CUBIST ALGEBRAS

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Abstract. We construct algebras from rhombohedral tilings of Euclidean space obtained as projections of certain cubical complexes. We show that these 'Cubist algebras' satisfy strong homological properties, such as Koszulity and quasi-heredity, reflecting the combinatorics of the tilings. We construct derived equivalences between Cubist algebras associated to local mutations in tilings. We recover as a special case the Rhombal algebras of Michael Peach and make a precise connection to weight 2 blocks of symmetric groups.

Contents

1. Introduction

1.1. **Homological algebras.** How many algebras are there which possess every strong homological property known to mathkind, yet are not semisimple ?

We define algebras $U_{\mathcal{X}}$, which generalise the Rhombal algebras of M. Peach, as well as the Brauer tree algebra associated to an infinite line. We prove that these algebras are Koszul, symmetric, supersymmetric algebras, whose projective modules are of identical odd Loewy length $2r-1$, for some natural number r. There are $r!$ different highest weight structures on $U_{\mathcal{X}}$-mod. Standard modules are Koszul, for any highest weight structure. Given a highest weight structure, there is a canonical choice of homogeneous cellular basis for $U_{\mathcal{X}}$.

The Koszul duals $V_{\mathcal{X}}$ of the $U_{\mathcal{X}}$ generalise the preprojective algebra associated to an infinite line. They have global dimension $2r-2$. Thinking of $V_{\mathcal{X}}$-mod as the category of quasi-coherent sheaves on a chimeric noncommutative affine algebraic variety of dimension $2r-2$, we may define a category of sheaves on the corresponding projective variety, which has dimension $2r-3$, and obeys Serre duality, with trivial canonical bundle. There are $r!$ different highest weight structures on $V_{\mathcal{X}}$-mod, dual to those on $U_{\mathcal{X}}$-mod. Again, standard modules are all Koszul. Given a highest weight structure, there is a canonical choice of homogeneous cellular basis for $V_{\mathcal{X}}$.

The only finite-dimensional algebras which enjoy these potent combinations of properties are semisimple algebras. Our examples are therefore necessarily infinite dimensional.

The combinatorics of our algebras is governed by collections of cubes in r-dimensional space, viewed from $(r-1)$-dimensional space. In homage to P. Picasso and G. Braque, who explored manifold possibilities of this geometric attitude when $r=3$, we call them *Cubist algebras.*

We expect there to be further examples of algebras $\mathcal{A}_\tau$ which satisfy many of the listed properties of the $U_{\mathcal{X}}$'s, and upon a suitable localisation, describe the Morita type of blocks of symmetric groups / Schur algebras. The Koszul duals of the algebras $\mathcal{A}_\tau$, upon localisation, will describe the Koszul duals of blocks of Schur algebras.

There are further similarities between the Cubist algebras $U_{\mathcal{X}}$, and blocks of symmetric groups. Large collections of them are derived equivalent, the tilting bimodule complexes being obtained by composing two-term complexes. Symmetric group blocks of weight two are largely similar to certain $U_{\mathcal{X}}$'s, in case $r=3$.

Differences between the algebras $U_{\mathcal{X}}$, and $\mathcal{A}_\tau$ are soon visible. On $U_{\mathcal{X}}$, there are $r!$ possible highest weight structures. However, $\mathcal{A}_\tau$ has only two alternative orderings on its simple objects, corresponding to the dominance ordering of partitions and its opposite. The combinatorics surrounding $\mathcal{A}_\tau$ is complicated, and mysterious, whilst numerical properties of $U_{\mathcal{X}}$ are elegant, and transparent.

Relations between matrices of composition multiplicities for $U_{\mathcal{X}}$ and $V_{\mathcal{X}}$ release beautiful combinatorial formulae, associated to Cubist views of Euclidean space.

Our proofs begin with the combinatorics of Cubist diagrams. Pursuing these combinatorics allows us to prove the existence of highest weight structure on $V_{\mathcal{X}}$-mod. Standard modules for $V_{\mathcal{X}}$ have linear projective resolutions, implying that $V_{\mathcal{X}}$ is standard Koszul, in the sense of Agoston, Dlab and Lukacs. This allows us to deduce Koszulity, and the existence of highest weight structures on $U_{\mathcal{X}}$-mod. The existence of cellular structures on $U_{\mathcal{X}}, V_{\mathcal{X}}$ are then apparent. Eventually, symmetry of $U_{\mathcal{X}}$ is won.

We proceed to prove that the derived categories of $U_{\mathcal{X}}, U_{\mathcal{X}'}$ are equivalent, whenever $\mathcal{X}'$ is obtained from $\mathcal{X}$ by a local flip. This equivalence of categories allows us identify portions of symmetric group blocks of weight two with portions of certain Cubist algebras, in case $r = 3$, following a path down from the Rouquier block.

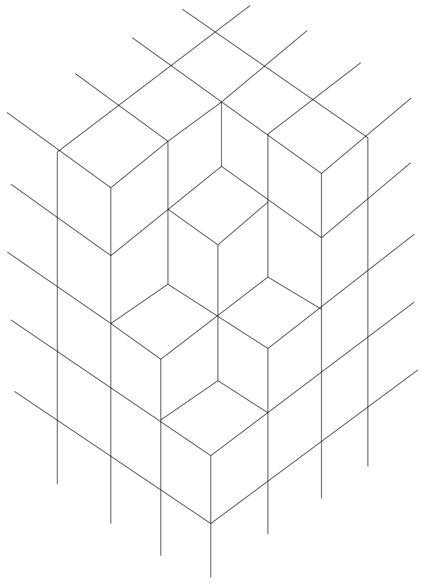
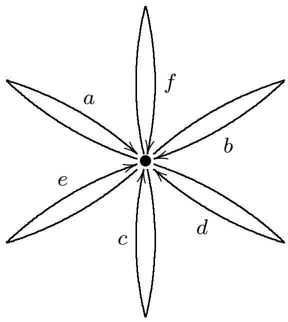


FIGURE 1. Part of a tiling Γ.

