamma)$. Find a two-sided Γ-ideal $I \subseteq \operatorname{rad} \Lambda$ of finite codimension (over k) and put $A = \Lambda/I$, $B = \Gamma/I$. We identify $\mathsf{Gr}(n\Phi, d)$ with the closed subscheme of $\mathsf{Gr}(nB, d)$ consisting of all subspaces containing nA. Then B also becomes a closed subscheme in $\mathsf{Gr}(nB, d)$. We consider the elements of nB as rows of length n with entries from B and identify $\mathrm{Aut}_B(nB)$ with the full linear group $\mathsf{G} = \mathrm{GL}(n, B)$ acting on nB by the rule: $g \cdot v = vg^{-1}$.

Each subspace $V \subseteq nB$ defines a Λ-submodule $M \subseteq n\Gamma$ and the following fact is quite obvious.

Proposition 3.2.1 *If* M *and* M' *are two* Λ*-submodules of* $n\Gamma$ *such that* $\Gamma M = \Gamma M' = n\Gamma$, *then* $M \simeq N$ *if and only if* $N = \sigma(M)$ *for some* $\sigma \in \mathrm{Aut}_\Gamma(n\Gamma)$.

Corollary 3.2.2 *Two subspaces* V *and* V' *belonging to* B *defines isomorphic* Λ*-modules if and only if* $V' = g \cdot V$ *for some* $g \in \mathrm{GL}(n, B)$.

Consider the elements of nV, where V is a subspace of nB, as $n \times n$ matrices (over B). Then the following is quite evident.

Proposition 3.2.3 *Let* $g \in \mathsf{G}$, $V \in \mathsf{B}$. *Then* $g \cdot V \in B$ *if and only if* $g \in \mathsf{G} \cap nV$. *Hence,* $(\mathsf{G} \cdot V) \cap \mathsf{B} \simeq (G \cap nV)/\mathrm{St}(V)$, *where* $\mathrm{St}(V) = \{ g \in \mathsf{G} \mid g \cdot V = V \}$.

As G is open in $\mathrm{Mat}\,(n\,,\,B)$, $\mathsf{G}\cap nV$ is open in nV. Therefore, $\mathsf{dim}\,(\mathsf{G}\cap nV)=\mathsf{dim}\,nV=n(bn-d)$, where $b=\mathsf{dim}\,B$, and $\mathsf{dim}\,((\mathsf{G}\cdot V)\cap\mathsf{B})=n(bn-d)-\mathsf{dim}\,\mathrm{St}\,(V)$. Now well-known properties of stabilizers in algebraic groups imply the following corollary.

Corollary 3.2.4 *The set* $\mathsf{B}_i=\{\,V\in\mathsf{B}\mid\mathsf{dim}\,((\mathsf{G}\cdot\mathsf{B})\cap\mathsf{B})\le i\,\}$ *is closed in* B *for each integer* i.

Using Corollary 3.1.6 and Remark 3.1.7, we get analoguos result for *any* family of Cohen-Macaulay modules.

Definition 3.2.5 *Let* $\mathcal{M}$ *be a family of Cohen-Macaulay* Λ*-modules with some base* X, M *some Cohen-Macaulay* Λ*-module. Denote:*

1. $X(M)=\{\,x\in X\mid\mathcal{M}(x)\simeq M(x)=M\otimes_{\mathsf{k}}\mathsf{k}(x)\,\}$.
2. $X_i=\bigcup_{\mathsf{dim}\,X(M)\le i}X(M)$, *where* M *runs through all possible Cohen-Macaulay* Λ*-modules.*
3. $\mathrm{par}\,(\mathcal{M})=\max\{\,\mathsf{dim}\,X_i-i\mid i\in\mathbf{N}\,\}$, the number of parametres *in the family* $\mathcal{M}$.
4. $\mathrm{par}\,(n,d;\Lambda,\Gamma)=\mathrm{par}\,(\mathcal{F}(n,d;\Lambda,\Gamma))$ *and*
   $\mathrm{par}\,(n;\Lambda,\Gamma)=\max\{\,\mathrm{par}\,(n,d;\Lambda,\Gamma)\mid d\in\mathbf{N}\,\}$, *where* Γ *is an over-ring of* Λ.
5. $\mathsf{p}\,(n,\Lambda)=\max\{\,\mathrm{par}\,(n;\Lambda,\Gamma)\mid\Gamma$ is an over-ring of $\Lambda\,\}$.

As $\mathrm{par}\,(n,d;\Lambda,\Gamma)\le\mathrm{par}\,(n,d+nc;\Lambda,\Gamma')$ for two over-rings $\Gamma\subseteq\Gamma'$ of Λ, where $c=\mathsf{dim}\,(\Gamma'/\Gamma)$, one can consider in Definition 3.2.5 *(3)* only *maximal* over-rings.

Corollary 3.2.6 *Under the notations of Definition 3.2.5:*

1. $X(M)$ *is a constructible subset in* X *(i.e. a finite union of locally closed subsets).*
2. *The subsets* X_i *are closed in* X.
   (This claim gives sense to Definition 3.2.5 *(3)*).

3. $\mathrm{par}\,(\mathcal{M}) \le \mathbf{p}(n,\Lambda)$, *where*

$$n = \max\left\{ \tilde{\ell}_\Lambda(\hat{\mathcal{M}}(x) \mid x \text{ is a closed point of } X \right\}$$

(if X is connected, the last number does not depend on the choice of x).

3.3 Families of algebras

In this section it is convenient to consider Cohen-Macaulay algebras not only over *rings* but also over *schemes*. The corresponnding definitions are quite evident. We fix a field k and consider only k-algebras and k-schemes.

Definition 3.3.1 *1. Call a scheme X* Cohen-Macaulay *if all its local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay .*

2. Call Cohen-Macaulay algebra over a Cohen-Macaulay scheme X *a coherent sheaf Λ of $\mathcal{O}_X$-algebras such that, for each point $x \in X$, the stalk Λ_x is a (maximal) Cohen-Macaulay $\mathcal{O}_{X,x}$-module.*

All definitions of Section 2.1 (e.g. those of *reduced, analitically reduced, maximal algebras, over-rings* etc.) can be applied to this more general situation. One can also construct Cohen-Macaulay modules and algebras *locally*, as in Proposition 2.1.4 (if $\mathsf{dim}\, X \le 2$). The following proposition follows easily from [3] and [7].

