On Representations of Algebraic Groups in Characteristic Two

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Abstract. Rational representations of Chevalley groups over an algebraically closed field of characteristic 2 are studied. For groups of rank ≤ 4 and their Lie algebras, all extensions of simple modules are computed.

Contents

Introduction

In this paper we investigate representations of simple algebraic groups over an algebraically closed field of characteristic 2 and of their Lie algebras. For the groups of rank 4 or less, we shall determine all of the extensions of simple modules. The central theme will be the study of some intimate connections among the groups of types B_l, C_l and D_l (and F_4 when $l = 4$). We also give calculations for those other groups of rank 4 or less which have not already been treated elsewhere ([Sin2], [An]), but this is primarily for the sake of completeness.

It has long been known that over an algebraically closed field k of characteristic 2 the groups of types B_l and C_l are isomorphic as abstract groups, but not as algebraic groups.

Thus, the relationship between them becomes more interesting when we take into account their structures as algebraic varieties. More precisely, if G and $\tilde{G}$ are simply connected groups of types B_l and C_l, defined over $\mathbf{F}_2$, then there exist *special isogenies* ([Ch], Exp. 23, Thm. 1; see also [St2], Thm. 28)

$$\sigma : G \to \tilde{G}, \qquad \tau : \tilde{G} \to G$$

such that the composites $\tau \circ \sigma$ and $\sigma \circ \tau$ are the Frobenius morphisms of G and $\tilde{G}$. The tangent maps $d\sigma$ and $d\tau$ between the Lie algebras have nonzero kernels $\mathfrak{g}_\sigma$ and $\tilde{\mathfrak{g}}_\tau$, which play an important role in the representation theory of the groups. They are the ideals generated by the short root spaces in the Lie algebras and in [St1] Steinberg showed how they can be used to sharpen his tensor product theorem, reducing the still unsolved problem of determining the characters of all simple (rational) modules for G and $\tilde{G}$ to the case of simple modules which remain simple as modules for $\mathfrak{g}_\sigma$ or $\tilde{\mathfrak{g}}_\tau$. Now $\mathfrak{g}_\sigma$ is a quotient of the direct sum of l copies of $\mathfrak{sl}_2$ (see I.5.3 below) and the only nontrivial G-module on which it acts irreducibly is the 2^l-dimensional *spin* module. The algebra $\tilde{\mathfrak{g}}_\tau$ is isomorphic to a quotient of the Lie algebra of a simply connected group of type D_l, so its simple modules are a subset of the restricted simple modules for type D_l. Thus, implicit in Steinberg's refinement is the interesting fact that all characters of simple modules for groups of types B_l, C_l and D_l will be known once they are known for D_l alone. In this way, the groups of type D_l enter naturally into our considerations. In fact, it emerges that a large part of the extension problem for simple modules for groups of types B_l, C_l and D_l reduces to questions purely about type D_l (cf. I.6.5 below).

In [Sin2], similar isogenies for the groups of types B_2 and F_4 in characteristic 2 and type G_2 in characteristic 3 were exploited to compute the simple module extensions. The kernel of the differential of the special isogeny for F_4 also happens to be isomorphic to a quotient of a Lie algebra of type D_4, and this link made it possible to draw on knowledge about D_4 in making calculations about F_4. This time, in our rank 4 calculations we shall transfer information in the opposite direction, from F_4 via D_4 to C_4 and B_4.

The first chapter of the paper is devoted to general results, valid for arbitrary ranks, about groups of types B_l, C_l and D_l. Mostly, we do nothing more than introduce notation and observe that many theorems about Frobenius kernels extend to the kernels of special isogenies. In formulating these easy generalizations one finds that when replacing the Frobenius kernel of G by $G_\sigma = \operatorname{Ker} \sigma$, the counterpart of the Steinberg module is the spin module, which leads to a variant of an important formula of Andersen and Haboush ([Ja1] II.3.19) in the cohomology of line bundles on the orthogonal flag variety. We also make some initial reductions in the calculation of extensions of simple modules.

Groups of ranks 2 and 3 are discussed in Chapter II. The calculations here have the secondary purpose of illustrating the general strategy in some cases which are relatively free of complications. The section on G_2 is independent of the rest of the paper. We have included a correction to [Sin1].

Chapter III on groups of rank 4 is where our major efforts are concentrated. The calculation of extensions of simple modules involves two main steps. The first consists of determining the action of $\tilde{G}/\tilde{G}_\tau \cong G$ on the spaces $\operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^0), \tilde{L}(\mu^0))$, where $\tilde{G}_\tau = \operatorname{Ker} \tau$ and the modules $\tilde{L}(\lambda^0)$ and $\tilde{L}(\mu^0)$ are simple modules for $\tilde{G}$ which remain simple for $\tilde{G}_\tau$.

If E denotes one of the Ext^1 groups above, then the second step is to compute the socle of the G-module $E \otimes L(\lambda)$ for all dominant weights λ. In practice, it is only necessary to consider those λ in a certain small range. There are points in both parts of the calculation which require detailed information about individual modules, usually Weyl modules and tensor products of Weyl modules or of simple modules. In an effort to avoid obscuring the general picture, these specific computations have been relegated to an appendix to Chapter III.

The methods used either require or else easily yield the structure of the groups of extensions of simple modules for the Frobenius kernels (or, equivalently, the restricted Lie algebras) as modules for the group. We describe these explicitly for the simply connected groups of rank 4 or less (II.1.4, II.2.5, II.3.3, III.1.3-4, III.2.4-5, III.3.5).

The section on A_4 does not require the preparatory material of Chapter I.

The results of this paper have been applied by the first author [Dw] to compute the 1-cohomology of the related finite groups with values in simple modules.

Throughout the paper, k will denote an algebraically closed field of characteristic 2.

Chapter I. General Results on Groups of Types B_l, C_l, and D_l.

§1. Roots. Let $e_1, \dots, e_l$ be an orthonormal basis for $\mathbf{Q}^l$ with the standard inner product $\langle \sum a_i e_i, \sum b_i e_i \rangle = \sum a_i b_i$. The sets of vectors

$$R = \{\pm e_i \pm e_j; \pm e_i |\ 1 \le i, j \le l, i \ne j\}$$

and

$$\tilde{R} = \{\pm e_i \pm e_j; \pm 2e_i |\ 1 \le i, j \le l, i \ne j\}$$

form root systems of types B_l and C_l respectively. For sets of simple roots we may take

$$S = \{\alpha_i = e_i - e_{i+1}\ (1 \le i \le l-1),\ \alpha_l = e_l\}$$

and

$$\tilde{S} = \{\tilde{\alpha}_i = e_i - e_{i+1}\ (1 \le i \le l-1),\ \tilde{\alpha}_l = 2e_l\}.$$

We shall continue with the notational convention of writing "$\tilde{x}$" for the object of type C_l corresponding to an object "x" of type B_l.

The long roots of R form a subsystem R_{long} of type D_l while the short roots form a system R_{short} of type ${A_1}^l$. For $\tilde{R}$ it is the other way around. We choose the sets of simple roots

$$S_{\text{long}} = \{\beta_i = e_i - e_{i+1}\ (1 \le i \le l-1),\ \beta_l = e_{l-1} + e_l\}, \qquad S_{\text{short}} = \{\gamma_i = e_i\ (1 \le i \le l)\},$$

$$\tilde{S}_{\text{short}} = \{\tilde{\beta}_i = e_i - e_{i+1}\ (1 \le i \le l-1),\ \tilde{\beta}_l = e_{l-1} + e_l\}, \text{ and } \tilde{S}_{\text{long}} = \{\tilde{\gamma}_i = 2e_i,\ (1 \le i \le l)\}$$

of these subsystems; they are sets of positive roots in the larger root systems.

The endomorphism $x \mapsto x^\vee = 2x/\langle x, x \rangle$ of $\mathbf{Q}^l$ interchanges R with $\tilde{R}$, swapping each simple root $\alpha_i \in S$ with $\tilde{\alpha}_i \in \tilde{S}$. This identifies $\tilde{R}$ with the dual root system $R^\vee$ of R, and R with $\tilde{R}^\vee$. We shall always regard R and $\tilde{R}$ as being dual to each other in this way.

The fundamental weights for the root systems with their simple roots described above will be defined by the equations (in which δ_{ij} is the Kronecker symbol)

$$\langle \omega_i, {\alpha_j}^\vee \rangle = \delta_{ij}, \quad \langle \delta_i, {\beta_j}^\vee \rangle = \delta_{ij}, \quad \langle \epsilon_i, {\gamma_j}^\vee \rangle = \delta_{ij},$$

and similarly for $\tilde{\omega}_i$, $\tilde{\delta}_i$ and $\tilde{\epsilon}_i$.

§2. Groups of type B_l and C_l and their Lie algebras.

2.1. Let G be a simply connected Chevalley group of type B_l over k. We may regard G as a k-group scheme, obtained by base change from a split $\mathbb{Z}$-group scheme $G_{\mathbb{Z}}$, so there is a maximal torus $T = (T_{\mathbb{Z}})_k$, defined and split over $\mathbf{F}_2$. If $X(T)$ denotes the character group of T then we may identify $X(T) \otimes_{\mathbb{Z}} \mathbf{Q}$ with $\mathbf{Q}^l$ in such a way that the root system R of §1 becomes identified with the roots of T (the nonzero characters of T afforded by the adjoint representation on the Lie algebra $\mathfrak{g}$ of G. For each $\alpha \in R$ there is an embedding (defined over $\mathbb{Z}$)

$$x_\alpha : G_a \to G$$

whose image U_α is a T-invariant subgroup with $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$, for $t \in T$, $u \in k$. Let $U = \langle U_\alpha \mid \alpha \in -R^+\rangle$. Then $B = TU$ is a Borel subgroup of G with unipotent radical U.

Let $\mathfrak{g}_\mathbb{Z} = \mathrm{Lie}(G_\mathbb{Z})$. The assumption of simple connectedness means that $\mathfrak{g}_\mathbb{Z}$ is spanned by a Chevalley basis $\{X_\alpha\ (\alpha \in R); H_\beta\ (\beta \in S)\}$. Since $G_\mathbb{Z}$ is smooth, we have $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$. We denote the images of the Chevalley basis elements by the same symbols; the element $X_\alpha \in \mathfrak{g}$ can also be described as $dx_\alpha(1)$, where dx_α is the differential of x_α at 0. We also set $\mathfrak{h} = \mathrm{Lie}\, T$.

Since the above discussion concerns general facts about simply connected semisimple groups, we can also consider a simply connected group $\tilde{G}$ of type C_l, with maximal torus $\tilde{T}$, Borel subgroup $\tilde{B}$, root homomorphisms $\tilde{x}_\alpha$ for $\alpha \in \tilde{R}$ and so on.

2.2. We define the subgroups

$$D = \langle x_\alpha(u), \alpha \in R_{\mathrm{long}}, u \in k\rangle \leq G$$

and

$$\tilde{A} = \langle \tilde{x}_\alpha(u), \alpha \in \tilde{R}_{\mathrm{long}}, u \in k\rangle \leq \tilde{G}.$$

Then D is of type D_l and $\tilde{A}$ is of type ${A_1}^l$. Both groups are simply connected. Their (scheme-theoretic) centres, represented by the group algebras over k of $X(T)/\mathbb{Z}R_{\mathrm{long}}$ and $X(\tilde{T})/\mathbb{Z}\tilde{R}_{\mathrm{long}}$ respectively (cf. [Ja1] II.1.6), are

$$Z(D) \cong \begin{cases} \mu_{(4)} & \text{if } l \text{ is odd} \\ \mu_{(2)} \times \mu_{(2)} & \text{if } l \text{ is even} \end{cases} \qquad \text{and} \qquad Z(\tilde{A}) = \tilde{T}_1 \cong {\mu_{(2)}}^l.$$

We set $\tilde{\mathfrak{a}} = \mathrm{Lie}(\tilde{A})$ and $\mathfrak{d} = \mathrm{Lie}(D)$.

2.3. All of the groups we have defined are stable under the Frobenius morphisms F_G of G and $F_{\tilde{G}}$ of $\tilde{G}$, so we also have Frobenius kernels $G_1 = \mathrm{Ker}\, F_G$, T_1, B_1, $\tilde{G}_1$, D_1 etc.

2.4. There is a well known equivalence between infinitesimal k-group schemes of height ≤ 1 and finite-dimensional restricted Lie algebras over k. Under this equivalence the Frobenius kernel of a group corresponds to the Lie algebra of the Frobenius kernel (which is the same as that of the group) in such a way that they have the same representation theory ([DG] II,§7,3.9-3.12, 4.1-1.3 and [Ja1], I.8.6). Thus, we will feel free switch between these languages, using whichever is convenient for the occasion.

§3. The special isogenies τ and σ.

3.1. The existence of isogenies between G and $\tilde{G}$ was proved in [Ch], Exp. 23-04, Th.1 and Exp 24-05. The isogenies $\sigma : G \to \tilde{G}$ and $\tau : \tilde{G} \to G$ are defined (see [St2], p146, Th. 28) by

$$\sigma : x_\alpha(t) \mapsto \begin{cases} \tilde{x}_{\alpha^\vee}(t) & \text{if } \alpha \in R_{\mathrm{long}} \\ \tilde{x}_{\alpha^\vee}(t^2) & \text{if } \alpha \in R_{\mathrm{short}} \end{cases} \tag{1}$$

and

$$\tau : \tilde{x}_\alpha(t) \mapsto \begin{cases} x_{\alpha^\vee}(t) & \text{if } \alpha \in \tilde{R}_{\text{long}} \\ x_{\alpha^\vee}(t^2) & \text{if } \alpha \in \tilde{R}_{\text{short}} \end{cases}. \tag{2}$$

The tangent maps are

$$d\sigma : X_\alpha \mapsto \begin{cases} \tilde{X}_{\alpha^\vee} & \text{if } \alpha \in R_{\text{long}} \\ 0 & \text{if } \alpha \in R_{\text{short}} \end{cases} \tag{3}$$

and

$$d\tau : \tilde{X}_\alpha \mapsto \begin{cases} X_{\alpha^\vee} & \text{if } \alpha \in \tilde{R}_{\text{long}} \\ 0 & \text{if } \alpha \in \tilde{R}_{\text{short}} \end{cases}. \tag{4}$$

We set $\mathfrak{g}_\sigma = \ker d\sigma$, $\tilde{\mathfrak{g}}_\tau = \ker d\tau$. These are restricted Lie algebras which correspond, under the equivalence described in (I.2.4), to the scheme-theoretic kernels G_σ of σ and $\tilde{G}_\tau$ of τ. Also, we may consider T_σ, $\mathfrak{h}_\sigma$, $\tilde{T}_\tau$, $\tilde{\mathfrak{h}}_\tau$, etc.

Note that $\tau \circ \sigma = F_G$ and $\sigma \circ \tau = F_{\tilde{G}}$, so G_σ is a normal subgroup scheme of G_1, $\mathfrak{g}_\sigma$ is an ideal of $\mathfrak{g}$, with $\mathfrak{g}/\mathfrak{g}_\sigma \cong \tilde{\mathfrak{g}}_\tau$, with parallel statements for $\tilde{G}_\tau$.

We let $\tilde{D} = \sigma(D)$ and $A = \tau(\tilde{A})$. These are the subgroups of $\tilde{G}$ and G, generated by the short root subgroups. The relationships among the groups we have discussed so far is given by the diagrams of group extensions (5)-(8) below. It is not difficult to check their correctness from (1)-(4) and by some elementary Lie algebra calculations. Consider, for example, the restriction of $d\sigma$ to $\mathfrak{d}$. It is immediate from (3) that $d\sigma$ maps $\mathfrak{d}$ onto $\tilde{\mathfrak{g}}_\tau$, and that the kernel lies inside $\mathfrak{h}$, which gives the third row of (7) below. We can describe the kernel explicitly. The basis element H_{β_j} is mapped to $\tilde{H}_{\tilde{\beta}_j}$. Expressing the latter in the basis $\{\tilde{H}_{\tilde{\alpha}_i}\}_{i=1}^l$ as $\tilde{H}_{\tilde{\beta}_j} = \sum_i \langle \tilde{\omega}_i, \tilde{\beta}_j^\vee \rangle \tilde{H}_{\tilde{\alpha}_i}$, we see that the matrix of $d\sigma|\mathfrak{h}$ is

$$\langle \tilde{\omega}_i, \tilde{\beta}_j^\vee \rangle = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 2 \end{pmatrix} \qquad \pmod 2.$$

Thus, the kernel $\mathfrak{d}_\sigma = \mathfrak{h}_\sigma$ is spanned by $H_{\beta_{l-1}} - H_{\beta_l}$.

REMARK.

Calculations of the same nature are also used to determine the centers of the various Lie algebras (cf. [Hu]). For example, an element $H \in \mathfrak{h}$ will belong to the center $z(\mathfrak{g})$ of $\mathfrak{g}$ if and only if it commutes with all of the elements X_α for $\alpha \in S$, and this condition is equivalent to being in the kernel of the linear map $\mathfrak{h} \to k^l$, $H \mapsto (\alpha_i(H))_{i=1}^l$, which has matrix equal to the Cartan matrix $\langle \alpha_i, {\alpha_j}^\vee \rangle$ mod 2. Thus, $z(\tilde{\mathfrak{g}}) = \langle \sum_{\substack{1 \le i \le l \\ i \text{ odd}}} \tilde{H}_{\tilde{\alpha}_i} \rangle$, and $z(\mathfrak{g}) = \langle H_{\alpha_l} \rangle = \mathfrak{h}_\sigma$. The Lie algebras $z(\mathfrak{g})$ and $z(\tilde{\mathfrak{g}})$ correspond to the centers $Z(G)$ and $Z(\tilde{G})$ of the groups as in 2.4, since the latter are both isomorphic to $\mu_{(2)}$. In particular, we see that $Z(G) = T_\sigma$. The induced morphism $G/Z(G) \to \tilde{G}$ and its differential are discussed in [Bo], V.23. (See also 5.6 below.)

Global picture.

$$
(5)\qquad
\begin{tikzcd}
1 \arrow[r] & G_\sigma \arrow[r] \arrow[d, equal] & A \arrow[r, "\sigma"] \arrow[d, "\cap"] & \tilde{A} \arrow[r] \arrow[d, "\cap"] & 1 \\
1 \arrow[r] & G_\sigma \arrow[r] & G \arrow[r, "\sigma"] & \tilde{G} \arrow[r] & 1 \\
1 \arrow[r] & T_\sigma \arrow[r] \arrow[u, "\cup"'] & D \arrow[r, "\sigma"] \arrow[u, "\cup"'] & \tilde{D} \arrow[r] \arrow[u, "\cup"'] & 1 \\
1 \arrow[r] & T_\sigma \arrow[r] \arrow[u, equal] & T \arrow[r, "\sigma"] \arrow[u, "\cup"'] & \tilde{T} \arrow[r] \arrow[u, "\cup"'] & 1
\end{tikzcd}
$$

$$
(6)\qquad
\begin{tikzcd}
1 \arrow[r] & \tilde{G}_\tau \arrow[r] \arrow[d, equal] & \tilde{D} \arrow[r, "\tau"] \arrow[d, "\cap"] & D \arrow[r] \arrow[d, "\cap"] & 1 \\
1 \arrow[r] & \tilde{G}_\tau \arrow[r] & \tilde{G} \arrow[r, "\tau"] & G \arrow[r] & 1 \\
1 \arrow[r] & \tilde{T}_\tau \arrow[r] \arrow[u, "\cup"'] & \tilde{A} \arrow[r, "\tau"] \arrow[u, "\cup"'] & A \arrow[r] \arrow[u, "\cup"'] & 1 \\
1 \arrow[r] & \tilde{T}_\tau \arrow[r] \arrow[u, equal] & \tilde{T} \arrow[r, "\tau"] \arrow[u, "\cup"'] & T \arrow[r] \arrow[u, "\cup"'] & 1
\end{tikzcd}
$$

Infinitesimal picture.

$$
(7)\qquad
\begin{tikzcd}
1 \arrow[r] & G_\sigma \arrow[r] \arrow[d, equal] & A_1 \arrow[r, "\sigma"] \arrow[d, "\cap"] & \tilde{T}_\tau \arrow[r] \arrow[d, "\cap"] & 1 \\
1 \arrow[r] & G_\sigma \arrow[r] & G_1 \arrow[r, "\sigma"] & \tilde{G}_\tau \arrow[r] & 1 \\
1 \arrow[r] & T_\sigma \arrow[r] \arrow[u, "\cup"'] & D_1 \arrow[r, "\sigma"] \arrow[u, "\cup"'] & \tilde{G}_\tau \arrow[r] \arrow[u, equal] & 1 \\
1 \arrow[r] & T_\sigma \arrow[r] \arrow[u, equal] & T_1 \arrow[r, "\sigma"] \arrow[u, "\cup"'] & \tilde{T}_\tau \arrow[r] \arrow[u, "\cup"'] & 1
\end{tikzcd}
$$

(8)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{G}_\tau & \longrightarrow & \tilde{D}_1 & \xrightarrow{\tau} & T_\sigma & \longrightarrow & 1 \\
 & & \| & & \downarrow\cap & & \downarrow\cap & & \\
1 & \longrightarrow & \tilde{G}_\tau & \longrightarrow & \tilde{G}_1 & \xrightarrow{\tau} & G_\sigma & \longrightarrow & 1 \\
 & & \uparrow\cup & & \uparrow\cup & & \| & & \\
1 & \longrightarrow & \tilde{T}_\tau & \longrightarrow & \tilde{A}_1 & \xrightarrow{\tau} & G_\sigma & \longrightarrow & 1 \\
 & & \| & & \uparrow\cup & & \uparrow\cup & & \\
1 & \longrightarrow & \tilde{T}_\tau & \longrightarrow & \tilde{T}_1 & \xrightarrow{\tau} & T_\sigma & \longrightarrow & 1
\end{array}$$

3.2. For any representation π of $\tilde{G}$, the composite $\pi \circ \sigma$ is a representation of G. If M is the module for π, we write $M^{(\sigma)}$ for the G-module thus obtained. Conversely, any G-module N on which G_σ acts trivially is of the form $M^{(\sigma)}$ for some $\tilde{G}$-module M, and we shall then write this module M as $N^{(\sigma^{-1})}$. Similar notation will be used for twisting and untwisting representations by τ. Twisting by the n-th power of the Frobenius map will be denoted by $M^{(2^n)}$ (many authors write $M^{(n)}$).

Since σ maps T onto $\tilde{T}$ and τ maps $\tilde{T}$ onto T we have injective homomorphisms

$$X(\tilde{T}) \xrightarrow{\sigma} X(T) \qquad \text{and} \qquad X(T) \xrightarrow{\tau} X(\tilde{T}).$$

(These maps should perhaps be called σ^* and τ^* but the chosen notation fits better with the standard notation for Frobenius twists.) The composites $\tau \circ \sigma$ and $\sigma \circ \tau$ are multiplication by 2.

Explicitly, we have (see §1)

(1)
$$\sigma : \tilde{\omega}_i \mapsto \omega_i \quad (1 \leq i \leq l-1), \tilde{\omega}_l \mapsto 2\omega_l,$$

(2)
$$\tau : \omega_i \mapsto 2\tilde{\omega}_i \quad (1 \leq i \leq l-1), \omega_l \mapsto \tilde{\omega}_l.$$

3.3. Let $X_+(\tilde{T})$ be the set of dominant weights $\{\lambda \in X(\tilde{T}) | \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \tilde{R}^+\}$. We define the *restricted* weights

$$X_1(\tilde{T}) = \{\lambda \in X_+(\tilde{T}) | \langle \lambda, \alpha^\vee \rangle < 2\}$$

and the *τ-restricted* weights

$$X_\tau = X_\tau(\tilde{T}) = \{\lambda \in X_1(\tilde{T}) | \langle \lambda, \tilde{\alpha}_l^\vee \rangle = 0 \}.$$

The sets $X_1(T)$ and $X_\sigma = X_\sigma(T)$ are defined analogously ($X_\sigma = X_\sigma(T) = \{\lambda \in X_1(T) | \langle \lambda, \alpha_j^\vee \rangle = 0 \text{ for all } j = 1, \ldots, l-1\}$); we see that X_τ is the set of those weights in $X_1(\tilde{T})$ which do not involve $\tilde{\omega}_l$, while $X_\sigma = \{0, \omega_l\}$.

§4. Simple modules.

4.1. The simple rational G-modules are parametrized by their highest weights $\lambda \in X_+(T)$, the simple module with highest weight λ being denoted by $L(\lambda)$ ([Ja1] II.2.7). Similarly, the simple $\tilde{G}$-modules will be denoted $\tilde{L}(\mu)$, $\mu \in X(\tilde{T})$. We shall also need corresponding notation for the simple modules of the subgroups D and $\tilde{A}$. We shall use $L_D(\lambda)$ for D and $\tilde{L}_{\tilde{A}}(\mu)$ for $\tilde{A}$, often dropping the subscripts unless there is danger of confusion, for example between the simple modules for G and D with the same highest weight.

It is well known that the modules $L(\lambda)$ for $\lambda \in X_1(T)$ and $\tilde{L}(\mu)$ for $\mu \in X_1(\tilde{T})$ are simple for the Frobenius kernels G_1 and $\tilde{G}_1$ respectively. In [St1] it is also shown that the $\tilde{L}(\lambda)$ for $\lambda \in X_\tau$ are simple for $\tilde{\mathfrak{g}}_\tau$ (or equivalently for $\tilde{G}_\tau$) (and that the $L(\mu)$, $\mu \in X_\sigma$ are simple $\mathfrak{g}_\sigma$-modules). This can also be seen as follows. For $\lambda \in X_\tau$, we have $\sigma\lambda \in X_1(T)$. Thus, the simple G-module $L(\sigma\lambda)$ is simple for $\mathfrak{g}$. It is the $\mathfrak{g}$-module obtained from the $\tilde{\mathfrak{g}}$-module $\tilde{L}(\lambda)$ by restriction along $d\sigma$. But $d\sigma(\mathfrak{g}) = \tilde{\mathfrak{g}}_\tau$, which shows that $\tilde{L}(\lambda)$ is simple for $\tilde{\mathfrak{g}}_\tau$.

Since the subalgebra $\mathfrak{d} \leq \mathfrak{g}$ is also mapped onto $\tilde{\mathfrak{g}}_\tau$ by $d\sigma$, we see that the simple G-module $L(\sigma\lambda)$ is also simple for $\mathfrak{d}$, hence for D as well, which in turn means that $\tilde{L}(\lambda)$ is simple for $\tilde{D}$. Thus, the τ-restricted simple $\tilde{G}$-modules afford irreducible representations of D, G, $\tilde{D}$, G_1, D_1, $\tilde{G}_1$, $\tilde{D}_1$ and $\tilde{G}_\tau$, with a similar conclusion for σ-restricted $\tilde{G}$-modules.

4.2. Now every $\lambda \in X_+(T)$ can be written as

$$\lambda = \lambda^0 + \sigma\lambda^1 + 2\lambda^2 + 2\sigma\lambda^3 + \cdots \tag{1}$$

with $\lambda^i \in X_\sigma$ for even indices i and $\lambda^i \in X_\tau$ for odd indices. The weights $\mu \in X_+(\tilde{T})$ have similar expressions with even and odd transposed and with τ in place of σ.

We adopt the convention that whenever a weight $\lambda \in X(T)$ is expanded as in (1), or partially so as

$$\lambda = \lambda^0 + \sigma\lambda^1 + 2\lambda^2 + 2\sigma\overline{\lambda}, \tag{2}$$

the λ^i will always lie in X_σ or X_τ (whichever makes sense) and $\overline{\lambda}$ will be in $X_+(T)$ or $X_+(\tilde{T})$ (depending on the previous term in the expansion). Similarly for characters of $\tilde{T}$.

A sharpened version of Steinberg's tensor product theorem ([St1], Theorem 11.1) states that for λ as in (1), we have a G-module isomorphism

$$L(\lambda) \cong L(\lambda^0) \otimes \tilde{L}(\lambda^1)^{(\sigma)} \otimes L(\lambda^2)^{(2)} \otimes \tilde{L}(\lambda^3)^{(2\sigma)} \otimes \cdots \tag{3}$$

and for $\mu \in X(\tilde{T})$ (expanded similarly) we have a $\tilde{G}$-module isomorphism

$$\tilde{L}(\mu) \cong \tilde{L}(\mu^0) \otimes L(\mu^1)^{(\tau)} \otimes \tilde{L}(\mu^2)^{(2)} \otimes L(\mu^3)^{(2\tau)} \otimes \cdots \tag{4}$$

Remark.

1) Recall that $X_\sigma = \{0, \omega_l\}$. The Weyl module $V(\omega_l)$ is the 2^l-dimensional *spin* module. Its weights $\{(\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2},)\}$ form a single orbit under the Weyl group of R so it is equal to $L(\omega_l)$. As we saw in (4.1), the modules $L(\sigma\lambda)$, $\lambda \in X_\tau$ are simple modules for

the group D, a simply connected group of type D_l. Therefore, the tensor product theorem tells us that once we know the characters of the restricted simple modules over k for type D_l, we shall know the characters of all simple modules for types B_l, C_l and D_l.

The following easy general observations, stated for G_σ are valid also for $\tilde{G}_\tau$ and all Frobenius kernels, with the obvious changes and will be used later for calculations with all of these groups.

Lemma. *Let M be a finite-dimensional G-module and let $\lambda = \lambda^0 + \sigma\hat{\lambda}$ and $\mu = \mu^0 + \sigma\hat{\mu}$ be dominant weights with λ^0, $\mu^0 \in X_\sigma$. Then the following hold.*

(a)

$$\mathrm{Hom}_G(L(\lambda), M \otimes L(\mu)) \cong \mathrm{Hom}_{\tilde{G}}(\mathrm{Hom}(L(\hat{\lambda}), \{\mathrm{Hom}_{G_\sigma}(L(\lambda^0), M \otimes L(\mu^0))\}^{\sigma^{-1}} \otimes L(\hat{\mu})))$$

In particular, if $\tilde{G}$ acts trivially on the first tensor factor (in braces), then $\mathrm{Hom}_G(L(\lambda), M \otimes L(\mu))$ is isomorphic to this factor if $\hat{\lambda} = \hat{\mu}$ and is zero otherwise.

(b) *Every submodule of $M^{(\sigma)} \otimes L(\mu^0)$ is of the form $N^{(\sigma)} \otimes L(\mu^0)$ for some submodule N of M.*

(c) *The evaluation map*

$$\mathrm{Hom}_{G_\sigma}(L(\mu^0), M) \otimes L(\mu^0) \to M$$

is injective

Proof: Part (a) is simply the fact the fixed points for G are the G/G_σ fixed points in the G_σ fixed points. Now apply (a) with $M^{(\sigma)}$ and μ^0 in place of M and $L(\mu)$. This shows that If $L \cong L(\lambda)$ is a simple submodule of $M^{(\sigma)} \otimes L(\mu^0)$, then $\lambda^0 = \mu^0$ and M has a submodule $\hat{L} \cong L(\hat{\lambda})$ with $L = \hat{L} \otimes L(\mu^0)$, so (b) holds for simple submodules. The general case follows by a straightforward induction on the composition length of the submodule. To prove (c), we apply (b) with $M^{(\sigma)}$ replaced by $\mathrm{Hom}_{G_\sigma}(L(\mu^0), M)$. The kernel of the evaluation map must be of the form $N \otimes L(\mu^0)$ for some submodule N of $\mathrm{Hom}_{G_\sigma}(L(\mu^0), M)$, and by definition of the evaluation map, we must have $N = 0$.

One consequence of (c) which will be used at several points later on is the following. (We discuss the D_1 case.) Let λ, μ, $\nu \in X_1$ (restricted weights for D). We want a simple sufficient condition for D to act trivially on $\mathrm{Hom}_{D_1}(L(\lambda), L(\mu) \otimes L(\nu))$. Let λ^* denote the highest weight of $\tilde{L}(\lambda)^*$ and let ρ be the half-sum of the positive roots (of R_L). Then $\rho = \rho^*$, so $\langle \lambda, \rho \rangle = \langle \lambda^*, \rho \rangle$. We may therefore assume without loss that

$$\langle \lambda, \rho \rangle \geq \langle \mu, \rho \rangle \geq \langle \nu, \rho \rangle. \tag{5}$$

The D_1 version of (c) says that if $L(2\omega)$ is a composition factor of $\mathrm{Hom}_{D_1}(L(\lambda), L(\mu) \otimes L(\nu))$, then $L(\lambda + 2\omega)$ is one of $L(\mu) \otimes L(\nu)$, hence

$$\langle \mu + \nu, \rho \rangle \geq \langle \lambda + 2\omega, \rho \rangle. \tag{6}$$

COROLLARY 4.2.1 (D_1 VERSION). *Let λ, μ, $\nu \in X_1$. Then D acts trivially on $\mathrm{Hom}_{D_1}(L(\lambda), L(\mu) \otimes L(\nu))$ if*

$$\langle \nu, \rho \rangle < \min_{\omega \in X_+(T) \setminus \{0\}} \langle 2\omega, \rho \rangle.$$

PROOF: If the inequality holds, then every composition factor must be trivial, so the module is trivial since $\mathrm{Ext}^1_D(k, k) = 0$. (Note that any self-dual weight β with the property that $\langle \alpha, \beta \rangle \geq 0$ for all positive roots α could have been used in place of ρ.)