1.2. **The case $r = 3$: Peach's rhombal algebras.** Let Γ be a tiling of the plane by congruent rhombi affixed to a hexagonal grid (see Figure 1). The rhombal algebra U_Γ associated to Γ is defined to be the path algebra of the quiver obtained from Γ by replacing every edge by two arrows in opposite directions, modulo the following quadratic relations:

- **Two rhombuses relation.** Any path of length two not bordering a single rhombus is zero.
- **Mirror relation.** The sum of the two paths of length two from one vertex of a rhombus to the opposite vertex is zero.
- **Star relation.** At each vertex x, there are six possible paths of length two from x to itself, which we label as follows:



We impose the relation

$$a + d = b + e = c + f,$$

where we read any of a, b, c, d, e, f as 0 if it is not present at the vertex x.

The choice of signs in this presentation is due to Turner [30, Definition 15]. The relationship with Peach's original presentation [22] is described in Remark 18 below.

We now list the good properties of U_Γ, together with references to results (proved in the generality of Cubist algebras) in the main body of the paper. Some of the properties were originally established by Peach; in those cases we also provide the reference in Peach's thesis.

- U_Γ is a locally finite dimensional graded algebra; each indecomposable projective module has radical length five ([22, Corollary 2.4.2], Corollary 69)
- U_Γ is a symmetric algebra, i.e. it possesses an invariant inner product. ([22, Theorem 2.5.1], Theorem 68)
- U_Γ is Koszul (Theorem 52)
- U_Γ is quasihereditary (Theorem 55)

Let V_Γ be the quadratic dual of U_Γ. Then

- V_Γ is Koszul (Theorem 46)
- V_Γ is quasihereditary (Theorem 40)
- V_Γ has global dimension 5 (Theorem 69)

The quasihereditary structure on U_Γ comes with a partial order $\preceq$ on simple modules and therefore on the vertices of Γ; the partial order for V_Γ is the opposite order. The partial order $\preceq$ may be described as follows. There exists a bijection λ from the set of vertices of Γ to the set of rhombi in Γ, given by Figure 2. Then $y \preceq x$ if y is a vertex in the rhombus λx, and this relation generates the partial order. The standard modules are pictured in Figure 3. Note that the standard module for U_Γ has four composition factors, while the standard module for V_Γ has infinitely many.

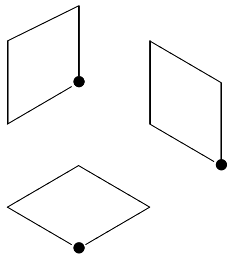


FIGURE 2. Bijection between vertices and rhombi.

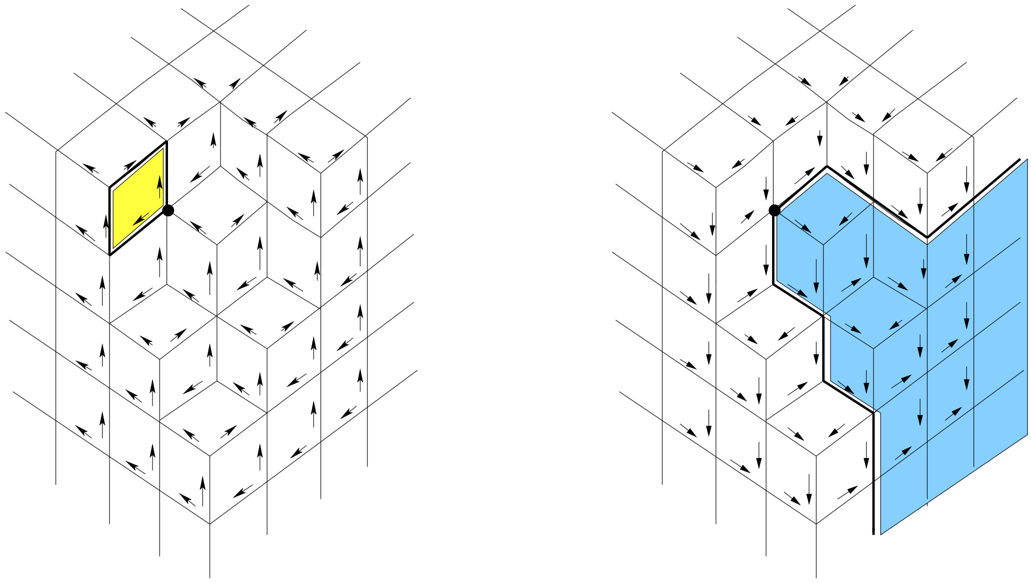


FIGURE 3. Standard modules in U_Γ and V_Γ.

Rotating Figure 2 by a multiple of $\pi/3$, we obtain 6 different partial orders on the vertices of Γ and therefore 6 different quasi-hereditary structures on U_Γ and V_Γ.

The quasi-hereditary and Koszul properties give rise to curious formulae for the inverse of a matrix recording distances in Γ. Indeed, given $x, y \in \Gamma$, let $d(x, y)$ be the length of the shortest path from x to y along edges in Γ. Let $\mathrm{Dist}(q)$ be the infinite matrix with rows and columns indexed by the vertices of Γ, whose (x, y)-entry is $q^{d(x,y)}$. Then

$$\mathrm{Dist}(q)\,\mathrm{Loc}(-q) = (1 - q^2)^2 I,$$

where I is the identity matrix, and $\mathrm{Loc}(q)$ is a matrix with a number of equivalent descriptions, each of which describe certain local configurations in Γ. The (x, y)-entry of $\mathrm{Loc}(q)$ can be written $\sum q^{d(x,z)+d(z,y)}$, where the sum is over all vertices z in Γ such that $x, y \in \lambda z$ (Corollary 57).