Proposition 3.3.2 *If X is an algebraic scheme, then any reduced Cohen-Macaulay algebra is also analitically reduced.*

Consider now the case of Cohen-Macaulay algebras over a *reduced curve* C. If Γ is an over-ring of such algebra Λ, then $\Gamma_c = \Lambda_c$ for almost all points $c \in C$. Hence, the following numbers are well-defined:

$$\begin{aligned} \mathrm{par}\,(n,d;\Lambda,\Gamma) &= \sum_{c\in C} \mathrm{par}\,(n,d;\Lambda_c,\Gamma_c)\,; \\ \mathrm{par}\,(n;\Lambda,\Gamma) &= \sum_{c\in C} \mathrm{par}\,(n;\Lambda_c,\Gamma_c)\,; \\ \mathsf{p}\,(n;\Lambda) &= \sum_{c\in C} \mathsf{p}\,(n;\Lambda_c)\,. \end{aligned}$$

We are really interested now in *families* of Cohen-Macaulay algebras in the following sense.

Definition 3.3.3 *Let $f : Y \to X$ be a morphism of schemes and $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_Y$-algebras. Call $(\mathcal{L}, f)$* family of (reduced) Cohen-Macaulay algebras with the base X *if the following conditions hold:*

1. *f is flat morphism and $f_*(\mathcal{L})$ is flat $\mathcal{O}_X$-module.*
2. *$Y(x) = f^{-1}(x)$ is a reduced curve over $\mathsf{k}(x)$ for each point $x \in X$.*
3. *$\mathcal{L}(x)$ is a (reduced) Cohen-Macaulay algebra over $Y(x)$ for each $x \in X$.*

Definition 3.3.4 *Let $(\mathcal{L}, f : Y \to X)$ be a family of Cohen-Macaulay algebras. A* family of over-rings of it *is*

, by definition, a family $(\mathcal{L}', f)$ (with the same f) such that $\mathcal{L}' \supseteq \mathcal{L}$ and $f_*(\Lambda'/\Lambda)$ is flat coherent $\mathcal{O}_X$-module.

The last condition also implies that, for each $x \in X$, $\mathcal{L}'(x)$ is an over-ring of $\mathcal{L}(x)$. Given such family of over-rings, we can define the following functions on X:

$$\begin{aligned} \operatorname{par}(x, n, d; \mathcal{L}, \mathcal{L}') &= \operatorname{par}(n, d; \mathcal{L}(x), \mathcal{L}'(x)); \\ \operatorname{par}(x, n; \mathcal{L}, \mathcal{L}') &= \operatorname{par}(n; \mathcal{L}, \mathcal{L}'). \end{aligned}$$

If the family of over-rings is fixed, we omit the letters $\mathcal{L}$ and $\mathcal{L}'$ in the notations of these functions.

Theorem 3.3.5 *For each family of over-rings $\mathcal{L}' \supseteq \mathcal{L}$ with the base X the functions $\operatorname{par}(x, n, d)$ and $\operatorname{par}(x, n)$ are upper semi-continuous, i.e. for each integer i the sets*

$$\begin{aligned} X_i(d) &= \{ x \in X \mid \operatorname{par}(x, n, d) \geq i \}; \\ X_i &= \{ x \in X \mid \operatorname{par}(x, n) \geq i \} \end{aligned}$$

are closed in X.

Proof. As $X_i = \bigcup_d X_i(d)$, we only have to prove that $X_i(d)$ is closed. Moreover, we may suppose X to be a smooth curve, hence $\mathsf{dim}\, Y = 2$. Let $\mathcal{N} = \mathcal{L}'/\mathcal{L}$. Consider the relative Grassmanian $\mathsf{Gr}(n\mathcal{N}, d)$ (over X) and its closed subscheme $\mathcal{B}$ (again over X) parametrizing the $\mathcal{L}$-submodules. Choose the biggest two-sided $\mathcal{L}'$-ideal $\mathcal{I} \subset \mathcal{L}$. Then $\mathcal{L}/\mathcal{I}$ is torsion-free over $\mathcal{O}_X$, hence, flat. Therefore, $\mathcal{J} = \mathcal{L}'/\mathcal{I}$ is also $\mathcal{O}_X$-flat. As in Section 3.2, identify $\mathcal{B}$ with a closed subscheme of $\mathsf{Gr}(n\mathcal{J}, d)$ and consider the group scheme $\mathrm{GL}(n, \mathcal{J})$ (over X) acting on the last Grassmanian. The same observations as in Section 3.2 show that $\mathcal{B}_j = \{ v \in \mathcal{B} \mid \mathsf{dim}\, \mathrm{St}(v) \geq j \}$ is closed in $\mathcal{B}$. As $\mathcal{B}$ is *proper* over X, its projection Z_j is also closed (in X). Hence, the sets $Z_{ij} = \{ x \in Z_{ij} \mid \mathsf{dim}\, \mathsf{B}_j(x) \geq i + j \}$ are also closed. But, by definition, $X_i(d) = \bigcup_j Z_{ij}$ **q.e.d.**

As for families of Cohen-Macaulay algebras, whose bases are smooth curves, one can construct over-rings locally, we get also the following corollary.

Corollary 3.3.6 *The function* $\mathsf{p}(n, \mathcal{L}(x))$ *is upper semi-continuous for any family* $\mathcal{L}$ *of Cohen-Macaulay algebras.*

For commutative case this result was obtained by Knörrer [17].

4 Tame and Wild Algebras

4.1 Definitions

From now on we suppose that k is an algebraically closed field, and all Cohen-Macaulay rings R are local complete k-algebras of Krull dimension 1 and such that $\mathsf{R}/\mathbf{m} = \mathsf{k}$ for the maximal ideal $\mathbf{m} \subset \mathsf{R}$. All Cohen-Macaulay algebras are supposed reduced. For each k-algebra A denote $A-\mathsf{mod}$ the category of finite-dimensional (over k) A-modules.