COROLLARY 4.2.2 (SPECIAL CASE). *Let $\tilde{\lambda}, \tilde{\mu} \in \tilde{X}_\tau$. Then $\tilde{G}$ acts trivially on*

$$\mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\tilde{\lambda}), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\tilde{\mu})).$$

PROOF: We check the hypotheses of (the $\tilde{G}_\tau$-version of) Corollary 4.2.1 with $\nu = \tilde{\omega}_1$ and using $\tilde{\rho} = (l, l-1, \ldots, 1)$ in place of ρ. Assume $l \geq 3$. For $\omega \in X_+(T) \setminus \{0\}$, the value of $\langle \tau\omega, \tilde{\rho} \rangle$ is minimized when $\tau\omega = 2\tilde{\omega}_1$, so the inequality in the hypothesis of Corollary 4.2.1 is satisfied. Therefore $\tilde{G}$ acts trivially on

$$\mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\tilde{\lambda}), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\tilde{\mu}))$$

when $l \geq 3$. It is easy to see that the action is also trivial when $l \leq 2$.

4.3. Set $\tilde{\rho}_\tau = \tilde{\omega}_1 + \cdots \tilde{\omega}_{l-1}$. Then $\sigma(\tilde{\rho}_\tau)$ is the weight of the first Steinberg module for D. It follows ([Ja1] II.10.2) that $L(\sigma(\tilde{\rho}_\tau))$ is injective for D_1 and that $\tilde{L}(\tilde{\rho}_\tau)$ is injective for $\tilde{G}_\tau$. Similarly, $L(\omega_l)$ is a injective simple module for G_σ. Its restriction along τ is the simple $\tilde{A}$-module which is the tensor product of l copies of the natural module for $\mathrm{SL}_2(k)$, one for each direct factor of $\tilde{A} \cong (\mathrm{SL}_2(k))^l$.

Thus, we see that when considering τ and σ the modules $\tilde{L}(\tilde{\rho}_\tau)$ and $L(\omega_l)$ assume the role usually played by the first Steinberg module when the Frobenius map is involved.

§5. Adjoint modules.

5.1. If $G^1 \xrightarrow{\phi} G^2$ is a morphism of k-group schemes, then the tangent map $d\phi : \mathrm{Lie}(G^1) \to \mathrm{Lie}(G^2)$ is a map of G^1-modules with respect to the adjoint actions of the groups on their Lie algebras, with $\ker d\phi = \mathrm{Lie}(\mathrm{Ker}\, \phi)$. In particular if G^1 is semisimple and V is a finite-dimensional G^1-module whose weights generate the character group of a maximal torus of G^1, then we have an embedding of G^1-modules

$$\mathrm{Lie}(G^1) \hookrightarrow \mathrm{End}_k(V) \cong V^* \otimes V. \tag{1}$$

We continue now with the notation of §§2-4. Our aim is to describe the submodule structure of Lie algebras $\mathfrak{g}$, $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}_\tau$, $\mathfrak{g}_\sigma$, $\mathfrak{d}$, $\tilde{\mathfrak{d}}$, $\tilde{\mathfrak{a}}$ and $\mathfrak{a}$ with respect to the adjoint actions of the various groups. These module structures can be found in [Hiß], which also treats the case of finite fields, but for our purposes we prefer to give a treatment in the context of the diagrams 3.1(5)-(8).

5.2. Consider the Lie algebra $\mathfrak{d}$, a Chevalley algebra of type D_l, $l \geq 3$. Its centre is (cf. remark in §3)

$$z(\mathfrak{d}) = \begin{cases} \langle H_{\beta_{l-1}} - H_{\beta_l} \rangle & \text{if } l \text{ is odd,} \\ \langle H_{\beta_{l-1}} - H_{\beta_l}, H_{\beta_1} + H_{\beta_3} + \cdots + H_{\beta_{l-1}} \rangle & \text{if } l \text{ is even.} \end{cases} \tag{1}$$

The D-module $\mathfrak{d}/z(\mathfrak{d})$ is simple (hence isomorphic to the simple module with highest weight equal to the maximal root $\beta_{\max} = (1, 1, 0, \ldots, 0)$ which is δ_2 if $l > 3$ and $\delta_1 + \delta_3$ if $l = 3$ (cf. I.1(1))). To see this, recall first the general fact the adjoint module is the Weyl module for the longest root. Its weights are the roots (each with multiplicity one) and 0. Thus, if a D-submodule of $\mathfrak{d}$ has a nonzero weight, then this weight is a root and since the Weyl group of R_{long} acts transitively on the roots, it follows that the submodule has the maximal root as a weight, hence is the whole module. Therefore, the only weight of any proper submodule is 0. Finally, since $\operatorname{Ext}^1_D(k,k) \cong H^1(D,k) = 0$, we see that any proper submodule of $\mathfrak{d}$ is trivial for D, hence contained in $z(\mathfrak{d})$. Note that this argument would also work for simply connected semisimple groups of type A_l and E_l, where all roots have the same length.

The third row of 3.1(7) gives a short exact sequence

$$0 \longrightarrow \mathfrak{h}_\sigma \longrightarrow \mathfrak{d} \xrightarrow{d\sigma} (\tilde{\mathfrak{g}}_\tau)^{(\sigma)} \longrightarrow 0 \tag{2}$$

of D-modules and Lie algebras. Now $\beta_{\max} = \sigma\tilde{\omega}_2$, so $\mathfrak{d}/z(\mathfrak{d}) \cong L_D(\beta_{\max})$ is the restriction along σ of the τ-restricted simple $\tilde{G}$-module $\tilde{L}(\tilde{\omega}_2)$. Also, it is easy to check (using (1) and the remark in §3) that $\mathfrak{h}_\sigma$ is the one-dimensional subspace $\langle H_{\beta_{l-1}} - H_{\beta_l} \rangle \leq z(\mathfrak{d})$ so that when l is even $d\sigma(z(\mathfrak{d})) = \langle \tilde{H}_{\tilde{\alpha}_1} + \tilde{H}_{\tilde{\alpha}_3} + \cdots + \tilde{H}_{\tilde{\alpha}_{l-1}} \rangle = z(\tilde{\mathfrak{g}})$ (cf. the remark in 3.1). Thus, we have the following conclusion from (2).

Lemma.

(a) *If l is odd then $\tilde{\mathfrak{g}}_\tau \cong \tilde{L}(\tilde{\omega}_2)$*

(b) *If l is even, then there is a nonsplit exact sequence*

$$0 \longrightarrow k \longrightarrow \tilde{\mathfrak{g}}_\tau \longrightarrow \tilde{L}(\tilde{\omega}_2) \longrightarrow 0,$$

where the trivial submodule is $\langle \tilde{H}_{\tilde{\alpha}_1} + \tilde{H}_{\tilde{\alpha}_3} + \cdots + \tilde{H}_{\tilde{\alpha}_{l-1}} \rangle = z(\tilde{\mathfrak{g}})$.

5.3. To describe $\mathfrak{g}_\sigma$, we consider the exact sequence of $\tilde{A}$-modules and Lie algebras

$$0 \longrightarrow \tilde{\mathfrak{h}}_\tau \longrightarrow \tilde{\mathfrak{a}} \xrightarrow{d\tau} (\mathfrak{g}_\sigma)^{(\tau)} \longrightarrow 0, \tag{1}$$

obtained from 3.1(8). The $\tilde{A}$-module $\tilde{\mathfrak{a}}$ is the direct sum of l copies $\langle \tilde{X}_{\tilde{\gamma}_i}, \tilde{H}_{\tilde{\gamma}_i}, \tilde{X}_{-\tilde{\gamma}_i} \rangle$ of $\mathfrak{sl}_2$. Now one easily checks that $\tilde{\mathfrak{h}}_\tau = \langle \tilde{H}_{\tilde{\gamma}_i} - \tilde{H}_{\tilde{\gamma}_l} \mid 1 \leq i \leq l-1 \rangle$, so $z(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{h}}$ is mapped by $d\sigma$ onto $\langle H_{\gamma_l} \rangle$, which is $z(\mathfrak{g})$. The quotient $\mathfrak{g}_\sigma/z(\mathfrak{g})$ is therefore a $2l$-dimensional module for $\tilde{G}$ (acting via τ as in (1)) with weights $\pm\tilde{\gamma}_i$, $1 \leq i \leq l$. Since $\tilde{\gamma}_1 = 2\tilde{\omega}_1$ and $\tilde{L}(\tilde{\omega}_1)$ is the natural $2l$-dimensional module for $\tilde{G} \cong \mathrm{Sp}_{2l}(k)$, we see that the quotient is isomorphic to $\tilde{L}(2\tilde{\omega}_1)$. Since $\tilde{H}_{\tilde{\gamma}_i} = [\tilde{X}_{\tilde{\gamma}_i}, \tilde{X}_{-\tilde{\gamma}_i}]$, it follows that $z(\mathfrak{g})$ lies in the kernel of every $\tilde{a}$-module homomorphism of $\mathfrak{g}_\sigma$ into k. We have proved the following result.

LEMMA. *There is a nonsplit exact sequence of G-modules*

$$0 \longrightarrow k \longrightarrow \mathfrak{g}_\sigma \longrightarrow L(\omega_1) \longrightarrow 0,$$

where the trivial submodule is $\langle H_{\alpha_l} \rangle = z(\mathfrak{g})$.

Finally, we note that since $\mathfrak{g}_\sigma$ has the same weights as the natural $(2l+1)$-dimensional orthogonal module for $G \cong \mathrm{Spin}_{2l+1}(k)$, it is clear that $\mathfrak{g}_\sigma$ is isomorphic to the Weyl module $V(\omega_1)$.

5.4. Set $V = \tilde{L}(\tilde{\omega}_1)(= \tilde{V}(\tilde{\omega}_1))$, the natural module for $\tilde{G}$. Since $V \cong V^*$ as $\tilde{G}$-modules, we have an embedding of $\tilde{G}$-modules $\tilde{\mathfrak{g}} \hookrightarrow \mathrm{End}(V) \cong V^* \otimes V \cong (V \otimes V)^*$. Since the maximal root of $\tilde{R}$ is $2\tilde{\omega}_1 = (2, 0, \ldots, 0)$, we have $\tilde{\mathfrak{g}} \cong \tilde{V}(2\tilde{\omega}_1)$, so we may identify $\tilde{\mathfrak{g}}$ with the submodule of $(V \otimes V)^*$ generated by the unique one-dimensional subspace of weight $2\tilde{\omega}_1$. This weight is clearly a weight of $S^2(V)^* \leq (V \otimes V)^*$, and now a comparison of dimensions shows that

$$\tilde{\mathfrak{g}} \cong S^2(V)^*. \tag{1}$$

Moreover, from the natural exact sequence

$$0 \longrightarrow \wedge^2(V)^* \longrightarrow S^2(V)^* \longrightarrow V^{(2)*} \longrightarrow 0 \tag{2}$$

we see that

$$\mathrm{rad}\, \tilde{\mathfrak{g}} \cong \wedge^2(V)^* \cong \wedge^2(V). \tag{3}$$

Now V carries a unique (up to scalar) nonzero $\tilde{G}$-invariant symplectic form (in $\wedge^2(V)^*$), whose kernel is therefore the unique $\tilde{G}$-submodule of codimension one in $\wedge^2(V) \cong \mathrm{rad}\, \tilde{\mathfrak{g}}$. On comparing dimensions (using 5.2(2)) we see that this submodule is isomorphic to $\tilde{\mathfrak{g}}_\tau$.

We now consider the two cases. First, when l is odd we have $\tilde{\mathfrak{g}}_\tau \cong \tilde{L}(\tilde{\omega}_2)$ by Lemma 5.2(a), so the self-dualtiy of $\wedge^2(V)$ forces

$$\mathrm{rad}\, \tilde{\mathfrak{g}} = z(\tilde{\mathfrak{g}}) \oplus \tilde{\mathfrak{g}}_\tau \cong k \oplus \tilde{L}(\tilde{\omega}_2). \tag{4}$$

In the even case, the self-duality of $\wedge^2(V)$ and the structure of $\tilde{\mathfrak{g}}_\tau$ given by Lemma 5.2(b) imply that $\mathrm{rad}\, \tilde{\mathfrak{g}} \cong \wedge^2(V)$ must be uniserial with composition factors in descending order k, $\tilde{L}(\tilde{\omega}_2)$, $k = z(\tilde{\mathfrak{g}})$. Thus, in this case $\tilde{\mathfrak{g}}$ is also uniserial. In fact, we can see from the the natural exact sequence

$$0 \longrightarrow S^2(V)^* \longrightarrow (V \otimes V)^* \longrightarrow \wedge^2(V)^* \longrightarrow 0 \tag{5}$$

that the self-dual module $V \otimes V$ is uniserial.

REMARKS.

1) The description above of $\tilde{\mathfrak{g}}_\tau$ as the unique $\tilde{G}$-submodule of codimension one in the exterior square of the natural module shows that it isomorphic to the Weyl module $\tilde{V}(\tilde{\omega}_2)$.
2) The results of this paragraph, together with the exact sequence (cf. 3.1(8))

$$0 \longrightarrow \tilde{\mathfrak{g}}_\tau \longrightarrow \tilde{\mathfrak{g}} \xrightarrow{d\tau} (\mathfrak{g}_\sigma)^{(\tau)} \longrightarrow 0 \tag{6}$$

provide an alternative proof of Lemma 5.3.

5.5. We now consider $\mathfrak{g}_\sigma \cong V(\omega_2)$. We have an exact sequence of G-modules (cf. 3.1(7))

$$0 \longrightarrow \mathfrak{g}_\sigma \longrightarrow \mathfrak{g} \xrightarrow{d\sigma} (\tilde{\mathfrak{g}}_\tau)^{(\sigma)} \longrightarrow 0. \tag{1}$$

As before, we have two cases.

If l is odd, then since $(\tilde{\mathfrak{g}}_\tau)^{(\sigma)} \cong L(\omega_2)$, by Lemma 5.2(a), we have $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}_\sigma$, whose structure is described in Lemma 5.3. Thus, $\mathfrak{g}$ is uniserial with composition factors $L(\omega_2)$, $L(\omega_1)$, $k = z(\mathfrak{g})$.

If l is even, we claim that $\mathfrak{g}$ is again uniserial, with composition factors in the order $L(\omega_2)$, k $L(\omega_1)$, $k = z(\mathfrak{g})$. By Lemma 5.2 and (1), it is clear that $\mathfrak{g}$ has such a composition series, and from the structures of $\tilde{\mathfrak{g}}_\tau$ and $\mathfrak{g}_\sigma$ given in Lemma 5.3, the uniseriality will be proved if we show $\mathfrak{g}/z(\mathfrak{g})^G = 0$. We can repeat an argument from (5.2): If this space were nonzero, its inverse image in $\mathfrak{g}$ would be a submodule in which all composition factors were k, so G would act trivially on it since $\mathrm{Ext}^1_G(k,k) \cong H^1(G,k) = 0$. But this would imply that the submodule is contained in $z(\mathfrak{g})$, a contradiction.

5.6. We end this section with a discussion of the Lie algebras arising from groups of type B_l, C_l, and D_l which are not simply connected. The Frobenius map of the simply connected groups induces multiplication by 2 on the characters of the maximal tori, and the special isogenies also correspond to maps of character groups which send roots to integer multiples of roots. Therefore, these morphisms induce morphisms of the quotients $\overline{G}$ etc. of the simply connected groups by arbitrary central subgroups, and we can consider the kernels $\overline{G}_1$, $\overline{G}_\sigma$ etc. of these induced morphisms. As in the simply connected case, the representation theory of these kernels may be be identified with that of their (restricted) Lie algebras. In the case of the Frobenius kernels, these are just the Lie algebras of the groups $\overline{G}$ etc. These Lie algebras are mod 2 reductions of $\mathbb{Z}$-Lie subalgebras of $\mathfrak{g}_\mathbb{Z} \otimes \mathbb{Q}$, $\tilde{\mathfrak{g}}_\mathbb{Z} \otimes \mathbb{Q}$, and $\mathfrak{d}_\mathbb{Z} \otimes \mathbb{Q}$ respectively. We may take Chevalley bases (with the notation of 2.1) for the rational Lie algebras in each case and then describe $\mathbb{Z}$-bases of the $\mathbb{Z}$-Lie algebras in terms of the Chevalley bases. In fact, the root spaces are always spanned by the root vectors X_α of a Chevalley basis, so each of the $\mathbb{Z}$-Lie algebras, hence the resulting k-Lie algebras, will be determined by their intersection with $\mathfrak{h}_\mathbb{Q}$, that is, by the Lie algebra of the split maximal torus in the corresponding $\mathbb{Z}$-group. These in turn can be found from the cocharacters, hence from the characters of the maximal torus.

We begin with groups of type D_l, $(l \geq 3)$, where we must separate the even and odd cases. We use the notation of 2.2, in which D is a simply connected k-group of type D_l, with maximal torus T, with root system $R = R_{\mathrm{long}}$, fundamental roots β_i and fundamental dominant weights δ_i.

Type D_l, l even. Here $X(T)/\mathbb{Z}R \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the nontrivial elements being the images of δ_1, δ_{l-1}, and δ_l. Let $X^{(1)}$, $X^{(l-1)}$ and $X^{(l)}$ are the sublattices of $X(T)$ generated by the roots and each of these three weights. They may be identified as the character groups of maximal tori $T^{(1)}$, $T^{(l-1)}$ and $T_{(l)}$ in groups $D^{(1)}$, $D^{(l-1)}$ and $D^{(l)}$ which are quotients of D by central subgroups isomorphic to $\mu_{(2)}$. $D^{(1)}$ comes from the orthogonal representation and the others from the two half-spin representations. We define the following three elements of $\mathfrak{h}_{\mathbb{Q}} = \mathrm{Lie}(T_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\begin{aligned} H^{(1)} &= \frac{1}{2}(H_{\beta_{l-1}} - H_{\beta_l}), \\ H^{(l-1)} &= \frac{1}{2}(\sum_{r=1}^{l-1} rH_{\beta_r}) + \frac{(l-2)}{4}(H_{\beta_{l-1}} - H_{\beta_l}), \\ H^{(l)} &= H^{(1)} + H^{(1-1)} = \frac{1}{2}(\sum_{r=1}^{l-1} rH_{\beta_r}) + \frac{l}{4}(H_{\beta_{l-1}} - H_{\beta_l}). \end{aligned} \tag{1}$$

Then the image of $2H^{(i)}$ in $\mathfrak{d} = \mathrm{Lie}(D)$ is central and spans the kernel of the map $\mathfrak{d} \to \mathfrak{d}^{(i)} = \mathrm{Lie}\, D^{(i)}$. Moreover, $\mathfrak{h}_{\mathbb{Z}}^{(i)} = \mathrm{Lie}(T_{\mathbb{Z}}^{(i)})$ is the lattice in $\mathfrak{h}_{\mathbb{Q}}$ spanned by the elements H_β, $\beta \in R$ together with $H^{(i)}$. The center of $\mathfrak{d}^{(i)}$ is spanned by the common image of $H^{(j)}$, $j \neq i$. For the adjoint group $\overline{D}$, we take the lattice spanned by all the H_βs and $H^{(i)}$s. The radical (=socle) strucures of the D-modules $\mathfrak{d}$, $\mathfrak{d}^{(i)}$ and $\overline{\mathfrak{d}}$ are now easily calculated, the results being depicted below.

$$\mathfrak{d} : \begin{matrix} L(\delta_2) \\ k \oplus k \end{matrix}, \qquad \mathfrak{d}^{(1)} : \begin{matrix} k \\ L(\delta_2) \\ k \end{matrix}, \qquad \mathfrak{d}^{(l-1)} : \begin{matrix} k \\ L(\delta_2) \\ k \end{matrix}, \qquad \mathfrak{d}^{(l)} : \begin{matrix} k \\ L(\delta_2) \\ k \end{matrix}, \qquad \overline{\mathfrak{d}} : \begin{matrix} k \oplus k \\ L(\delta_2) \end{matrix}. \tag{2}$$

Type D_l, l odd. Here $X(T)/\mathbb{Z}R \cong \mathbb{Z}/4\mathbb{Z}$. We denote by $\overline{D}$ the adjoint group and by $\tilde{D}$ the intermediate group. The latter is isomorphic to the subgroup of $\tilde{G}$ which we have already named $\tilde{D}$. The lattice $\overline{\mathfrak{h}}_{\mathbb{Z}}$ is spanned by the H_βs and the element $\overline{H} = \frac{1}{2}(\sum_{r=1}^{l-1} rH_{\beta_r}) + \frac{l}{4}(H_{\beta_{l-1}} - H_{\beta_l})$, while $\tilde{\mathfrak{h}}_{\mathbb{Z}}$ is spanned by the H_βs and $\tilde{H} = 2\overline{H}$. The image in $\mathfrak{d}$ of $2\tilde{H}$ spans $z(\mathfrak{d})$, which is the kernel of the map $\mathfrak{d} \to \tilde{\mathfrak{d}}$. The image of this map is an ideal isomorphic to $\mathfrak{g}_\tau$ (as Lie algebras and D-modules). The image of $\tilde{H}$ in $\tilde{\mathfrak{d}}$ spans $z(\tilde{\mathfrak{d}})$, which is the kernel of the map $\tilde{\mathfrak{d}} \to \overline{\mathfrak{d}}$. It follows that $\tilde{\mathfrak{d}} \cong z(\tilde{\mathfrak{d}}) \oplus \mathfrak{g}_\tau$ as Lie algebras and D-modules.

Again, we give pictures of the module structures. For $l = 3$, replace δ_2 by $\delta_2 + \delta_3$. (Recall that for $l = 3$ the centre node of the Dynkin diagram is labelled "1").

$$\mathfrak{d} : \begin{matrix} L(\delta_2) \\ k \end{matrix}, \qquad \tilde{\mathfrak{d}} : k \oplus L(\delta_2), \qquad \overline{\mathfrak{d}} : \begin{matrix} k \\ L(\delta_2) \end{matrix}. \tag{3}$$

Remark.The module $\tilde{\mathfrak{d}}$ provides and example in which the reduction mod p of an admissible lattice in a simple module for a complex simple Lie algebra is decomposable as a module for the algebraic group in characteristic p. (See [Lin]. For arbitrary p, examples can be derived in an analogous way from the group $G = \mathrm{SL}_{p^2}(K)$, where K is an algebraically closed field of characteristic p. The center, which is isomorphic to $\mu_{(p^2)}$, contains a subgroup $Z \cong \mu_{(p)}$, and the G-module $\mathrm{Lie}(G/Z)$ is the direct sum of the center and the commutator subalgebra.)

We now discuss the Lie algebras related to the adjoint groups $\overline{G}$ and $\overline{\tilde{G}}$ of types B_l and C_l. Let $\overline{\mathfrak{g}} = \mathrm{Lie}(\overline{G})$, $\overline{\tilde{\mathfrak{g}}}$, $\overline{\mathfrak{g}_\sigma}$ etc. denote the objects for the adjoint groups corresponding to the unbarred objects for the simply connected groups. By arguments used in the discussions of type D_l above and the simply connected groups of type B_l and C_l earlier, we can determine the structures of the modules $\overline{\mathfrak{g}}$, $\overline{\mathfrak{g}}_\sigma$, $\overline{\tilde{\mathfrak{g}}}$ and $\overline{\tilde{\mathfrak{g}}_\tau}$. We omit the details, but in the figures below, we show the radical (=socle) filtrations of these modules, together with those already described in the simply connected case, for comparison.

l odd.

$$\tilde{\mathfrak{g}}_\tau : \tilde{L}(\tilde{\omega}_2), \qquad \mathfrak{g}_\sigma : \begin{matrix} L(\omega_1) \\ k \end{matrix}, \qquad \tilde{\mathfrak{g}} : \begin{matrix} \tilde{L}(2\tilde{\omega}_1) \\ k \oplus \tilde{L}(\tilde{\omega}_2) \end{matrix}, \qquad \mathfrak{g} : \begin{matrix} L(\omega_2) \\ L(\omega_1) \\ k \end{matrix}. \tag{4}$$

$$\overline{\tilde{\mathfrak{g}}}_\tau : \tilde{L}(\tilde{\omega}_2), \qquad \overline{\mathfrak{g}}_\sigma : \begin{matrix} k \\ L(\omega_1) \end{matrix}, \qquad \overline{\tilde{\mathfrak{g}}} : \begin{matrix} k \\ \tilde{L}(2\tilde{\omega}_1) \\ \tilde{L}(\tilde{\omega}_2) \end{matrix}, \qquad \overline{\mathfrak{g}} : \begin{matrix} k \oplus L(\omega_2) \\ L(\omega_1) \end{matrix}. \tag{5}$$

l even.

$$\tilde{\mathfrak{g}}_\tau : \begin{matrix} \tilde{L}(\tilde{\omega}_2) \\ k \end{matrix}, \qquad \mathfrak{g}_\sigma : \begin{matrix} L(\omega_1) \\ k \end{matrix}, \qquad \tilde{\mathfrak{g}} : \begin{matrix} \tilde{L}(2\tilde{\omega}_1) \\ k \\ \tilde{L}(\tilde{\omega}_2) \\ k \end{matrix}, \qquad \mathfrak{g} : \begin{matrix} L(\omega_2) \\ k \\ L(\omega_1) \\ k \end{matrix}. \tag{6}$$

$$\overline{\tilde{\mathfrak{g}}}_\tau : \begin{matrix} k \\ \tilde{L}(\tilde{\omega}_2) \end{matrix}, \qquad \overline{\mathfrak{g}}_\sigma : \begin{matrix} k \\ L(\omega_1) \end{matrix}, \qquad \overline{\tilde{\mathfrak{g}}} : \begin{matrix} k \\ \tilde{L}(2\tilde{\omega}_1) \\ k \\ \tilde{L}(\tilde{\omega}_2) \end{matrix}, \qquad \overline{\mathfrak{g}} : \begin{matrix} k \\ L(\omega_2) \\ k \\ L(\omega_1) \end{matrix}. \tag{7}$$

§**6. Cohomology.** In each row of the diagrams 3.1(5)-(8) we have an affine k-group scheme with a normal subgroup scheme such that the quotient is an affine k-group scheme. Therefore there are Lyndon-Hochschild-Serre spectral sequences ([Ja1] I.6.6) for the rows.

6.1. When the normal subgroup scheme is of multiplicative type, the spectral sequences collapse. Thus, the maps σ and τ induce the following isomorphisms.

$$H^*(D, M) \cong H^*(\tilde{D}, M) \qquad \text{for any } \tilde{D}\text{-module } M; \tag{1}$$

$$H^*(\tilde{A}, M) \cong H^*(A, M) \qquad \text{for any } A\text{-module } M; \tag{2}$$

$$H^*(\tilde{A}_1, M) \cong H^*(G_\sigma, M) \qquad \text{for any } G_\sigma\text{-module } M; \tag{3}$$

$$H^*(D_1, M) \cong H^*(\tilde{G}_\tau, M) \qquad \text{for any } \tilde{G}_\tau\text{-module } M. \tag{4}$$

These are isomorphisms of graded k-algebras. Furthermore, (3) is an isomorphism of $\tilde{A}$-modules and (4) is one of D-modules.

By the same token, the cohomology of an adjoint group with values in any module is isomorphic to that of its simply connected covering group.

6.2. With the convention of 4.2, let $\lambda = \lambda^0 + \sigma\overline{\lambda}$, $\mu = \mu^0 + \sigma\overline{\mu} \in X_+(T)$ and $\tilde{\lambda} = \tilde{\lambda}^0 + \tau\overline{\overline{\lambda}}$, $\tilde{\mu} = \tilde{\mu}^0 + \tau\overline{\overline{\mu}} \in X_+(\tilde{T})$ be given.

Taking into account the tensor product theorem 4.2(3),(4), we see that the second rows of 3.1(5) and 3.1(6) yield the spectral sequences

$$\mathrm{Ext}^i_{\tilde{G}}(\tilde{L}(\overline{\lambda}),\ \mathrm{Ext}^j_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\overline{\mu})) \Rightarrow \mathrm{Ext}^{i+j}_G(L(\lambda), L(\mu)) \tag{1}$$

and

$$\mathrm{Ext}^i_G(L(\overline{\overline{\lambda}}),\ \mathrm{Ext}^j_{\tilde{G}_\tau}(\tilde{L}(\tilde{\lambda}^0), \tilde{L}(\tilde{\mu}^0))^{(\tau^{-1})} \otimes L(\overline{\overline{\mu}})) \Rightarrow \mathrm{Ext}^{i+j}_{\tilde{G}}(\tilde{L}(\tilde{\lambda}), \tilde{L}(\tilde{\mu})). \tag{2}$$

The 5-term sequences, which will shall use repeatedly, are

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\overline{\lambda}),\ \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\overline{\mu})) \longrightarrow \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \\ &\longrightarrow \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\overline{\lambda}),\ \mathrm{Ext}^1_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\overline{\mu})) \\ &\longrightarrow \mathrm{Ext}^2_{\tilde{G}}(\tilde{L}(\overline{\lambda}),\ \mathrm{Hom}_{G_\sigma}(L(\lambda^0), L(\mu^0))^{(\sigma^{-1})} \otimes \tilde{L}(\overline{\mu})) \longrightarrow \mathrm{Ext}^2_G(L(\lambda), L(\mu)) \end{aligned} \tag{3}$$

and

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1_G(L(\overline{\overline{\lambda}}),\ \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\tilde{\lambda}^0), \tilde{L}(\tilde{\mu}^0))^{(\tau^{-1})} \otimes L(\overline{\overline{\mu}})) \longrightarrow \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\tilde{\lambda}), \tilde{L}(\tilde{\mu})) \\ &\longrightarrow \mathrm{Hom}_G(L(\overline{\overline{\lambda}}),\ \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\lambda}^0), \tilde{L}(\tilde{\mu}^0))^{(\tau^{-1})} \otimes L(\overline{\overline{\mu}})) \\ &\longrightarrow \mathrm{Ext}^2_G(L(\overline{\overline{\lambda}}),\ \mathrm{Hom}^j_{\tilde{G}_\tau}(\tilde{L}(\tilde{\lambda}^0), \tilde{L}(\tilde{\mu}^0))^{(\tau^{-1})} \otimes L(\overline{\overline{\mu}})) \longrightarrow \mathrm{Ext}^2_{\tilde{G}}(\tilde{L}(\tilde{\lambda}), \tilde{L}(\tilde{\mu})). \end{aligned} \tag{4}$$

We shall not discuss the spectral sequences for the other extensions in 3.1(5)-(8) until they are needed, except to note that in the infinitesimal cases such as the second row of 3.1(7), there is an action of a global group, in this case G. Thus, for any G-module M, we obtain a spectral sequence of modules for $G \cong G/G_1$

$$H^i(\tilde{G}_\tau, H^j(G_\sigma, M)^{(\sigma^{-1})})^{(\tau^{-1})} \Rightarrow H^{i+j}(G_1, M)^{(2^{-1})}. \tag{5}$$

6.3. We can now make some reductions in the problem of computing simple module extensions for $\tilde{G}$ and G.

Let $\tilde{\lambda}$, $\tilde{\mu} \in X(\tilde{T})$. Then since $\mathrm{Ext}^j_{G_\sigma}(L(\omega_l), L(\omega_l)) = 0$ for $j \geq 1$, we obtain from 6.2(1) the isomorphisms

$$\mathrm{Ext}^i_{\tilde{G}}(\tilde{L}(\tilde{\lambda}), \tilde{L}(\tilde{\mu})) \cong \mathrm{Ext}^i_G(L(\omega_l + \sigma\tilde{\lambda}), L(\omega_l + \sigma\tilde{\mu})), \quad i \geq 0. \tag{1}$$

(See also 7.2(2) below.) Next, write λ, $\mu \in X_+(T)$ as $\lambda = \lambda^0 + \sigma\lambda^1 + 2\overline{\overline{\lambda}}$ and $\mu = \mu^0 + \sigma\mu^1 + 2\overline{\overline{\mu}}$. Then, in the same way, the 5-term sequence for (G, G_1) ([Ja1] II.10.17) shows that if $\lambda^0 + \sigma\lambda^1 = \mu^0 + \sigma\mu^1$, then

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\overline{\overline{\lambda}}), L(\overline{\overline{\mu}})). \tag{2}$$

These two reductions show that in order to compute all extensions of simple modules for G and $\tilde{G}$, it is enough to compute the groups $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$, with $\lambda^0 + \sigma\lambda^1 \neq \mu^0 + \mu^1$. These reductions apply also to extensions of simple modules for the Frobenius kernels.