An alternative formula for the (x, y)-entry of $\mathrm{Loc}(q)$, which does not depend on a choice of highest weight structure is $\sum_{z\in I_\Gamma(x)\cap I_\Gamma(y)} q^2[3 - d(z, x) - d(z, y)]_q$, where $I_\Gamma(x)$ is a set describing a local configuration about x (Proposition 86), and $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. Let H be a tiled hexagon, in which the six internal lines/rhombi are placed in correspondence with the six external vertices, as in Figure 4.

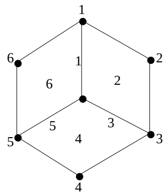


FIGURE 4. The hexagon H.

Let $H(x)$ be the hexagon H, centered at x, and dilated so its external vertices coincide with the six vertices of Γ closest to x. Note that the rhombic tiles in $H(x)$ cannot necessarily be thought of as tiles in Γ. The set $I_\Gamma(x)$ consists of the vertex x, as well as those vertices $1, ..., 6$ in $H(x)$ corresponding to lines/rhombi in $H(x)$ which can also be thought of as lines/rhombi in Γ.

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2. Cubist combinatorics

We define and study certain subsets of integer lattices that correspond to tilings of Euclidean space by rhombohedra.

2.1. **Cubist subsets.** Given $x, y \in \mathbb{R}^r$, we write $x \leq y$ if $y - x \in \mathbb{R}^r_{\geq 0}$. This defines a partial order on $\mathbb{R}^r$. We denote by $\epsilon_1, \ldots, \epsilon_r$ the standard basis of $\mathbb{R}^r$. For $x \in \mathbb{R}^r$ and $\zeta \in \mathbb{R}$, let $x[\zeta] = x + \zeta(\epsilon_1 + \cdots + \epsilon_r) \in \mathbb{R}^r$.

Definition 1. *A subset $\mathcal{X} \subset \mathbb{Z}^r$ is* Cubist*, if $\mathcal{X} = \mathcal{X}^- \backslash \mathcal{X}^-[-1]$, where $\mathcal{X}^-$ is a nonempty proper ideal of $\mathbb{Z}^r$ (with respect to the partial order $\leq$).*

Note that $\mathcal{X}^-$ is uniquely determined by $\mathcal{X}$; it is the ideal of $\mathbb{Z}^r$ generated by $\mathcal{X}$. An ideal of $\mathbb{Z}^2$ is an infinite version of (the Ferrers diagram of) a partition, and of $\mathbb{Z}^3$ is an infinite version of a plane partition (see, e.g., [28, p.371]). So Cubist subsets may be regarded as higher-dimensional generalisations of infinite partitions.

We have the following easy inductive characterisation of ideals in $\mathbb{Z}^r$.

Lemma 2. *$\mathcal{X}^-$ is an ideal of $\mathbb{Z}^r$ if, and only if,*
(1) *$\mathcal{X}^-_i = \{x \in \mathbb{Z}^{r-1} \mid (x, i) \in \mathcal{X}\}$ is an ideal of $\mathbb{Z}^{r-1}$ for $i \in \mathbb{Z}$, and*
(2) *$\mathcal{X}^-_{i+1} \subset \mathcal{X}^-_i$, for all $i \in \mathbb{Z}$.* $\square$

This leads to an useful inductive description of Cubist subsets.

Lemma 3. *If $\mathcal{X} \subset \mathbb{Z}^r$ is Cubist, then $\mathcal{X}_i = \{x \in \mathbb{Z}^{r-1} \mid (x, i), (x, i-1) \in \mathcal{X}\}$ is either a Cubist subset of $\mathbb{Z}^{r-1}$ or empty, for $i \in \mathbb{Z}$.* $\square$

Proof. By Lemma 2, $\mathcal{X}_i^- = \{x \in \mathbb{Z}^{r-1} \mid (x,i) \in \mathcal{X}^-\}$ is an ideal in $\mathbb{Z}^{r-1}$, and

$$\begin{aligned}\mathcal{X}_i^- \setminus \mathcal{X}_i^-[-1] &= \{x \in \mathbb{Z}^{r-1} \mid (x,i) \in \mathcal{X}^-,\ (x[1],i) \notin \mathcal{X}^-\} \\ &= \{x \in \mathbb{Z}^{r-1} \mid (x,i),\ (x,i-1) \in \mathcal{X}^-,\ (x[1],i),\ (x[1],i+1) \notin \mathcal{X}^-\} \\ &= \{x \in \mathbb{Z}^{r-1} \mid (x,i) \in \mathcal{X},\ (x,i-1) \in \mathcal{X}\}.\end{aligned}$$

□

The set of Cubist subsets of $\mathbb{Z}^r$ is invariant under translations and under the action of the symmetric group Σ_r permuting coordinates, as well as under the involution $x \mapsto -x$. The latter is true because of the following easy lemma.

Lemma 4. *A subset $\mathcal{X}$ of $\mathbb{Z}^r$ is Cubist if, and only if, $\mathcal{X} = \mathcal{X}^+ \backslash \mathcal{X}^+[1]$, where $\mathcal{X}^+$ is a nonempty proper coideal of $\mathbb{Z}^r$.* □

2.2. **Rhombohedral tilings.** In case $r \leq 3$, it is easy to understand Cubist subsets: one only needs to draw a picture. In higher dimensions, Cubist combinatorics are not so easy, but a topological perspective can be helpful. Here we associate to any Cubist subset $\mathcal{X}$ of $\mathbb{Z}^r$ a polytopal complex $\mathcal{C}_\mathcal{X}$ of dimension $r-1$ inside $\mathbb{R}^r$, whose faces are all cubes. This cubical complex projects homeomorphically onto a hyperplane, inducing a tiling of Euclidean $(r-1)$-space by rhombohedra.