Definition 4.1.1 *Let* Λ *be a Cohen-Macaulay* R*-algebra and* A *be a* k*-algebra. Denote by* $\mathcal{CM}(\Lambda, A)$ *the category of finitely generated* Λ*-*A*-bimodules* $\mathcal{M}$ *satisfying the following conditions:*

1. $\mathcal{M}$ *is torsion-free as* R*-module.*
2. $\mathcal{M}$ *is flat over* A.

3. $\mathcal{M}(L) = \mathcal{M} \otimes_A L$ *is Cohen-Macaulay* Λ*-module for each* $L \in A-\mathsf{mod}$ *.*

Put then $|\mathcal{M}| = \{\, \mathcal{M}(L) \mid L \in A - \mathsf{mod} \,\}$ *.*

If A/I is finite-dimensional for each maximal ideal $I \subset A$, the last condition is equivalent to the following one:

3′. For any non-zero-divisor $a \in \mathsf{R}$ *the* A*-module* $\mathcal{M}/a\mathcal{M}$ *is also flat.*

Call the bimodules from $\mathcal{CM}(\Lambda, A)$ *families of Cohen-Macaulay* Λ*-modules over* A. Of course, if A is commutative, this notion coincides with that of families of Cohen-Macaulay Λ-modules on $\operatorname{Spec} A$ (cf. Section 3.1).

Definition 4.1.2 *1. Call a family* $\mathcal{M} \in \mathcal{CM}(\Lambda, A)$ strict *if the following conditions hold:*

(a) If $L \in A - \mathsf{mod}$ *is indecomposable, then also* $\mathcal{M}(L)$ *is indecomposable.*

(b) If $L \not\simeq L'$ *for some modules* $L, L' \in A - \mathsf{mod}$*, then also* $\mathcal{M}(L) \not\simeq \mathcal{M}(L')$ *.*

2. Call a Cohen-Macaulay algebra Λ CM-wild *(or simply* wild*) if, for any finitely generated* k*-algebra* A*, there exists a strict family* $\mathcal{M} \in \mathcal{CM}(\Lambda, A)$ *.*

It is well-known that really, to prove the wildness, one needs only to construct a strict family over one of "standard" algebras. The most known and used of them are:

- the free algebra $\mathsf{k}\langle x, y\rangle$ in two generators;
- the polynomial algebra $\mathsf{k}[x, y]$ in two generators;
- the formal power series algebra $\mathsf{k}[[x, y]]$ in two generators.

Definition 4.1.3 *Call* rational family *of Cohen-Macaulay* Λ*-modules any one, whose base is a smooth rational curve over* k *.*

Definition 4.1.4 *Let* $\mathcal{S} = \{\, \mathcal{M}_i \mid i \in I,\ \mathcal{M}_i \in \mathcal{CM}(\Lambda, A_i) \,\}$ *be a set of families of Cohen-Macaulay* Λ*-modules (possibly, over different bases* A_i*) indexed by some set* I *.*

1. *Denote* $|\mathcal{S}| = \bigcup_{i\in I} |\mathcal{M}|$.

2. *Call* $\mathcal{S}$ exhaustive *if, for any possible* $\mathbf{r}$ *, almost all (up to isomorphism) indecomposable Cohen-Macaulay* Λ*-modules of vector-rank* $\mathbf{r}$ *belong to* $|\mathcal{S}|$ *.*

3. *Call* $\mathcal{S}$ locally finite *if, for any possible* $\mathbf{r}$ *, the set*

$$\mathcal{S}(\mathbf{r}) = \{\, i \in I \mid |\mathcal{M}_i| \cap \mathcal{CM}\,(\mathbf{r},\Lambda) \neq \emptyset \,\}$$

is finite.

4. *Call* $\mathcal{S}$ strict *if each* $\mathcal{M}_i$ *is strict and* $\mathcal{M}_i(L) \not\simeq \mathcal{M}_j(L')$ *for any* $i \neq j$ *and any* $L \in A_i - \mathsf{mod}$ *,* $L' \in A_j - \mathsf{mod}$ *.*

Theorem-Definition 4.1.5 *Call a Cohen-Macaulay algebra* Λ CM-tame *(or simply* tame*), if one of the following equivalent conditions (and hence any of them) holds:*

1. Λ *is not CM-wild.*

2. *For any possible* $\mathbf{r} = (r_1, r_2, \ldots, r_s)$ *there exists a* d*-parametre family* $\mathcal{M}$ *of Cohen-Macaulay* Λ*-modules, with* $d \leq |\mathbf{r}|$ *, such that any Cohen-Macaulay* Λ*-module of vector-rank* $\mathbf{r}$ *belongs to* $|\mathcal{M}|$ *.*

3. *For any possible* $\mathbf{r}$ *there exists a* 1*-parametre family* $\mathcal{M}$ *of Cohen-Macaulay* Λ*-modules such that any* indecomposable *Cohen-Macaulay* Λ*-module of rank* $\mathbf{r}$ *belongs to* $|\mathcal{M}|$ *.*

4. *There exists a strict locally finite exhaustive set* $\mathcal{S}$ *of rational families of Cohen-Macaulay* Λ*-modules.*

It is quite clear that *4* $\Rightarrow$ *3* $\Rightarrow$ *2* $\Rightarrow$ *1*. In the next section we will prove that also *1* $\Rightarrow$ *4*.

Remark 4.1.6 *One can easily see that, if* Λ *is CM-wild, then there is infinitely many ranks* $\mathbf{r}$ *such that there exists a strict family* $\mathcal{M}$ *of Cohen-Macaulay* Λ*-modules with* $|\mathcal{M}| \cap \mathcal{CM}\,(\mathbf{r},\Lambda) \neq \emptyset$ *and* $\mathrm{par}\,(\mathcal{M}) \geq c|\mathbf{r}|^2$ *for some constant* c*. Moreover, one can even suppose that* $|\mathcal{M}| \cap \mathcal{CM}\,(\mathbf{r},\Lambda)$ *consists only of indecomposable modules.*

4.2 Tame/wild dichotomy

To prove Theorem 4.1.5, we need some results from the "matrix problems" theory (cf. [6],[9]).