6.4. The reductions of 6.3 lead us to consider the spectral sequence 6.2(1). We need the following result.

LEMMA. *We have an isomorphism of graded $\tilde{G}$-algebras*

$$(1) \qquad H^*(G_\sigma, k)^{(\sigma^{-1})} \cong S(\tilde{L}(\tilde{\omega}_1)),$$

where right hand side is the symmetric algebra with its usual grading.

PROOF: We have $Z(G) = T_\sigma$ and from Lemma 5.2 it follows that $G_\sigma/Z(G) \cong (L(\omega_1))_{a,1}$ (Frobenius kernel of the additive group of the module), and this isomorphism is compatible with the action of G. Therefore,

$$(2) \qquad H^*(G_\sigma, k) \cong H^*(G_\sigma/Z(G), k) \cong H^*((L(\omega_1))_{a,1}, k)$$

as graded G-algebras. The result now follows from the known isomorphism ([Ja1], I.4.27)

$$H^*((L(\omega_1))_{a,1}, k) \cong S(L(\omega_1)).$$

Thus, if $\lambda^0 = \mu^0 = 0$ in the 5-term sequence 6.2(3) we have:

$$(3) \quad 0 \longrightarrow \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\overline{\mu})) \longrightarrow \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \\ \longrightarrow \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\overline{\mu})) \longrightarrow \mathrm{Ext}^2_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\overline{\mu})).$$

6.5. We return to the situation left by 6.3. Thus, we have $\lambda = \lambda^0 + \sigma\overline{\lambda} = \lambda^0 + \sigma\lambda^1 + 2\overline{\overline{\lambda}}$ (with $\overline{\overline{\lambda}} = \lambda^2 + \sigma\lambda^3 + \cdots$, etc.), $\mu = \mu^0 + \sigma\overline{\mu} = \mu^0 + \sigma\mu^1 + 2\overline{\overline{\mu}} \in X_+(T)$, and we shall assume

$$(1) \qquad \lambda^0 + \sigma\lambda^1 \neq \mu^0 + \sigma\mu^1.$$

Recall that $X_\sigma = \{0, \omega_l\}$ and that $L(\omega_l)$ is injective for G_σ. There are three cases:

(a) $\{\lambda^0, \mu^0\} = \{0, \omega_l\}$. Here, we have $\mathrm{Ext}^j_{G_\sigma}(L(\lambda^0), L(\mu^0)) = 0$ for all j, so $\mathrm{Ext}^j_G(L(\lambda), L(\mu)) = 0$ for all j, by 6.2(1).
(b) $\lambda^0 = \mu^0 = 0$ and $\overline{\lambda} - \overline{\mu} \in \mathbb{Z}\tilde{R}$, or $\lambda^0 = \mu^0 = \omega_l$;
(c) $\lambda^0 = \mu^0 = 0$ and $\overline{\lambda} - \overline{\mu} \notin \mathbb{Z}\tilde{R}$.

REMARK: Since $2X(\tilde{T}) \leq \mathbb{Z}\tilde{R}$, the conditions on $\overline{\lambda} - \overline{\mu}$ in (b) and (c) depend only on the first two terms of the expansions of $\overline{\lambda}$ and $\overline{\mu}$. The precise conditions are as follows.
If l is even then $\tau X(T) = \mathbb{Z}\tilde{R}$, so

$$\overline{\lambda} - \overline{\mu} \in \mathbb{Z}\tilde{R} \iff \lambda^1 - \mu^1 \in \mathbb{Z}\tilde{R}.$$

If l is odd then $2X(\tilde{T}) \leq \mathbb{Z}\tilde{R}$ (but $\tau X(T) \not\subseteq \mathbb{Z}\tilde{R}$). Then

$$\overline{\lambda} - \overline{\mu} \in \mathbb{Z}\tilde{R} \iff \begin{cases} \lambda^1 - \mu^1 \in \mathbb{Z}\tilde{R} & \text{and } \lambda^2 = \mu^2, \quad \text{or} \\ \lambda^1 - \mu^1 \notin \mathbb{Z}\tilde{R} & \text{and } \lambda^2 \neq \mu^2. \end{cases}$$

(b) Suppose first that $\lambda^0 = \mu^0 = \omega_l$. This has already been discussed in 6.3; we have $\mathrm{Ext}^j_{G_\sigma}(L(\lambda^0), L(\mu^0)) = 0$ for $j \geq 1$, so by 6.2(1) $\mathrm{Ext}^j_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^j_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\overline{\mu}))$ for all j. (Note thus that if l is even, $\lambda^0 = \mu^0 = \omega_l$, and $\overline{\lambda} - \overline{\mu} \notin \mathbb{Z}\tilde{R}$, then $\mathrm{Ext}^j_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^j_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\overline{\mu})) \cong 0$ for all j.) Now suppose $\lambda^0 = \mu^0 = 0$ and $\overline{\lambda} - \overline{\mu} \in \mathbb{Z}\tilde{R}$. Then in 6.4(3), since $\tilde{\omega}_1 \notin \mathbb{Z}\tilde{R}$, the third term is zero, and so the first two terms are isomorphic. Thus, in case (b) we have

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\overline{\mu})). \tag{2}$$

Now $\overline{\lambda} = \lambda^1 + \tau\overline{\overline{\lambda}}$ $\overline{\mu} = \mu^1 + \tau\overline{\overline{\mu}}$. In view of (1), the first and fourth terms of 6.2(4) (with $\tilde{\lambda} = \overline{\lambda}$ and $\tilde{\mu} = \overline{\mu}$) are zero, and combining the resulting isomorphism with (2), we obtain

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(L(\overline{\overline{\lambda}}), \mathrm{Ext}^1_{G_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})} \otimes L(\overline{\overline{\mu}})). \tag{3}$$

(c) In this case, the first and fourth terms of 6.2(3) are zero, so, using Lemma 6.4, we obtain

$$\begin{aligned} \mathrm{Ext}^1_G(L(\lambda), L(\mu)) &\cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\overline{\lambda}), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\overline{\mu})) \\ &\cong \Big[\mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^1))^{(\tau^{-1})} \otimes \mathrm{Hom}(L(\overline{\overline{\lambda}}), L(\overline{\overline{\mu}})\Big]^G \\ &\cong \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^1))^{(\tau^{-1})} \otimes \mathrm{Hom}_G(L(\overline{\overline{\lambda}}), L(\overline{\overline{\mu}})), \end{aligned} \tag{4}$$

by Corollary 4.2.2.

Thus, in case (c), we have:

(5)

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \begin{cases} \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^1)) \cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\lambda^1), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^1)) \\ \qquad \text{if } \overline{\overline{\lambda}} = \overline{\overline{\mu}}, \\ 0 \qquad \text{otherwise} \end{cases}.$$

We have reduced the problem of computing extensions of simple modules for G and $\tilde{G}$ to the computation of the spaces in (3) and (5). In (5), the simple modules involved in the right hand side are τ-restricted, so they are simple for D_1 and D, acting through σ (cf. 4.2). Moreover, D also acts trivially on the Hom-space. Thus, since $\tilde{L}(\tilde{\omega}_1)^{(\sigma)} \cong L(\delta_1)$ as D-modules, the right hand side of (5) can be computed as

$$\mathrm{Hom}_D(L(\sigma\lambda^1), L(\delta_1) \otimes L(\sigma\mu^1)), \tag{6}$$

which is a problem purely about D.

The isomorphism 6.1(4) shows that results for D are very useful in the computation of the module $\mathrm{Ext}^1_{G_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})}$, appearing in (3). However, the D-module structure is not quite sufficient to compute the right hand side of (3) in all cases; one needs to know the G-action.

6.6. We turn now to the G-module structure of the spaces $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda), \tilde{L}(\mu))$ $(\lambda, \mu \in X_\tau)$, such as appear in 6.5(3). We do not present the parallel results for Frobenius kernels, which we shall also use later, as the notational changes involved are clear. For $\mu \in X_\tau$, let $\tilde{Q}(\mu)$ denote the $\tilde{G}_\tau$-injective hull (= projective cover) of the simple module $\tilde{L}(\mu)$. Denote by μ^* the highest weight of $\tilde{L}(\mu)^*$. Then one has for $\lambda \in X_\tau$

$$(1) \qquad \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\mu), \tilde{L}(\rho_\tau - \mu^*) \otimes \tilde{L}(\rho_\tau)) = \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\mu), \tilde{L}(\rho_\tau - \mu^*) \otimes \tilde{L}(\rho_\tau)) \cong k,$$

as can be seen by considering weights, using the injectivity of $\tilde{L}(\rho_\tau)$. Therefore the G-module $\tilde{L}(\rho_\tau - \mu^*) \otimes \tilde{L}(\rho_\tau)$ has a unique copy of $\tilde{L}(\mu)$ in its socle and when considered as a $\tilde{G}_\tau$-module is injective, with $\tilde{Q}(\mu)$ appearing exactly once as a direct summand. Thus,

$$(2) \qquad \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda), \tilde{L}(\mu))^{(\tau^{-1})} \cong \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda), (\tilde{L}(\rho_\tau - \mu^*) \otimes \tilde{L}(\rho_\tau))/\tilde{L}(\mu))^{(\tau^{-1})}$$

as G-modules. It now follows from Lemma 4.2 that the multiplicity of $L(\nu)$ as a composition factor of (2) is no greater than the composition multiplicity

$$(3) \qquad \left[\tilde{L}(\rho_\tau - \mu^*) \otimes \tilde{L}(\rho_\tau) : \tilde{L}(\lambda + \tau\nu)\right]_{\tilde{G}}.$$

This multiplicity can be computed from the weights alone, so in practice, this can be used as a rough first estimate in finding the module structure. Lemma 4.2 can also sometimes be applied to determine the G-socle of (2) if certain G-extensions have already been computed, by whatever means.

§7. Cohomology of Frobenius kernels.

In this section, which is not required for the computations for G and $\tilde{G}$, we discuss the cohomology of the Frobenius kernels G_1 and $\tilde{G}_1$. We shall concentrate on G_1, which is more interesting.

7.1. If M is a G-module, the second row of 3.1(7) gives rise to the following spectral sequence of modules for $G \cong G/G_1$. (Note that we must keep track of twisting.)

$$(1) \qquad H^i(\tilde{G}_\tau, H^j(G_\sigma, M)^{(\sigma^{-1})})^{(\tau^{-1})} \Rightarrow H^{i+j}(G_1, M)^{(2^{-1})}.$$

Let $\lambda = \lambda^0 + \sigma\lambda^1$, $\mu = \mu^0 + \sigma\mu^1 \in X_1(T)$. We will consider the consequences of (1) for the various choices of λ^0 and μ^0.

Suppose first that $\{\lambda^0, \mu^0\} = \{0, \omega_l\}$. Then as $L(\omega_l)$ is injective for G_σ, we see that

$$(2) \qquad \mathrm{Ext}^i_{G_1}(L(\lambda), L(\mu)) = 0 \qquad \text{for } i \geq 0.$$

Next, suppose $\lambda^0 = \mu^0 = \omega_l$. Then we obtain isomorphisms

$$(3) \qquad \mathrm{Ext}^i_{G_1}(L(\lambda), L(\mu))^{(2^{-1})} \cong \mathrm{Ext}^i_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})} \qquad \text{for } i \geq 0.$$

Now we come to the most complicated case $\lambda^0 = \mu^0 = 0$. We have by Lemma 6.4,

$$(4) \qquad \mathrm{Ext}^i_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), S^j(\tilde{L}(\tilde{\omega}_1)) \otimes \tilde{L}(\mu^1))^{(\tau^{-1})} \Rightarrow \mathrm{Ext}^{i+j}_{G_1}(L(\lambda), L(\mu))^{(2^{-1})}$$

To compute $\mathrm{Ext}^1_{G_1}(L(\lambda),L(\mu))^{(2^{-1})}$, we consider the commutative diagram with exact rows:

(5)
$$\begin{array}{ccccccc}
0\to \mathrm{Ext}^1_{\tilde G}(\tilde L(\lambda^1),\tilde L(\mu^1)) & \longrightarrow & \mathrm{Ext}^1_G(L(\lambda),L(\mu)) & \longrightarrow & \mathrm{Hom}_{\tilde G}(\tilde L(\lambda^1),\ \tilde L(\tilde\omega_1)\otimes\tilde L(\mu^1)) \to \\
\downarrow & & \downarrow & & \| \\
0\to \mathrm{Ext}^1_{\tilde G_\tau}(\tilde L(\lambda^1),\tilde L(\mu^1)) & \longrightarrow & \mathrm{Ext}^1_{G_1}(L(\lambda),L(\mu)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{\tilde G_\tau}(\tilde L(\lambda^1),\ \tilde L(\tilde\omega_1)\otimes\tilde L(\mu^1))
\end{array}$$

The equality between the third terms follows by Corollary 4.2.2.

If $\lambda^1-\mu^1\in\mathbb{Z}\tilde R$, then the third term is zero, since $\tilde\omega_1\notin\mathbb{Z}\tilde R$. Thus:

$$\mathrm{Ext}^1_{G_1}(L(\lambda),L(\mu))^{(2^{-1})}\cong\mathrm{Ext}^1_{\tilde G_\tau}(\tilde L(\lambda^1),\tilde L(\mu^1))^{(\tau^{-1})}\qquad \text{if } \lambda^1-\mu^1\in\mathbb{Z}\tilde R. \tag{6}$$

If $\lambda^1-\mu^1\notin\mathbb{Z}\tilde R$, then the first term and the missing fourth term $\mathrm{Ext}^2_{\tilde G}(\tilde L(\lambda^1),\tilde L(\mu^1))$ in the top row are zero. It follows that the restriction map in the bottom row is surjective in this case. We must now separate the even and odd rank cases.

l even. In this case $\tau X(T)=\mathbb{Z}\tilde R$, and so the first term in the bottom row is zero. Thus, we have shown:

$$\mathrm{Ext}^1_{G_1}(L(\lambda),L(\mu))^{(2^{-1})}\cong\mathrm{Hom}_{\tilde G_\tau}(\tilde L(\lambda^1),\ \tilde L(\tilde\omega_1)\otimes\tilde L(\mu^1))^{(\tau^{-1})}. \tag{7}$$

l odd. We claim in this case that no weight of $\mathrm{Ext}^1_{\tilde G_\tau}(\tilde L(\lambda^1),\tilde L(\mu^1))^{(\tau^{-1})}$ belongs to $\mathbb{Z}R$. Now since $\lambda^1\neq\mu^1$, we have by 6.2(4)

$$\mathrm{Hom}_G(L(\nu),\mathrm{Ext}^1_{\tilde G_\tau}(\tilde L(\lambda^1),\tilde L(\mu^1))^{(\tau^{-1})})\cong\mathrm{Ext}^1_{\tilde G}(\tilde L(\lambda^1+\tau\nu),\tilde L(\mu^1)),$$

which, since we are assuming $\lambda^1-\mu^1\notin\mathbb{Z}\tilde R$, will be zero unless $\tau\nu\notin\mathbb{Z}\tilde R$, or equivalently $\nu\notin\mathbb{Z}R$. Our claim now follows. As we have already mentioned, the restriction map in the bottom row is surjective and G acts trivially on the image. It now follows that

$$\mathrm{Ext}^1_{G_1}(L(\lambda),L(\mu))^{(2^{-1})}\cong\mathrm{Ext}^1_{\tilde G_\tau}(\tilde L(\lambda^1),\tilde L(\mu^1))^{(\tau^{-1})}\oplus\mathrm{Hom}_{\tilde G_\tau}(\tilde L(\lambda^1),\ \tilde L(\tilde\omega_1)\otimes\tilde L(\mu^1)), \tag{8}$$

with trivial G action on the second summand.

The computation of the $\tilde G$-modules $\mathrm{Ext}^1_{\tilde G_1}(\tilde L(\lambda),\tilde L(\mu))$ for $\lambda,\ \mu\in X_1(\tilde T)$ by means of the 5-term "inflation-restriction" sequence arising from the second row of 3.1(8) is less complicated, since either the inflation map or the restriction map will be an isomorphism, according to whether or not λ^0 and μ^0 are equal. The precise result is:

$$\mathrm{Ext}^1_{\tilde G_1}(\tilde L(\lambda),\tilde L(\mu))^{(2^{-1})}\cong\begin{cases}\tilde L(\tilde\omega_1) & \text{if } \lambda=\mu=\lambda^0;\\ \mathrm{Hom}_{G_\sigma}(L(\lambda^1),\mathrm{Ext}_{\tilde G_\tau}(\tilde L(\lambda^0),\tilde L(\mu^0))^{(\tau^{-1})}\otimes L(\mu^1))^{(\sigma^{-1})} & \\ \text{if } \lambda^0\neq\mu^0; & \\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

7.2. In the same way, it is possible to compute $\overline{G}_\sigma$ and $\overline{G}_1$ extensions between simple $\overline{G}$-modules and $\overline{\tilde{G}}_\tau$ and $\overline{\tilde{G}}_1$-extensions between simple $\overline{\tilde{G}}$-modules. The same applies to the Frobenius kernels of the various groups of type D_l. The various group extensions from which the appropriate spectral sequences arise can be derived from the descriptions of their Lie algebras in 5.6. For example, when l is even, we have corresponding to 5.6(7) the group extensions

$$(10)\quad 1 \to G_\sigma/Z(G) \to \overline{G}_\sigma \to Z(G/G_\sigma) \to 1, \quad 1 \to \tilde{G}_\tau/Z(\tilde{G}) \to \overline{\tilde{G}}_\tau \to Z(\tilde{G}/\tilde{G}_\tau) \to 1,$$

$$(11)\qquad 1 \to \overline{G}_\sigma \to \overline{G}_1 \xrightarrow{\sigma} \overline{\tilde{G}_\tau} \to 1 \qquad \text{and} \qquad 1 \to \overline{\tilde{G}}_\tau \to \overline{\tilde{G}}_1 \xrightarrow{\tau} \overline{G}_\sigma \to 1.$$

(See also [AJ], 6.4, where a similar situation is considered.)

§8. Cohomology of sheaves on G/B and $\tilde{G}/\tilde{B}$.

Various formulae in the cohomology of line bundles on the flag varieties G/B and $\tilde{G}/\tilde{B}$ can be extended immediately to the present context with the Frobenius map replaced by σ and τ. We omit the proofs, which are formally the same as in the usual case. Each B-module M defines a G-linearized sheaf $\mathcal{L}(M)$ on G/B ([Ja1] I.5.8). The i-th cohomology group $H^i(M) = H^i(G/B, \mathcal{L}(M))$ is therefore a G-module. Similarly, we have $\tilde{G}$-modules $\tilde{H}^i(N)$ for each $\tilde{B}$-module N. For $i = 0$, we obtain the induced modules $\mathrm{ind}_B^G(M)$ and $\mathrm{ind}_{\tilde{B}}^{\tilde{G}}(N)$.

8.1. For λ, $\mu \in X_+(T)$, we have a $\tilde{G}$-module isomorphism

$$(1)\qquad \mathrm{Ext}^1_{G_\sigma}(L(\lambda), H^0(\mu))^{(\sigma^{-1})} \cong \mathrm{ind}_{\tilde{B}}^{\tilde{G}}(\mathrm{Ext}^1_{B_\sigma}(L(\lambda), \mu)^{(\sigma^{-1})}),$$

and for λ, $\mu \in X_+(\tilde{T})$, we have a G-module isomorphism

$$(2)\qquad \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda), \tilde{H}^0(\mu))^{(\tau^{-1})} \cong \mathrm{ind}_B^G(\mathrm{Ext}^1_{\tilde{B}_\tau}(\tilde{L}(\lambda), \mu)^{(\tau^{-1})}).$$

(As usual, one-dimensional modules will be denoted by their characters.) The usual version of these formulae were first proved in [AJ] (See [Ja1], II.12.8.). We will often use the case $\lambda = 0$.

8.2. (cf. [Ja1] II.3.19.) For any B-module M, we have a $\tilde{G}$-module isomorphism

$$(1)\qquad \tilde{H}^i(\tilde{\rho}_\tau \otimes M^{(\tau)}) \cong \tilde{L}(\tilde{\rho}_\tau) \otimes H^i(M)^{(\tau)}$$

and for any $\tilde{B}$-module N, we have a G-module isomorphism

$$(2)\qquad H^i(\omega_l \otimes N^{(\sigma)}) \cong L(\omega_l) \otimes \tilde{H}^i(N)^{(\sigma)}.$$

These formulae are variations of one in [An] and [Ha].

REMARK. We have $\omega_l = (\frac{1}{2}, \frac{1}{2}, \dots, , \frac{1}{2})$ and $\sigma X(\tilde{T}) \leq \mathbb{Z}^l$. Therefore, (2) shows that if the coordinates of $\lambda \in X_+(T)$ are half-integers, then $H^0(\lambda)$ is divisible by the spin module. This statement remains true in arbitrary characteristic at the level of formal characters, as can also be seen using Weyl's character formula.

§**9. Good filtrations.** In this final paragraph, G will denote any reductive group over an algebraically closed field k. Our purpose is to collect together some important facts which will be needed later. An ascending filtration of a G-module is said to be *good* if the subquotients are isomorphic to induced modules $H^0(\lambda)$ for various $\lambda \in X_+$. There is also the dual notion of a Weyl filtration. The important facts for our purposes about a finite-dimensional G-module M with a good filtration are the following:

(1) The multiplicity of $H^0(\lambda)$ as a subquotient is $\dim_k \mathrm{Hom}_G(V(\lambda), M)$.
(2) If $H^0(\lambda)$ and $H^0(\mu)$ are both good filtration factors and $\lambda \not> \mu$, then M has a good filtration in which the factor $H^0(\mu)$ appears above the factor $H^0(\lambda)$.
(3) If the module M' also has a good filtration, then so does $M \otimes M'$.

A proof of (1) can be found in ([Ja1], II.4.18), and (2) follows from a standard property of Weyl modules ([Ja1], II.2.14). The deeper fact (3) is proved in ([Do2], 7.3.1) with a few exceptions and in general in [Ma]. Given a module with a good filtration, the multiplicities of the subquotients can be determined from the weight multiplicities in the module. Thus, for modules of the form $M = H^0(\lambda) \otimes H^0(\mu)$, it is routine to calculate $\dim_k \mathrm{Hom}_G(V(\nu), M)$ for any $\nu \in X_+$.

II. Groups of Rank 2 and 3

§1. Type $A_3 = D_3$.

1.1. Throughout this section, D will denote the simply connected group of type D_3 over the field k, obtained from a split $\mathbb{Z}$-group $D_{\mathbb{Z}}$. We shall use notation of I.1, so for example the simple roots are β_i, i=1, 2, 3, with corresponding fundamental weights δ_i. Note that in our notation β_1 belongs to the middle node of the Dynkin graph.

1.2. We have $D \cong \mathrm{SL}_4(k)$ and $V = V(\delta_2)$ may be identified with the natural 4-dimensional module. The graph automorphism J of D of order 2 may then be identified with the map sending a matrix to its inverse-transpose. Twisting a simple module by J yields the dual module. We have D-module isomorphisms $V(\delta_1) \cong \wedge^2(V)$ and $V(\delta_3) \cong \wedge^3(V) \cong V^*$. Also, the module $H^0(r\delta_2)$ is isomorphic to the r-th symmetric power $S^r(V)$. We have the following natural exact sequences:

$$0 \longrightarrow \wedge^2(V) \longrightarrow V \otimes V \longrightarrow S^2(V) \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow V^{(2)} \longrightarrow S^2(V) \longrightarrow \wedge^2(V) \longrightarrow 0 \tag{2}$$

and

$$0 \longrightarrow \wedge^3(V) \xrightarrow{\phi_3} V \otimes \wedge^2(V) \xrightarrow{\phi_2} V \otimes S^2(V) \xrightarrow{\phi_1} S^3(V) \longrightarrow 0. \tag{3}$$

Since $\operatorname{char} k \neq 3$, the maps in (3) all split naturally over k. The 20-dimensional D-module $\operatorname{Coker}\phi_3 = \ker\phi_1$ is $H^0(\delta_1+\delta_2)$. It is clear that V and $\wedge^2(V)$ are simple modules as they each contain a single orbit of weights under the Weyl group. Thus, we have $(V^* \otimes \wedge^2(V^*))^{(J)} \cong (V^* \otimes \wedge^2(V^*))^* \cong V \otimes \wedge^2(V)$, from which it follows that $H^0(\delta_1+\delta_2) \cong V(\delta_1+\delta_2)$ so it is also simple. As usual, the first Steinberg module $H^0(\delta_1+\delta_2+\delta_3) = H^0(\rho)$ is simple, and is injective for the first Frobenius kernel D_1. Finally, the adjoint module $\mathfrak{d} = V(\delta_2+\delta_3)$ can be identified with the Lie algebra of endomorphisms of V with trace zero. The centre consists of the scalar multiplications and the quotient by the centre is simple, being a self-dual quotient of a Weyl module.

Remark. It is interesting to note that the Lie algebra $\mathfrak{d}/z(\mathfrak{d})$ is isomorphic to the simple Chevalley algebra of type G_2, as pointed out to the second author in 1990 by I. Kaplansky. Using this isomorphism, we could make use of results on type D_3 to prove results about type G_2 or vice versa. For example, the cohomology rings of the Frobenius kernels are isomorphic. However, we are interested in the action of the global groups D and $G_2(k)$ and the isomorphism above does not take this fully into account. Therefore, we prefer to treat the two groups separately (see §4 for G_2), now that we have pointed out the source of many coincidences in the two sets of results.

1.3. For the rest of this section let λ and μ be dominant weights. We fix the notation $\lambda = \sum 2^i\lambda^i = \lambda^0 + 2\hat{\lambda} = \lambda^0 + 2\lambda^1 + 4\overline{\lambda}$, and similarly for μ.

Our aim is to compute $\mathrm{Ext}^1_D(L(\lambda), L(\mu))$ using the 5-term sequence for (D, D_1):

$$(1)\quad 0 \to \mathrm{Ext}^1_D(L(\hat\lambda), \mathrm{Hom}_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\hat\mu)) \to \mathrm{Ext}^1_D(L(\lambda), L(\mu))$$
$$\to \mathrm{Hom}_D(L(\hat\lambda), \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\hat\mu))$$
$$\to \mathrm{Ext}^2_D(L(\hat\lambda), \mathrm{Hom}_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\hat\mu)) \to 0$$

If $\lambda^0 = \mu^0$, then the third term is zero by [Ja1], II.12.9, and the resulting isomorphism of the first two terms shows that eventually we are reduced to the case $\lambda^0 \neq \mu^0$. Assume from now on that $\lambda^0 \neq \mu^0$. Then the first and fourth terms are zero, yielding

$$(2)\quad \mathrm{Ext}^1_D(L(\lambda), L(\mu)) \cong \mathrm{Hom}_D(L(\hat\lambda), \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\hat\mu))$$
$$\cong \mathrm{Hom}_D(L(\overline{\lambda}), \mathrm{Hom}_{D_1}(L(\lambda^1), \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\mu^1))^{(2^{-1})} \otimes L(\overline{\mu})).$$

1.4. We compute the D-modules $\mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))$. The number of cases which must be considered is reduced by some general properties. The group of extensions is trivial if either λ^0 or μ^0 is equal to ρ (since $L(\rho)$ is injective, hence also projective for D_1) or if $\lambda^0 = \mu^0$ (by [Ja1] Proposition II.12.9), or if $\lambda^0 - \mu^0 \notin 2X(T)$ (by [Ja1] Lemma II.9.16(b)). Furthermore we have D-module isomorphisms

$$(1)\qquad \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0)) \cong \mathrm{Ext}^1_{D_1}(L(\mu^0)^*, L(\lambda^0)^*) \cong [\mathrm{Ext}^1_{D_1}(L(\mu^0), L(\lambda^0))]^{(J)},$$

where J is the inverse transpose automorphism (See 1.2.).

Lemma. *For λ^0, $\mu^0 \in X_1(T)$, we have $\mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0)) = 0$ except in the following cases (up to duality and J):*

(a) $\mathrm{Ext}^1_{D_1}(k, L(\delta_1))^{(2^{-1})} \cong L(\delta_2) \oplus L(\delta_3)$;
(b) $\mathrm{Ext}^1_{D_1}(k, L(\delta_2 + \delta_3))^{(2^{-1})} \cong L(\delta_1) \oplus k$;
(c) $\mathrm{Ext}^1_{D_1}(L(\delta_3), L(\delta_2))^{(2^{-1})} \cong L(\delta_3)$;
(d) $\mathrm{Ext}^1_{D_1}(L(\delta_3), L(\delta_1 + \delta_3))^{(2^{-1})} \cong L(\delta_2)$.

Proof: The computations for (a) and (b) are [Ja2], 4.1(b) and 6.8(c) respectively. We next consider (c). In the notation of §1.2, this group is $H^1(D_1, V \otimes V)^{(2^{-1})}$. From 1.2(1) and (2) we obtain

$$(2)\qquad 0 \to V^{(2)} \to H^1(D_1, \wedge^2(V)) \to H^1(D, V \otimes V) \to H^1(D_1, S^2(V))$$

and

$$(3)\qquad 0 = H^1(D_1, V^{(2)}) \to H^1(D_1, S^2(V)) \to H^1(D_1, \wedge^2(V)).$$

Then, since $\wedge^2(V) \cong L(\delta_1)$, we obtain from (2), (3) and (a) the exact sequence

$$(4)\qquad 0 \to L(2\delta_3) \to H^1(D_1, V \otimes V) \to L(2\delta_2) \oplus L(2\delta_3)$$

This shows that the possible composition factors of $H^1(D_1, V \otimes V)^{(2^{-1})}$ are $L(2\delta_2)$ and $L(2\delta_3)$. Since there are no extensions among these composition factors, the module is semisimple. We have

$$\mathrm{Hom}_D(L(\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_3), L(\delta_2))^{(2^{-1})} \cong \mathrm{Ext}^1_D(L(\delta_3 + 2\delta_2), L(\delta_2)) \tag{5}$$

and

$$\mathrm{Hom}_D(L(\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_3), L(\delta_2))^{(2^{-1})} \cong \mathrm{Ext}^1_D(L(\delta_3 + 2\delta_2), L(\delta_2)). \tag{6}$$

Elementary arguments show that $\mathrm{rad}\, V(\delta_3 + 2\delta_2) \cong L(\delta_1 + \delta_3)$ and that $\mathrm{rad}\, V(3\delta_3) \cong L(\delta_2)$, and now (c) follows.

For the rest of this lemma, we want to apply I.6.6 repeatedly. Let $Q(\nu)$ denote the D_1-injective hull of the restricted simple module $L(\nu)$. We have $Q(\delta_1 + \delta_3) \leq L(\delta_3) \otimes L(\rho)$ and $Q(\delta_2 + \delta_3) \leq L(\delta_1) \otimes L(\rho)$ (as direct summands). We list the composition factors of the tensor products (with multiplicites):

$$L(\delta_3) \otimes L(\rho): \quad L(\delta_1 + \delta_2 + 2\delta_3), 2L(\delta_2 + 2\delta_1), 3L(\delta_3 + 2\delta_2), 4\Lambda(\delta_1 + \delta_3). \tag{7}$$

$$\begin{aligned} L(\delta_1) \otimes L(\rho): \quad & L(\delta_2 + \delta_3 + 2\delta_1), 2L(2(\delta_2 + \delta_3)), 3L(\delta_1 + 2\delta_3), \\ & 3L(\delta_1 + 2\delta_2), 6L(2\delta_1), 6L(\delta_2 + \delta_3), 8k. \end{aligned} \tag{8}$$

We can now prove (d). By I.6.6 and (7), the module $\mathrm{Ext}^1_{D_1}(L(\delta_3), L(\delta_1 + \delta_3))^{(2^{-1})}$ is either zero or $L(\delta_2)$. We have

$$\mathrm{Hom}_D(L(\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_3), L(\delta_1 + \delta_3))^{(2^{-1})}) \cong \mathrm{Ext}^1_D(L(\delta_3 + 2\delta_2), L(\delta_1 + \delta_3)) \cong k,$$

from the description of $V(\delta_3 + 2\delta_2)$ given above, hence (d).