Let us spell this out it detail. We define a *cube* in $\mathbb{Z}^r$ to be a subset of the form

$$C = x + \left\{ \sum_{j \in S} a_j \epsilon_j, \quad a_j = 0, 1 \right\},$$

where $x \in \mathbb{Z}^r$ and S is a subset of $\{1, \ldots, r\}$. We say C is a *d-cube* if S has size d. These cubes define a polytopal decomposition of $\mathbb{R}^r$ in which the d-dimension faces are the convex hulls of the d-cubes in $\mathbb{Z}^r$. For $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$, the unique face of smallest dimension containing x is $\{y \in \mathbb{R}^r \mid \lfloor x \rfloor \leq y \leq \lceil x \rceil\}$, where $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_r \rfloor)$ and $\lceil x \rceil = (\lceil x_1 \rceil, \ldots, \lceil x_r \rceil)$.

Fix a Cubist subset $\mathcal{X}$ of $\mathbb{Z}^r$.

Definition 5. *We define $\mathcal{C}_\mathcal{X}$ to be the smallest subcomplex of $\mathbb{R}^r$ containing $\mathcal{X}$. Equivalently $\mathcal{C}_\mathcal{X}$ is the subcomplex consisting of all faces which are convex hulls of cubes contained in $\mathcal{X}$.*

For all $x \in \mathbb{R}^r$, we have $x \in \mathcal{C}_\mathcal{X}$ if and only if $\lfloor x \rfloor \in \mathcal{X}$ and $\lceil x \rceil \in \mathcal{X}$, i.e., if and only if $\lceil x \rceil \in \mathcal{X}^-$ and $\lfloor x \rfloor[1] \notin \mathcal{X}^-$. We define $\mathcal{C}_{\mathcal{X}^-}$ much as we did $\mathcal{C}_\mathcal{X}$: as the smallest subcomplex of $\mathbb{Z}^r$ containing $\mathcal{X}^-$. We note that if x is in the ideal of $\mathbb{R}^r$ generated by $\mathcal{X}^-$, then $\lceil x \rceil$ is as well. It follows that this ideal is equal to $\mathcal{C}_{\mathcal{X}^-}$.

Let L be an affine line in $\mathbb{R}^r$ parallel to the vector $\epsilon_1 + \ldots + \epsilon_r$. Because $\mathcal{X}^-$ is a nonempty proper subset of $\mathbb{Z}^r$, the intersection of L with $\mathcal{C}_{\mathcal{X}^-}$ is a half-line: there exists $x_L \in L$ such that for all $y \in L$, $y \in \mathcal{C}_{\mathcal{X}^-}$ if and only if $y \leq x_L$. Clearly, x_L is on the boundary of $\mathcal{C}_{\mathcal{X}^-}$. Conversely any point on the boundary is of the form x_L for some L. Indeed, if $x \in \mathcal{C}_{\mathcal{X}^-}$ and $x \neq x_L$, where L is the affine line parallel to $\epsilon_1 + \ldots + \epsilon_r$ containing x, then $\{y \in \mathbb{R}^r \mid y < x_L\}$ is a neighborhood of x contained in $\mathcal{C}_{\mathcal{X}^-}$.

We claim that the boundary of $\mathcal{C}_{\mathcal{X}^-}$ is $\mathcal{C}_\mathcal{X}$. Suppose that $x \in \mathcal{C}_\mathcal{X}$ is not on the boundary of $\mathcal{C}_{\mathcal{X}^-}$. By the discussion above, $x[\varepsilon] \in \mathcal{C}_{\mathcal{X}^-}$ for some $\varepsilon > 0$. Then $\lfloor x \rfloor \in \mathcal{X}$ and $\lfloor x \rfloor[1] \leq \lceil x[\varepsilon] \rceil \in \mathcal{X}^-$, contradicting the assumption that $\mathcal{X}$ is Cubist. On the other hand if x is on the boundary of $\mathcal{C}_{\mathcal{X}^-}$, then $\lceil x \rceil \in \mathcal{X}^-$ and $\lfloor x \rfloor[1] \notin \mathcal{X}^-$, the latter because $x[\varepsilon] < \lfloor x \rfloor[1]$ for some $\varepsilon > 0$.

Proposition 6. *Let $\mathbb{R}^r_0$ be the hyperplane $\{(x_1, \dots, x_r) | \sum x_i = 0\}$ of $\mathbb{R}^r$. Let π be the orthogonal projection of $\mathbb{R}^r$ onto $\mathbb{R}^r_0$. Then the restriction of π to $\mathcal{C}_\mathcal{X}$ is a homeomorphism onto $\mathbb{R}^r_0$.*

Proof. In the discussion above we proved that $\mathcal{C}_\mathcal{X}$ is the boundary of $\mathcal{C}_{\mathcal{X}^-}$ and thus meets each fiber of π in exactly one point. So the restriction of π to $\mathcal{C}_\mathcal{X}$ is a continuous bijection. In fact it is a homeomorphism, since the restriction to any cell of $\mathcal{C}_\mathcal{X}$ is a homeomorphism onto a closed subset of $\mathbb{R}^r_0$. □

The homeomorphism between $\mathcal{C}_\mathcal{X}$ and $\mathbb{R}^r_0$ induces a rhombotopal decomposition of $\mathbb{R}^r_0$, which Linde, Moore and Nordahl call 'a configuration of the $(r-1)$-dimensional tiling' [17, §2]. When $r = 3$ we obtain the rhombus tilings in $\mathbb{R}^2$ considered in § 1.2.

From Proposition 6 we deduce some useful properties of Cubist subsets.

Definition 7. *A subset $\mathcal{S}$ of $\mathbb{Z}^r$ is* connected*, if for any $x, y \in \mathcal{S}$, there exists a sequence $x = x^0, x^1, \dots, x^l = y$, such that $x^m \in \mathcal{S}$ and for each $m \geq 1$, we have $x^{m+1} = x^m \pm \epsilon_j$, for some $n = n(m)$.*

Let $\mathcal{S}$ be a connected subset of $\mathbb{Z}^r$. For $x, y \in \mathcal{S}$, let $d_\mathcal{S}(x, y)$ be the smallest number l for which there exists a sequence $x = x^0, x^1, \dots, x^l = y$, such that $x^m \in \mathcal{S}$ and for each $m \geq 1$, we have $x^{m+1} = x^m \pm \epsilon_n$, for some $n = n(m)$.