Let A and B be k-algebras and U be an A-B-bimodule. For a (finitely generated) projective A-module P and a (finitely generated) projective B-module Q put $U(P,Q) = \mathrm{Hom}_A(P, U \otimes_B Q)$.

Definition 4.2.1 *Define the* category $\mathsf{El} = \mathsf{El}(U)$ of elements of the bimodule U *in the following way:*

> *The set of objects* $\mathsf{Ob}\,\mathsf{El}(U)$ *(called* elements of U *) is the union* $\bigcup_{P,Q} U(P,Q)$ *.*
>
> *The set of morphisms* $\mathsf{El}(u,v)$ *, for* $u \in U(P,Q)$ *and* $v \in U(P',Q')$ *, is the set of all pairs* (α,β) *, where* $\alpha \in \mathrm{Hom}_A(P,P')$ *,* $\beta \in \mathrm{Hom}_B(Q,Q')$ *, such that* $v\alpha = \beta u$ *.*
>
> $(\alpha,\beta)(\alpha',\beta') = (\alpha\alpha',\beta\beta')$ *, where* $(\alpha,\beta) \in \mathsf{El}(u,v)$ *,* $(\alpha',\beta') \in \mathsf{El}(w,u)$ *.*

For any k-algebra Σ we can define the $(A \otimes_{\mathsf{k}} \Sigma)$-$(B \otimes_{\mathsf{k}} \Sigma)$-bimodule $U_\Sigma = \Sigma \otimes_{\mathsf{k}} U \otimes_{\mathsf{k}} \Sigma$. Therefore, we can define, just as above, *families of elements* of the bimodule U, then *wild* and *tame* bimodules. In [6] the analogue of Theorem 4.1.5 is proved for finite-dimensional bimodules. In order to apply this result to Cohen-Macaulay algebras, we need a slight generalization of it, namely, to the *open subcategories* in the following sense.

Definition 4.2.2 *Let* U *be a finite-dimensional bimodule.*

1. *A full subcategory* $\mathcal{X} \subseteq \mathsf{El}(U)$ *is said to be* open *if it satisfies the following conditions:*
   - *(a) If* $u \in \mathcal{X}$ *and* $v \simeq u$ *, then* $v \in \mathcal{X}$ *.*
   - *(b)* $u \oplus v \in \mathcal{X}$ *if and only if* $u \in \mathcal{X}$ *and* $v \in \mathcal{X}$ *.*
   - *(c) For each* P *and* Q *the intersection* $U(P,Q) \cap \mathcal{X}$ *is open in* $U(P,Q)$ *(in Zariski topology).*
2. *Given an open subcategory* $\mathcal{X} \subseteq \mathsf{El}(U)$ *, put, for any* k*-algebra* Σ *,*

$$\mathcal{X}(\Sigma) = \{\, u \in \mathsf{El}(U_\Sigma) \mid u(L) \in \mathcal{X} \text{ for any } L \in \Sigma - \mathsf{mod} \,\} \,.$$

Now we are able to define *wild* and *tame* open subcategories just as it has been done for Cohen-Macaulay algebras. In [9] the analogue of Theorem 4.1.5 is proved for open subcategories.[2] To apply this result to Cohen-Macaulay algebras, use the following rather simple and more or less standard observations (cf. e.g. [14] or [19]). Fix an over-ring Γ of a Cohen-Macaulay algebra Λ and a set $\mathbf{M}$ of indecomposable Cohen-Macaulay Γ-modules. For the sake of simplicity we suppose $\mathbf{M}$ to be finite. Denote $\mathsf{add}\,\mathbf{M}$ the *additive hull* of M, i.e. the category of all (finite) direct sums of modules from $\mathbf{M}$, and $\mathcal{CM}(\Lambda,\mathbf{M})$ the full subcategory of $\mathcal{CM}(\Lambda)$ consisting of all such modules M that $\Gamma M \in \mathsf{add}\,\mathbf{M}$. Of course, if $\mathbf{M}$ consists of all indecomposable Cohen-Macaulay Γ-modules, then $\mathcal{CM}(\Lambda,\mathbf{M}) = \mathcal{CM}(\Lambda)$. Find a two-sided Γ-ideal $I \subseteq \operatorname{rad}\Lambda$ such that $\dim_{\mathsf{k}}\Gamma/I < \infty$. Then $IM \subset M \subseteq \Gamma M$ for any Cohen-Macaulay Λ-module M and any homomorphism $\varphi : M \to M'$ can be uniquely prolonged to the homomorphism $\Gamma\varphi : \Gamma M \to \Gamma M'$. Put

$$A = \Lambda/I\,,\;\; N = \sum_{L\in\mathbf{M}} L\,,\;\; B = \operatorname{End}_\Gamma(N)/\operatorname{Hom}_\Gamma(N, IN) \text{ and } U = N/IN\,.$$

Consider U as A-B-bimodule. Then the category $\mathsf{El}(U)$ is well defined. Consider also a new category Sub, whose objects are pairs $(N/IN, V)$, where $N \in \mathbf{M}$ and V is an A-submodule in N/IN. Then the following two functors are defined:

$$\begin{array}{lcl} \mathsf{T} : \mathcal{CM}(\Lambda,\mathbf{M}) \to \mathsf{Sub} & : & \mathsf{T}(M) = (\Gamma M/IM\,,\, M/IM)\,; \\ \mathsf{Im} : \mathsf{El}(U) \to \mathsf{Sub} & : & \mathsf{Im}(v) = (Q \otimes_B U\,,\, \operatorname{Im} v) \text{ for } v \in U(P,Q)\,. \end{array}$$