After taking various symmetries into account, the proof of the lemma will be complete once we have shown that $\mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} = 0$ for the pairs

$$(\lambda^0, \mu^0) = \quad (i)\ (\delta_1 + \delta_2, \delta_1 + \delta_3) \quad (ii)\ (\delta_2, \delta_1 + \delta_3) \quad (iii)\ (\delta_1, \delta_1 + \delta_3).$$

In case (i), we see from (7)and I.6.6 that the group is either zero or else $L(\delta_3)$. If the latter were true we would have

$$\mathrm{Hom}_D(L(\delta_1 + \delta_2 + 2\delta_3), [L(\delta_3) \otimes L(\rho)]/L(\delta_1 + \delta_3)) \neq 0. \tag{10}$$

which would mean that the submodule of $L(\delta_3) \otimes L(\rho)$ generated by its highest weight space, isomorphic to $V(\delta_1 + \delta_2 + 2\delta_3)$ would have a simple radical $L(\delta_1 + \delta_3)$. But by considering dimensions, one sees that this is not so. Therefore, case (i) is done.

For (ii), all possibilities for composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_2), L(\delta_1 + \delta_3))^{(2^{-1})}$ except for $L(\delta_1)$ are ruled out by I.6.6 and (7). We have

$$\mathrm{Hom}_D(L(\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_2), L(\delta_1 + \delta_3))^{(2^{-1})}) \cong \mathrm{Ext}^1_D(L(\delta_2 + 2\delta_1), L(\delta_1 + \delta_3)).$$

We claim that $\operatorname{rad} V(\delta_2 + 2\delta_1)$ is a nonsplit extension of $L(\delta_3 + 2\delta_2)$ by $L(\delta_1 + \delta_3)$. It is routine to check that the composition factors are correct. Since $V(\delta_2 + 2\delta_1)$ is isomorphic to a submodule of $L(\delta_1 + \delta_2) \otimes L(\delta_1)$, the claim follows from the calculation
(12)
$$\operatorname{Hom}_D(L(\delta_3)\otimes L(\delta_2)^{(2)}, L(\delta_1+\delta_2)\otimes L(\delta_1)) \cong \operatorname{Hom}_D(L(\delta_1+\delta_3)\otimes L(\delta_2)^{(2)}, L(\delta_2)\otimes L(\delta_1)) = 0.$$

Case (ii) is finished.

For (iii),the possible composition factors of $\operatorname{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3))^{(2^{-1})}$ are restricted by I.6.6 and (8) to $L(\delta_2)$ and $L(\delta_3)$. Therefore, as this module is isomorphic to its J-twist, it suffices to show $Ext^1_D(L(\delta_1 + 2\delta_2), L(\delta_2 + \delta_3)) = 0$. In order to show this, we will use some detailed facts about certain Weyl modules (see Table II.A.2). These can be checked by elementary computations. Now $V(\delta_1+2\delta_2) \leq V(\delta_1)\otimes V(2\delta_2)$, and since both $V(\delta_1)$ and $V(\delta_1)\otimes L(2\delta_2)$ are simple, we have $\operatorname{rad} V(\delta_1)\otimes V(2\delta_2) \leq V(\delta_1)\otimes \operatorname{rad} V(2\delta_2) \cong V(\delta_1)\otimes V(\delta_1)$. Also $V(\delta_1)\otimes V(\delta_1)$ contains a unique copy of $V(2\delta_1)$ and since $[V(\delta_1+2\delta_2) : L(2\delta_1)]_D \neq 0$, this copy lies inside $\operatorname{rad} V(\delta_1+2\delta_2)$. The result now follows from the fact that $[V(\delta_1+2\delta_2) : L(\delta_2 + \delta_3)] = [V(2\delta_1) : L(\delta_2 + \delta_3)] = 1$. This completes the proof of the lemma.

1.5. Let E be one of the modules $\operatorname{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}$ in Lemma 1.4. Then E is a direct sum of simple modules k and $L(\delta_i)$, $i = 1, 2, 3$. We wish to apply Corollary I.4.2 with $\nu = 0$, δ_1, δ_2 and δ_3. The minimum value of $\langle 2\omega, \rho\rangle$ for $\omega \in X_1(T) \setminus \{0\}$ is 3 whereas $\langle\delta_1, \rho\rangle = 2$ and $\langle\delta_2, \rho\rangle = \langle\delta_3, \rho\rangle = 3/2$, so the hypotheses of the corollary are satisfied. It follows that D acts trivially on the modules $\operatorname{Hom}_{D_1}(L(\lambda^1), E \otimes L(\mu^1))^{(2^{-1})}$ appearing in 1.3(2). Therefore, we can rewrite 1.3(2) in the form

(1)
$$\operatorname{Ext}^1_D(L(\lambda), L(\mu)) \cong \begin{cases} \operatorname{Hom}_D(L(\lambda^1), E \otimes L(\mu^1))^{(2^{-1})} & \text{if } \overline{\lambda} = \overline{\mu} \\ 0 \quad \text{otherwise} \end{cases}$$

Thus in order to compute $Ext^1_D(L(\lambda), L(\mu))$, we are reduced to computing the D_1-socles of the tensor products $E \otimes L(\mu^1)$, which, as we have just seen, are equal to the D-socles. The D-socles can be found in the following way. Each such E is a direct sum of restricted simple modules $L(\delta)$, so can be embedded to a direct sum of modules of the form $H^0(\delta)$ for the various δs. As a preliminary step, we find the factors in a good filtration of $H^0(\delta) \otimes H^0(\mu^1)$. This narrows down the possibilities for the D-socles of $L(\delta) \otimes L(\mu^1)$. The final answer can then be obtained by elementary calculations.

The results are given in the following table. The notation $M_1//[M_2 \oplus M_3]//M_4$ indicates a module with a descending filtration with subquotients M_1, $M_2 \oplus M_3$ and M_4.

Table II.1.5

(ν,ν')	$H^0(\nu)\otimes H^0(\nu')$	$\mathrm{soc}_{D_1}(L(\nu)\otimes L(\nu'))$
(δ_2,δ_2)	$H^0(2\delta_2)//H^0(\delta_1)$	$L(\delta_1)$
(δ_2,δ_1)	$H^0(\delta_1+\delta_2)\oplus H^0(\delta_3)$	$L(\delta_1+\delta_2)\oplus L(\delta_3)$
(δ_2,δ_3)	$H^0(\delta_2+\delta_3)//k$	k
(δ_2,ρ)	$Q(\delta_1+\delta_2)$	$L(\delta_1+\delta_2)$
$(\delta_2,\delta_1+\delta_3)$	$H^0(\delta_1+2\delta_2)//H^0(2\delta_1)//H^0(\delta_2+\delta_3)$	$L(\rho)\oplus L(\delta_1)$
$(\delta_2,\delta_1+\delta_2)$	$H^0(\rho)\oplus[H^0(2\delta_3),H^0(\delta_1)]$	$L(\delta_2+\delta_3)$
$(\delta_2,\delta_2+\delta_3)$	$H^0(2\delta_2+\delta_3)//[H^0(\delta_1+\delta_3)\oplus H^0(\delta_2)]$	$L(\delta_1+\delta_3)$
(δ_1,δ_1)	$H^0(2\delta_1)//H^0(\delta_2+\delta_3)//k$	$k\oplus L(\delta_2+\delta_3)$
(δ_1,ρ)	$Q(\delta_2+\delta_3)$	$L(\delta_2+\delta_3)$
$(\delta_1,\delta_2+\delta_3)$	$H^0(\rho)\oplus[[H^0(2\delta_2)\oplus H^0(2\delta_3)]//H^0(\delta_1)]$	$L(\rho)\oplus L(\delta_1)$
$(\delta_1,\delta_1+\delta_2)$	$H^0(\delta_2+2\delta_1)//H^0(\delta_3+2\delta_2)//[H^0(\delta_1+\delta_3)\oplus H^0(\delta_2)]$	$L(\delta_2)\oplus L(\delta_1+\delta_3)$

1.6. We summarize our results. First, by 1.5(1) the space $Ext^1_D(L(\lambda),L(\mu))$ is zero unless $\overline{\lambda}=\overline{\mu}$). If $\overline{\lambda}=\overline{\mu}$) then there are (up to duality and J) essentially 4 choices of (λ^0,μ^0) where is not zero, corresponding to the four cases of Lemma 1.4. The dimension of $Ext^1_D(L(\lambda),L(\mu))$ for the different choices of (λ^1,μ^1) can then be read off from Table II.1.5 and are given in the tables below

Table II.1.6(a) $(\lambda^0,\mu^0)=(0,\delta_1)$

(λ^1,μ^1)	0	δ_1	δ_2	δ_3	$\delta_1+\delta_2$	$\delta_1+\delta_3$	$\delta_2+\delta_3$	ρ
0	0	0	1	1	0	0	0	0
δ_1	0	0	1	1	1	1	0	0
δ_2	1	1	0	0	0	0	0	0
δ_3	1	1	0	0	0	0	0	0
$\delta_1+\delta_2$	0	1	0	0	0	0	1	1
$\delta_1+\delta_3$	0	1	0	0	0	0	1	1
$\delta_2+\delta_3$	0	0	0	0	1	1	0	0
ρ	0	0	0	0	1	1	0	0

Table II.1.6(b) $(\lambda^0, \mu^0) = (0, \delta_2 + \delta_3)$

(λ^1, μ^1)	0	δ_1	δ_2	δ_3	$\delta_1 + \delta_2$	$\delta_1 + \delta_3$	$\delta_2 + \delta_3$	ρ
0	1	1	0	0	0	0	0	0
δ_1	1	1	0	0	0	0	1	0
δ_2	0	0	1	1	1	0	0	0
δ_3	0	0	1	1	0	1	0	0
$\delta_1 + \delta_2$	0	0	1	0	1	1	0	0
$\delta_1 + \delta_3$	0	0	0	1	1	1	0	0
$\delta_2 + \delta_3$	0	1	0	0	0	0	1	1
ρ	0	0	0	0	0	0	1	1

Table II.1.6(c) $(\lambda^0, \mu^0) = (\delta_3, \delta_2)$

(λ^1, μ^1)	0	δ_1	δ_2	δ_3	$\delta_1 + \delta_2$	$\delta_1 + \delta_3$	$\delta_2 + \delta_3$	ρ
0	0	0	0	1	0	0	0	0
δ_1	0	0	0	1	1	0	0	0
δ_2	0	1	0	0	0	0	0	0
δ_3	1	0	0	0	0	0	0	0
$\delta_1 + \delta_2$	0	0	0	0	0	0	1	0
$\delta_1 + \delta_3$	0	1	0	0	0	0	0	1
$\delta_2 + \delta_3$	0	0	0	0	0	1	0	0
ρ	0	0	0	0	1	0	0	0

Table II.1.6(d) $(\lambda^0, \mu^0) = (\delta_3, \delta_1 + \delta_3)$

(λ^1, μ^1)	0	δ_1	δ_2	δ_3	$\delta_1+\delta_2$	$\delta_1+\delta_3$	$\delta_2+\delta_3$	ρ
0	0	0	0	1	0	0	0	0
δ_1	0	0	1	0	0	1	0	0
δ_2	1	0	0	0	0	0	0	0
δ_3	0	1	0	0	0	0	0	0
$\delta_1+\delta_2$	0	1	0	0	0	0	0	1
$\delta_1+\delta_3$	0	0	0	0	0	0	1	0
$\delta_2+\delta_3$	0	0	0	0	1	0	0	0
ρ	0	0	0	0	0	1	0	0

§2. Types B_3 and C_3.

2.1. By I.6.5, the problem of computing extensions of simple modules for G and $\tilde{G}$ has been reduced to computing the groups on the right of I.6.5(6) and I.6.5(3). The first computation can be read off from Table II.1.5. Here is the relevant part of that table, with notation suitably changed.

Table II.2.1 $\tilde{G}_\tau$-socles of $\tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\nu)$, $\nu \in X_\tau$

$\nu \in X_\tau$	$\mathrm{soc}_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\nu))$
$\tilde{\omega}_1$	$k \oplus \tilde{L}(\tilde{\omega}_2)$
$\tilde{\omega}_2$	$\tilde{L}(\tilde{\rho}_\tau) \oplus \tilde{L}(\tilde{\omega}_1)$
$\tilde{\rho}_\tau$	$\tilde{L}(\tilde{\omega}_2)$

The next two paragraphs are devoted to the second.

2.2. Extensions for $\tilde{G}_\tau$.

Lemma. *Up to switching λ^1 and $\mu^1 \in X_\tau$, the only nonzero groups $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})}$ are:*

(a) $\mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_1))^{(\tau^{-1})} \cong L(\omega_3)$;

(b) $\mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_2))^{(\tau^{-1})} \cong H^0(\omega_1)$. *Further, $H^0(\omega_1)/L(\omega_1) \cong k$.*

Proof: By I.6.1(4), the module $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})}$ are zero unless the corresponding D_1-extension $\mathrm{Ext}^1_{D_1}(L(\sigma\lambda^1), L(\sigma\mu^1))^{(2^{-1})}$ is listed in Lemma II.1.4 (with $(\lambda^0, \mu^0) = (\lambda^1, \mu^1)$), up to switching λ^1 with μ^1. and in that case, the Lemma gives the D-module structure, with D acting though σ. Only the first two parts of Lemma II.1.4 describe extensions of simple modules which come from simple $\tilde{G}_\tau$-modules, namely, $L(\sigma\tilde{\omega}_1) \cong L(\delta_1)$ and $L(\sigma\tilde{\omega}_2) \cong L(\delta_2+\delta_3)$. Therefore Lemma II.1.4(a) yields (a) of this lemma immediately, while part (b) of that lemma at least tells us the composition factors for (b) of the present lemma. The module structure in (b) can then be found, either by elementary arguments, or by applying formula I.7.1(2).

2.3. Socles of tensor products.

Let E denote one of the G-modules in Lemma 2.2. To apply I.6.5(3), we need to find the G-socles of $L(\mu^2) \otimes E$ for $\mu^2 \in X_\sigma$.

LEMMA. *We have G-module isomorphisms:*

(a) $\mathrm{soc}_{G_\sigma}(L(\omega_3) \otimes L(\omega_3)) \cong k$;

(b) $\mathrm{soc}_{G_\sigma}(H^0(\omega_1)) \cong L(\omega_1)$;

(c) $L(\omega_3) \otimes H^0(\omega_1) \cong L(\omega_3) \oplus (L(\omega_3) \otimes L(\omega_1))$.

In (a) and (b) the G-socles are the same.

PROOF: Every composition factor of $L(\omega_3) \otimes L(\omega_3)$ is trivial for G_σ, so (a) holds. Next, G_σ acts trivially on $L(\omega_1)$, so in (b), the G_σ-socle equals the space of fixed points of G_σ, which is therefore the σ-twist of a $\tilde{G}$-module. Now $H^0(\omega_1)$ is not such a module because $\tilde{H}^0(\tilde{\omega}_1)$ is simple. This proves (b). For (c), we note that $L(\omega_3)$ is injective for G_σ. Hence, for $\nu \in X_+(T)$,

$$\begin{aligned}\mathrm{Hom}_G(L(\nu), L(\omega_3) \otimes H^0(\omega_1)) &\cong \mathrm{Hom}_{G/G_\sigma}(k, \mathrm{Hom}_{G_\sigma}(L(\nu), L(\omega_3) \otimes H^0(\omega_1)))\\ &\cong \mathrm{Hom}_{G/G_\sigma}(k, \mathrm{Hom}_{G_\sigma}(L(\nu), L(\omega_3) \oplus L(\omega_3) \otimes L(\omega_1)))\\ &\cong \mathrm{Hom}_G(L(\nu), L(\omega_3)) \oplus \mathrm{Hom}_G(L(\nu), L(\omega_3) \otimes L(\omega_1)).\end{aligned} \tag{1}$$

The G_σ-socles in (a) and (b) are simple G-modules, hence must be equal to the G-socles.

In the notation of I.6.5(3), let $\overline{\overline{\lambda}} = \lambda^2 + \sigma\lambda^3 + 2\widehat{\lambda}$ and $\overline{\overline{\mu}} = \mu^2 + \sigma\mu^3 + 2\widehat{\mu}$. Then by the lemma, the only G-composition factors of the modules $M^{(\sigma)} = \mathrm{Hom}_{G_\sigma}(L(\lambda^2), E \otimes L(\mu^2))$ are k and $L(\omega_1)$, both 2-restricted.

Hence by I.6.5(3),

$$\begin{aligned}\mathrm{Ext}^1(L(\lambda), L(\mu)) &\cong \mathrm{Hom}_G(L(\overline{\overline{\lambda}}), E \otimes L(\overline{\overline{\mu}}))\\ &\cong \mathrm{Hom}_{G_1}(L(\lambda^2 + \sigma\lambda^3), E \otimes L(\mu^2 + \sigma\mu^3))^{(2^{-1})} \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}))\\ &\cong \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda^3), M \otimes \tilde{L}(\mu^3))^{((2\tau)^{-1})} \otimes \mathrm{Hom}_G(L(\widehat{\lambda}), L(\widehat{\mu}))\end{aligned} \tag{2}$$

Since M is a direct sum of copies of k and $\tilde{L}(\tilde{\omega}_1)$, we are reduced to computing the dimension of $\mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\lambda^3), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\mu^3))$, which has already been given in Table II.2.1

2.4. Extensions of simple modules for G and $\tilde{G}$.

Recall that by I.6.3, the calculation of all extensions of simple modules for G and $\tilde{G}$ is reduced to the calculation of $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$ for λ, $\mu \in X_+(T)$ satisfying $\lambda^0 + \sigma\lambda^1 \neq \mu^0 + \sigma\mu^1$. Then in I.6.5, the calculation was further reduced to the computation of the groups in I.6.5(3) and I.6.5(6), which has been given (for $l = 3$) in the preceding three sections. We collect these results together in the following statement, from which all extension groups of simple modules for G and $\tilde{G}$ can be computed.

Proposition. *Let* $\lambda = \lambda^0 + \sigma\lambda^1 + 2\lambda^2 + 2\sigma\lambda^3 + 4\widehat{\lambda}$, $\mu = \mu^0 + \sigma\mu^1 + 2\mu^2 + 2\sigma\mu^3 + 4\widehat{\mu} \in X_+(T)$, *with* $\lambda^0 + \sigma\lambda^1 \neq \mu^0 + \sigma\mu^1$. *Then* $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$ *unless* $\lambda^0 = \mu^0$ *and* $\widehat{\lambda} = \widehat{\mu}$. *If these conditions are satisfied then the dimension* $d(\lambda,\mu)$ *of* $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$ *is given as follows:*

(a) $\{\lambda^1, \mu^1\} = \{0, \tilde{\omega}_1\}$, $\lambda^2 \neq \mu^2$ *and* $\lambda^3 = \mu^3$. *In this case, we have* $d(\lambda,\mu) = 1$.

(b) $\{\lambda^1, \mu^1\} = \{0, \tilde{\omega}_2\}$, $\lambda^2 = \mu^2$. *In this case,* $d(\lambda,\mu)$ *is given by Table II.2.4(*a*) below, in which the symbol "(1)" means 1 if* $\lambda^2 = \omega_3$ *and 0 if* $\lambda^2 = 0$.

(c) $\lambda^0 = \mu^0 = 0$ *and* $(\lambda^1 + \tau\lambda^2) - (\mu^1 - \tau\mu^2 \notin \mathbb{Z}\tilde{R})$. *In this case,*

$$d(\lambda,\mu) = \begin{cases} 1 & \text{if } \lambda^2 = \mu^2,\ \lambda^3 = \mu^3 \text{ and } \{\lambda^1, \mu^1\} \neq \{0, \tilde{\rho}_\tau\}, \\ 0 & \text{otherwise.} \end{cases}$$

In all other cases we have $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$.

Note that parts (a) and (b) of the proposition correspond to the case (b) in I.6.5, while (c) corresponds to (c) in I.6.5.

Table II.2.4(a)

(λ^3,μ^3)	0	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\rho}_\tau$
0	(1)	1	0	0
$\tilde{\omega}_1$	1	(1)	1	0
$\tilde{\omega}_2$	0	1	(1)	1
$\tilde{\rho}_\tau$	0	0	1	(1)

2.5. Extensions for G_1 and $\tilde{G}_1$. These were computed in section I.6.7 from the spectral sequence I.6.2(5). We display the specific results for rank $l = 3$ in the tables below. Extensions of 2^r-restricted simple modules for the higher Frobenius kernels G_r and $\tilde{G}_r$ are the same as for the global groups. Now note that for G, we have that the result is zero if $\lambda^0 \neq \mu^0$ (in the notation of I.6.5), eliminating the need to include that part of the table. Furthermore, we can utilize the notational convention introduced in the preceding section. Let $\lambda = \lambda^0 + \sigma\lambda^1$ and $\mu = \mu^0 + \sigma\mu^1$. In the next table, the notation (x) will mean x if $\lambda^0 = \mu^0 = 0$ and 0 if $\lambda^0 = \mu^0 = \omega_3$.

Table II.2.5(a) $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))^{(2^{-1})}$

(λ_1,μ_1)	0	ω_1	ω_2	$\omega_1+\omega_2$
0	0	$(k) \oplus L(\omega_3)$	$H^0(\omega_1)$	0
ω_1	$(k) \oplus L(\omega_3)$	0	(k)	0
ω_2	$H^0(\omega_1)$	(k)	0	(k)
$\omega_1+\omega_2$	0	0	(k)	0

Table II.2.5(b) $\mathrm{Ext}^1_{\tilde G_1}(\tilde L(\lambda),\tilde L(\mu))^{(2^{-1})}$

(λ,μ)	0	$\tilde\omega_1$	$\tilde\omega_2$	$\tilde\omega_1+\tilde\omega_2$	$\tilde\omega_3$	$\tilde\omega_1+\tilde\omega_3$	$\tilde\omega_2+\tilde\omega_3$	$\tilde\omega_1+\tilde\omega_2+\tilde\omega_3$
0	$\tilde L(\tilde\omega_1)$	0	$\tilde L(\tilde\omega_1)$	0	0	k	0	0
$\tilde\omega_1$	0	$\tilde L(\tilde\omega_1)$	0	0	k	0	0	0
$\tilde\omega_2$	$\tilde L(\tilde\omega_1)$	0	$\tilde L(\tilde\omega_1)$	0	0	0	0	0
$\tilde\omega_1+\tilde\omega_2$	0	0	0	$\tilde L(\tilde\omega_1)$	0	0	0	0
$\tilde\omega_3$	0	k	0	0	0	0	$k\oplus\tilde L(\tilde\omega_1)$	0
$\tilde\omega_1+\tilde\omega_3$	k	0	0	0	0	0	0	0
$\tilde\omega_2+\tilde\omega_3$	0	0	0	0	$k\oplus\tilde L(\tilde\omega_1)$	0	0	0
$\tilde\omega_1+\tilde\omega_2+\tilde\omega_3$	0	0	0	0	0	0	0	0

§3. Type G_2.

3.1. Throughout this section, G will denote the simple group of type G_2 over the field k, obtained from a split $\mathbb{Z}$-group $G_{\mathbb{Z}}$. We shall use notation similar to [Ja1]. Thus, $T=(T_{\mathbb{Z}})_k$ is a maximal torus , $R\subset X(T)$ its root system and $S=\{\alpha_1,\alpha_2\}$ a set of simple roots, with α_1 being the short one. Let ω_1 and ω_2 be the corresponding fundamental weights. In order to compute the group of extensions between the simple modules $L(\lambda)$ and $L(\mu)$, for λ, $\mu\in X_+(T)$, we use the 5-term sequence of the Lyndon-Hochschild Serre spectral sequence. Thus if $\lambda=\lambda^0+2\overline{\lambda}$ and $\mu=\mu^0+2\overline{\mu}$, with λ^0, $\mu^0\in X_1(T)=\{0,\omega_1,\omega_2,\omega_1+\omega_2\}$ and $\overline{\lambda}$, $\overline{\mu}\in X_+(T)$, we have an exact sequence

$$\begin{aligned}(1)\quad 0\longrightarrow \mathrm{Ext}^1_G(L(\overline{\lambda}),\mathrm{Hom}_{G_1}(L(\lambda^0),L(\mu^0))^{(2^{-1})}\otimes L(\overline{\mu}))\longrightarrow \mathrm{Ext}^1_G(L(\lambda),L(\mu))\\ \longrightarrow \mathrm{Hom}_G(L(\overline{\lambda}),\mathrm{Ext}^1_{G_1}(L(\lambda^0),L(\mu^0))^{(2^{-1})}\otimes L(\overline{\mu}))\\ \longrightarrow \mathrm{Ext}^2_G(L(\overline{\lambda}),\mathrm{Hom}_{G_1}(L(\lambda^0),L(\mu^0))^{(2^{-1})}\otimes L(\overline{\mu}))\longrightarrow 0\end{aligned}$$

If $\lambda^0=\mu^0$, then the third term is zero by a theorem of Andersen ([Ja1], II.12.9), and the resulting isomorphism of the first two terms shows that eventually we are reduced to the case $\lambda^0\neq\mu^0$. In this case the second and third terms are isomorphic and so we are interested in the G-module structure of the groups $\mathrm{Ext}^1_{G_1}(L(\lambda^0),L(\mu^0))$. The groups $H^1(G_1,L(\lambda^0))$ have been computed in [Ja2], and it is clear how to obtain all the $H^1(G,L(\lambda))$ from this. Also, the 1-cohomology of simple modules has been found for the finite groups $G_2(2^n)$, $n\geq 6$ in [Sin1]. We shall need some results from these papers in the calculations below.

Remark.We would like to correct some errors in the statement of the main theorem on p. 2654 in [Sin1]. The triples for which there is nonzero cohomology are the conjugates of $(\{0\},\emptyset,\emptyset)$ and $(\{1\},\{0\},\emptyset)$. (In [Sin1], the second triple contained a misprint.) Also, for the algebraic group, conjugation should be by the powers of the Frobenius map. (The assertion $\mathrm{Gal}(F/\mathbb{F}_2) \cong \mathbb{Z}$ is incorrect.) However, it is the corrected statement which is proved in [Sin1].

3.2. We shall describe some small modules here. We omit the details, which consist of elementary computations, and can be found or easily derived from [Sin1]. $L(\omega_1)$ and $L(\omega_2)$ have dimensions 6 and 14 respectively. The latter module is equal to $V(\omega_2)$, the adjoint module, while $V(\omega_1)$ has a trivial one-dimensional radical. Set $\rho = \omega_1 + \omega_2$. The module $L(\rho)$ is the first Steinberg module, hence equal to $V(\rho)$. We shall also need the structure of $V(2\omega_1)$. Its radical is isomorphic to the direct sum of $L(\omega_2)$ and $H^0(\omega_1)$. Finally, the tensor product $L(\omega_1) \otimes L(\omega_2)$ is isomorphic to the direct sum of $L(\rho)$ and a uniserial module with composition factors $L(\omega_1)$, k, $L(2\omega_1)$, k, $L(\omega_1)$.

3.3. In this paragraph we determine the groups $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$ for λ^0, $\mu^0 \in X_1(T)$. There are no self-extensions and $L(\rho)$ is injective for G_1, so the following result gives the complete answer.

LEMMA.

(a) $H^1(G_1, L(\omega_2))^{(2^{-1})} \cong H^0(\omega_1)$.

(b) $H^1(G_1, L(\omega_1))^{(2^{-1})}$ *is uniserial, having a trivial socle with quotient isomorphic to* $H^0(\omega_1)$.

(c) $\mathrm{Ext}^1_{G_1}(L(\omega_1), L(\omega_2)) = 0$.

PROOF: Part (a) is [Ja2], 5.3 (2), and for (b), formula 5.3(3) of the same paper gives the exact sequence

$$0 \to k \to H^1(G_1, L(\omega_1))^{(2^{-1})} \to H^0(\omega_1) \to 0,$$

so we only have to show that the middle term has no submodule isomorphic to $L(\omega_1)$. Now,

$$\mathrm{Hom}_G(L(\omega_1), H^1(G_1, L(\omega_1))^{(2^{-1})}) \cong \mathrm{Ext}^1_G(L(2\omega_1), L(\omega_1)),$$

which is zero, from the structure of $V(2\omega_1)$.

To prove (c), we note that the group there is isomorphic to $H^1(G_1, U)$, where U is the uniserial summand of $L(\omega_1) \otimes L(\omega_2))$ in 3.2. We have exact sequences

$$0 \to H^1(G_1, \mathrm{rad}\, U) \to H^1(G_1, U) \to H^1(G_1, L(\omega_1)) \tag{1}$$

and (since $H^1(G_1, k) = 0$)

$$0 \to \mathrm{rad}\, U / \mathrm{soc}\, U \to H^1(G_1, L(\omega_1)) \to H^1(G_1, \mathrm{rad}\, U) \to 0. \tag{2}$$

By (b), the first map of (2) is an isomorphism, so the last map in (1) is an embedding of G-modules. If the image were nonzero, it would contain a trivial submodule, by (b), but this would contradict the fact that $\mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(L(\omega_1), L(\omega_2))) \cong \mathrm{Ext}^1_G(L(\omega_1), L(\omega_2)) = 0$, since $V(\omega_2)$ is simple.

3.4. The next lemma describes the G_1-socles of $L(\omega_1)$ with the restricted simple modules.

LEMMA.

(a) $\mathrm{soc}_{G_1}(L(\omega_1)\otimes L(\omega_1))\cong k\oplus L(\omega_2)$.
(b) $\mathrm{soc}_{G_1}(L(\omega_1)\otimes L(\omega_2))\cong L(\rho)\oplus L(\omega_1)$.
(c) $\mathrm{soc}_{G_1}(L(\omega_1)\otimes L(\rho))\cong L(\omega_2)$.

The G-socles coincide with these.

PROOF: The G-socles are given in [Sin1] Lemmas 2.5 and 2.6. To show that the G_1-socles are the same as the G-socles we need to show that G acts trivially on $\mathrm{Hom}_{G_1}(L(\nu), L(\omega_1)\otimes L(\nu'))$ for any restricted simple modules $L(\nu)$ and $L(\nu')$. At this point we could apply Corollary I.4.2, or else argue directly as follows. The result is clear if $\nu'=0$, so we assume $\nu'\neq 0$. In (a) and (b), Lemma 2.1 of [Sin1] shows that all nontrivial simple G_1-modules occur with multiplicity less than 6 as G_1 composition factors of the tensor products on the left of the isomorphisms. Since 6 is the smallest dimension of a nontrivial G-module, the result follows for (a) and (b). For (c), it is enough to note that $\mathrm{Hom}_{G_1}(L(\nu), L(\omega_1)\otimes L(\rho))\cong \mathrm{Hom}_{G_1}(L(\rho), L(\omega_1)\otimes L(\nu))$ and the multiplicity of $L(\rho)$ as a G_1 composition factor of $L(\omega_1)\otimes L(\nu)$ is at most 1, by [Sin1], Lemma 2.1.

3.5. Let $\lambda=\lambda^0+2\lambda^1+4\tilde{\lambda}$ and $\mu=\mu^0+2\mu^1+4\tilde{\mu}$, with $\lambda^0\neq\mu^0$. ¿From 3.4 and 3.3, it follows that $\mathrm{Hom}_{G_1}(L(\lambda^1), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}\otimes L(\lambda^1))$ has dimension less than 6, hence must be trivial for G. Therefore, from 3.1, we obtain

(1)

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu))\cong\begin{cases}\mathrm{Hom}_G(L(\lambda^1), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}\otimes L(\lambda^1)) \\ \cong \mathrm{Hom}_{G_1}(L(\lambda^1), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})}\otimes L(\lambda^1)) & \text{if } \tilde{\lambda}=\tilde{\mu}, \\ 0 \quad \text{otherwise.}\end{cases}$$

LEMMA.

(a) $\mathrm{soc}_{G_1}(H^1(G_1, L(\omega_1))^{(2^{-1})})\cong k$.
(b) $\mathrm{soc}_{G_1}(H^1(G_1, L(\omega_2))^{(2^{-1})})\cong L(\omega_1)$.
(c) $\mathrm{soc}_{G_1}(L(\omega_1)\otimes H^1(G_1, L(\omega_1))^{(2^{-1})})\cong L(\omega_1)\oplus L(\omega_2)$.
(d) $\mathrm{soc}_{G_1}(L(\omega_2)\otimes H^1(G_1, L(\omega_1))^{(2^{-1})})\cong L(\omega_1)\oplus L(\omega_2)\oplus L(\rho)$.
(e) $\mathrm{soc}_{G_1}(L(\rho)\otimes H^1(G_1, L(\omega_1))^{(2^{-1})})\cong L(\omega_2)\oplus L(\rho)\oplus L(\rho)$.
(f) $\mathrm{soc}_{G_1}(L(\omega_1)\otimes H^1(G_1, L(\omega_2))^{(2^{-1})})\cong k\oplus L(\omega_2)$.
(g) $\mathrm{soc}_{G_1}(L(\omega_2)\otimes H^1(G_1, L(\omega_2))^{(2^{-1})})\cong L(\omega_1)\oplus L(\rho)$.
(h) $\mathrm{soc}_{G_1}(L(\rho)\otimes H^1(G_1, L(\omega_2))^{(2^{-1})})\cong L(\omega_2)\oplus L(\rho)$.