We write $d(x, y) = d_{\mathbb{Z}^r}(x, y)$.

Corollary 8. *Let $\mathcal{X}$ be a Cubist subset of $\mathbb{Z}^r$. Then*

1. *$\mathcal{X}$ is a connected subset of $\mathbb{Z}^r$, such that $\mathbb{Z}^r = \amalg_\mathbb{Z} \mathcal{X}[m]$.*
2. *No r-cube is contained in $\mathcal{X}$.*
3. *For any $x \in \mathcal{X}$ the intersection of the $(r-1)$-cubes in $\mathcal{X}$ containing x is $\{x\}$.*

Given a polytopal complex $\mathcal{C}$ which is everywhere locally homeomorphic to $\mathbb{R}^w$, for some w, one can form its dual complex $\mathcal{C}'$. The dual $\mathcal{C}'$ is a polytopal complex homeomorphic to $\mathcal{C}$, whose d-dimensional faces are in bijection with the $(w-d)$-dimensional faces of $\mathcal{C}$. The poset of faces of $\mathcal{C}'$ is opposite to the poset of faces of $\mathcal{C}$.

Definition 9. *Let x be an element of a cubist subset $\mathcal{X}$ of $\mathbb{Z}^r$. We define $\mathcal{P}_x$ to be the face of $\mathcal{C}'_\mathcal{X}$ which corresponds to the vertex $x \in \mathcal{C}_\mathcal{X}$.*

Thus, $\mathcal{P}_x$ is an $r-1$-dimensional polytope, which describes the configuration of $\mathcal{X}$ about x.

In case $r = 3$, the polytope $\mathcal{P}_x$ can be a triangle, a square, a pentagon, or a hexagon. For general r, the number of d_2-dimensional faces containing any given d_1-dimensional face of $\mathcal{P}_x$ is $\binom{r-1-d_1}{d_2-d_1}$, whenever $d_1 \leq d_2$.

2.3. **Vertex-facet bijection.** While we have defined d-cubes in $\mathbb{Z}^r$ for all d, Proposition 6 shows that $(r-1)$-cubes are particularly relevant. We call them *facets*. Let $\mathcal{F}$ be the set of facets in $\mathbb{Z}^r$ and $\mathcal{F}_\mathcal{X}$ the set of facets contained in a Cubist subset $\mathcal{X}$. Any $F \in \mathcal{F}$ can be written as

$$F = x + F_i,$$

where

$$F_i = \left\{ \sum_{j<i} a_j \epsilon_j - \sum_{j>i} a_j \epsilon_j, \quad a_j = 0, 1 \right\},$$

for a unique choice of $x \in \mathbb{Z}^r$ and $i \in \{1, \ldots, r\}$.

Proposition 10. *For each $x \in \mathcal{X}$, there is a unique i such that $x + F_i \subseteq \mathcal{X}$. We have*

$$i = \max\{j \mid x + \epsilon_1 + \ldots + \epsilon_{j-1} \in \mathcal{X}\},$$

and putting $\lambda x = x + F_i$, we obtain a bijection

$$\lambda = \lambda_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{F}_{\mathcal{X}}.$$

Proof. Let $i = \max\{j \mid x + \epsilon_1 + \ldots + \epsilon_{j-1} \in \mathcal{X}\}$. Then $x + F_i \subseteq \mathcal{X}$. Indeed if $y \in x + F_i$, then $y \leq x + \epsilon_1 + \ldots + \epsilon_{i-1} \in \mathcal{X}^-$ and $y[1] \geq x + \epsilon_1 + \ldots + \epsilon_i \notin \mathcal{X}^-$. On the other hand, if $j > i$ (resp. $j < i$) then $x + \epsilon_1 + \ldots + \epsilon_i$ (resp. $x - \epsilon_i - \ldots - \epsilon_r$) is in $x + F_j$ but not in $\mathcal{X}$. Once the map λ is defined it is clearly a bijection. □

Definition 11. *For $x \in \mathcal{X}$, let μx be the cone opposite x. Thus, if $\lambda x = x + F_i$, then $\mu x = x + C_i$, where $C_i = \mathbb{Z}^{i-1}_{\leq 0} \times \mathbb{Z} \times \mathbb{Z}^{r-i}_{\geq 0}$.*

2.4. **Basic examples.** Let $1 \leq j \leq r$. Then the lattice $\mathcal{H}_j = \{(x_1, \ldots, x_n) \mid x_j = 0\}$ is a Cubist subset of $\mathbb{Z}^r$. The corresponding tiling of $\mathbb{R}^r_0$ is a regular tiling composed of translates of a fixed $(r-1)$-dimensional rhombohedron. A Cubist subset $\mathcal{X} \subset \mathbb{Z}^r$ may look locally like one of these 'flat' Cubist subsets H_j.

Definition 12. *Let $\mathcal{X} \subset \mathbb{Z}^r$ be Cubist. The subset $\mathcal{X}_{flat}$ of* flat elements *is defined to be the subset of elements $x \in \mathcal{X}$ for which there exists $i = i(x)$ such that $x + \epsilon_i, x - \epsilon_i \notin \mathcal{X}$. The subset $\mathcal{X}_{crooked}$ of* crooked elements *is the complement $\mathcal{X} \backslash \mathcal{X}_{flat}$.*

Lemma 13. *Let $\mathcal{X} \subset \mathbb{Z}^r$ be a Cubist subset.*

1. *If $x \in \mathcal{X}$ is crooked, then for all j, either $x + \epsilon_j \in \mathcal{X}$, or $x - \epsilon_j \in \mathcal{X}$.*
2. *If $x \in \mathcal{X}$ is flat, then for all $j \neq i(x)$, we have $x + \epsilon_j, x - \epsilon_j \in \mathcal{X}$.* □

Proof. We proceed by induction on r. The lemma is obvious for small r. Suppose the lemma holds in dimensions $< r$, and $\mathcal{X}$ is a cubist subset of $\mathbb{Z}^r$. We may assume that $x = 0$. If $x \in \mathcal{X}_0$, then $x - \epsilon_r \in \mathcal{X}$, and the statement of the lemma holds by the inductive hypothesis. Similarly, if $x \in \mathcal{X}_1 - \epsilon_r$, then $x + \epsilon_r \in \mathcal{X}$, and the statement of the lemma holds by the inductive hypothesis. Otherwise, $x \pm \epsilon_r \notin \mathcal{X}$, in which case $x[-1], x[1] \notin \mathcal{X}$. Therefore, $x + \sum_{j<r} \epsilon_j, x - \sum_{j<r} \epsilon_j \in \mathcal{X}$, and $x \pm \epsilon_j \in \mathcal{X}$, for $1 \leq j \leq r - 1$. □

Another example is the 'corner configuration'.