Proposition 4.2.3 *1. Denote* Sub_0 *the full subcategory of* Sub *consisting of such pairs* (W,V) *that* $\Gamma V = W$*. Then* $\mathsf{T}(M) \in \mathsf{Sub}_0$ *for any* M *and the functor* $\mathsf{T} : \mathcal{CM}(\Lambda,\mathbf{M}) \to \mathsf{Sub}_0$ *is full, dense, reflects isomorphisms and indecomposability.*

2. Denote $\mathcal{X}$ *the full subcategory of* $\mathsf{El}(U)$ *consisting of all such* $v \in U(P,Q)$ *that* $\operatorname{Ker} v \subseteq \operatorname{rad} P$ *and* $\Gamma\operatorname{Im} v = U \otimes_B Q$*. Then* $\mathsf{Im}\, v \in \mathsf{Sub}_0$ *for any* $v \in \mathcal{X}$ *and the functor* $\mathsf{Im} : \mathcal{X} \to \mathsf{Sub}_0$ *is full, dense, reflects isomorphisms and indecomposability.*

---

[2] Really, in [9] the "tame/wild dichotomy" is proved for open subcategories in the representation categories of *bocses* in the sense of [6]. But we are not going to precise the corresponding definitions here, as the case of bimodules is quite enough for our purpose.

One can easily see that $\mathcal{X}$ is an open subcategory in $\mathsf{El}(U)$.

Corollary 4.2.4 *1. If the open subcategory $\mathcal{X}$ is wild, then Λ is also CM-wild.*

2. If $\mathbf{M}$ contains all indecomposable Cohen-Macaulay Γ-modules and the condition 4 of Theorem 4.1.5 holds for the open subcategory $\mathcal{X}$, then it also holds for Λ.

As we can always choose for Γ some hereditary (e.g. maximal) over-ring and hence for $\mathbf{M}$ the set of all indecomposable Γ-modules, we get the implication *4* $\Rightarrow$ *1* of Theorem 4.1.5 for any Cohen-Macaulay algebra Λ.

Using Corollary 3.3.6 and Remark 4.1.6, we get also the following useful result.

Corollary 4.2.5 *Let $\mathcal{L}$ be a family of Cohen-Macaulay algebras over a base X. Then the set* $\{ x \in X \mid \mathcal{L}(x) \text{ is wild } \}$ *is a countable union of closed subsets.*

It looks very likely that this set is really *closed* itself, but we are not able to prove it till now.

4.3 Commutative case

Till now, criteria of tameness are known only for a few classes of Cohen-Macaulay algebras. The widest is, perhaps, that of *commutative algebras* (i.e. of Cohen-Macaulay rings themselves). The following result was inspired by the work of Greuel and Knörrer [15], who had remarked that in the "geometrical situation" the known criterion of finiteness of the number of indecomposable Cohen-Macaulay modules [16],[12] can be reformulated in the following way:

A Cohen-Macaulay ring R has only finitely many indecomposable Cohen-Macaulay modules (up to isomorphism) if and only if it dominates *(i.e. is over-ring of) one of* simple plane curve singularities *in the sense of [1].*

It so happened that the criterion of tameness is of the same form. Remind that here all Cohen-Macaulay rings are local complete reduced algebras over an algebraically closed field k, which coincides with the residue field. Denote T_{pq} the Cohen-Macaulay ring

$$\mathsf{k}[[x,y]]/(x^p + \lambda x^2 y^2 + y^q), \quad \text{where } 1/p + 1/q \leq 1/2 \text{ and } \lambda \in \mathsf{k} \setminus \{0,1\}.$$

Remark that actually, except for the cases $(p,q)=(4,4)$ and $(p,q)=(3,6)$, all values of λ (including 1) lead to isomorphic rings, so λ can be omitted. On the contrary, for $(4,4)$ and $(3,6)$ cases different values of λ give non-isomorphic rings and the restriction $\lambda \neq 1$ is also indispensable.

Theorem 4.3.1 *Let R be a Cohen-Macaulay ring of infinite Cohen-Macaulay type (i.e. with infinitely many non-isomorphic indecomposable Cohen-Macaulay modules). Then R is CM-tame if and only if it dominates one of the rings T_{pq}.*

The proof of this theorem consists of several steps of rather different nature.

Step 1 Consider first some other "standard" Cohen-Macaulay rings, namely, the rings

$$\mathsf{P}_{pq} = \mathsf{k}[[x,y,z]]/(xy,\ x^p+y^q+z^2) \quad \text{where } p,q \geq 2 \text{ and } (p,q) \neq (2,2).$$

Proof that all these rings are tame. We use the method of the preceding section, where this question has been reduced to that for bimodules. Namely, each of the rings P_{pq} is Gorenstein. Hence, it has the only minimal over-ring P'_{pq} and all indecomposable Cohen-Macaulay P_{pq}-modules, except P_{pq} itself, are also P'_{pq}-module. Hence, we have to consider $\Lambda = \mathsf{P}'_{pq}$. Put $I = \operatorname{rad}\Lambda$ and $\Gamma = \operatorname{End}_\Lambda \mathbf{m}$. It is known (cf. [23]) that Λ is isomorphic to the subring of $\mathsf{k}[[t]]^s$ generated by the elements a_1, a_2, b_1, b_2, where s, a_1, a_2, b_1, b_2 depend on the parity of p and q, namely:

p	q	s	a_1	a_2	b_1	b_2
odd	odd	2	$(t^2,0)$	$(0,t^2)$	$(t^p,0)$	$(0,t^q)$
odd	even	3	$(t,t,0)$	$(0,0,t^2)$	$(0,0,t^p)$	$(t^{q/2},0,0)$
even	even	4	$(t,t,0,0)$	$(0,0,t,t)$	$(t^{p/2},0,0,0)$	$(0,0,t^{q/2},0)$

Then $\Gamma = \Gamma_1 \times \Gamma_2$, where both of Γ_i are *Bass* rings in the sense of [12], hence, all indecomposable Γ_i-modules are either over-rings of Γ_i or, if $\mathsf{Q}\Gamma_i$ is not a field, the direct summands of its maximal over-ring. Then the concise computa

tion (cf. [9]) shows that the classification of elements of the bimodule U introduced in the last section is really a sort of the so called "Gelfand Problem" (in the variant described in [2]). Therefore U and, according to Theorem 4.1.5, also Λ are tame.