The G-socles are the same.

PROOF: We have already noted above, that the G-socles are equal to the G_1-socles, so we shall compute whichever one is the most convenient in each case. First, (a) and (b) are clear from 3.3. Also, (e) and (h) follow easily from [Sin1], Lemma 2.1, using the injectivity of $L(\rho)$. Next, we prove (g). We have $\mathrm{Hom}_G(k, L(\omega_2)\otimes H^0(\omega_1))=0$ and from the structure of $L(\omega_1)\otimes L(\omega_2)$ (3.2) we see that $\mathrm{Hom}_G(L(\omega_1), L(\omega_2)\otimes H^0(\omega_1))\cong k$. Also, $\mathrm{Hom}_G(L(\omega_2), L(\omega_2)\otimes H^0(\omega_1))\cong \mathrm{Hom}_G(V(\omega_2), H^0(\omega_2)\otimes H^0(\omega_1))=0$, as the latter group has dimension equal to the multiplicity of $H^0(\omega_2)$ as a good filtration factor of the tensor product. For (f), it is clear that the trivial module appears once in the socle and one of the computations in the proof of (g) shows that $L(\omega_2)$ appears once. There is no $L(\omega_1)$ in the socle since $[L(\omega_1)\otimes L(\omega_1) : L(\omega_1)] = 0$, so that $\mathrm{Hom}_G(L(\omega_1)\otimes L(\omega_1), H^0(\omega_1)) = 0$. To show that $L(\rho)$ does not appear in the socle one applies ([Sin1], Lemma 2.1) and the injectivity of $L(\rho)$ for G_1. (c) Set $M = H^1(G_1, L(\omega_1)^{(2^{-1})}$. It is clear that $\mathrm{Hom}_G(k, L(\omega_1)\otimes M) = 0$. Since $M/k \cong H^0(\omega_1)$ (3.3 (b)), it follows from (f) that $\mathrm{Hom}_G(L(\omega_1), L(\omega_1)\otimes(M/k)) = 0$, hence $\mathrm{Hom}_G(L(\omega_1), L(\omega_1)\otimes M) = k$. From the structure of $L(\omega_1)\otimes L(\omega_2)$, we obtain $\mathrm{Hom}_G(L(\omega_2), L(\omega_1)\otimes M) = k$, and from consideration of composition factors, we find $\mathrm{Hom}_G(L(\rho), L(\omega_1)\otimes M) = k$. Finally, we prove (d). The only computation which has not already been done above and does not follow from consideration of composition factors is that of $\mathrm{Hom}_G(L(\omega_2), L(\omega_2)\otimes M)$. By (g), we have $\mathrm{Hom}_G(L(\omega_2), L(\omega_2)\otimes(M/k)) = 0$, so the result is clear.

3.6. We summarize the results of 3.1-3.5.

PROPOSITION. *Let $\lambda = \lambda^0 + 2\lambda^1 + 4\tilde{\lambda}$ and $\mu = \mu^0 + 2\mu^1 + 4\tilde{\mu}$, where λ^i, $\mu^i \in X_1(T)$ and $\tilde{\lambda}$, $\tilde{\mu} \in X_+(T)$. Let $\overline{\lambda} = \lambda^1 + \tilde{\lambda}$ and $\overline{\mu} = \mu^1 + \tilde{\mu}$. Then if $\lambda^0 = \mu^0$, we have $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\overline{\lambda}), L(\overline{\mu}))$. If $\lambda^0 \neq \mu^0$ then $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$ unless $\tilde{\lambda} = \tilde{\mu}$ and $\{\lambda^0, \mu^0\}$ is (a) $\{0, \omega_1)$ or (b) $\{0, \omega_2\}$, in which case the dimension is given in the tables below.*

TABLE II.3.6(a)

(λ^1, μ^1)	0	ω_1	ω_2	$\omega_1+\omega_2$
0	1	0	0	0
ω_1	0	1	1	0
ω_2	0	1	1	1
$\omega_1+\omega_2$	0	0	1	2

TABLE II.3.6(b)

(λ^1, μ^1)	0	ω_1	ω_2	$\omega_1+\omega_2$
0	0	1	0	0
ω_1	1	0	1	0
ω_2	0	1	0	1
$\omega_1+\omega_2$	0	0	1	1

II.A Appendix

Table II.A.1. Weight multiplicities (Type D_3)

	dim	0	010	100	020 002	011	101	ρ
W-orbit size		1	4	6	4	12	12	24
$L(0)$	1	1						
$L(010)$	4	·	1					
$L(100)$	6	·	·	1				
$L(011)$	14	2	·	·	·	1		
$L(101)$	20	·	2	·	·	·	1	
$L(\rho)$	64	·	·	4	2	·	·	1

Table II.A.2 Weyl module composition factors (D_3)

Weyl module	radical (composition factors)
$V(\delta_1)$	$\emptyset$
$V(\delta_2)$	$\emptyset$
$V(\delta_2+\delta_3)$	k
$V(\delta_1+\delta_3)$	$\emptyset$
$V(2\delta_1)$	$L(\delta_2+\delta_3)$
$V(2\delta_2)$	$L(\delta_1)$
$V(3\delta_3)$	$L(\delta_2)$
$V(\delta_3+2\delta_2)$	$L(\delta_1+\delta_3)$
$V(\delta_2+2\delta_1)$	$L(\delta_1+\delta_3), L(2\delta_2+\delta_3)$
$V(\delta_1+2\delta_2)$	$k, L(\delta_2+\delta_3), L(2\delta_1)$
$V(\delta_1+\delta_2+2\delta_3)$	$L(\delta_1+\delta_3), L(2\delta_2+\delta_3), L(\delta_2+2\delta_1)$

Table II.A.3 Weyl module composition factors (B_3)

Weyl module	radical (composition factors)
$V(\omega_1)$	k
$V(\omega_2)$	$k, L(\omega_1)$
$V(\omega_3)$	$\emptyset$
$V(2\omega_1)$	$k, L(\omega_1), L(\omega_2)$

Table II.A.4 Weyl module composition factors (C_3)

Weyl module	radical (composition factors)
$V(\tilde{\omega}_1)$	$\emptyset$
$V(\tilde{\omega}_2)$	$\emptyset$
$V(2\tilde{\omega}_1)$	$k, L(\tilde{\omega}_2)$

Chapter III. Groups of Rank 4

§1. Preliminary remarks.

1.1. The goal of this chapter is to compute the extensions of simple modules for the simply connected semisimple algebraic groups of types A_4, B_4, C_4, and D_4 and for their Lie algebras. Much of the effort in this chapter is directed towards the computation of the quantities $E = \operatorname{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$ for all pairs of 2-restricted highest weights (λ^0, μ^0) in the case where G is simply connected of type A_4 or D_4. The socle of E can be determined by using an isomorphism derived from the Lyndon-Hochschild-Serre spectral sequence (cf. II.1.3). A modification of this idea can be used to determine the higher socle layers of E. If P, Q are simple G-modules, the long exact sequence in cohomology gives

$$0 \longrightarrow \operatorname{Hom}_G(P, \operatorname{soc}(E) \otimes Q) \longrightarrow \operatorname{Hom}_G(P, E \otimes Q) \longrightarrow \operatorname{Hom}_G(P, (E/\operatorname{soc}(E)) \otimes Q)$$

$$\longrightarrow \operatorname{Ext}^1_G(P, \operatorname{soc}(E) \otimes Q) \longrightarrow \dots \tag{1}$$

We will apply this repeatedly in situations where $\operatorname{Ext}^1_G(P, \operatorname{soc}(E) \otimes Q) \cong 0$. (It will turn out that for simply connected A_4 and D_4, we are able to show that all of the quantities $E = \operatorname{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$ have zero second socle layers, i.e., that E is semisimple.)

We use the resulting information to compute all of the groups $\operatorname{Ext}^1_G(L, M)$ for the (simply connected) algebraic groups of types A_4 and D_4 using the 5-term sequence. We then use the Lyndon-Hochschild-Serre spectral sequence for the pair $(\tilde{G}, \tilde{G}_\tau)$ (where $\tilde{G}_\tau$ is the kernel of the special isogeny $\tau : \tilde{G} \to G$ introduced in section I.3.1), together with the information gathered about the extension modules for the Frobenius kernel of D_4, to calculate the extension modules for $\tilde{G}_\tau$. Finally, we combine this information with known results (I.6.4) about the extension modules for G_σ (the kernel of $\sigma : G \to \tilde{G}$ discussed in I.3.1) to calculate all of the extension modules for (simply connected) B_4 and C_4, and their Lie algebras.

§2. Type A_4.

2.1. Throughout this section, G will denote the simply connected group of type A_4 over the field k, obtained from a split $\mathbb{Z}$-group $G_{\mathbb{Z}}$. We shall use notation similar to that of I.1, so for example the simple roots are α_i, i=1, 2, 3, 4, with corresponding fundamental weights λ_i.

2.2. We have $G \cong \mathrm{SL}_5(k)$ and $V = V(\lambda_1) = L(\lambda_1)$ may be identified with the natural 5-dimensional module. It is well known that $\Lambda^i(V) \cong V(\lambda_i)$ for $i = 2, 3, 4$. We observe also that $V(\lambda_1 + \lambda_4) \cong L(\lambda_1 + \lambda_4)$ may be identified with the adjoint module.

Unlike the situation in chapter II, we need first to calculate some socles of tensor products of simple restricted modules, as this information is needed in order to compute the structure of the $\operatorname{Ext}^1_{G_1}$ modules. As in chapter II, we list the factors in a good filtration of $H^0(\nu) \otimes H^0(\nu')$, narrowing down the possibilities for the G- socles of $L(\nu) \otimes L(\nu')$. These are listed in Table III.A.3 in the Appendix. The final answer can then be obtained by straightforward calculations. For future reference, we also record a table (Table III.A.5) of composition factors of certain Weyl modules which will be used throughout the paper. These can easily be computed using Jantzen's Sum Formula ([Ja1], II.8.19), together with

Freudenthal's Formula (or by utilizing the tables of weight multiplicities of Bremner et al. [BMP]). To save space, we do not repeat information which can be obtained directly from that already recorded by duality or a graph automorphism. Occasionally, we need to know the multiplicities of certain simple modules appearing as composition factors of certain tensor products of restricted simple modules, especially, for example, when utilizing the computations outlined in section I.6.6 for narrowing down the sructure of the $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$'s. For this information, we refer the reader to Table 2-2 of [Dw]. (The weight multiplicities of the restricted simple modules can be completely determined from those of the Weyl modules [BMP] using the Jantzen Sum Formula ([Ja1], II.8.19). The tensor products are then computed by calculating the weight orbits under the Weyl group and then multiplying the appropriate formal characters.)

2.3 Extensions for $(A_4)_1$. In this section we compute, along the lines described previously, all of the quantities $\mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$, where $L(\lambda^0)$ and $L(\mu^0)$ are simple G-modules with 2-restricted highest weights. Using the result of I.6.6 together with Table 2-2 of [Dw] and taking into account duality and the graph automorphism of A_4, we need to list 42 such computations. We also use the fact that for $G = A_4$, $\mathrm{Ext}^1_{G_1}(L, L) = 0$ for any restricted simple module L.

a) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_1 + \lambda_2 + \lambda_4)) \cong L(2\lambda_1)$.
From the fact ([Ja1], II.12.8) that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1, \lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_1 + \lambda_2 + \lambda_4))^{(2^{-1})} \cong 0$. From the Jantzen Sum Formula we obtain that $H^0(\lambda_1 + \lambda_2 + \lambda_4)/L(\lambda_1 + \lambda_2 + \lambda_4)$ has a filtration $L(\lambda_2)//L(2\lambda_1)$. Thus, the short exact sequence

$$0 \to L(\lambda_1 + \lambda_2 + \lambda_4) \to H^0(\lambda_1 + \lambda_2 + \lambda_4) \to H^0(\lambda_1 + \lambda_2 + \lambda_4)/L(\lambda_1 + \lambda_2 + \lambda_4) \to 0$$

yields the isomorphism

$$0 \to L(2\lambda_1) \to H^1(G_1, L(\lambda_1 + \lambda_2 + \lambda_4)) \to 0,$$

by taking G_1 -fixed points, and considering the long exact sequence in cohomology.

b) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_2 + \lambda_3)) \cong k$.
From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1, \lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_2 + \lambda_3))^{(2^{-1})} \cong 0$. Then from the short exact sequence

$$0 \to L(\lambda_2 + \lambda_3) \to H^0(\lambda_2 + \lambda_3) \to k \to 0,$$

we obtain

$$0 \to k \to H^1(G_1, L(\lambda_2 + \lambda_3)) \to 0,$$

by taking G_1 -fixed points, and considering the long exact sequence in cohomology.

c) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_2+\lambda_4)) \cong L(2\lambda_3)$.
From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_2+\lambda_4))^{(2^{-1})} \cong L(\lambda_3)$. Then from the short exact sequence

$$0 \to L(\lambda_2+\lambda_4) \to H^0(\lambda_2+\lambda_4) \to L(\lambda_1) \to 0,$$

we obtain

$$0 \to H^1(G_1, L(\lambda_2+\lambda_4)) \to L(2\lambda_3) \to 0,$$

by taking G_1 -fixed points, and considering the long exact sequence in cohomology.
d) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_3+\lambda_4)) \cong 0$.
From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_3+\lambda_4))^{(2^{-1})} \cong 0$. However, $H^0(\lambda_3+\lambda_4) \cong L(\lambda_3+\lambda_4)$.
e) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_1+\lambda_4)) \cong 0$.
From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_1+\lambda_4))^{(2^{-1})} \cong 0$. However, $H^0(\lambda_1+\lambda_4) \cong L(\lambda_1+\lambda_4)$.
f) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_2)) \cong L(2\lambda_1)$.
From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_2))^{(2^{-1})} \cong L(\lambda_1)$. However, $H^0(\lambda_2) \cong L(\lambda_2)$.
g) $\mathrm{Ext}^1_{G_1}(k, L(\lambda_1)) \cong 0$.
From the fact that

$$H^1(G_1, H^0(\lambda))^{(2^{-1})} \cong \mathrm{ind}_B^G(H^1(B_1,\lambda)^{(2^{-1})}),$$

we obtain $H^1(G_1, H^0(\lambda_1))^{(2^{-1})} \cong 0$. However, $H^0(\lambda_1) \cong L(\lambda_1)$.
h) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3)) \cong 0$.
By section I.6.6 and Table 2-2 of [Dw], the only simple G-module that could appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3))$ is $L(2\lambda_3)$.

$$\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_1+\lambda_2+\lambda_3)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_4), L(\lambda_3) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_4+2\lambda_4), L(\lambda_1+\lambda_2+\lambda_3+2\lambda_1)) \cong 0,$$

since $\lambda_1 + 3\lambda_4$ is not comparable with $3\lambda_1 + \lambda_2 + \lambda_3$ in the (usual) partial order.

i) $\mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

$$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + \lambda_4 + 2\lambda_3), L(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_1)) \cong 0,$$

since $3\lambda_3 + \lambda_4$ is not comparable with $3\lambda_1 + \lambda_2 + \lambda_3$ in the (usual) partial order.

j) $\mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_1 + \lambda_2 + \lambda_3)) \cong L(2\lambda_4)$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_4)$.

$$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_2), L(\lambda_1 + \lambda_2 + \lambda_3)))$$

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_2 + 2\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \cong k,$$

by considering the Weyl module $V(\lambda_1 + \lambda_2 + 2\lambda_4)$.

k) $\mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_1)$.

$$\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_3 + \lambda_4), L(\lambda_1 + \lambda_2 + \lambda_3)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + \lambda_3 + \lambda_4 + 2\lambda_2), L(\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_1)) \cong 0,$$

since $3\lambda_2 + \lambda_3 + \lambda_4$ is not comparable with $3\lambda_1 + \lambda_2 + \lambda_3$ in the (usual) partial order.

l) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong L(2\lambda_3)$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_3)$.

$$\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1 + \lambda_2 + \lambda_4)))$$

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + 2\lambda_3), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong k,$$

by considering the Weyl module $V(\lambda_1 + 2\lambda_3)$.

m) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong 0$.

The only simple G-module isomorphism types that could appear as a composition factors are $L(2\lambda_4), L(2[\lambda_1 + \lambda_3])$.

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_4)) \cong 0,$$

as $3\lambda_4$ is not comparable with $\lambda_1 + \lambda_2 + \lambda_4$ in the (usual) partial order.

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1 + \lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1 + \lambda_2 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1+\lambda_2), L(\lambda_1+\lambda_3)\otimes L(\lambda_4))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4+2[\lambda_1+\lambda_2]), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4))\cong 0,$$

since $2\lambda_1+2\lambda_2+\lambda_4$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order.

n) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1+\lambda_2+\lambda_4))\cong 0$.

The only simple G-module isomorphism types that could appear as a composition factors are $k, L(2[\lambda_1+\lambda_4])$.

i) $\mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_2), L(\lambda_1+\lambda_2+\lambda_4))\cong 0,$$

by considering the Weyl module $V(\lambda_1+\lambda_2+\lambda_4)\subseteq V(\lambda_1+\lambda_2)\otimes V(\lambda_4)\cong L(\lambda_1+\lambda_2)\otimes L(\lambda_4)$. (See Table III.A.5.) Since $\mathrm{Hom}_G(L(2\lambda_1), L(\lambda_1+\lambda_2)\otimes L(\lambda_4))\cong \mathrm{Hom}_G(L(2\lambda_1+\lambda_3+\lambda_4), L(\lambda_4))\cong 0$, we must have that the single composition factor of $V(\lambda_1+\lambda_2+\lambda_4)$ isomorphic to $L(\lambda_2)$ lies in (in fact, equals) $\mathrm{rad}^2(V(\lambda_1+\lambda_2+\lambda_4))$.

ii) $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1+\lambda_2+\lambda_4))\otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_1+\lambda_4)\otimes L(\lambda_2))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+2[\lambda_3+\lambda_4]), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_2))\cong 0,$$

since $\lambda_2+2\lambda_3+2\lambda_4$ is not comparable with $\lambda_1+3\lambda_2+\lambda_4$ in the (usual) partial order.

o) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_1+\lambda_2+\lambda_4))\cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_1+\lambda_2+\lambda_4))\otimes L(2[\lambda_1+\lambda_4]))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_2)\otimes L(\lambda_1+\lambda_4))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+2[\lambda_3+\lambda_4]), L(\lambda_1+\lambda_2+\lambda_4+2[\lambda_1+\lambda_4]))\cong 0,$$

since $3\lambda_3+2\lambda_4$ is not comparable with $3\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order.

p) $\mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4))\cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_3)$.

$\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)\otimes L(\lambda_4))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+\lambda_4+2\lambda_2), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4))\cong 0,$$

since $3\lambda_2+\lambda_4$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order.

q) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$.
The only simple G-module that could appear as a composition factor is $L(2\lambda_4)$.

$$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+2\lambda_3), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+3\lambda_3$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order.

r) $\mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$.
The only simple G-module that could appear as a composition factor is $L(2\lambda_1)$.

$$\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2+\lambda_3), L(\lambda_1+\lambda_2+\lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+\lambda_3+2\lambda_2), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_1)) \cong 0,$$

since $3\lambda_2+\lambda_3$ is not comparable with $3\lambda_1+\lambda_2+\lambda_4$ in the (usual) partial order.

s) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \cong 0$.
The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

$$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3+\lambda_4), L(\lambda_1+\lambda_2+\lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+\lambda_4+2\lambda_1), L(\lambda_1+\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $3\lambda_1+\lambda_3+\lambda_4$ is not comparable with $\lambda_1+\lambda_2+3\lambda_4$ in the (usual) partial order.

t) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)) \cong 0$.
We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_4) \otimes L(\lambda_2+\lambda_3)) \cong H^1(G_1, L(\lambda_4) \otimes L(\lambda_2+\lambda_3)).$$

By considering the composition factors of $L(\lambda_4) \otimes L(\lambda_2+\lambda_3)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_4) \otimes L(\lambda_2+\lambda_3))$ is $H^1(G_1, L(\lambda_1+\lambda_3)) \cong L(2\lambda_2)$, by 2.3(c).

$$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_3)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + 2\lambda_1), L(\lambda_2 + \lambda_3 + 2\lambda_4)) \cong 0,$$

since $3\lambda_1$ is not comparable with $\lambda_2 + \lambda_3 + 2\lambda_4$ in the (usual) partial order.

u) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_3)) \cong 0$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_4)$, $L(2[\lambda_1 + \lambda_3])$.

i) $\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_3)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2\lambda_4), L(\lambda_2 + \lambda_3)) \cong 0,$$

by examining the Weyl modules $V(\lambda_2 + 2\lambda_4)$ and $V(2\lambda_1)$. From $\mathrm{Ext}^1_G(k, L(\lambda_2 + 2\lambda_4)) \cong \mathrm{Ext}^1_G(L(2\lambda_1), L(\lambda_2)) \cong k$, we must have that the radical of $V(\lambda_2 + 2\lambda_4)$ is uniserial with a filtration $k//L(\lambda_2 + \lambda_3)//k$.

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_3)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1 + \lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2 + \lambda_3)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1 + \lambda_2), L(\lambda_1 + \lambda_3) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2[\lambda_1 + \lambda_2]), L(\lambda_2 + \lambda_3 + 2\lambda_4)) \cong 0,$$

since $2\lambda_1 + 3\lambda_2$ is not comparable with $\lambda_2 + \lambda_3 + 2\lambda_4$ in the (usual) partial order.

v) $\mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_4), L(\lambda_2 + \lambda_3)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_4), L(\lambda_2 + \lambda_3)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_2 + \lambda_4), L(\lambda_2 + \lambda_3)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + \lambda_4 + 2\lambda_1), L(\lambda_2 + \lambda_3 + 2\lambda_4)) \cong 0,$$

since $2\lambda_1 + \lambda_2 + \lambda_4$ is not comparable with $\lambda_2 + \lambda_3 + 2\lambda_4$ in the (usual) partial order.

w) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)) \cong k$.

The only possible isomorphism types of summands of the socle are k, $L(2[\lambda_1 + \lambda_4])$, and $L(2[\lambda_2 + \lambda_3])$.

i) $\mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_1), L(\lambda_2 + \lambda_4)) \cong k,$$

by considering the Weyl module $V(\lambda_2 + \lambda_4)$.

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4)) \otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_1+\lambda_4)\otimes L(\lambda_2))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2[\lambda_3+\lambda_4]), L(\lambda_2+\lambda_4+2\lambda_2))\cong 0,$$

since $\lambda_1+2\lambda_3+2\lambda_4$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order.

iii) $\mathrm{Hom}_G(L(2[\lambda_2+\lambda_3]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4))\otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3+\lambda_4), L(\lambda_2+\lambda_3)\otimes L(\lambda_2))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2[\lambda_3+\lambda_4]), L(\lambda_2+\lambda_4+2\lambda_2))\cong 0,$$

since $\lambda_1+2\lambda_3+2\lambda_4$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order.

We have thus shown that

$$\mathrm{soc}_G(\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4)))\cong k.$$

Now, the only type of simple G-module which can appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2+\lambda_4))$, and which extends k, is $L(2[\lambda_2+\lambda_3])$. However, the calculation in part (iii) together with the sequence 1.1(1) shows that $L(2[\lambda_2+\lambda_3])$ cannot be a summand of the second socle layer because $L(\lambda_3+\lambda_4)$ does not extend $k\otimes L(\lambda_2)\cong L(\lambda_2)$.

x) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2+\lambda_4))\cong L(2\lambda_1)$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2+\lambda_4))\cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1)\otimes L(\lambda_2+\lambda_4)).$$

$$\cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_3+\lambda_4)\oplus L(\lambda_1+\lambda_2+\lambda_4))$$

$$\cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_3+\lambda_4))\oplus \mathrm{Ext}^1_{G_1}(k, L(\lambda_1+\lambda_2+\lambda_4))$$

$$\cong L(2\lambda_1).$$

(by 2.3(a), (d).)

y) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_4))\cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_4))\cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_3)\otimes L(\lambda_2+\lambda_4))\cong H^1(G_1, L(\lambda_3)\otimes L(\lambda_2+\lambda_4)).$$

By considering the composition factors of $L(\lambda_3)\otimes L(\lambda_2+\lambda_4)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_3)\otimes L(\lambda_2+\lambda_4))$ is $H^1(G_1, L(\lambda_1+\lambda_3))\cong L(2\lambda_2)$, by 2.3(c).

$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_2+\lambda_4))\otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2)\otimes L(\lambda_1))\cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_1))\cong 0,$$

since $\lambda_2+2\lambda_3$ is not comparable with $2\lambda_1+\lambda_2+\lambda_4$ in the (usual) partial order.

z) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2+\lambda_4)) \cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2+\lambda_4)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_2)\otimes L(\lambda_2+\lambda_4)) \cong H^1(G_1, L(\lambda_2)\otimes L(\lambda_2+\lambda_4)).$$

By considering the composition factors of $L(\lambda_2)\otimes L(\lambda_2+\lambda_4)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_2)\otimes L(\lambda_2+\lambda_4))$ is $H^1(G_1, L(\lambda_3)) \cong L(2\lambda_4) \cong H^1(G_1, L(\lambda_1+\lambda_3+\lambda_4))$, by 2.3(a), (f).

$$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $3\lambda_3$ is not comparable with $\lambda_2+3\lambda_4$ in the (usual) partial order.

aa) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_2+\lambda_4)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_3)$.

$$\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_4+2\lambda_2), L(\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+2\lambda_2+\lambda_4$ is not comparable with $\lambda_2+3\lambda_4$ in the (usual) partial order.

bb) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4)) \cong 0$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_1)$, $L(2[\lambda_2+\lambda_4])$.

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4))\otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_1)\otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_2)) \cong 0,$$

since $\lambda_1+3\lambda_3$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order.

ii) $\mathrm{Hom}_G(L(2[\lambda_2+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_3), L(\lambda_2+\lambda_4))\otimes L(2\lambda_2))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2+\lambda_4)\otimes L(\lambda_2)) \cong \mathrm{Hom}_G(L(\lambda_1+\lambda_3), L(\lambda_2)\otimes L(\lambda_2)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_3+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_2)) \cong 0,$$

since $\lambda_1+3\lambda_3$ is not comparable with $3\lambda_2+\lambda_4$ in the (usual) partial order.

cc) $\mathrm{Ext}^1_{G_1}(L(\lambda_3+\lambda_4), L(\lambda_2+\lambda_4)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_2)$.

$$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3+\lambda_4), L(\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3+\lambda_4), L(\lambda_2+\lambda_4))\otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2)\otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3+\lambda_4+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_1)) \cong 0,$$

since $3\lambda_3+\lambda_4$ is not comparable with $2\lambda_1+\lambda_2+\lambda_4$ in the (usual) partial order.

dd) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_2+\lambda_4)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_4)$.

$$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_2+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_2+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_2+2\lambda_3), L(\lambda_2+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+\lambda_2+2\lambda_3$ is not comparable with $\lambda_2+3\lambda_4$ in the (usual) partial order.

ee) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3+\lambda_4)) \cong 0$.

The only simple G-module that could appear as a composition factor is $L(2\lambda_3)$.

$$\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3+\lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2\lambda_2), L(\lambda_3+\lambda_4+2\lambda_4)) \cong 0,$$

since $\lambda_1+2\lambda_2$ is not comparable with $\lambda_3+3\lambda_4$ in the (usual) partial order.

ff) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3+\lambda_4)) \cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3+\lambda_4)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1)\otimes L(\lambda_3+\lambda_4)) \cong H^1(G_1, L(\lambda_1)\otimes L(\lambda_3+\lambda_4)).$$

By considering the composition factors of $L(\lambda_1)\otimes L(\lambda_3+\lambda_4)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_1)\otimes L(\lambda_3+\lambda_4))$ is $H^1(G_1, L(\lambda_3)) \cong L(2\lambda_4) \cong H^1(G_1, L(\lambda_1+\lambda_3+\lambda_4))$, by 2.3(a), (f).

$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_3 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2\lambda_4), L(\lambda_3 + \lambda_4)) \cong 0,$$

by considering the Weyl module $V(3\lambda_4)$.

gg) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3 + \lambda_4)) \cong 0$.

The only simple G-modules that could appear as a composition factors are k and $L(2[\lambda_1 + \lambda_4])$.

i) $\mathrm{Hom}_G(k, \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_2), L(\lambda_3 + \lambda_4)) \cong 0,$$

by considering the Weyl module $V(\lambda_3 + \lambda_4) \cong L(\lambda_3 + \lambda_4)$.

ii) $\mathrm{Hom}_G(L(2[\lambda_1 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1 + \lambda_2]), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_3 + \lambda_4)) \otimes L(2\lambda_3))$$

(as $\mathrm{Hom}_G(L(\lambda_1 + \lambda_2), L(\lambda_1 + \lambda_4) \otimes L(\lambda_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2[\lambda_1 + \lambda_2]), L(\lambda_3 + \lambda_4 + 2\lambda_3)) \cong 0,$$

since $2\lambda_1 + 3\lambda_2$ is not comparable with $3\lambda_3 + \lambda_4$ in the (usual) partial order.

hh) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_3 + \lambda_4)) \cong 0$.

The only simple G-module type that could appear as a composition factor is $L(2\lambda_2)$.

$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_3 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + 2\lambda_2), L(\lambda_3 + \lambda_4)) \cong 0,$$

by considering the Weyl module $V(2\lambda_2 + \lambda_3)$.

ii) $\mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_3 + \lambda_4)) \cong 0$.

The only simple G-module types that could appear as composition factors are $L(2\lambda_1)$, $L(2[\lambda_2 + \lambda_4])$.

i) $\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_3 + \lambda_4)))$

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_4 + 2\lambda_1), L(\lambda_3 + \lambda_4)) \cong k,$$

by considering the Weyl module $V(3\lambda_1 + \lambda_4)$.

ii) $\mathrm{Hom}_G(L(2[\lambda_2 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_3 + \lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_3 + \lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1 + \lambda_4), L(\lambda_3 + \lambda_4)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3 + \lambda_4), L(\lambda_2 + \lambda_4) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + \lambda_4 + 2[\lambda_3 + \lambda_4]), L(\lambda_3 + \lambda_4 + 2\lambda_1)) \cong 0,$$

by considering the Weyl module $V(\lambda_1 + 2\lambda_3 + 3\lambda_4])$.

We have thus shown that

$$\mathrm{soc}_G(\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4))) \cong L(2\lambda_1).$$

Now, the only type of simple G-module which can appear as a composition factor of $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_4), L(\lambda_3+\lambda_4))$, and which extends $L(2\lambda_1)$, is $L(2[\lambda_2+\lambda_4])$. However, the calculation in (ii) together with the sequence 1.1(1) shows that $L(2[\lambda_2+\lambda_4])$ cannot be a summand of the second socle layer because $L(\lambda_3+\lambda_4)$ does not extend any of the composition factors of $L(\lambda_1)\otimes L(\lambda_1)$.

jj) $\mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_3+\lambda_4)) \cong 0$.
The only simple G-module types that could appear as composition factors are $L(2\lambda_2)$, $L(2[\lambda_3+\lambda_4])$.

i) $\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_3+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_3+\lambda_4))\otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_1), L(\lambda_2)\otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_2+2\lambda_1), L(\lambda_3+\lambda_4+2\lambda_4)) \cong 0,$$

since $3\lambda_1+\lambda_2$ is not comparable with $\lambda_3+3\lambda_4$ in the (usual) partial order.

ii) $\mathrm{Hom}_G(L(2[\lambda_3+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_3+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2[\lambda_1+\lambda_4]), \mathrm{Ext}^1_{G_1}(L(\lambda_1+\lambda_2), L(\lambda_3+\lambda_4))\otimes L(2\lambda_3))$$

(as $\mathrm{Hom}_G(L(2[\lambda_1+\lambda_4]), L(\lambda_3+\lambda_4)\otimes L(\lambda_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+\lambda_2+2[\lambda_1+\lambda_4]), L(\lambda_3+\lambda_4+2\lambda_3)) \cong 0,$$

since $3\lambda_1+\lambda_2+2\lambda_4$ is not comparable with $3\lambda_3+\lambda_4$ in the (usual) partial order.

kk) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1+\lambda_4)) \cong 0$.
We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1+\lambda_4)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_4)\otimes L(\lambda_1+\lambda_4)) \cong H^1(G_1, L(\lambda_4)\otimes L(\lambda_1+\lambda_4)).$$

By considering the composition factors of $L(\lambda_4)\otimes L(\lambda_1+\lambda_4)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_4)\otimes L(\lambda_1+\lambda_4))$ is $H^1(G_1, L(\lambda_1+\lambda_3)) \cong L(2\lambda_2)$, by 2.3(c).