Definition 14. *The* Corner configuration *is the Cubist subset*

$$\begin{aligned}\mathcal{X}_{CC} &= \mathbb{Z}^r_{\leq 0} \backslash \mathbb{Z}^r_{\leq 0}[-1] \\ &= \left\{(x_1, \ldots, x_n) \in \mathbb{Z}^r_{\leq 0} \mid x_i = 0 \textit{ for some } i\right\}\end{aligned}$$

of $\mathbb{Z}^r$.

Lemma 15. *Any Cubist subset $\mathcal{X}$ can be approximated in an arbitrarily large finite region, by removing a finite number of r-cubes from the Corner configuration.*

Precisely, given $x \in \mathcal{X}$, and $N \geq 0$, there exists $z \in \mathbb{Z}^r$, and a Cubist set $\mathcal{X}(x,N) \subset \mathbb{Z}^r$, such that $\mathcal{X}(x,N)^-$ is obtained from $(z + \mathcal{X}_{CC})^-$ by removing a finite number of elements, and

$$y \in \mathcal{X} \Leftrightarrow y \in \mathcal{X}(x,N),$$

for all $y \in \mathbb{Z}^r$ such that $d(y,x) \leq N$.

Proof. Let $\mathcal{X}(x,N)^-$ be the ideal of $(\mathbb{Z}^r, \leq)$ generated by

$$(\mathcal{X} \cap (x + [-N,N]^r)) \cup \{x[N] - 2N\epsilon_i \mid i = 1, \ldots, r\}.$$

Let $z = x[N]$. The statement of the lemma is satisfied for this pair $(z, \mathcal{X}(x,N)^-)$. □

3. Some algebraic preliminaries

Let k be a field. We shall be working with associative k-algebras A graded over the integers. So $A = \oplus_{i\in\mathbb{Z}} A_i$ and $A_i A_j \subset A_{i+j}$. While not assuming the existence of a unit, we require A to be equipped with a set of mutually orthogonal idempotents $\{e_s \mid s \in \mathcal{S}\} \subset A_0$ such that $A = \oplus_{s,s'\in\mathcal{S}} e_s A e_{s'}$. It will be useful to allow some of the idempotents to be zero. Unless stated otherwise, all A-modules M are assumed to be *graded* left modules, so that $M = \oplus_{i\in\mathbb{Z}} M_i$ and $A_i M_j \subset M_{i+j}$, and to be quasi-unital, i.e., $M = \oplus_{s\in S} e_s M$. Given $n \in \mathbb{Z}$, we let $M\langle n\rangle$ be the A-module obtained by shifting the grading by n, so that $M\langle n\rangle_i = M\langle n-i\rangle$.

Suppose that $A_i = 0$ for all but finitely many negative integers i, and that $e_s A_i e_{s'}$ is finite dimensional for all $s, s' \in \mathcal{S}$ and $i \in \mathbb{Z}$. We define the graded Cartan matrix $C_A(q)$ to be the matrix with rows and columns labelled by $\mathcal{S}$ and entries

$$C_A(q)_{s,s'} = \sum_{i\in\mathbb{Z}} (\dim e_s A_i e_{s'})\, q^i$$

in the ring of Laurent power series in an indeterminate q. If $e_s = 0$ then the entries in the row and column labelled by s are zero. So we will often regard $C_A(q)$ as an $\mathcal{R} \times \mathcal{R}$ matrix, where $\mathcal{R} = \{s \in S \mid e_s \neq 0\}$.

Now suppose that A is positively graded, i.e. $A_i = 0$ for $i < 0$, and that $\{e_s \mid s \in \mathcal{R}\}$ is a basis for A_0. Let us also impose the finiteness condition $\dim e_s A_i < \infty$ for all $s \in \mathcal{R}$ and $i \in \mathbb{Z}$. Let A-Mod be the category of all graded A-modules, where the space of morphisms between graded modules M and N, which we denote $\mathrm{Hom}_A(M,N)$, consists of A-module homomophisms preserving degree. We denote by A-mod the full subcategory consisting of modules M such that $\dim e_s M_i < \infty$ for all $s \in \mathcal{R}$ and $i \in \mathbb{Z}$, and that $M_i = 0$ for $i << 0$.

Then Ae_s is a projective A-module (=projective indecomposable object in A-mod) for each $s \in \mathcal{R}$, and every projective indecomposable A-module is isomorphic to $Ae_s\langle n\rangle$, for a unique $s \in \mathcal{R}$ and $n \in \mathbb{Z}$. Similarly, every simple A-module is isomorphic to $L_A(s)\langle n\rangle$ for a unique $s \in \mathcal{R}$ and $n \in \mathbb{Z}$, where $L_A(s) = Ae_s / A_{>0} e_s$. The category A-mod contains enough projective objects.