Step 2 Now we are able to prove that all rings T_{pq} and, hence, all their over-rings are tame. To do it, consider the algebra $\mathcal{L}$ over the polynomial ring $A = \mathsf{k}[\lambda]$:

$$\mathcal{L} = A[x,y]/(xy - \lambda z\,,\; x^p + y^q + z^2)\,.$$

Of course, it is a family of Cohen-Macaulay rings on the affine line. Evidently, the only singular point of each ring $\mathcal{L}(\lambda)$, for $\lambda \in \mathsf{k}$, is $x = y = 0$. Moreover, the completion of this only singular local ring of $\mathcal{L}(\lambda)$ is isomorphic to T_{pq} if $\lambda \neq 0$ and to P_{pq} if $\lambda = 0$. As we have remarked above, for $(p,q) \neq (4,4)$ or $(3,6)$, all rings T_{pq} are isomorphic. Were they wild, then $\mathsf{P}_{pq} = \mathcal{L}(0)$ were also wild by Corollary 4.2.5. But we have just proved that P_{pq} are tame, a contradiction.

For the remaining cases $(p,q) = (4,4)$ and $(3,6)$ the tameness has been proved in [4],[5] (cf. also [24]). Thus, we have proved the *sufficiency* in Theorem 4.3.1.

Step 3 To prove the *necessity*, we introduce some other conditions for R to be tame, in terms of *over-rings*. They are rather alike the conditions for finiteness of Cohen-Macaulay type from [12].

Definition 4.3.2 *Let* Γ *be an over-ring of a Cohen-Macaulay ring* R, $\mathbf{m} = \operatorname{rad}\mathsf{R}$ *and* $\Gamma/\Gamma\mathbf{m} = \prod_{i=1}^m D_i$ *with local algebras* D_i. *Denote:*

1. $\mathbf{d}(\Gamma) = (d_1, d_2, \ldots, d_m)$, *where* $d_i = \dim_{\mathsf{k}} D_i$, *the* multiplicity vector of R with respect to Γ.

2. $d(\Gamma) = |\mathbf{d}(\Gamma)| = \sum_{i=1}^m d_i$, *the* (total) multiplicity of R with respect to Γ.

We always arrange the numbers d_i in the increasing order: $d_1 \leq d_2 \leq \ldots \leq d_m$.

Definition 4.3.3 *For a Cohen-Macaulay algebra* R *denote:*

1. R_0 *its* normalization, *i.e. its maximal over-ring.*
   As k is algebraically closed, $\mathsf{R}_0 = \prod_{i=1}^s \Delta_i$ with $\Delta_i \simeq \mathsf{k}[[t_i]]$.

2. e_i *the idempotent of* R_0 *such that* $\Delta_i = e_i\mathsf{R}_0$.

3. $t = (t_1, t_2, \ldots, t_s) \in \mathsf{R}_0$ *and* $\theta \in \mathbf{m}$ *such element that* $\mathbf{m}\mathsf{R}_0 = \theta\mathsf{R}_0$.

4. $\mathsf{R}' = t\mathsf{R}_0 + \mathsf{R}$, *the* weak normalization of R, *i.e. its maximal local over-ring, and* $\mathsf{R}'_i = \mathsf{k}e_i + \mathsf{R}'$.

5. $\mathsf{R}'' = \theta t\mathsf{R}_0 + \mathsf{R}$.

Theorem 4.3.4 *Let* R *be a Cohen-Macaulay ring of infinite Cohen-Macaulay type. The following conditions are necessary and sufficient for* R *to be CM-tame:*

1. $d(\mathsf{R}_0) \le 4$ *and* $\mathbf{d}(\mathsf{R}) \notin \{(4), (1,3), (3)\}$.

2. $d(\mathsf{R}') \le 3$ *and* $\mathbf{d}(\mathsf{R}'_i) \ne (1,3)$ *for each* i.

3. *If* $d(\mathsf{R}_0) = 3$, *then* $d(\mathsf{R}'') \le 2$.

Remark 4.3.5 *Really one has to check the condition for* $\mathbf{d}(\mathsf{R}'_i)$ *only for such idempotents* e_i *that* $e_i\mathbf{m} \subseteq \mathbf{m} + \theta t\mathsf{R}_0$.

The *necessity* of these conditions can be proved by rather straightforward calculations using the methods of Section 4.2. Namely, first we prove the following lemma.

Lemma 4.3.6 *Suppose that* R *has an over-ring* Γ *with either* $d(\Gamma) > 4$ *or* $\mathbf{d}(\Gamma) = (4)$ *or* $(1,3)$. *Then* R *is wild.*

Proof. Note that $\mathbf{d}(\Gamma)$ does not change if we replace R by $\mathbf{m}\Gamma + \mathsf{R}$. Hence, we may suppose that $\Gamma\mathbf{m} = \mathbf{m}$ and use Corollary 4.2.4 with $I = \mathbf{m}$. If $d(\mathsf{R}) > 4$, one can easily count that the number of parametres $\operatorname{par}(n; \mathsf{R}, \Gamma)$ grows quadratically with n (cf. [6]), so R is wild. If $\mathbf{d}(\Gamma) = (4)$, we construct explicitely a strict family of elements of the bimodule U over $\mathsf{k}\langle x, y\rangle$. Here we take always $\mathbf{M} = \{\Gamma\}$, hence, $B = \Gamma/\mathbf{m}$. Remark also, that in our case $A = \mathsf{R}/\mathbf{m} = \mathsf{k}$. Therefore, a map $v : P \to U \otimes_B Q$, where $P = n\mathsf{k}$ and $Q = m\mathsf{B}$, can be identified with a set of n elements of mU. It is convinient to consider it as an $m \times n$ matrix with entries from U. For instance, if $\Gamma/\mathbf{m} = \mathsf{k}[a]$ with $a^4 = 0$ (the most complicated case), then we take $m = 5,\ n = 9$ and

$$v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a & 0 & a^2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & a & 0 & a^2 \\ 0 & 0 & 1 & 0 & 0 & a^2 & 0 & 0 & a^3 \\ 0 & 0 & 0 & 1 & 0 & 0 & a^2 & a^3x & a^3y \\ 0 & 0 & 0 & 0 & 1 & a^3 & 0 & 0 & 0 \end{pmatrix}$$

Then one can check, by a straightforward, though rather cumbersome, calculation, that v is really a strict element. Quite analogously the strict elements are constructed in other possible cases (cf. [9]) **q.e.d.**

Certainly, this lemma implies the necessity of the over-rings conditions *1* and *2*, except $\mathbf{d}(\mathsf{R}) \neq (3)$. The necessity of other conditions is proved in analogous way (cf. [9]).