$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1+\lambda_4)))$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_1+\lambda_4))\otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2)\otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1+2\lambda_3), L(\lambda_1+\lambda_4+2\lambda_1)) \cong 0,$$

since $\lambda_1 + 2\lambda_3$ is not comparable with $3\lambda_1 + \lambda_4$ in the (usual) partial order.

ll) $\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)) \cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_3) \otimes L(\lambda_1 + \lambda_4)) \cong H^1(G_1, L(\lambda_3) \otimes L(\lambda_1 + \lambda_4)).$$

By considering the composition factors of $L(\lambda_3) \otimes L(\lambda_1 + \lambda_4)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_3) \otimes L(\lambda_1 + \lambda_4))$ is $H^1(G_1, L(\lambda_3)) \cong L(2\lambda_4) \cong H^1(G_1, L(\lambda_1 + \lambda_3 + \lambda_4))$, by 2.3(a), (f).

$$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_2), L(\lambda_1 + \lambda_4)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_4) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_2 + 2\lambda_3), L(\lambda_1 + \lambda_4 + 2\lambda_4)) \cong 0,$$

since $\lambda_2 + 2\lambda_3$ is not comparable with $\lambda_1 + 3\lambda_4$ in the (usual) partial order.

mm) $\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_4) \otimes L(\lambda_2)) \cong H^1(G_1, L(\lambda_4) \otimes L(\lambda_2)).$$

By considering the composition factors of $L(\lambda_4) \otimes L(\lambda_2)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_4) \otimes L(\lambda_2))$ is $H^1(G_1, L(\lambda_2 + \lambda_4)) \cong L(2\lambda_3)$, by 2.3(c).

$$\mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2)) \otimes L(2\lambda_4))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_3) \otimes L(\lambda_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_1 + 2\lambda_2), L(\lambda_2 + 2\lambda_4)) \cong 0,$$

since $\lambda_1 + 2\lambda_2$ is not comparable with $\lambda_2 + 2\lambda_4$ in the (usual) partial order.

nn) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)) \cong L(2\lambda_4)$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1) \otimes L(\lambda_2)) \cong H^1(G_1, L(\lambda_1) \otimes L(\lambda_2)).$$

By considering the composition factors of $L(\lambda_1) \otimes L(\lambda_2)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_1) \otimes L(\lambda_2))$ is $H^1(G_1, L(\lambda_3)) \cong L(2\lambda_4)$, by 2.3(f).

$$\mathrm{Hom}_G(L(2\lambda_4), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2)))$$

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2\lambda_4), L(\lambda_2)) \cong k,$$

by considering the Weyl module $V(3\lambda_4)$.

oo) $\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_2) \otimes L(\lambda_2)) \cong H^1(G_1, L(\lambda_2) \otimes L(\lambda_2)).$$

By considering the composition factors of $L(\lambda_2) \otimes L(\lambda_2)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_2) \otimes L(\lambda_2))$ is $H^1(G_1, L(\lambda_1 + \lambda_3)) \cong L(2\lambda_2)$,by 2.3(c).

$$\mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_3), \mathrm{Ext}^1_{G_1}(L(\lambda_3), L(\lambda_2)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_3), L(\lambda_2) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_3 + 2\lambda_3), L(\lambda_2 + 2\lambda_1)) \cong 0,$$

since $3\lambda_3$ is not comparable with $2\lambda_1 + \lambda_2$ in the (usual) partial order.

pp) $\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)) \cong 0$.

We have

$$\mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1) \otimes L(\lambda_1)) \cong H^1(G_1, L(\lambda_1) \otimes L(\lambda_1)).$$

By considering the composition factors of $L(\lambda_1) \otimes L(\lambda_1)$, and the long exact sequence in cohomology, we observe that the only simple G-module type that could appear as a composition factor of $H^1(G_1, L(\lambda_1) \otimes L(\lambda_1))$ is $H^1(G_1, L(\lambda_2)) \cong L(2\lambda_1)$, by 2.3(f).

$$\mathrm{Hom}_G(L(2\lambda_1), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)))$$

$$\precsim \mathrm{Hom}_G(L(2\lambda_2), \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1)) \otimes L(2\lambda_1))$$

(as $\mathrm{Hom}_G(L(\lambda_2), L(\lambda_1) \otimes L(\lambda_1)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\lambda_4 + 2\lambda_2), L(\lambda_1 + 2\lambda_1)) \cong 0,$$

since $2\lambda_2 + \lambda_4$ is not comparable with $3\lambda_1$ in the (usual) partial order.

2.4 Ext Computation Tables for A_4. We are now ready to compute the Ext^1_Gs for the simply connected algebraic group of type A_4. We use the same notation as in II.1.3, letting λ and μ be dominant weights, with $\lambda = \sum 2^i\lambda^i = \lambda^0 + 2\hat{\lambda} = \lambda^0 + 2\lambda^1 + 4\overline{\lambda}$, and similarly for μ. As in II.1.3, if $\lambda^0 = \mu^0$, we obtain the isomorphism

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\hat{\lambda}), L(\hat{\mu})),$$

because of the vanishing third term of the 5-term sequence for (G, G_1). This reduces us again to the case $\lambda^0 \neq \mu^0$. Then the vanishing of the first and fourth terms of the 5-term sequence yields

$$\begin{aligned}(1)\quad \mathrm{Ext}^1_G(L(\lambda), L(\mu)) &\cong \mathrm{Hom}_G(L(\hat{\lambda}), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\hat{\mu})) \\ &\cong \mathrm{Hom}_G(L(\overline{\lambda}), \mathrm{Hom}_{G_1}(L(\lambda^1), \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\mu^1))^{(2^{-1})} \otimes L(\overline{\mu})).\end{aligned}$$

By the results of the preceding section, all the modules $E = \mathrm{Ext}^1_{G_1}(L(\lambda^0), L(\mu^0))$ are simple modules of the form k or $L(\lambda_i)$, $i = 1, 2, 3, 4$. We wish to apply Corollary I.4.2.1 with $\nu = 0$, λ_i. The minimum value of $\langle 2\omega, \rho\rangle$ for $\omega \in X_1(T) \setminus \{0\}$ is 4 whereas $\langle \lambda_1, \rho\rangle = \langle \lambda_4, \rho\rangle = 2$, and $\langle \lambda_2, \rho\rangle = \langle \lambda_3, \rho\rangle = 3$, so the hypotheses of the corollary are satisfied. It follows that G acts trivially on modules of the form $\mathrm{Hom}_{G_1}(L(\lambda^1), E \otimes L(\mu^1))^{(2^{-1})}$. Therefore, we obtain the same result as in II.1.5(1), namely,

$$(2) \qquad \mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \begin{cases} \mathrm{Hom}_G(L(\lambda^1), E \otimes L(\mu^1))^{(2^{-1})} & \text{if } \overline{\lambda} = \overline{\mu} \\ 0 \quad \text{otherwise} \end{cases}$$

This observation, together with the information gathered in the preceding section allow us now to read off all of the extensions for simply connected A_4 from appropriate tables of tensor product socles:

Table III.2.4(a): $E = L(\lambda_1) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1 + \lambda_2 + \lambda_4))^{(2^{-1})}$ (2.3(a).)
$\cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_2))^{(2^{-1})}$ (2.3(f).)
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_2 + \lambda_4))^{(2^{-1})}$ (2.3(x).)
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_3 + \lambda_4), L(\lambda_2 + \lambda_3 + \lambda_4))^{(2^{-1})}$ (2.3(j).)
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_3))^{(2^{-1})}$ (2.3(nn).)

Table III.2.4(b): $E = L(\lambda_2) \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_1 + \lambda_3))^{(2^{-1})}$ (2.3(c).)
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_4), L(\lambda_1 + \lambda_3 + \lambda_4))^{(2^{-1})}$ (2.3(l).)

Of course, no table is needed for $E = k \cong \mathrm{Ext}^1_{G_1}(k, L(\lambda_2 + \lambda_3))^{(2^{-1})}$ (2.3(b).)
$\cong \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2 + \lambda_4))^{(2^{-1})}$ (2.3(w).)

All other $\mathrm{Ext}^1_{G_1}$ modules for A_4 are either zero or duals of those listed above.

Table III.2.4(a) $E = L(\lambda_1)$

product	socle
$E \otimes L(\lambda_1)$	$L(\lambda_2)$
$E \otimes L(\lambda_4)$	$k \oplus L(\lambda_1 + \lambda_4)$
$E \otimes L(\lambda_2)$	$L(\lambda_3) \oplus L(\lambda_1 + \lambda_2)$
$E \otimes L(\lambda_3)$	$L(\lambda_4)$
$E \otimes L(\lambda_1 + \lambda_4)$	$L(\lambda_1)$
$E \otimes L(\lambda_2 + \lambda_4)$	$L(\lambda_3 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3)$	$L(\lambda_2 + \lambda_3)$
$E \otimes L(\lambda_3 + \lambda_4)$	$L(\lambda_3)$
$E \otimes L(\lambda_1 + \lambda_2)$	$L(\lambda_1 + \lambda_3)$
$E \otimes L(\lambda_2 + \lambda_3)$	$L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_4)$	$L(\lambda_1 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3 + \lambda_4)$	$L(\lambda_1 + \lambda_3) \oplus L(\lambda_2 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_3)$	$L(\lambda_1 + \lambda_2 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_3 + \lambda_4)$	$L(\lambda_2 + \lambda_3) \oplus L(\rho)$
$E \otimes L(\rho)$	$L(\lambda_1 + \lambda_2 + \lambda_3)$

TABLE III.2.4(b) $E = L(\lambda_2)$

product	socle
$E \otimes L(\lambda_1)$	$L(\lambda_3) \oplus L(\lambda_1 + \lambda_2)$
$E \otimes L(\lambda_4)$	$L(\lambda_1)$
$E \otimes L(\lambda_2)$	$L(\lambda_4) \oplus L(\lambda_1 + \lambda_3)$
$E \otimes L(\lambda_3)$	$k \oplus L(\lambda_1 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_4)$	$L(\lambda_2) \oplus L(\lambda_3 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_4)$	$L(\lambda_3) \oplus L(\lambda_1 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3)$	$L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$
$E \otimes L(\lambda_3 + \lambda_4)$	$L(\lambda_4) \oplus L(\lambda_2 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_2)$	$L(\lambda_2 + \lambda_3) \oplus L(\lambda_1 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_3)$	$L(\lambda_3 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_4)$	$L(\lambda_1 + \lambda_3) \oplus L(\lambda_2 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_1 + \lambda_3 + \lambda_4)$	$L(\lambda_2 + \lambda_3) \oplus L(\rho)$
$E \otimes L(\lambda_1 + \lambda_2 + \lambda_3)$	$L(\lambda_1 + \lambda_2) \oplus L(\lambda_1 + \lambda_3 + \lambda_4)$
$E \otimes L(\lambda_2 + \lambda_3 + \lambda_4)$	$L(\lambda_2 + \lambda_4) \oplus L(\lambda_1 + \lambda_2 + \lambda_3)$
$E \otimes L(\rho)$	$L(\lambda_1 + \lambda_2 + \lambda_4)$

§3. Type D_4.

3.1. Throughout this section, D will denote the simply connected group of type D_4 over the field k, obtained from a split $\mathbb{Z}$-group $D_{\mathbb{Z}}$. We shall use the notation of I.1, so for example the simple roots are β_i, i=1, 2, 3, 4, with corresponding fundamental weights δ_i.

3.2. We have $D \cong \mathrm{Spin}_8(k)$ and $V(\delta_1) \cong L(\delta_1)$ may be identified with the natural 8-dimensional orthogonal module, while $V(\delta_3)$ and $V(\delta_4)$ may be identified with the half-spin modules. (They are conjugate by the triality automorphism of order 3.) We have already observed (I.5.6) that $V(\delta_2)$ may be identified with the adjoint module.

As in III.2, we need first to calculate some socles of tensor products of simple restricted modules, (as this information is needed in order to compute the structure of the $\mathrm{Ext}^1_{D_1}$ modules.) As usual, we list the factors in a good filtration of $H^0(\nu) \otimes H^0(\nu')$, narrowing down the possibilities for the G- socles of $L(\nu) \otimes L(\nu')$. These are listed in Table III.A.4 in the Appendix. The final answer can then be obtained by straightforward calculations in all cases except the following: $L(\delta_2) \otimes L(\delta_2), L(\delta_2) \otimes L(\delta_1 + \delta_3 + \delta_4)$ (which were already computed in [Sin2]) and $L(\delta_2) \otimes L(\delta_2 + \delta_3 + \delta_4)$, which is handled in the next subsection along with a couple of other miscellaneous lemmas about tensor product socles which are needed for the computation of the $\mathrm{Ext}^1_{D_1}$ structures. For future reference, we also record a table (Table III.A.6) of composition factors of certain Weyl modules which will be used throughout the paper. (These can easily be computed using Jantzen's Sum Formula ([Ja1], II.8.19), together with Freudenthal's Formula or by using the tables of weight multiplicities of Bremner et al. [BMP]). We do not repeat information which can be obtained directly from that already recorded through duality or an automorphism.

3.3. We include here some miscellaneous lemmas involving some socles of tensor products which will be used in the next chapter. In the following, denote by G the simply connected group of type B_4, by $\tilde{G}$ the simply connected group of type C_4, and by D the subgroup of G generated by the long root subgroups (which is simply connected of type D_4).

LEMMA 3.3.1.

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)) \cong k.$$

PROOF: First we observe that

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)) \neq 0. \tag{1}$$

This follows from the fact that

$$\mathrm{Hom}_D(L, L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))$$

can be shown to be zero for all other composition factors of $L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)$ in the same linkage class as $L(\delta_2+\delta_3+\delta_4)$.

Also, we have

$$\mathrm{Hom}_D(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4))$$
$$\cong \mathrm{Hom}_{D_1}(L(\delta_2+\delta_3+\delta_4), L(\delta_2)\otimes L(\delta_2+\delta_3+\delta_4)),$$

(by Cor. I.4.2.1 with $\nu=\delta_2$)

$$\cong \mathrm{Hom}_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3), \tilde{L}(\tilde{\omega}_2)\otimes\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))$$

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3), \tilde{L}(\tilde{\omega}_2)\otimes\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)),$$

(by the $\tilde{G}_\tau$ version of Corollary I.4.2.1 with $\nu=\tilde{\omega}_2$,)

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)\otimes\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)).$$

The last space is isomorphic to a subspace of

$$\mathrm{Hom}_{\tilde{G}}(\tilde{V}(\tilde{\omega}_2), \tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)\otimes\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)) \tag{2}$$

$$\cong \mathrm{Hom}_{\tilde{G}}(\tilde{V}(\tilde{\omega}_2+\tilde{\omega}_3), \tilde{H}^0(\tilde{\omega}_2)\otimes\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)),$$

which has dimension 2, as can be seen by means of a good filtration of $\tilde{H}^0(\tilde{\omega}_2)\otimes\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)$:

$\tilde{H}^0(2\tilde{\omega}_2+\tilde{\omega}_3)//\tilde{H}^0(\tilde{\omega}_1+2\tilde{\omega}_3)//\tilde{H}^0(\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_4)//\tilde{H}^0(\tilde{\omega}_3+\tilde{\omega}_4)//\tilde{H}^0(\tilde{\omega}_1+2\tilde{\omega}_2)//$
$\tilde{H}^0(2\tilde{\omega}_1+\tilde{\omega}_3)//2\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)//\tilde{H}^0(\tilde{\omega}_1+\tilde{\omega}_4)//\tilde{H}^0(\tilde{\omega}_1+\tilde{\omega}_2)//\tilde{H}^0(\tilde{\omega}_3)$.

Thus, since $\tilde{V}(\tilde{\omega}_2)\not\cong\tilde{L}(\tilde{\omega}_2)$, the lemma will follow from (1) once we show that the 2-dimensional homomorphism space (2) contains an embedding. We have $\tilde{V}(\tilde{\omega}_2)\cong\tilde{\mathfrak{g}}_\tau\subseteq\tilde{\mathfrak{g}}$, (I.5.4, Remark 1) and there is an embedding $\tilde{\mathfrak{g}}\hookrightarrow\mathrm{End}_k(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))$ by I.5.1(1), since the weights of $\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$ generate the character group of $\tilde{T}$. Finally, we have

$$\begin{aligned}\mathrm{End}_k(\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))\cong\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)^*\otimes\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)\cong\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)\otimes\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)\\ \precsim\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3)\otimes\tilde{H}^0(\tilde{\omega}_2+\tilde{\omega}_3).\end{aligned}$$

Lemma 3.3.2.

$$\mathrm{Hom}_D(L(\delta_2+\delta_4),L(\delta_1+\delta_4)\otimes L(\delta_1+\delta_2))\not\cong 0.$$

Proof: This follows from the fact that

$$\mathrm{Hom}_D(L,L(\delta_1+\delta_4)\otimes L(\delta_1+\delta_2))$$

can be shown to be zero for all other composition factors of $L(\delta_1+\delta_4)\otimes L(\delta_1+\delta_2)$ in the same linkage class as $L(\delta_2+\delta_4)$.

Lemma 3.3.3.

$$\mathrm{Hom}_D(L(\delta_1+\delta_2+\delta_4),L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3))\not\cong 0.$$

Proof: This follows from the fact that

$$\mathrm{Hom}_D(L,L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3))$$

can be shown to be zero for all other composition factors of $L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3)$ in the same linkage class as $L(\delta_1+\delta_2+\delta_4)$. This follows immediately from Table 2-1 of [Dw] (and the identity $\mathrm{Hom}_D(U,V\otimes W)\cong\mathrm{Hom}_D(W^*,V\otimes U^*)$), except for the composition factors $L(\delta_1+\delta_2+2\delta_3+\delta_4)$ (Remark: $L(\delta_2+3\delta_3)$ is in a different linkage class.) However, we obtain

$$\mathrm{Hom}_D(L(\delta_1+\delta_2+2\delta_3+\delta_4),L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3))\cong 0,$$

from the inclusion

$$V(\delta_1+\delta_2+2\delta_3+\delta_4)\subseteq V(\delta_1+\delta_3+\delta_4)\otimes V(\delta_2+\delta_3),$$

which has a filtration

$L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3)//\mathrm{rad}[V(\delta_1+\delta_3+\delta_4)]\otimes L(\delta_2+\delta_3)$.

We then observe from Table 2-1 of [Dw], and the composition factors of $\mathrm{rad}[V(\delta_1+\delta_3+\delta_4)]$ that $\mathrm{rad}[V(\delta_1+\delta_3+\delta_4)]\otimes L(\delta_2+\delta_3)$ contains no composition factor isomorphic to $L(\delta_1+2\delta_2+\delta_4)$, which is a composition factor of $\mathrm{rad}[V(\delta_1+\delta_2+2\delta_3+\delta_4)]$. Therefore, the quotient of $V(\delta_1+\delta_2+2\delta_3+\delta_4)$ inside $L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3)$ has nonzero radical and so the unique composition factor of $L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3)$ isomorphic to $L(\delta_1+\delta_2+2\delta_3+\delta_4)$ cannot appear in the socle of $L(\delta_1+\delta_3+\delta_4)\otimes L(\delta_2+\delta_3)$.

3.4 Extensions for $(D_4)_1$. We now compute, using the same method as in section 2.3, all of the quantities $\mathrm{Ext}^1_{D_1}(L, M)$, where L and M are restricted simple modules. Utilizing the result of I.6.6 together with Table 2-1 of [Dw] and exploiting the symmetry of D_4, we need to list only 9 such computations. We also use the fact that for $G = D_4$, $\mathrm{Ext}^1_{D_1}(L, L) = 0$ for any restricted simple module L. Occasionally, we need to know the multiplicities of certain simple modules appearing as composition factors of certain tensor products of restricted simple modules, especially, for example, when utilizing the computations outlined in section I.6.6 for narrowing down the sructure of the $\mathrm{Ext}^1_{D_1}(L, M)$'s. For this information, we refer the reader to Table 2-1 of [Dw]. (The weight multiplicities of the restricted simple modules can be completely determined from those of the Weyl modules [BMP] using the Jantzen Sum Formula ([Ja1], II.8.19). The tensor products are then computed by calculating the weight orbits under the Weyl group and then multiplying the appropriate formal characters.)

The following nonzero $\mathrm{Ext}^1_{D_1}$ modules for D_4 were calculated in [Sin2]:

a) $\mathrm{Ext}^1_{D_1}(k, L(\delta_2)) \cong 2k \oplus L(2\delta_1) \oplus L(2\delta_3) \oplus L(2\delta_4)$.

b) $\mathrm{Ext}^1_{D_1}(k, L(\delta_1 + \delta_3 + \delta_4)) \cong k \oplus L(2\delta_2)$.

c) $\mathrm{Ext}^1_{D_1}(L(\delta_2), L(\delta_1 + \delta_3 + \delta_4)) \cong k$.

Note that (a) and (b) are the only nonzero modules of the form $\mathrm{Ext}^1_{D_1}(k, L(\nu))$ for restricted ν, as $2X(T) \subseteq \mathbb{Z}R$.

d) $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)) \cong k$.

By section I.6.6 and Table 2-1 of [Dw], the only simple D-modules that could appear as composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4))$ are k, $L(2\delta_1)$, $L(2[\delta_3 + \delta_4])$, and $L(2\delta_2)$.

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)) \cong k$$

by the structure of the Weyl module $V(\delta_2 + \delta_3 + \delta_4)$. (See Table III.A.6)

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1 + 2\delta_1, L(\delta_2 + \delta_3 + \delta_4)) \cong 0$$

as $3\delta_1$ is incomparable with $\delta_2 + \delta_3 + \delta_4$ (in the usual partial order.)

iii) $\mathrm{Hom}_D(L(2[\delta_3 + \delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_2 + \delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)) \otimes L(2\delta_3))$$

(as $\mathrm{Hom}_D(L(\delta_2 + \delta_4), L(\delta_3) \otimes L(\delta_3 + \delta_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_D(L(\delta_1 + 2[\delta_2 + \delta_4]), L(\delta_2 + \delta_3 + \delta_4 + 2\delta_3)) \cong 0,$$

as $\delta_1 + 2[\delta_2 + \delta_4]$ is incomparable with $\delta_2 + \delta_3 + \delta_4 + 2\delta_3$.

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_1 + \delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4)) \otimes L(2[\delta_3 + \delta_4]))$$

(as $\mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_2)\otimes L(\delta_3+\delta_4))\cong k$,)

$$\cong \mathrm{Ext}^1_D(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_2+\delta_3+\delta_4+2[\delta_3+\delta_4]))\cong 0$$

as $\delta_1+2[\delta_1+\delta_2])$ is incomparable with $\delta_2+\delta_3+\delta_4+2[\delta_3+\delta_4]$.

Thus, we have $\mathrm{soc}(\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)))\cong k$. Now, the only simple D-modules that could appear in the second socle layer are isomorphs of $L(2\delta_2)$ (since $\mathrm{Ext}^1_D(k,k)$, $\mathrm{Ext}^1_D(L(\delta_1),k)$, and $\mathrm{Ext}^1_D(L(\delta_3+\delta_4),k)$ are all zero); however, the calculation above together with the sequence 1.1(1) shows that

$$\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4))/\mathrm{soc}(\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2+\delta_3+\delta_4)))\cong 0,$$

because $\mathrm{Ext}^1_D(L(\delta_1+\delta_2), L(\delta_3+\delta_4))\cong 0$. (Observe that $\delta_1+\delta_2$ and $\delta_3+\delta_4$ are in different linkage classes.)

e) $\mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4))\cong 0$.

The only simple D-modules that can appear as composition factors are $k, L(2\delta_1)$, and $L(2\delta_2)$.

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_D(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4))\cong 0,$$

by the structure of the Weyl module

$$V(\delta_2+\delta_3+\delta_4)\subseteq V(\delta_2+\delta_3)\otimes V(\delta_4)\cong L(\delta_2+\delta_3)\otimes L(\delta_4):$$

$$\mathrm{Hom}_D(L(\delta_1), L(\delta_2+\delta_3)\otimes L(\delta_4))\cong \mathrm{Hom}_D(L(\delta_2+\delta_3), L(\delta_1)\otimes L(\delta_4))\cong 0,$$

so that the unique composition factor of $V(\delta_2+\delta_3+\delta_4)$ isomorphic to $L(\delta_3+\delta_4)$ must equal $\mathrm{rad}^2(V(\delta_2+\delta_3+\delta_4))$, i.e.,

$$V(\delta_2+\delta_3+\delta_4)\cong L(\delta_2+\delta_3+\delta_4)//L(\delta_1)//L(\delta_3+\delta_4)$$

is uniserial. (See Table III.A.6)

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_D(L(\delta_3+\delta_4+2\delta_1), L(\delta_2+\delta_3+\delta_4))\cong 0,$$

by the structure of the Weyl module $V(\delta_3+\delta_4+2\delta_1)\subseteq V(\delta_3+2\delta_1)\otimes V(\delta_4)$, which has a filtration $(L(\delta_3)\otimes L(\delta_4))\otimes L(2\delta_1)//L(\delta_2+\delta_3)\otimes L(\delta_4)$. The multiplicity of $L(\delta_2+\delta_3+\delta_4)$ as a composition factor of this filtration is easily checked to be equal to one, while the multiplicity of $L(\delta_1)$ is 2. On the other hand, we have $V(\delta_2+\delta_3+\delta_4)\subseteq V(\delta_2+\delta_3)\otimes V(\delta_4)\cong L(\delta_2+\delta_3)\otimes L(\delta_4)$. As $L(\delta_2+\delta_3+\delta_4)$ is indeed a composition factor of $V(\delta_3+\delta_4+2\delta_1)$, and since its multiplicity in $V(\delta_3+2\delta_1)\otimes V(\delta_4)$ is one, we must have $V(\delta_2+\delta_3+\delta_4)\subseteq V(\delta_3+\delta_4+2\delta_1)$. Since the multiplicity of $L(\delta_1)$ in $V(\delta_3+\delta_4+2\delta_1)$ is exactly two, we obtain that the first Jantzen layer of $V(\delta_3+\delta_4+2\delta_1)$ (which consists of 2 composition factors of $L(\delta_1)$, and one factor of $L(\delta_2+\delta_3+\delta_4)$, by the Jantzen Sum Formula) must be uniserial, by self-duality of Jantzen layers.

iii) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_2+\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_D(L(\delta_3+\delta_4+2\delta_2), L(\delta_2+\delta_3+\delta_4)) \cong 0.$$

We consider the Weyl module $V(\delta_3+\delta_4+2\delta_2)$. The multiplicity of $L(\delta_2+\delta_3+\delta_4)$ as a composition factor is equal to one, as is that of $L(\delta_1+\delta_2+2\delta_3)$. We also have

$$V(\delta_3+\delta_4+2\delta_2) \subseteq V(\delta_3)\otimes V(\delta_4+2\delta_2) \cong L(\delta_3)\otimes V(\delta_4+2\delta_2).$$

A check of the filtration factors of $L(\delta_3)\otimes V(\delta_4+2\delta_2)$ that result from the composition factors of $V(\delta_4+2\delta_2)$ shows the multiplicities of $L(\delta_2+\delta_3+\delta_4)$ and $L(\delta_1+\delta_2+2\delta_3)$ to be 4 and 1 respectively, all occurring inside of a submodule which is isomorphic to $L(\delta_1+\delta_2+\delta_3)\otimes L(\delta_3)$. We also have

$$V(\delta_1+\delta_2+2\delta_3) \subseteq V(\delta_1+\delta_2+\delta_3)\otimes V(\delta_3) \cong V(\delta_1+\delta_2+\delta_3)\otimes L(\delta_3),$$

which has $L(\delta_1+\delta_2+\delta_3)\otimes L(\delta_3)$ as a homomorphic image. Let M be the image of $V(\delta_1+\delta_2+2\delta_3)$ under this homomorphism. In particular, $M \subseteq V(\delta_3)\otimes V(\delta_4+2\delta_2)$. We note that $M \not\cong 0$, since the kernel contains no composition factors isomorphic to $L(\delta_1+\delta_2+2\delta_3)$. Also, M has a composition factor isomorphic to $L(\delta_2+\delta_3+\delta_4)$ (since the kernel contains no such composition factor.) Because the multiplicity of $L(\delta_1+\delta_2+2\delta_3)$ as a composition factor of $V(\delta_3)\otimes V(\delta_4+2\delta_2)$ is one, we have that

$$M \subseteq V(\delta_3+\delta_4+2\delta_2).$$

This implies that the unique composition factor of $V(\delta_3+\delta_4+2\delta_2)$ isomorphic to $L(\delta_2+\delta_3+\delta_4)$ must occur inside $\mathrm{rad}^2(V(\delta_3+\delta_4+2\delta_2))$.

f) $\mathrm{Ext}^1_{D_1}(L(\delta_1+\delta_2), L(\delta_2+\delta_3+\delta_4)) \cong 0$.
The only candidates for summands of the socle are $L(2\delta_3)$ and $L(2\delta_4)$.

$$\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_1+\delta_2), L(\delta_2+\delta_3+\delta_4)))$$

$$\precsim \mathrm{Hom}_D(L(2[\delta_2+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1+\delta_2), L(\delta_2+\delta_3+\delta_4))\otimes L(2[\delta_3+\delta_4])),$$

(as $\mathrm{Hom}_D(L(\delta_2+\delta_4), L(\delta_3)\otimes L(\delta_3+\delta_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_D(L(\delta_1+\delta_2+2[\delta_2+\delta_4]), L(\delta_2+\delta_3+\delta_4+2[\delta_3+\delta_4])) \cong 0,$$

as $\delta_1+\delta_2+2[\delta_2+\delta_4]$ is incomparable with $\delta_2+\delta_3+\delta_4+2[\delta_3+\delta_4]$ in the usual partial order. ($L(2\delta_4)$ is handled similarly.)

g) $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)) \cong L(2\delta_3)\oplus L(2\delta_4)$.
The only possible isomorphism types of summands of the socle are $k, L(2\delta_1), L(2\delta_3)$, $L(2\delta_4), L(2\delta_2), L(2[\delta_3+\delta_4]), L(2[\delta_1+\delta_4]), L(2[\delta_1+\delta_3]), L(2[\delta_2+\delta_3])$, and $L(2[\delta_2+\delta_4])$.

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(\delta_1+\delta_2) \cong L(\delta_1+\delta_2)$.

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_1), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(3\delta_1)$.

iii) $\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_3), L(\delta_1+\delta_2)) \cong k,$$

by considering $V(\delta_1+2\delta_3)$. (Similarly for $L(2\delta_4)$.)

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_2), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(\delta_1+2\delta_2)$.

v) $\mathrm{Hom}_D(L(2[\delta_3+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2[\delta_3+\delta_4]), L(\delta_1+\delta_2)) \cong 0,$$

by considering $V(\delta_1+2[\delta_3+\delta_4])$.

vi) $\mathrm{Hom}_D(L(2[\delta_1+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)) \otimes L(2\delta_4)),$$

(as $\mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_4) \otimes L(\delta_1+\delta_4)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_1+\delta_2+2\delta_4)) \cong 0,$$

by considering $V(\delta_1+2[\delta_1+\delta_2])$. (Similarly for $L(2[\delta_1+\delta_3])$.)

vii) $\mathrm{Hom}_D(L(2[\delta_2+\delta_3]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2)) \otimes L(2\delta_4)),$$

(as $\mathrm{Hom}_D(L(\delta_1+\delta_2), L(\delta_4) \otimes L(\delta_2+\delta_3)) \cong k$,)

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_1+\delta_2+2\delta_4)) \cong 0,$$

by considering $V(\delta_1+2[\delta_1+\delta_2])$. (Similarly for $L(2[\delta_2+\delta_4])$.)

We have thus shown that

$$\mathrm{soc}_D(\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2))) \cong L(2\delta_3) \oplus L(2\delta_4).$$

Now, the only types of simple D-modules which can appear as composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1+\delta_2))$, that extend $L(2\delta_3)$ or $L(2\delta_4)$, are $L(2[\delta_1+\delta_4])$ and $L(2[\delta_1+\delta_3])$. However, the calculation in part (vi) together with the sequence 1.1(1) shows that $L(2[\delta_1+\delta_4])$ (similarly for $L(2[\delta_1+\delta_3])$) cannot be a summand of the second socle layer because $L(\delta_1+\delta_2)$ does not extend any of the composition factors of $L(\delta_4)\otimes[L(\delta_3)\oplus L(\delta_4)]$.

h) $\mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2))\cong 0$.

The only possible isomorphism types that can appear in the socle are $k, L(2\delta_1)$, $L(2\delta_3), L(2\delta_4), L(2\delta_2), L(2[\delta_1+\delta_4])$, and $L(2[\delta_1+\delta_3])$.