Fix a partial order $\preceq$ on $\mathcal{R}$. For each $s \in \mathcal{R}$, the standard module $\Delta_A(s) = \frac{Ae_s}{\sum_{t\succ s} Ae_t A e_s}$ is the largest quotient of Ae_s which does not contain $L(t)\langle n\rangle$ as a composition factor for $t \succ s$. We define the graded decomposition matrix $D_A(q)$ of A to be the $\mathcal{R} \times \mathcal{R}$ matrix with entries

$$D_A(q)_{st} = \sum_{i\in\mathbb{Z}} (\dim e_t \Delta_A(s)_i)\, q^i.$$

If A has an antiautomorphism fixing each e_s, then $D_A(q) = D_{A^{\mathrm{op}}}(q)$, where A^{op} is the opposite algebra.

We say that A is a graded quasi-hereditary algebra, or that A-mod is a graded highest weight category, if for all $s \in \mathcal{R}$,

- $D_A(q)_{st} = 0$ for all $t \not\preceq s$,
- $ker(Ae_s \twoheadrightarrow \Delta_A(s))$ has a filtration in which each section is isomorphic to $\Delta_A(t)\langle n\rangle$ for some $t \succ s$ and $n \in \mathbb{Z}$.

This differs from the original notion of quasiheredity introduced by Cline, Parshall and Scott in that A is allowed to be infinite dimensional; in particular A may have infinite global dimension. We are also using a slightly different notion of (graded) highest weight category than that introduced by Cline, Parshall and Scott. We filter projective objects by standard modules, rather than injectives by costandards. Furthermore, we do not assume the finite interval property holds with respect to our partial order $\preceq$.

If A-mod is a highest weight category, then according to [8, Theorem 3.1.11] we have Brauer-Humphreys reciprocity:

$$C_A(q) = D_A(q)^T D_{A^{op}}(q).$$

We will need a version of Rickard's Morita theorem for derived categories [24] adapted to our graded algebras (see, e.g., [10, §2]). Let us assume that Ae_s is finite dimensional for each $s \in \mathcal{S}$. Let $\{\Gamma_t \mid t \in \mathcal{T}\}$ be a collection of bounded complexes of projective A-modules. Denote by $D^b(A\text{-mod})$ the derived category of bounded complexes in A-mod. Our formulation is rather clumsy; for the general theory it is better to think in terms of functor categories.

Theorem 16 (Rickard). *Suppose that*

- *for $t, t' \in \mathcal{T}$ and $m, n \in \mathbb{Z}$ with $m \neq 0$,*

$$\mathrm{Hom}_{D^b(A\text{-mod})}(\Gamma_t\langle n\rangle, \Gamma_{t'}[m]) = 0.$$

- *The triangulated subcategory of $D^b(A\text{-mod})$ generated by all summands of $\Gamma_t\langle n\rangle$, $t \in \mathcal{T}$, $n \in \mathbb{Z}$, contains Ae_s for all $s \in \mathcal{S}$.*

Then, the graded endomorphism ring $E = \oplus_{n\in\mathbb{Z}} E_n$ with components

$$E_n = \oplus_{t,t'\in\mathcal{X}} \mathrm{Hom}_{D^b(A\text{-mod})}(\Gamma_t\langle n\rangle, \Gamma_{t'}),$$

comes equipped with idempotents $e_t = \mathrm{id}_{\Gamma_t} \in E_0$, and there exists an equivalence

$$F : D^b(E^{op}\text{-mod}) \xrightarrow{\sim} D^b(A\text{-mod})$$

such that $F(E^{op}e_t) \cong \Gamma_t$ for all $t \in \mathcal{T}$, and $F(X\langle n\rangle) \cong F(X)\langle n\rangle$ for $X \in D^b(A\text{-mod})$ and $n \in \mathbb{Z}$.

4. Definitions

Let r be a natural number. In this section, we define algebras U_r, and V_r by quiver and relations. The Cubist algebras are defined to be quotients, or subalgebras of these.

Motivation. Before stating the generators and relations which define U_r and V_r, we explore their conception.

Indeed, let J be a $2r$-dimensional orthogonal vector space over an algebraically closed field $\bar{k}$, with non-degenerate bilinear form $\langle , \rangle$.

Let H be the Heisenberg Lie superalgebra of dimension $2r+1$ associated to J, with $H_0 = \bar{k}, H_1 = J$, and bracket

$$[(\lambda, x), (\mu, y)] = (\langle x, y\rangle, 0).$$

Let T be a maximal torus in $SO(J)$. Then T acts on H as automorphisms, via $(\lambda, x)^t = (\lambda, x^t)$.

In the sequel, we define an algebra U_r over an arbitrary field k. Over $\bar{k}$, this algebra has the same finite-dimensional graded complex representations as the crossed product $U(H) \rtimes T$ of the universal enveloping algebra of H with T.

The Koszul dual of this crossed product is $S(J^*)/\delta \rtimes T$, where $S(J^*)$ is the symmetric algebra on J, and δ is the quadratic form on J, identified as an element of $S^2(J^*) \cong S^2(J)$. This Koszul dual algebra has the same finite-dimensional graded representations as the algebra V_r.

Definition 17. *We define a graded associative algebra U_r by quiver and relations, over any field k. The quiver Q has vertices*

$$\{e_x \mid x \in \mathbb{Z}^r\},$$

and arrows

$$\{a_{x,i}, b_{x,i} \mid x \in \mathbb{Z}^r, 1 \leq i \leq r\}.$$

The arrow $a_{x,i}$ is directed from e_x to $e_{x+\epsilon_i}$, and $b_{x,i}$ is directed from e_x to $e_{x-\epsilon_i}$. U_r is defined to be the path algebra kQ of Q, modulo square relations,

$$\text{(U0)} \qquad \begin{aligned} a_{x,i}a_{x+\epsilon_i,i} &= 0, \\ b_{x,i}b_{x-\epsilon_i,i} &= 0, \end{aligned}$$

for $x \in \mathbb{Z}^r,\ 1 \leq i \leq r$, as well as supercommutation relations,

$$\text{(U1)} \qquad \begin{aligned} a_{x,i}a_{x+\epsilon_i,j} + a_{x,j}a_{x+\epsilon_j,i} &= 0, \\ b_{x,i}b_{x-\epsilon_i,j} + b_{x,j}b_{x-\epsilon_j,i} &= 0, \\ a_{x,i}b_{x+\epsilon_i,j} + b_{x,j}a_{x-\epsilon_j,i} &= 0, \end{aligned}$$

for $x \in \mathbb{Z}^r,\ 1 \leq i, j \leq r,\ i \neq j$, and Heisenberg relations*,*

$$\text{(U2)} \qquad b_{x,i}a_{x-\epsilon_i,i} + a_{x,i}b_{x+\epsilon_i,i} = b_{x,i+1}a_{x-\epsilon_{i+1},i+1} + a_{x,i+1}b_{x+\epsilon_{i+1},i+1},$$

for $x \in \mathbb{Z}^r,\ 1 \leq i < r$.