Remark 4.3.7 *By the way, it follows from the condition* 1 *that if* R *is a* domain *(i.e.* $s = 1$*), then it is either of finite Cohen-Macaulay type or CM-wild.*

Step 4 To accomplish the proof of Theorems 4.3.1 and 4.3.4, one only has now to check that any ring R, which is of infinite Cohen-Macaulay type and satisfies the over-rings conditions *1–3* is really an over-ring of some T_{pq}. It can also be done by some straightforward observations (using the precise parametrization of T_{pq}) and we omit these calculations, referring to [9]. Now both theorems are completely proved.

4.4 Further results

4.4.1 Rings of finite growth

We keep all suppositions of preceding sections.

Definition 4.4.1 *Let* $\mathcal{S} = \{\, \mathcal{M}_i \mid \mathcal{M}_i \in \mathcal{CM}\,(\Lambda, A_i)\,\}$ *be a locally finite set of families of Cohen-Macaulay modules over a Cohen-Macaulay algebra* Λ *(cf. Definition 4.1.4).*

1. *Denote* $\nu(\mathcal{S}, \mathbf{r})$ *the number of elements in* $\mathcal{S}(\mathbf{r})$.
2. *Say that the set* $\mathcal{S}$ is of finite growth *if there exists a constant* c *such that* $\nu(\mathcal{S}, \mathbf{r}) \leq c$ *for all* $\mathbf{r}$.
3. *Say that* Λ is of finite growth *if it has an exhaustive set of rational families of Cohen-Macaulay modules, which is of finite growth.*
4. *Call* Λ (CM-) domestic *if it has a* finite *exhaustive set of rational families of Cohen-Macaulay modules.*

Of course, any CM-domestic algebra is of finite growth and any Cohen-Macaulay algebra of finite growth is tame. Moreover, it follows from Theorem 4.1.5 and simple geometrical observations that, for such algebra, any locally finite strict set of rational families is really of finite growth.

Following accurately the calculations in the proof of Theorem 4.3.1, one can also get criteria for a commutative algebra to be of finite growth or domestic (cf. [9]).

Theorem 4.4.2 *Let* R *be a Cohen-Macaulay ring. Then:*

1. R *is of finite growth if and only if it dominates one of the singularities* T_{44} *or* T_{36} *.*

2. R *is domestic if and only if it properly dominates the minimal over-ring of one of the singularities* T_{44} *or* T_{36} (as T_{pq} is Gorenstein, it has the only minimal over-ring).

4.4.2 2×2 matrix algebras

Another class of algebras, for which a criterion of tameness is known, is that of 2×2 *matrix algebras*, i.e. such Cohen-Macaulay algebras Λ that $\mathsf{Q}\Lambda = \mathrm{Mat}\,(2\,,\,\mathsf{D})$, where D is a skewfield. As we consider the *comlete* case, there is the only maximal order Δ in D and we always suppose that

Λ *contains* ΔE *, where* E *is the identity matrix.*

Moreover, Δ is a (non-commutative) local principal ideal ring. Denote π the generator of its only maximal ideal. For such algebras the following criterion of tameness was obtained in [13].

Definition 4.4.3 1. *Write* $\Lambda \subseteq_C \Lambda'$ *if there exists an invertible matrix* S *such that* $S^{-1}\Lambda S \subseteq \Lambda'$ *.*

2. *Denote* Λ_w *the order in* $\mathrm{Mat}\,(2\,,\,\mathsf{D})$ *having the* Δ*-bases:*

$$\left\{ E\,,\ \pi^2 e_{11}\,,\ \pi^2 e_{12}\,,\ \pi e_{21} \right\},$$

where e_{ij} *are usual matrix units.*

Theorem 4.4.4 1. *A* 2×2 *matrix algebra* Λ *(containing* ΔE *) is wild if and only if* $\Lambda \subseteq_C \Lambda_w$ *.*

2. *If* Λ *is local and non-Gorenstein, then it is tame if and only if* $\Gamma/\mathrm{rad}\,\Gamma \neq \mathsf{k}$ *, where* $\Gamma = \mathrm{End}_\Lambda(\mathrm{rad}\,\Lambda)$ *.*

The proof is alike that for commutative algebras (though simpler). First, by a straightforward construction, using the methods of Section 4.2, we show that Λ_w is wild. Then the same procedure, applied to an algebra satisfying the condition *2*, leads to a sort of "Gelfand problem" (hence, all these algebras are tame). At last, we check that whenever this condition does not hold, then $\Lambda \subseteq_C \Lambda_w$, hence, is also wild.

Moreover, in [13] a complete list of tame 2×2 matrix algebras is given.

4.4.3 Families of ideals

The "standard" tame rings T_{pq} of Theorem 4.3.1 are the "serial" part of the so called *unimodal plane curve singularities* in the sense of [1]. There are also 14 "exceptional" ones, which happen to be wild. Of course, the *bimodal* plane curve singularities in the sense of [1] are also wild. Nevertheless, this class of "good" singularities can also be characterized in terms of Cohen-Macaulay modules. Namely, the following result has been proved in [21],[11].