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4), L(\delta_1+\delta_2))\cong 0,$$

by considering $V(\delta_1+\delta_2)\cong L(\delta_1+\delta_2)$.

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4+2\delta_1), L(\delta_1+\delta_2))\cong 0,$$

by considering $V(\delta_3+\delta_4+2\delta_1)$.

iii) $\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4+2\delta_3), L(\delta_1+\delta_2))\cong 0,$$

by considering $V(\delta_3+\delta_4+2\delta_3)$. We have

$$V(3\delta_3+\delta_4)\subseteq V(2\delta_3+\delta_4)\otimes V(\delta_3)\cong V(2\delta_3+\delta_4)\otimes L(\delta_3),$$

which has a filtration $(L(\delta_3)\otimes L(\delta_4))\otimes L(2\delta_3)//L(\delta_3)\otimes L(\delta_2+\delta_4)$. Now,

$$\mathrm{Hom}_D(L(\delta_1+2\delta_4), L(\delta_3)\otimes L(\delta_2+\delta_4))\cong \mathrm{Hom}_D(L(\delta_2+\delta_4+2\delta_4), L(\delta_1)\otimes L(\delta_3))\cong 0,$$

while $L(\delta_3)\otimes L(\delta_4))\otimes L(2\delta_3)$ has no composition factor isomorphic to $L(\delta_1+2\delta_4)$. Thus $\mathrm{Hom}_D(L(\delta_1+2\delta_4), V(\delta_3+\delta_4+2\delta_3))\cong 0$, which implies that $\mathrm{rad}^2(V(\delta_3+\delta_4+2\delta_3))\cong L(\delta_1+\delta_2)$.

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4+2\delta_2), L(\delta_1+\delta_2))\cong 0,$$

by considering $V(\delta_3+\delta_4+2\delta_2)$.

v) $\mathrm{Hom}_D(L(2[\delta_1+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_2+\delta_4]), \mathrm{Ext}^1_{D_1}(L(\delta_3+\delta_4), L(\delta_1+\delta_2))\otimes L(2[\delta_1+\delta_2]),$$

(since $\mathrm{Hom}_D(L(\delta_2+\delta_4), L(\delta_1+\delta_4)\otimes L(\delta_1+\delta_2))\not\cong 0$, by Lemma 3.3.2)

$$\cong \mathrm{Ext}^1_G(L(\delta_3+\delta_4+2[\delta_2+\delta_4]), L(\delta_1+\delta_2+2[\delta_1+\delta_2]))\cong 0,$$

as $\delta_3+\delta_4+2[\delta_2+\delta_4]$ is incomparable (in the usual partial order) with $\delta_1+\delta_2+2[\delta_1+\delta_2]$.

i) $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \cong k \oplus L(2\delta_1)$.
We have

$$\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)) \cong \mathrm{Ext}^1_{D_1}(k, L(\delta_1)\otimes L(\delta_3+\delta_4)) \cong H^1(G_1, L(\delta_1)\otimes L(\delta_3+\delta_4)).$$

By considering the composition factors of $L(\delta_1)\otimes L(\delta_3+\delta_4)$, and the long exact sequence in cohomology, we observe that the only simple D-module types that could appear as composition factors of $H^1(G_1, L(\delta_1)\otimes L(\delta_3+\delta_4))$ are those of $H^1(G_1, L(\delta_2)) \cong 2k \oplus L(2\delta_1)\oplus L(2\delta_3)\oplus L(2\delta_4)$, and $H^1(G_1, L(\delta_1+\delta_3+\delta_4)) \cong k \oplus 2\delta_2$, by 3.4(a), (b).

i) $\mathrm{Hom}_D(k, \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1), L(\delta_3+\delta_4)) \cong k,$$

by considering $V(\delta_3+\delta_4)$.

ii) $\mathrm{Hom}_D(L(2\delta_1), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(3\delta_1), L(\delta_3+\delta_4)) \cong k,$$

by considering $V(3\delta_1)$.

iii) $\mathrm{Hom}_D(L(2\delta_3), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2\delta_3), L(\delta_3+\delta_4)) \cong 0,$$

by considering $V(\delta_1+2\delta_3)$. The calculation for $L(2\delta_4)$ is similar.

iv) $\mathrm{Hom}_D(L(2\delta_2), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4)))$

$$\precsim \mathrm{Hom}_D(L(2[\delta_1+\delta_2]), \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))\otimes L(2\delta_1)),$$

(as $L(\delta_2)\otimes L(\delta_1) \cong L(\delta_3+\delta_4)\oplus L(\delta_1+\delta_2)$.)

$$\cong \mathrm{Ext}^1_G(L(\delta_1+2[\delta_1+\delta_2]), L(\delta_3+\delta_4+2\delta_1)) \cong 0,$$

by considering $V(\delta_1+2[\delta_1+\delta_2])$.

We have thus shown

$$\mathrm{soc}_D(\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))) \cong k \oplus L(2\delta_1).$$

Therefore, the only candidates for summands of the second socle layer are possible composition factors of $\mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3+\delta_4))$ that extend either k or $L(2\delta_1)$. The only possibilities are isomorphs of $L(2\delta_2)$. However, the calculation in (iv) together with the sequence 1.1(1) shows that $L(2\delta_2)$ does not appear in the second socle layer, because $L(2[\delta_1+\delta_2])$ does not extend any of the composition factors of $[k\oplus L(2\delta_1)]\otimes L(2\delta_1)$.

3.5 Ext Computation Tables for D_4. We are now ready to compute the Ext^1_Ds for the simply connected algebraic group of type D_4. We use the same notation as in II.1.3 and III.2.4 letting λ and μ be dominant weights, with $\lambda = \sum 2^i\lambda^i = \lambda^0 + 2\hat{\lambda} = \lambda^0 + 2\lambda^1 + 4\overline{\lambda}$, and similarly for μ. As in II.1.3, if $\lambda^0 = \mu^0$, we obtain the isomorphism

$$\mathrm{Ext}^1_D(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_D(L(\hat{\lambda}), L(\hat{\mu})),$$

because of the vanishing third term of the 5-term sequence for (D, D_1). This reduces us again to the case $\lambda^0 \neq \mu^0$. Then the vanishing of the first and fourth terms of the 5-term sequence yields

$$\begin{aligned}(1)\quad \mathrm{Ext}^1_D(L(\lambda), L(\mu)) &\cong \mathrm{Hom}_D(L(\hat{\lambda}), \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\hat{\mu})) \\ &\cong \mathrm{Hom}_D(L(\overline{\lambda}), \mathrm{Hom}_{D_1}(L(\lambda^1), \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))^{(2^{-1})} \otimes L(\mu^1))^{(2^{-1})} \otimes L(\overline{\mu})).\end{aligned}$$

By the results of the preceding section, all the modules $E = \mathrm{Ext}^1_{D_1}(L(\lambda^0), L(\mu^0))$ are direct sums of simple modules of the form k and $L(\delta_i), i = 1, 2, 3, 4$. We wish to apply Corollary I.4.2.1 with $\nu = 0$, δ_i. The minimum value of $\langle 2\omega, \rho\rangle$ for $\omega \in X_1(T) \setminus \{0\}$ is 6 whereas $\langle \delta_1, \rho\rangle = \langle \delta_3, \rho\rangle = \langle \delta_4, \rho\rangle = 3$, and $\langle \delta_2, \rho\rangle = 5$, so the hypotheses of the corollary are satisfied. It follows that D acts trivially on modules of the form $\mathrm{Hom}_{D_1}(L(\lambda^1), E \otimes L(\mu^1))^{(2^{-1})}$. Therefore, we obtain the same result as in II.1.5(1) and III.2.4(2), namely,

$$(2)\qquad \mathrm{Ext}^1_D(L(\lambda), L(\mu)) \cong \begin{cases} \mathrm{Hom}_D(L(\lambda^1), E \otimes L(\mu^1))^{(2^{-1})} & \text{if } \overline{\lambda} = \overline{\mu} \\ 0 \quad \text{otherwise} \end{cases}$$

This observation, together with the information gathered in the preceding section allow us now to read off all of the extensions for simply connected D_4 from appropriate tables of tensor product socles:

Table III.3.5(a): $E = L(\delta_3) \oplus L(\delta_4) \cong \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_1 + \delta_2))^{(2^{-1})}$ (3.4(g))

Table III.3.5(b): $E = k \oplus L(\delta_1) \cong \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_3 + \delta_4))^{(2^{-1})}$ (3.4(i))

Table III.3.5(c): $E = 2k \oplus L(\delta_1) \oplus L(\delta_3) \oplus L(\delta_4) \cong \mathrm{Ext}^1_{D_1}(k, L(\delta_2))^{(2^{-1})}$ (3.4(a))

Table III.3.5(d): $E = k \oplus L(\delta_2) \cong \mathrm{Ext}^1_{D_1}(k, L(\delta_1 + \delta_3 + \delta_4))^{(2^{-1})}$ (3.4(b))

Of course, no table is needed for $E = k \cong \mathrm{Ext}^1_{D_1}(L(\delta_2), L(\delta_1 + \delta_3 + \delta_4))^{(2^{-1})}$ (3.4(c)) $\cong \mathrm{Ext}^1_{D_1}(L(\delta_1), L(\delta_2 + \delta_3 + \delta_4))^{(2^{-1})}$ (3.4(d))

All other $\mathrm{Ext}^1_{D_1}$ modules for D_4 are either zero or images under the graph automorphisms of those listed above.

Table III.3.5(a) $E = L(\delta_3) \oplus L(\delta_4)$

product	socle
$E \otimes L(\delta_1)$	$L(\delta_4) \oplus L(\delta_3)$
$E \otimes L(\delta_3)$	$k \oplus L(\delta_1)$
$E \otimes L(\delta_2)$	$L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_3) \oplus L(\delta_1+\delta_3) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_3+\delta_4)$	$L(\delta_2+\delta_4) \oplus L(\delta_2+\delta_3)$
$E \otimes L(\delta_1+\delta_4)$	$L(\delta_2) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_1+\delta_2)$	$L(\delta_1+\delta_3) \oplus L(\delta_2+\delta_4) \oplus L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_3)$
$E \otimes L(\delta_2+\delta_3)$	$L(\delta_2) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$L(\delta_1+\delta_2+\delta_3) \oplus L(\delta_1+\delta_2+\delta_4)$
$E \otimes L(\delta_1+\delta_2+\delta_4)$	$L(\delta_1+\delta_3+\delta_4) \oplus L(\rho) \oplus L(\delta_2+\delta_3+\delta_4)$
$E \otimes L(\rho)$	$L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$

Table III.3.5(b) $E = k \oplus L(\delta_1)$

product	socle
$E \otimes L(\delta_1)$	$L(\delta_1) \oplus k$
$E \otimes L(\delta_3)$	$L(\delta_3) \oplus L(\delta_4)$
$E \otimes L(\delta_2)$	$L(\delta_2) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_3+\delta_4)$	$L(\delta_3+\delta_4) \oplus L(\delta_2)$
$E \otimes L(\delta_1+\delta_4)$	$L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_1+\delta_2)$	$L(\delta_1+\delta_2) \oplus L(\delta_2)$
$E \otimes L(\delta_2+\delta_3)$	$L(\delta_2+\delta_3) \oplus L(\delta_1+\delta_3) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$L(\delta_1+\delta_3+\delta_4) \oplus L(\delta_2+\delta_3+\delta_4)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$L(\delta_2+\delta_3+\delta_4) \oplus L(\delta_1+\delta_3+\delta_4) \oplus L(\rho)$
$E \otimes L(\delta_1+\delta_2+\delta_4)$	$L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$
$E \otimes L(\rho)$	$L(\rho) \oplus L(\delta_2+\delta_3+\delta_4)$

TABLE III.3.5(c) $E = 2k \oplus L(\delta_1) \oplus L(\delta_3) \oplus L(\delta_4)$

product	socle
$E \otimes L(\delta_1)$	$2L(\delta_1) \oplus k \oplus L(\delta_4) \oplus L(\delta_3)$
$E \otimes L(\delta_2)$	$2L(\delta_2) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2) \oplus L(\delta_1+\delta_4) \oplus$ $L(\delta_2+\delta_3) \oplus L(\delta_1+\delta_3) \oplus L(\delta_2+\delta_4)$
$E \otimes L(\delta_3+\delta_4)$	$2L(\delta_3+\delta_4) \oplus L(\delta_2) \oplus$ $L(\delta_2+\delta_4) \oplus L(\delta_2+\delta_3)$
$E \otimes L(\delta_1+\delta_2)$	$2L(\delta_1+\delta_2) \oplus L(\delta_2) \oplus L(\delta_1+\delta_3) \oplus$ $L(\delta_2+\delta_4) \oplus L(\delta_1+\delta_4) \oplus L(\delta_2+\delta_3)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$2L(\delta_1+\delta_3+\delta_4) \oplus L(\delta_2+\delta_3+\delta_4) \oplus$ $L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$2L(\delta_2+\delta_3+\delta_4) \oplus L(\delta_1+\delta_3+\delta_4) \oplus$ $L(\rho) \oplus L(\delta_1+\delta_2+\delta_3) \oplus L(\delta_1+\delta_2+\delta_4)$
$E \otimes L(\rho)$	$2L(\rho) \oplus L(\delta_2+\delta_3+\delta_4) \oplus$ $L(\delta_1+\delta_2+\delta_4) \oplus L(\delta_1+\delta_2+\delta_3)$

TABLE III.3.5(d) $E = k \oplus L(\delta_2)$

product	socle
$E \otimes L(\delta_1)$	$L(\delta_1) \oplus L(\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_2)$	$2L(\delta_2) \oplus k$
$E \otimes L(\delta_3+\delta_4)$	$L(\delta_3+\delta_4) \oplus L(\delta_1) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\delta_1+\delta_2)$	$2L(\delta_1+\delta_2) \oplus L(\delta_1) \oplus L(\delta_3+\delta_4) \oplus L(\delta_2+\delta_3+\delta_4)$
$E \otimes L(\delta_1+\delta_3+\delta_4)$	$2L(\delta_1+\delta_3+\delta_4) \oplus L(\rho)$
$E \otimes L(\delta_2+\delta_3+\delta_4)$	$2L(\delta_2+\delta_3+\delta_4) \oplus L(\delta_1+\delta_2)$
$E \otimes L(\rho)$	$3L(\rho) \oplus L(\delta_1+\delta_3+\delta_4)$

§4. Types B_4 and C_4.

We now use the information obtained about module extensions for D_1, together with the isomorphism I.6.1(4), to calculate the ext groups for $\tilde{G}_\tau$ in rank $l = 4$.

4.1 Extensions for $(C_4)_\tau$.

LEMMA 4.1.1. *The* $\mathrm{Ext}^1_{\tilde{G}_\tau}$*s for* $\tilde{G} = C_4$ *are as follows:*
a) $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3))^{(\tau^{-1})} \cong k$.
b) $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2))^{(\tau^{-1})} \cong L(\omega_4)$.
c) $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3))^{(\tau^{-1})} \cong H^0(\omega_1)$.

d) $\mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})} \cong k$.
e) $\mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})} \cong k \oplus L(\omega_2)$.
f) $\mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_2))^{(\tau^{-1})} \cong M \oplus L(\omega_4)$,
where M is the unique (up to isomorphism) uniserial G- module with filtration

$$k//L(\omega_1)//k.$$

PROOF: The right hand sides of (a), (b), and (d) are the unique G- modules that restrict to the appropriate $\mathrm{Ext}^1_{D_1}$s. Result (c) follows from consideration of the Weyl module $\tilde{V}(\tilde{\omega}_3)$ (i.e., that $\mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3)) \cong 0$,) together with the five-term sequence. The argument for result (f) is similar, by considering the Weyl modules $\tilde{V}(\tilde{\omega}_2)$ and $\tilde{V}(2\tilde{\omega}_1)$. Finally, to prove (e), we observe that if $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})}$ were uniserial, then it would be the σ-twist of a $\tilde{G}$-module (namely, either $\tilde{H}^0(\tilde{\omega}_2)^{(\sigma)}$ or $\tilde{V}(\tilde{\omega}_2)^{(\sigma)}$). This is because $\mathrm{Ext}^1_G(k, L(\omega_2)) \cong k$ and thus there is a unique (up to isomorphism) uniserial G-module with filtration $k//L(\omega_2)$, resp., $L(\omega_2)//k$.

We now use the fact that

$$\mathrm{Hom}_{D_1}(k, M^{(\sigma)}) \cong (\mathrm{Hom}_{\tilde{G}_\tau}(k, M))^{(\sigma)} \; [\cong \mathrm{Hom}_{G_1}(k, M^{(\sigma)}] \tag{1}$$

as D-modules (for arbitrary $\tilde{G}$-modules M) because σ maps D_1 (as well as G_1) onto $\tilde{G}_\tau$. However, we know that the restriction of E to D is split.

4.2 Socles of Tensor Products for B_4 and C_4.

We need several more lemmas before we can obtain the Ext^1_Gs (and $\mathrm{Ext}^1_{\tilde{G}}$s) for B_4 and C_4.

LEMMA 4.2.1. *Let M be any composition factor of $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\mu^0), \tilde{L}(\nu^0))^{(\tau^{-1})}$, where μ^0, ν^0 are any two τ-restricted weights. Then if λ, β is any pair of 2-restricted weights for G, then*

$$\mathrm{Hom}_{G_1}(L(\lambda), M \otimes L(\beta))$$

is G-trivial.

PROOF: By Lemma 4.1.1, $M = k, L(\omega_1), L(\omega_2)$, or $L(\omega_4)$, all of whose highest weights satisfy the hypothesis of Corollary I.4.2.

LEMMA 4.2.2. *Let $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\mu^0), \tilde{L}(\nu^0))^{(\tau^{-1})}$, where μ^0, ν^0 are any two τ-restricted weights. Then if λ, β is any pair of 2-restricted weights for G, then*

$$\mathrm{Hom}_{G_1}(L(\lambda), E \otimes L(\beta))$$

is G-trivial.

PROOF: The G-composition factors of $\mathrm{Hom}_{G_1}(L(\lambda), E \otimes L(\beta))$ form a subset of the composition factors of $\mathrm{Hom}_{G_1}(L(\lambda), M \otimes L(\beta))$ as M ranges over the composition factors of E. Now use the preceding lemma and the fact that $\mathrm{Ext}^1_G(k, k) = 0$.

This lemma, when combined with the results of section I.6.5, shows that we can reduce the computation of $\operatorname{Ext}^1_{\tilde{G}}(L(\tilde{\lambda}), L(\tilde{\mu}))$ to the case where $\tilde{\lambda} = \tilde{\lambda}^0 + \tau\tilde{\lambda}^1 + 2\tilde{\lambda}^2 + 2\tau\bar{\bar{\bar{\tilde{\lambda}}}}$ and $\tilde{\mu} = \tilde{\mu}^0 + \tau\tilde{\mu}^1 + 2\tilde{\mu}^2 + 2\tau\bar{\bar{\bar{\tilde{\mu}}}}$, where $\bar{\bar{\bar{\tilde{\lambda}}}} = \bar{\bar{\bar{\tilde{\mu}}}}$, and $\operatorname{Ext}^1_G(L(\lambda), L(\mu))$ to the case where $\lambda = \lambda^0 + \sigma\lambda^1 + 2\lambda^2 + 2\sigma\lambda^3 + 4\bar{\bar{\lambda}}$ and $\mu = \mu^0 + \sigma\mu^1 + 2\mu^2 + 2\sigma\mu^3 + 4\bar{\bar{\mu}}$, where $\bar{\bar{\lambda}} = \bar{\bar{\mu}}$, and $\lambda^0 = \mu^0$.

4.3 Ext Computation Tables for B_4 and C_4. The following tables allow us to use Lemma 4.2.2 and §I.6.5 to compute all of the extensions for simply connected B_4 and C_4. The result of §I.6.5 reduces the computation of all of these extensions to the case $\operatorname{Ext}^1_G(L(\lambda), L(\mu))$, with $\lambda = \lambda^0 + \sigma\bar{\lambda} = \lambda^0 + \sigma\lambda^1 + 2\bar{\bar{\lambda}} = \lambda^0 + \sigma\lambda^1 + 2\lambda^2 + \sigma\lambda^3 + 4\bar{\bar{\bar{\lambda}}}$, etc., where $\lambda^0 + \sigma\lambda^1 \neq \mu^0 + \sigma\mu^1$. Thus, because of Lemma 4.2.2, we obtain (in the notation of §I.6.5):

(a) If $\{\lambda^0, \mu^0\} = \{0, \omega_l\}$, we have $\operatorname{Ext}^j_G(L(\lambda), L(\mu)) = 0$ for all j.

(b) If $\lambda^0 = \mu^0 = 0$ and $\lambda^1 - \mu^1 \in \mathbb{Z}\tilde{R}$ (note that l is even), or $\lambda^0 = \mu^0 = \omega_l$, we have

$$\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \cong \begin{cases} \operatorname{Hom}_G(L(\lambda^2 + \sigma\lambda^3), E \otimes L(\mu^2 + \sigma\mu^3))^{(2^{-1})} & \text{if } \bar{\bar{\bar{\lambda}}} = \bar{\bar{\bar{\mu}}} \\ 0 \quad \text{otherwise} \end{cases}$$

where $E = \operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})}$.

(c) If $\lambda^0 = \mu^0 = 0$ and $\lambda^1 - \mu^1 \notin \mathbb{Z}\tilde{R}$, we have

$$\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \cong \begin{cases} \operatorname{Hom}_{\tilde{G}}(\tilde{L}(\lambda^1), E \otimes \tilde{L}(\mu^1))^{(\tau^{-1})} & \text{if } \bar{\bar{\lambda}} = \bar{\bar{\mu}} \\ 0 \quad \text{otherwise} \end{cases}$$

where $E = \operatorname{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})} \cong \tilde{L}(\tilde{\omega}_1)$ (by Lemma I.6.4).

Thus we only need to list the G -socles of tensor products of $E = \operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\lambda^1), \tilde{L}(\mu^1))^{(\tau^{-1})}$ with 2-restricted G-modules and the $\tilde{G}$ -socles of tensor products of $\operatorname{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})}$ with τ-restricted $\tilde{G}$-modules: (Lemma 4.1.1)

Table III.4.3(a): $E = \operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), L(\tilde{\omega}_1 + \tilde{\omega}_2))^{(\tau^{-1})}$

Table III.4.3(b): $E = \operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3))^{(\tau^{-1})}$

Table III.4.3(c): $E = \operatorname{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_2))^{(\tau^{-1})}$

Table III.4.3(d): $E = \operatorname{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})}$

Table III.4.3(e): $E = \operatorname{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})} \cong \tilde{L}(\tilde{\omega}_1)$ (Lemma I.6.4)

Of course, no table is needed for $E = k \cong \operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_2), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})}$
$\cong \operatorname{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_2 + \tilde{\omega}_3))^{(\tau^{-1})}$.

All other $\operatorname{Ext}^1_{\tilde{G}_\tau}$ modules for C_4 are zero. The tables in this section are computed using the results of sections 2.2 and 2.3 (see Table III.A.4 in the Appendix), the fact (4.1(1)) that

$$\mathrm{Hom}_{D_1}(k, M^{(\sigma)}) \cong \mathrm{Hom}_{G_1}(k, M^{(\sigma)})$$

as D-modules (for arbitrary $\tilde{G}$-modules M) together with the following lemma. (We will also use the fact that $L(\omega_4)$ is injective for G_σ.)

LEMMA 4.3.1. *Let M be one of the non-semisimple $\mathrm{Ext}^1_{\tilde{G}_\tau}$ modules listed in Lemma 4.1.1. Let μ, ν be 2-restricted weights for $G = B_4$ which are σ-twisted (i.e., in the span of the fundamental dominant weights corresponding to the long simple roots). Then*

$$\mathrm{Hom}_G(L(\nu), M \otimes L(\mu)) \cong \mathrm{Hom}_G(L(\nu), soc_G(M) \otimes L(\mu))$$

PROOF: We have

$$\mathrm{Hom}_G(L(\nu), M \otimes L(\mu)) \cong \mathrm{Hom}_G(L(\nu) \otimes L(\mu)^*, M).$$

However, $L(\nu) \otimes L(\mu)^*$ is a σ-twist of some $\tilde{G}$-module, whereas M has no non-semisimple submodules which are σ-twists.

TABLE III.4.3(a) $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_2))^{(\tau^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	$L(\omega_1 + \omega_4)$
$E \otimes L(\omega_2)$	$L(\omega_2 + \omega_4)$
$E \otimes L(\omega_3)$	$L(\omega_3 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2)$	$L(\omega_1 + \omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_3)$	$L(\omega_1 + \omega_3 + \omega_4)$
$E \otimes L(\omega_2 + \omega_3)$	$L(\omega_2 + \omega_3 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2 + \omega_3)$	$L(\rho)$
$E \otimes L(\omega_4)$	k
$E \otimes L(\omega_1 + \omega_4)$	$L(\omega_1)$
$E \otimes L(\omega_2 + \omega_4)$	$L(\omega_2)$
$E \otimes L(\omega_3 + \omega_4)$	$L(\omega_3)$
$E \otimes L(\omega_1 + \omega_2 + \omega_4)$	$L(\omega_1 + \omega_2)$
$E \otimes L(\omega_1 + \omega_3 + \omega_4)$	$L(\omega_1 + \omega_3)$
$E \otimes L(\omega_2 + \omega_3 + \omega_4)$	$L(\omega_2 + \omega_3)$
$E \otimes L(\rho)$	$L(\omega_1 + \omega_2 + \omega_3)$

Table III.4.3(b) $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(\tilde{L}(\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_3))^{(\tau^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	k
$E \otimes L(\omega_2)$	$L(\omega_1 + \omega_2) \oplus L(\omega_3)$
$E \otimes L(\omega_3)$	$L(\omega_2)$
$E \otimes L(\omega_1 + \omega_2)$	$L(\omega_2)$
$E \otimes L(\omega_1 + \omega_3)$	$L(\omega_2 + \omega_3)$
$E \otimes L(\omega_2 + \omega_3)$	$L(\omega_1 + \omega_3) \oplus L(\omega_1 + \omega_2 + \omega_3)$
$E \otimes L(\omega_1 + \omega_2 + \omega_3)$	$L(\omega_2 + \omega_3)$
$E \otimes L(\omega_4)$	$L(\omega_4) \oplus L(\omega_1 + \omega_4)$
$E \otimes L(\omega_1 + \omega_4)$	$L(\omega_1 + \omega_4) \oplus L(\omega_4)$
$E \otimes L(\omega_2 + \omega_4)$	$L(\omega_2 + \omega_4) \oplus L(\omega_1 + \omega_2 + \omega_4) \oplus L(\omega_3 + \omega_4)$
$E \otimes L(\omega_3 + \omega_4)$	$L(\omega_3 + \omega_4) \oplus L(\omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2 + \omega_4)$	$L(\omega_1 + \omega_2 + \omega_4) \oplus L(\omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_3 + \omega_4)$	$L(\omega_1 + \omega_3 + \omega_4) \oplus L(\omega_2 + \omega_3 + \omega_4)$
$E \otimes L(\omega_2 + \omega_3 + \omega_4)$	$L(\omega_2 + \omega_3 + \omega_4) \oplus L(\omega_1 + \omega_3 + \omega_4) \oplus L(\rho)$
$E \otimes L(\rho)$	$L(\rho) \oplus L(\omega_2 + \omega_3 + \omega_4)$

Table III.4.3(c) $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_2))^{(\tau^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	$L(\omega_1) \oplus L(\omega_1 + \omega_4)$
$E \otimes L(\omega_2)$	$L(\omega_2) \oplus L(\omega_2 + \omega_4)$
$E \otimes L(\omega_3)$	$L(\omega_3) \oplus L(\omega_3 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2)$	$L(\omega_1 + \omega_2) \oplus L(\omega_1 + \omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_3)$	$L(\omega_1 + \omega_3) \oplus L(\omega_1 + \omega_3 + \omega_4)$
$E \otimes L(\omega_2 + \omega_3)$	$L(\omega_2 + \omega_3) \oplus L(\omega_2 + \omega_3 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2 + \omega_3)$	$L(\omega_1 + \omega_2 + \omega_3) \oplus L(\rho)$
$E \otimes L(\omega_4)$	$k \oplus 2L(\omega_4) \oplus L(\omega_1 + \omega_4)$
$E \otimes L(\omega_1 + \omega_4)$	$L(\omega_1) \oplus 2L(\omega_1 + \omega_4) \oplus L(\omega_4)$
$E \otimes L(\omega_2 + \omega_4)$	$L(\omega_2) \oplus 2L(\omega_2 + \omega_4) \oplus$ $L(\omega_1 + \omega_2 + \omega_4) \oplus L(\omega_3 + \omega_4)$
$E \otimes L(\omega_3 + \omega_4)$	$L(\omega_3) \oplus 2L(\omega_3 + \omega_4) \oplus L(\omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_2 + \omega_4)$	$L(\omega_1 + \omega_2) \oplus 2L(\omega_1 + \omega_2 + \omega_4) \oplus L(\omega_2 + \omega_4)$
$E \otimes L(\omega_1 + \omega_3 + \omega_4)$	$L(\omega_1 + \omega_3) \oplus 2L(\omega_1 + \omega_3 + \omega_4) \oplus L(\omega_2 + \omega_3 + \omega_4)$
$E \otimes L(\omega_2 + \omega_3 + \omega_4)$	$L(\omega_2 + \omega_3) \oplus 2L(\omega_2 + \omega_3 + \omega_4) \oplus$ $L(\omega_1 + \omega_3 + \omega_4) \oplus L(\rho)$
$E \otimes L(\rho)$	$L(\omega_1 + \omega_2 + \omega_3) \oplus 2L(\rho) \oplus L(\omega_2 + \omega_3 + \omega_4)$

Table III.4.3(d) $E = \mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3))^{(\tau^{-1})}$

product	G-socle
$E \otimes L(\omega_1)$	$L(\omega_1) \oplus L(\omega_3) \oplus L(\omega_1 + \omega_2)$
$E \otimes L(\omega_2)$	$2L(\omega_2) \oplus k$
$E \otimes L(\omega_3)$	$L(\omega_3) \oplus L(\omega_1) \oplus L(\omega_1 + \omega_2)$
$E \otimes L(\omega_1 + \omega_2)$	$2L(\omega_1 + \omega_2) \oplus L(\omega_1) \oplus L(\omega_3) \oplus L(\omega_2 + \omega_3)$

TABLE III.4.3(d) (CONTINUED)

product	G-socle
$E \otimes L(\omega_1+\omega_3)$	$2L(\omega_1+\omega_3) \oplus L(\omega_1+\omega_2+\omega_3)$
$E \otimes L(\omega_2+\omega_3)$	$2L(\omega_2+\omega_3) \oplus L(\omega_1+\omega_2)$
$E \otimes L(\omega_1+\omega_2+\omega_3)$	$3L(\omega_1+\omega_2+\omega_3) \oplus L(\omega_1+\omega_3)$
$E \otimes L(\omega_4)$	$L(\omega_4) \oplus L(\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_4)$	$L(\omega_1+\omega_4) \oplus L(\omega_3+\omega_4) \oplus L(\omega_1+\omega_2+\omega_4)$
$E \otimes L(\omega_2+\omega_4)$	$2L(\omega_2+\omega_4) \oplus L(\omega_4)$
$E \otimes L(\omega_3+\omega_4)$	$L(\omega_3+\omega_4) \oplus L(\omega_1+\omega_4) \oplus L(\omega_1+\omega_2+\omega_4)$
$E \otimes L(\omega_1+\omega_2+\omega_4)$	$2L(\omega_1+\omega_2+\omega_4) \oplus L(\omega_1+\omega_4)\oplus$ $L(\omega_3+\omega_4) \oplus L(\omega_2+\omega_3+\omega_4)$
$E \otimes L(\omega_1+\omega_3+\omega_4)$	$2L(\omega_1+\omega_3+\omega_4) \oplus L(\rho)$
$E \otimes L(\omega_2+\omega_3+\omega_4)$	$2L(\omega_2+\omega_3+\omega_4) \oplus L(\omega_1+\omega_2+\omega_4)$
$E \otimes L(\rho)$	$3L(\rho) \oplus L(\omega_1+\omega_3+\omega_4)$

TABLE III.4.3(e) $E = \mathrm{Ext}^1_{G_\sigma}(k,k)^{(\sigma^{-1})} \cong \tilde{L}(\tilde{\omega}_1)$

product	$\tilde{G}$-socle
$E \otimes \tilde{L}(\tilde{\omega}_1)$	k
$E \otimes \tilde{L}(\tilde{\omega}_2)$	$\tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2) \oplus \tilde{L}(\tilde{\omega}_3)$
$E \otimes \tilde{L}(\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_2)$
$E \otimes \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2)$	$\tilde{L}(\tilde{\omega}_2)$
$E \otimes \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$
$E \otimes \tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3) \oplus \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_3)$
$E \otimes \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_2+\tilde{\omega}_3)$	$\tilde{L}(\tilde{\omega}_2+\tilde{\omega}_3)$

4.4 Examples. In this subsection, we compute two specific examples of module extensions for simply connected C_4, to illustrate the application of §I.6.5 and Tables III.4.3(a)-(e).

Example 4.4.1 $E = \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(2\tilde{\omega}_1+2\tilde{\omega}_2+2\tilde{\omega}_3), \tilde{L}(3\tilde{\omega}_1+2\tilde{\omega}_2+3\tilde{\omega}_3))$.

By I.6.3, we obtain first $E \cong \mathrm{Ext}^1_G(L(\omega_4+2\omega_1+2\omega_2+2\omega_3), L(\omega_4+3\omega_1+2\omega_2+3\omega_3))$. This puts us in case (b) of I.6.5. Therefore, we have

$$E \cong \mathrm{Hom}_G(L(\omega_1+\omega_2+\omega_3), \mathrm{Ext}^1_{\tilde{G}_\tau}(k, \tilde{L}(\tilde{\omega}_1+\tilde{\omega}_3))^{(\tau^{-1})} \otimes L(\omega_1+\omega_2+\omega_3)) \cong 3k,$$

by Table III.4.3(d).

Example 4.4.2 $E = \mathrm{Ext}^1_{\tilde{G}}(\tilde{L}(2\tilde{\omega}_1 + 2\tilde{\omega}_3), \tilde{L}(2\tilde{\omega}_2 + 2\tilde{\omega}_3))$.

By I.6.3, we obtain first $E \cong \mathrm{Ext}^1_G(L(\omega_4 + 2\omega_1 + 2\omega_3), L(\omega_4 + 2\omega_2 + 2\omega_3)) \cong \mathrm{Ext}^1_G(L(\omega_1 + \omega_3), L(\omega_2 + \omega_3))$. This puts us in case (c) of I.6.5. Therefore, we have

$$E \cong \mathrm{Hom}_{\tilde{G}}(\tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3), \tilde{L}(\tilde{\omega}_1) \otimes \tilde{L}(\tilde{\omega}_2 + \tilde{\omega}_3)) \cong k,$$

by Table III.4.3(e).

4.5 Cohomology of Frobenius Kernels for B_4 and C_4.

Finally, we use the 5-term sequences (as in §I.7) for the pairs (G_1, G_σ) and $(\tilde{G}_1, \tilde{G}_\tau)$ to compute the quantities $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))^{(2^{-1})}$ and $\mathrm{Ext}^1_{\tilde{G}_1}(\tilde{L}(\lambda), \tilde{L}(\mu))^{(2^{-1})}$ for pairs of 2-restricted weights λ, μ. Since these quantities are symmetric in λ and μ (as all simple modules for G and $\tilde{G}$ are self-dual), we display only the lower left half of each table, (each in 3 seperate pieces due to space contraints.) Now note that for G, we have that the result is zero if $\lambda^0 \neq \mu^0$ (in the notation of I.6.5), eliminating the need to display the 2nd piece. Furthermore, we can use the notational reduction introduced in Proposition II.2.4. Let $\lambda = \lambda^0 + \sigma\lambda^1$ and $\mu = \mu^0 + \sigma\mu^1$. In the next table, the notation (x) will mean x if $\lambda^0 = \mu^0 = 0$ and 0 if $\lambda^0 = \mu^0 = \omega_4$. In the following, M is the unique (up to isomorphism) uniserial G- module with composition series $k//L(\omega_1)//k$.

Table III.4.5(a) $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))^{(2^{-1})}$

(λ^1, μ^1)	0	ω_1	ω_2	ω_3	$\omega_1+\omega_2$	$\omega_1+\omega_3$	$\omega_2+\omega_3$	$\omega_1+\omega_2+\omega_3$
0	0							
ω_1	(k)	0						
ω_2	$M \oplus L(\omega_4)$	0	0					
ω_3	0	$H^0(\omega_1)$	(k)	0				
$\omega_1+\omega_2$	0	$L(\omega_4)$	(k)	0	0			
$\omega_1+\omega_3$	$k \oplus L(\omega_2)$	0	k	0	0	0		
$\omega_2+\omega_3$	0	k	0	0	0	(k)	0	
$\omega_1+\omega_2+\omega_3$	0	0	0	0	0	0	(k)	0

TABLE III.4.5(b) $\mathrm{Ext}^1_{\tilde{G}_1}(\tilde{L}(\lambda), \tilde{L}(\mu))^{(2^{-1})}$

(λ, μ)	0	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_1 + \tilde{\omega}_2$	$\tilde{\omega}_1 + \tilde{\omega}_3$	$\tilde{\omega}_2 + \tilde{\omega}_3$	$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3$
0	$L(\tilde{\omega}_1)$							
$\tilde{\omega}_1$	0	$L(\tilde{\omega}_1)$						
$\tilde{\omega}_2$	k	0	$L(\tilde{\omega}_1)$					
$\tilde{\omega}_3$	0	$L(\tilde{\omega}_1)$	0	$L(\tilde{\omega}_1)$				
$\tilde{\omega}_1 + \tilde{\omega}_2$	0	0	0	0	$L(\tilde{\omega}_1)$			
$\tilde{\omega}_1 + \tilde{\omega}_3$	$k \oplus L(\tilde{\omega}_2)$	0	k	0	0	$L(\tilde{\omega}_1)$		
$\tilde{\omega}_2 + \tilde{\omega}_3$	0	k	0	0	0	0	$L(\tilde{\omega}_1)$	
$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3$	0	0	0	0	0	0	0	$L(\tilde{\omega}_1)$

TABLE III.4.5(b) (CONT.) $\mathrm{Ext}^1_{\tilde{G}_1}(\tilde{L}(\lambda), \tilde{L}(\mu))^{(2^{-1})}$

(λ, μ)	0	$\tilde{\omega}_1$	$\tilde{\omega}_2$	$\tilde{\omega}_3$	$\tilde{\omega}_1 + \tilde{\omega}_2$	$\tilde{\omega}_1 + \tilde{\omega}_3$	$\tilde{\omega}_2 + \tilde{\omega}_3$	$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3$
$\tilde{\omega}_4$	0	0	k	0	0	0	0	0
$\tilde{\omega}_1 + \tilde{\omega}_4$	0	0	0	0	k	0	0	0
$\tilde{\omega}_2 + \tilde{\omega}_4$	k	0	0	0	0	0	0	0
$\tilde{\omega}_3 + \tilde{\omega}_4$	0	0	0	0	0	0	0	0
$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_4$	0	k	0	0	0	0	0	0
$\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4$	0	0	0	0	0	0	0	0
$\tilde{\omega}_2 + \tilde{\omega}_3 + \tilde{\omega}_4$	0	0	0	0	0	0	0	0
ρ	0	0	0	0	0	0	0	0

TABLE III.4.5(b) (CONT.) $\mathrm{Ext}^1_{\tilde{G}_1}(\tilde{L}(\lambda), \tilde{L}(\mu))^{(2^{-1})}$

(λ, μ)	$\tilde{\omega}_4$	$\tilde{\omega}_1 + \tilde{\omega}_4$	$\tilde{\omega}_2 + \tilde{\omega}_4$	$\tilde{\omega}_3 + \tilde{\omega}_4$	$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_4$	$\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4$	$\tilde{\omega}_2 + \tilde{\omega}_3 + \tilde{\omega}_4$	ρ
$\tilde{\omega}_4$	0							
$\tilde{\omega}_1 + \tilde{\omega}_4$	0	0						
$\tilde{\omega}_2 + \tilde{\omega}_4$	$2k \oplus L(\tilde{\omega}_1)$	0	0					
$\tilde{\omega}_3 + \tilde{\omega}_4$	0	$k \oplus L(\tilde{\omega}_1)$	0	0				
$\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_4$	0	0	0	0	0			
$\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4$	$k \oplus L(\tilde{\omega}_2)$	0	k	0	0	0		
$\tilde{\omega}_2 + \tilde{\omega}_3 + \tilde{\omega}_4$	0	k	0	0	0	0	0	
ρ	0	0	0	0	0	0	0	0

III.A Appendix

Table III.A.1(a) Weight multiplicities (A_4)

	dim	0	1001	0110	0102 2010	0021 1200	2002	ρ
W-orbit size		1	20	30	30	20	20	120
$L(0)$	1	1						
$L(1001)$	24	4	1					
$L(0110)$	74	4	2	1				
$L(\rho)$	1024	24	14	8	4	2	2	1

Table III.A.1(b) Weight multiplicities (A_4)

	dim	1000	0101	0020	2001	1110
W-orbit size		5	30	10	20	60
$L(1000)$	5	1				
$L(0101)$	40	2	1			
$L(1110)$	280	8	4	2	2	1

Table III.A.1(c) Weight multiplicities (A_4)

	dim	0100	2000	0011	1101
W-orbit size		10	5	20	60
$L(0100)$	10	1			
$L(0011)$	40	2	·	1	
$L(1101)$	160	5	2	2	1

Table III.A.2(a) Weight multiplicities (D_4)

	dim	0	0100	2000 0020 0002	1011	0200	2100 0120 0102	0022 2002 2020	ρ
W-orbit size		1	24	8	96	24	48	32	192
$L(0)$	1	1							
$L(0100)$	26	2	1						
$L(1011)$	246	6	4	2	1				
$L(\rho)$	4096	64	40	24	14	8	4	2	1

Table III.A.2(b) Weight multiplicities (D_4)

	dim	1000	0011	1100	1020 1002	0111
W-orbit size		8	32	48	32	96
$L(1000)$	8	1				
$L(0011)$	48	2	1			
$L(1100)$	160	6	2	1		
$L(0111)$	784	14	8	4	2	1

Table III.A.3 (Type A_4 tensor products)

(ν, ν')	$H^0(\nu) \otimes H^0(\nu')$	$\mathrm{soc}_{G_1}(L(\nu) \otimes L(\nu'))$
(λ_1, λ_1)	$H^0(2\lambda_1)//H^0(\lambda_2)$	$L(\lambda_2)$
(λ_1, λ_4)	$H^0(\lambda_1 + \lambda_4) \oplus k$	$L(\lambda_1 + \lambda_4) \oplus k$
(λ_1, λ_2)	$H^0(\lambda_1 + \lambda_2) \oplus H^0(\lambda_3)$	$L(\lambda_1 + \lambda_2) \oplus L(\lambda_3)$
(λ_1, λ_3)	$H^0(\lambda_1 + \lambda_3)//H^0(\lambda_4)$	$L(\lambda_4)$
$(\lambda_1, \lambda_1 + \lambda_4)$	$H^0(2\lambda_1 + \lambda_4)//H^0(\lambda_2 + \lambda_4)//H^0(\lambda_1)$	$L(\lambda_1)$
$(\lambda_1, \lambda_2 + \lambda_4)$	$H^0(\lambda_1 + \lambda_2 + \lambda_4)//H^0(\lambda_3 + \lambda_4)//H^0(\lambda_2)$	$L(\lambda_1 + \lambda_2 + \lambda_4)$ $\oplus L(\lambda_3 + \lambda_4)$
$(\lambda_1, \lambda_1 + \lambda_3)$	$[H^0(2\lambda_1 + \lambda_3)//H^0(\lambda_2 + \lambda_3)]$ $\oplus H^0(\lambda_1 + \lambda_4)$	$L(\lambda_2 + \lambda_3)$
$(\lambda_1, \lambda_3 + \lambda_4)$	$H^0(\lambda_1 + \lambda_3 + \lambda_4)//H^0(2\lambda_4)//H^0(\lambda_3)$	$L(\lambda_3)$
$(\lambda_1, \lambda_1 + \lambda_2)$	$H^0(2\lambda_1 + \lambda_2)//H^0(2\lambda_2)//H^0(\lambda_1 + \lambda_3)$	$L(\lambda_1 + \lambda_3)$
$(\lambda_1, \lambda_2 + \lambda_3)$	$H^0(\lambda_1 + \lambda_2 + \lambda_3)//H^0(2\lambda_3)//H^0(\lambda_2 + \lambda_4)$	$L(\lambda_1 + \lambda_2 + \lambda_3)$ $\oplus L(\lambda_2 + \lambda_4)$
$(\lambda_1, \lambda_1 + \lambda_2 + \lambda_4)$	$H^0(2\lambda_1 + \lambda_2 + \lambda_4)//H^0(2\lambda_2 + \lambda_4)//$ $H^0(\lambda_1 + \lambda_3 + \lambda_4)//H^0(\lambda_1 + \lambda_2)$	$L(\lambda_1 + \lambda_3 + \lambda_4)$
$(\lambda_1, \lambda_1 + \lambda_3 + \lambda_4)$	$[H^0(2\lambda_1 + \lambda_3 + \lambda_4)//H^0(\lambda_2 + \lambda_3 + \lambda_4)]$ $\oplus[H^0(\lambda_1 + 2\lambda_4)//H^0(\lambda_1 + \lambda_3)]$	$L(\lambda_2 + \lambda_3 + \lambda_4)$ $\oplus L(\lambda_1 + \lambda_3)$
$(\lambda_1, \lambda_1 + \lambda_2 + \lambda_3)$	$H^0(2\lambda_1 + \lambda_2 + \lambda_3)//H^0(2\lambda_2 + \lambda_3)//$ $H^0(\lambda_1 + 2\lambda_3)//H^0(\lambda_1 + \lambda_2 + \lambda_4)$	$L(\lambda_1 + \lambda_2 + \lambda_4)$
$(\lambda_1, \lambda_2 + \lambda_3 + \lambda_4)$	$H^0(\rho) \oplus [H^0(2\lambda_3 + \lambda_4)//$ $H^0(\lambda_2 + 2\lambda_4)//H^0(\lambda_2 + \lambda_3)]$	$L(\rho) \oplus L(\lambda_2 + \lambda_3)$
(λ_1, ρ)	$Q(\lambda_1 + \lambda_2 + \lambda_3)$	$L(\lambda_1 + \lambda_2 + \lambda_3)$
(λ_2, λ_2)	$H^0(2\lambda_2)//H^0(\lambda_1 + \lambda_3)//H^0(\lambda_4)$	$L(\lambda_1 + \lambda_3) \oplus L(\lambda_4)$
(λ_2, λ_3)	$H^0(\lambda_2 + \lambda_3)//H^0(\lambda_1 + \lambda_4)//k$	$L(\lambda_1 + \lambda_4) \oplus k$
$(\lambda_2, \lambda_1 + \lambda_4)$	$H^0(\lambda_1 + \lambda_2 + \lambda_4)//H^0(\lambda_3 + \lambda_4)//$ $H^0(2\lambda_1)//H^0(\lambda_2)$	$L(\lambda_3 + \lambda_4) \oplus L(\lambda_2)$
$(\lambda_2, \lambda_2 + \lambda_4)$	$H^0(2\lambda_2 + \lambda_4)//H^0(\lambda_1 + \lambda_3 + \lambda_4)//$ $H^0(\lambda_1 + \lambda_2)//H^0(2\lambda_4)//H^0(\lambda_3)$	$L(\lambda_1 + \lambda_3 + \lambda_4) \oplus L(\lambda_3)$
$(\lambda_2, \lambda_1 + \lambda_3)$	$H^0(\lambda_1 + \lambda_2 + \lambda_3)//H^0(2\lambda_1 + \lambda_4)//$ $H^0(2\lambda_3)//H^0(\lambda_2 + \lambda_4)//H^0(\lambda_1)$	$L(\lambda_1 + \lambda_2 + \lambda_3)$ $\oplus L(\lambda_2 + \lambda_4)$

Table III.A.3 (Type A_4 tensor products)(cont.)

(ν,ν')	$H^0(\nu)\otimes H^0(\nu')$	$\mathrm{soc}_{G_1}(L(\nu)\otimes L(\nu'))$
$(\lambda_2,\lambda_3+\lambda_4)$	$H^0(\lambda_2+\lambda_3+\lambda_4)//H^0(\lambda_1+2\lambda_4)//$ $H^0(\lambda_1+\lambda_3)//H^0(\lambda_4)$	$L(\lambda_2+\lambda_3+\lambda_4)\oplus L(\lambda_4)$
$(\lambda_2,\lambda_1+\lambda_2)$	$[H^0(\lambda_1+2\lambda_2)//H^0(2\lambda_1+\lambda_3)//$ $H^0(\lambda_2+\lambda_3)]\oplus H^0(\lambda_1+\lambda_4)$	$L(\lambda_2+\lambda_3)$ $\oplus L(\lambda_1+\lambda_4)$
$(\lambda_2,\lambda_2+\lambda_3)$	$[H^0(2\lambda_2+\lambda_3)//H^0(\lambda_1+2\lambda_3)//$ $H^0(\lambda_1+\lambda_2+\lambda_4)//H^0(\lambda_2)]\oplus H^0(\lambda_3+\lambda_4)$	$L(\lambda_1+\lambda_2+\lambda_4)$ $\oplus L(\lambda_3+\lambda_4)$
$(\lambda_2,\lambda_1+\lambda_2+\lambda_4)$	$H^0(\lambda_1+2\lambda_2+\lambda_4)//H^0(2\lambda_1+\lambda_3+\lambda_4)//$ $H^0(\lambda_2+\lambda_3+\lambda_4)//H^0(2\lambda_1+\lambda_2)//$ $H^0(\lambda_1+2\lambda_4)//H^0(2\lambda_2)//H^0(\lambda_1+\lambda_3)$	$L(\lambda_2+\lambda_3+\lambda_4)$ $\oplus L(\lambda_1+\lambda_3)$
$(\lambda_2,\lambda_1+\lambda_3+\lambda_4)$	$H^0(\rho)\oplus[H^0(2\lambda_1+2\lambda_4)//H^0(2\lambda_3+\lambda_4)//$ $H^0(2\lambda_1+\lambda_3)//H^0(\lambda_2+2\lambda_4)//$ $H^0(\lambda_2+\lambda_3)]\oplus H^0(\lambda_1+\lambda_4)$	$L(\rho)\oplus L(\lambda_2+\lambda_3)$
$(\lambda_2,\lambda_1+\lambda_2+\lambda_3)$	$[H^0(\lambda_1+2\lambda_2+\lambda_3)//H^0(2\lambda_1+2\lambda_3)//$ $H^0(2\lambda_1+\lambda_2+\lambda_4)//H^0(\lambda_2+2\lambda_3)//$ $H^0(2\lambda_2+\lambda_4)//H^0(\lambda_1+\lambda_3+\lambda_4)]$ $\oplus H^0(\lambda_1+\lambda_2)$	$L(\lambda_1+\lambda_3+\lambda_4)$ $\oplus L(\lambda_1+\lambda_2)$
$(\lambda_2,\lambda_2+\lambda_3+\lambda_4)$	$H^0(2\lambda_2+\lambda_3+\lambda_4)$ $H^0(\lambda_1+2\lambda_3+\lambda_4)//H^0(\lambda_1+\lambda_2+2\lambda_4)//$ $H^0(\lambda_1+\lambda_2+\lambda_3)//H^0(\lambda_3+2\lambda_4)//$ $H^0(2\lambda_3)//H^0(\lambda_2+\lambda_4)$	$L(\lambda_1+\lambda_2+\lambda_3)$ $\oplus L(\lambda_2+\lambda_4)$
(λ_2,ρ)	$Q(\lambda_1+\lambda_2+\lambda_4)$	$L(\lambda_1+\lambda_2+\lambda_4)$

Table III.A.4 (Type D_4 tensor products)

(ν,ν')	$H^0(\nu)\otimes H^0(\nu')$	$\mathrm{soc}_{G_1}(L(\nu)\otimes L(\nu'))$
(δ_1,δ_1)	$H^0(2\delta_1)//H^0(\delta_2)//k$	k
(δ_1,δ_3)	$H^0(\delta_1+\delta_3)//H^0(\delta_4)$	$L(\delta_4)$
(δ_1,δ_2)	$H^0(\delta_1+\delta_2)\oplus[H^0(\delta_3+\delta_4)//H^0(\delta_1)]$	$L(\delta_1+\delta_2)\oplus L(\delta_3+\delta_4)$
$(\delta_1,\delta_3+\delta_4)$	$H^0(\delta_1+\delta_3+\delta_4)//H^0(2\delta_4)//$ $H^0(2\delta_3)//H^0(\delta_2)$	$L(\delta_2)$
$(\delta_1,\delta_1+\delta_4)$	$H^0(2\delta_1+\delta_4)//H^0(\delta_2+\delta_4)//$ $H^0(\delta_1+\delta_3)//H^0(\delta_4)$	$L(\delta_2+\delta_4)$

Table III.A.4 (Type D_4 tensor products)(cont.)

(ν,ν')	$H^0(\nu)\otimes H^0(\nu')$	$\mathrm{soc}_{G_1}(L(\nu)\otimes L(\nu'))$
$(\delta_1,\delta_1+\delta_2)$	$H^0(2\delta_1+\delta_2)//H^0(2\delta_2)//$ $H^0(\delta_1+\delta_3+\delta_4)//H^0(2\delta_1)//H^0(\delta_2)$	$L(\delta_2)$
$(\delta_1,\delta_2+\delta_3)$	$[H^0(2\delta_3+\delta_4)//H^0(\delta_2+\delta_4)]$ $\oplus[H^0(\delta_1+\delta_2+\delta_3)//H^0(\delta_1+\delta_3)]$	$L(\delta_2+\delta_4)$ $\oplus L(\delta_1+\delta_3)$
$(\delta_1,\delta_1+\delta_3+\delta_4)$	$[H^0(\delta_1+2\delta_4)//H^0(\delta_1+2\delta_3)//$ $H^0(\delta_1+\delta_2)]\oplus[H^0(2\delta_1+\delta_3+\delta_4)//$ $H^0(\delta_2+\delta_3+\delta_4)//H^0(\delta_3+\delta_4)]$	$L(\delta_2+\delta_3+\delta_4)$
$(\delta_1,\delta_2+\delta_3+\delta_4)$	$H^0(\rho)\oplus[H^0(2\delta_3+2\delta_4)//$ $H^0(\delta_2+2\delta_3)//H^0(\delta_2+2\delta_4)//$ $H^0(2\delta_2)//H^0(\delta_1+\delta_3+\delta_4)]$	$L(\rho)\oplus L(\delta_1+\delta_3+\delta_4)$
$(\delta_1,\delta_1+\delta_2+\delta_4)$	$[H^0(2\delta_1+\delta_2+\delta_4)//H^0(2\delta_2+\delta_4)//$ $H^0(\delta_1+\delta_3+2\delta_4)//H^0(\delta_1+\delta_2+\delta_3)]$ $\oplus[H^0(2\delta_1+\delta_4)//H^0(\delta_2+\delta_4)]$	$L(\delta_1+\delta_2+\delta_3)$
(δ_1,ρ)	$Q(\delta_2+\delta_3+\delta_4)$	$L(\delta_2+\delta_3+\delta_4)$
(δ_2,δ_2)	$H^0(2\delta_2)//H^0(\delta_1+\delta_3+\delta_4)//$ $H^0(2\delta_1)//H^0(2\delta_3)//H^0(2\delta_4)//$ $H^0(\delta_2)//k$	$L(\delta_2)\oplus k$
$(\delta_2,\delta_3+\delta_4)$	$[H^0(\delta_2+\delta_3+\delta_4)//2H^0(\delta_3+\delta_4)//$ $H^0(\delta_1)]\oplus[H^0(\delta_1+2\delta_3)//$ $H^0(\delta_1+2\delta_4)//H^0(\delta_1+\delta_2)]$	$L(\delta_1)\oplus L(\delta_1+\delta_2)$
$(\delta_2,\delta_1+\delta_2)$	$[H^0(\delta_1+2\delta_2)//H^0(2\delta_1+\delta_3+\delta_4)//$ $H^0(\delta_2+\delta_3+\delta_4)//H^0(3\delta_1)//$ $H^0(\delta_3+\delta_4)//H^0(\delta_1)]\oplus[H^0(\delta_1+2\delta_3)//$ $H^0(\delta_1+2\delta_4)//H^0(\delta_1+\delta_2])$	$L(\delta_2+\delta_3+\delta_4)$ $\oplus L(\delta_3+\delta_4)\oplus L(\delta_1)$ $\oplus L(\delta_1+\delta_2)$
$(\delta_2,\delta_1+\delta_3+\delta_4)$	$H^0(\rho)\oplus[H^0(2\delta_3+2\delta_4)//H^0(2\delta_1+2\delta_4)//$ $H^0(2\delta_1+2\delta_3)//H^0(\delta_2+2\delta_1)//$ $H^0(\delta_2+2\delta_3)//H^0(\delta_2+2\delta_4)//$ $H^0(2\delta_2)//3H^0(\delta_1+\delta_3+\delta_4)//H^0(2\delta_1)//$ $H^0(2\delta_3)//H^0(2\delta_4)//H^0(\delta_2)]$	$L(\rho)\oplus L(\delta_1+\delta_3+\delta_4)$
$(\delta_2,\delta_2+\delta_3+\delta_4)$	$[H^0(2\delta_2+\delta_3+\delta_4)//H^0(\delta_1+2\delta_3+2\delta_4)//$ $H^0(\delta_1+\delta_2+2\delta_3)//H^0(\delta_1+\delta_2+2\delta_4)//$ $H^0(\delta_1+2\delta_2)//H^0(2\delta_1+\delta_3+\delta_4)//$ $3H^0(\delta_2+\delta_3+\delta_4)//H^0(\delta_3+\delta_4)]$ $\oplus[H^0(\delta_3+3\delta_4)//H^0(3\delta_3+\delta_4)//$ $H^0(\delta_1+2\delta_3)//H^0(\delta_1+2\delta_4)//$ $H^0(\delta_1+\delta_2)]$	$L(\delta_2+\delta_3+\delta_4)$ $\oplus L(\delta_1+\delta_2)$
(δ_2,ρ)	$Q(\delta_1+\delta_3+\delta_4)\oplus 2Q(\rho)$	$L(\delta_1+\delta_3+\delta_4)\oplus 2L(\rho)$

Table III.A.5 Weyl module composition factors (A_4)

Weyl module	radical (composition factors)
$V(\lambda_1)$	$\emptyset$
$V(\lambda_2)$	$\emptyset$
$V(\lambda_1+\lambda_2)$	$\emptyset$
$V(\lambda_1+\lambda_4)$	$\emptyset$
$V(\lambda_2+\lambda_4)$	$L(\lambda_1)$
$V(\lambda_2+\lambda_3)$	k
$V(\lambda_1+\lambda_2+\lambda_4)$	$L(\lambda_2), L(2\lambda_1)$
$V(\lambda_1+\lambda_2+\lambda_3)$	$\emptyset$
$V(2\lambda_1)$	$L(\lambda_2)$
$V(2\lambda_3)$	$L(\lambda_2+\lambda_4)$
$V(3\lambda_4)$	$L(\lambda_2)$
$V(2\lambda_1+\lambda_4)$	$L(\lambda_1), L(\lambda_2+\lambda_4)$
$V(\lambda_2+2\lambda_4)$	$2k, L(\lambda_2+\lambda_3)$
$V(3\lambda_1+\lambda_4)$	$L(\lambda_3+\lambda_4)$
$V(\lambda_1+2\lambda_3)$	$L(\lambda_1+\lambda_2+\lambda_4)$
$V(2\lambda_2+\lambda_3)$	$L(2\lambda_1), L(\lambda_1+\lambda_2+\lambda_4), L(\lambda_1+2\lambda_3)$
$V(\lambda_1+\lambda_2+2\lambda_4)$	$L(\lambda_1+\lambda_2+\lambda_3)$
$V(2\lambda_2+3\lambda_4)$	$L(\lambda_2+\lambda_4), L(2\lambda_3), 2L(2\lambda_1+\lambda_4), L(3\lambda_2), L(\lambda_1+\lambda_3+3\lambda_4)$
$V(3\lambda_1+2\lambda_2+\lambda_4)$	$2L(2\lambda_2+\lambda_3+\lambda_4), L(2\lambda_1+\lambda_2+\lambda_3+\lambda_4), L(4\lambda_1+\lambda_3+\lambda_4)$
$V(3\lambda_2+2\lambda_3)$	$2L(\lambda_2), 2L(2\lambda_1), 2L(3\lambda_4), 2L(2\lambda_2+2\lambda_4),$ $L(\lambda_2+2\lambda_3+\lambda_4), 2L(2\lambda_1+2\lambda_3+\lambda_4), L(4\lambda_3),$ $L(\lambda_1+2\lambda_2+\lambda_3+\lambda_4), L(4\lambda_2+\lambda_4)$

Table III.A.6 Weyl module composition factors (D_4)

Weyl module	radical (composition factors)
$V(\delta_1)$	$\emptyset$
$V(\delta_2)$	$2k$
$V(\delta_1+\delta_2)$	$\emptyset$
$V(\delta_3+\delta_4)$	$L(\delta_1)$
$V(\delta_1+\delta_3+\delta_4)$	$2k, 3L(\delta_2), L(2\delta_1), L(2\delta_3), L(2\delta_4)$
$V(\delta_2+\delta_3+\delta_4)$	$L(\delta_1), L(\delta_3+\delta_4)$
$V(3\delta_1)$	$L(\delta_3+\delta_4)$
$V(\delta_1+2\delta_3)$	$L(\delta_1+\delta_2)$
$V(2\delta_1+\delta_3+\delta_4)$	$2L(\delta_1), L(\delta_3+\delta_4), L(3\delta_1), L(\delta_2+\delta_3+\delta_4)$
$V(3\delta_1+\delta_4)$	$L(\delta_1+\delta_2), L(\delta_1+2\delta_3), L(\delta_1+2\delta_4)$
$V(\delta_1+2\delta_2)$	$3L(\delta_1), L(\delta_2+\delta_3+\delta_4), L(2\delta_1+\delta_3+\delta_4)$
$V(\delta_1+\delta_2+2\delta_3)$	$4L(\delta_1), L(\delta_3+\delta_4), L(3\delta_1),$ $L(\delta_2+\delta_3+\delta_4), L(2\delta_1+\delta_3+\delta_4), L(\delta_1+2\delta_2)$
$V(5\delta_1)$	$2L(\delta_1), L(2\delta_1+\delta_3+\delta_4), L(\delta_1+2\delta_2)$
$V(3\delta_1+2\delta_4)$	$L(\delta_1+\delta_2), 2L(\delta_1+2\delta_3), L(\delta_1+2\delta_4),$ $L(\delta_3+3\delta_4), L(3\delta_1+\delta_2)$
$V(\delta_1+2\delta_3+2\delta_4)$	$5L(\delta_1), L(\delta_3+\delta_4), 2L(3\delta_1),$ $L(\delta_2+\delta_3+\delta_4), L(2\delta_1+\delta_3+\delta_4), L(\delta_1+2\delta_2),$ $L(\delta_1+\delta_2+2\delta_3), L(\delta_1+\delta_2+2\delta_4)$
$V(2\delta_2+\delta_3+\delta_4)$	$6L(\delta_1), 3L(\delta_3+\delta_4), 3L(3\delta_1), L(\delta_2+\delta_3+\delta_4),$ $L(2\delta_1+\delta_3+\delta_4), 2L(\delta_1+2\delta_2), L(\delta_1+\delta_2+2\delta_3),$ $L(\delta_1+\delta_2+2\delta_4), L(\delta_1+2\delta_3+2\delta_4)$
$V(2\delta_1+\delta_2+\delta_3+\delta_4)$	$7L(\delta_1), 4L(\delta_3+\delta_4), 4L(3\delta_1), L(\delta_2+\delta_3+\delta_4),$ $2L(2\delta_1+\delta_3+\delta_4), 2L(\delta_1+2\delta_2), L(\delta_1+\delta_2+2\delta_3),$ $L(\delta_1+\delta_2+2\delta_4), L(\delta_1+2\delta_3+2\delta_4), L(2\delta_2+\delta_3+\delta_4)$
$V(3\delta_1+2\delta_2)$	$4L(\delta_3+\delta_4), 2L(3\delta_1), 2L(2\delta_2+\delta_3+\delta_4),$ $L(2\delta_1+\delta_2+\delta_3+\delta_4), L(4\delta_1+\delta_3+\delta_4)$

Table III.A.7 Weyl module composition factors (C_4)

Weyl module	radical (composition factors)
$\tilde{V}(\tilde{\omega}_1)$	$\emptyset$
$\tilde{V}(\tilde{\omega}_2)$	k
$\tilde{V}(\tilde{\omega}_3)$	$\emptyset$
$\tilde{V}(2\tilde{\omega}_1)$	$2k, \tilde{L}(\tilde{\omega}_2)$

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