Theorem 4.4.5 *Let* R *be a Cohen-Macaulay ring (local, complete, reduced and of Krull dimension 1). Then* $\mathbf{p}n(1,\mathsf{R}) \leq 1$ *((i.e. there exist not more than* 1*-parametre families of* ideals *of* R *) if and only if* R *dominates one of the unimodal or bimodal plane curve singularities in the sense of [1].*[3]

Really, the proof of this theorem (cf. [11]) is also based on some "over-rings conditions", alike those of Theorem 4.3.4.

Theorem 4.4.6 *Under conditions of Theorem 4.4.5, put* $\mathsf{R}_i = \mathbf{m}^i\mathsf{R}_0 + \mathsf{R}$*, where* $\mathbf{m}$ *is the maximal ideal and* R_0 *the normalization of* R*. Then* $\mathsf{p}\,(1,\mathsf{R}) \leq 1$ *if and only if the following conditions hold:*

1. $d(\mathsf{R}_0) \leq 4$ *.*
2. $d(\mathsf{R}_1) \leq 3$ *.*

[3] In [21],[11] these singulariries are called *strictly unimodal*, using the terminology of [23].

3. *$d(\mathsf{R}_1 + e\mathsf{R}) \leq 3$ for each idenpotent $e \in \mathsf{R}_0$ such that $d(e\mathsf{R}_0) = 1$ (provided it exists).*

4. *If $d(\mathsf{R}_0) = 3$, then $d(\mathsf{R}_2) \leq 2$.*

(here $d(M)$, as in Theorem 4.3.4, denotes the number of generatores of the R-module M).

The proof of Theorems 4.4.5 and 4.4.6 follows the same scheme as that of Theorems 4.3.1 and 4.3.4. Namely, in [21] it was proved that $\mathsf{p}(1, \mathsf{R}) \leq 1$ for each uni- or bimodal plane curve singularity. Now, a rather straightforward calculation shows that $\mathsf{p}(1, \mathsf{R}) \geq 2$, whenever the conditions *1–4* do not hold. At last, we show that these conditions imply that R dominates one of the uni- or bimodal plane curve singularities.

References

[1] Arnold V. I. Varchenko A. N., Gusein-Zade S. M., *Singularities of Differentiable Maps*, Vol. 1, Birkhäuser, 1985.

[2] Bondarenko V. M., *Bands of semi-chain sets and their representations*, Preprint, Kiev, Institute of Mathematics, 1988.

[3] Bourbaki N., *Algèbre commutative*, Paris, Hermann, 1964–1969.

[4] Dieterich E., *Solution of a non-dommestic tame classification problem from integral representation theory of finite groups*, Mem. Amer. Math. Soc., **450**, 1991.

[5] Dieterich E., *Lattice categories over curve singularities with large conductor*, Preprint 92-069, SFB 343, Universität Bielefeld, 1992.

[6] Drozd Yu. A., *Tame and wild matrix problems*, In: *Representations and Quadratic Forms*, Kiev, Institute of Mathematics, 1979, 39–74 (English translation: Amer. Math. Soc. Transl. (2) **128**, 1986, 31–55).

[7] Drozd Yu. A., *Existence of maximal orders*, Mat. Zametki, **37** (1985) 313–316.

[8] Drozd Yu. A., Greuel G.-M., *Tame-wild dichotomy for Cohen-Macaulay algebras*, Math. Ann., **294** (1992) 387–394.

[9] Drozd Yu. A., Greuel G.-M., *Cohen-Macaulay module type*, Compositio Math., **89** (1993) 315–338.

[10] Drozd Yu. A., Greuel G.-M., *Semicontinuity for representations of Cohen-Macaulay rings*, Preprint Nr. 247, Universität Kaiserslautern, Fachbereich Mathematik, 1993.

[11] Drozd Yu. A., Greuel G.-M., *On Schappert's characterization of strictly unimodal plane curve singularities*, to appear.

[12] Drozd Yu. A. Roiter A. V., *Commutative rings with a finite number of indecomposable integral representations*, Izv. Akad. Nauk SSSR. Ser. Mat., **31** (1967) 783–798.

[13] Drozd Yu. A., Shapiro V. B., *Tame and wild two by two matrix orders*, J. Algebra, **163** (1994) 258–264.

[14] Green E. L., Reiner I., *Integral representations and diagrams*, Michigan Math. J., **25** (1978) 53–84.

[15] Greuel G.-M., Knörrer H., *Einfache Kurvesingularitäten und torsionfreie Moduln*, Math. Ann., **270** (1985) 417–425.

[16] Jacobonski H., *Sur les ordres commutatifs avec un nombre fini de réseaux indécomposables*, Acta Math., **118** (1967) 1–31.

[17] Knörrer H., *Torsionfreie Moduln bei Deformation von Kurvensingularitäten*, In: *Singularities, Representations of algebras and Vector Bundles*, Lambrecht 1985, Springer-Verlag, 1987, 150–155.

[18] Mumford D., *Lectures on Curves on Algebraic Surfaces*, Princeton Univ. Press, 1966.

[19] Ringel C. M., Roggenkamp K. W., *Diagrammatic methods in representation theory of orders*, J. Algebra, **60** (1979) 11–42.

[20] Roggenkamp K. W., *Lattices over Orders*, II, Springer-Verlag, 1970.

[21] Schappert A., *A characterization of strict unimodular plane curve singularities*, In: *Singularities, Representations of Algebras and Vector Bundles*, Lambrecht 1985, Springer-Verlag, 1987, 168–177.

[22] Serre J.-P., *Algèbre local. Multiplicités*, Springer-Verlag, 1965.

[23] Wall C. T. C., *Classification of unimodal isolated singularities of complete intersections*, In: *Singularities*, Arcata 1981, Proc. Symp. Pure Math. **40**(2) (1983) 625–640.

[24] Yakovlev A. V., *Classification of 2-adique representations of the cyclic group of the eighth order*, Zap. Nauch. Semin. LOMI, **28** (1972) 93–129.

Faculty of Mechanics & Mathematics
Kiev Taras Shevchenko University
Vladimirskaya, 64, Kiev 252 017, Ukraine