Remark 18. Applying the automorphism τ of kQ defined by $\tau(e_x) = e_x,\ \tau(a_{x,i}) = (-1)^{\sum_{\zeta=1}^{i} x_\zeta} a_{x,i}$ and $\tau(b_{x,i}) = (-1)^{\sum_{\zeta=1}^{i-1} x_\zeta} b_{x,i}$, we obtain an alternative presentation for U_r, in which the relations (U1) and (U2) are replaced by

$$\text{(U1')} \qquad \begin{aligned} a_{x,i}a_{x+\epsilon_i,j} - a_{x,j}a_{x+\epsilon_j,i} &= 0, \\ b_{x,i}b_{x-\epsilon_i,j} - b_{x,j}b_{x-\epsilon_j,i} &= 0, \\ a_{x,i}b_{x+\epsilon_i,j} - b_{x,j}a_{x-\epsilon_j,i} &= 0, \end{aligned}$$

for $x \in \mathbb{Z}^r,\ 1 \leq i, j \leq r,\ i \neq j$, and

$$\text{(U2')} \qquad (-1)^{x_i}\left(b_{x,i}a_{x-\epsilon_i,i} - a_{x,i}b_{x+\epsilon_i,i}\right) = (-1)^{x_{i+1}}\left(b_{x,i+1}a_{x-\epsilon_{i+1},i+1} - a_{x,i+1}b_{x+\epsilon_{i+1},i+1}\right),$$

for $x \in \mathbb{Z}^r,\ 1 \leq i < r$. This presentation coincides with that used by Peach for his rhombal algebras (the $r = 3$ case).

Remark 19. Let R_r be the algebra generated by indeterminates $a_i, b_i, i = 1, \dots, r$, modulo relations

$$a_i^2 = b_i^2 = 0,$$
$$a_i a_j + a_j a_i = b_i b_j + b_j b_i = 0,$$
$$a_i b_j + b_j a_i = 0, \ i \neq j,$$
$$a_i b_i + b_i a_i = a_j b_j + b_j a_j, \ i \neq j,$$

for $i, j = 1, \dots, r$. The algebra R_r acts on the right of U_r, via

$$u \circ a_i = \sum_{x \in \mathbb{Z}^r} u a_{x,i},$$
$$u \circ b_i = \sum_{x \in \mathbb{Z}^r} u b_{x,i},$$

for $u \in U_r$. Similarly, R_r acts on the left of U_r. Let c be the element $a_1 b_1 + b_1 a_1$ of U_r. By the supercommutation relations, the left and right actions of c commute: $c \circ u = u \circ c$, for $u \in U_r$. Therefore, by a combination of the left and right actions, U_r attains the structure of an $R_r \otimes_{k[c]} R_r^{op}$-module.

Definition 20. *We define V_r to be the quadratic dual of U_r. It is the path algebra of the quiver Q' with vertices*

$$\{f_x \mid x \in \mathbb{Z}^r\},$$

and arrows,

$$\{\alpha_{x,i}, \beta_{x,i} \mid x \in \mathbb{Z}^r, 1 \leq i \leq r\},$$

modulo commutation relations*,*

$$\alpha_{x,i}\alpha_{x+\epsilon_i,j} - \alpha_{x,j}\alpha_{x+\epsilon_j,i} = 0,$$
$$\beta_{x,i}\beta_{x-\epsilon_i,j} - \beta_{x,j}\beta_{x-\epsilon_j,i} = 0,$$
$$x \in \mathbb{Z}^r, 1 \leq j < i \leq r,$$
$$\alpha_{x,i}\beta_{x+\epsilon_i,j} - \beta_{x,j}\alpha_{x-\epsilon_j,i} = 0,$$

for $x \in \mathbb{Z}^r, 1 \leq j \leq i \leq r$, and the Milnor relation*,*

$$\sum_{i=1}^{r} \beta_{x,i}\alpha_{x-\epsilon_i,i} = 0,$$

for $x \in \mathbb{Z}^r$.

Remark 21. Let α_i, β_i $(i = 1, \dots, r)$ be indeterminates. Let $\gamma_i = \alpha_i \beta_i$. Let

$$\Lambda_r = k[\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r]/(\sum_{i=1}^{r} \gamma_i).$$

The algebra Λ_r acts on the right of V_r, via

$$v \circ \alpha_i = \sum_{x \in \mathbb{Z}^r} v \alpha_{x,i},$$
$$v \circ \beta_i = \sum_{x \in \mathbb{Z}^r} v \beta_{x,i},$$

for $v \in V_r$. Similarly, Λ_r acts on the left of V_r. Let Γ_r be the subalgebra $k[\gamma_1, \dots, \gamma_r]/(\sum \gamma_i)$ of Λ_r. By the commutation relations, the right and left actions of Γ_r commute: $\gamma_i \circ v = v \circ \gamma_i$ for

$v \in V_r$. Therefore, by a combination of the left and right actions, V_r attains the structure of a $\Lambda_r \otimes_{\Gamma_r} \Lambda_r^{op}$-module.

Remark 22. Let $x, y \in \mathbb{Z}^r$. Any two paths in Q' of length $d(x,y)$ represent the same element of V_r, by the commutation relations. We define p_{xy} to be the element of V_r representing a path in Q' of length $d(x,y)$.

Lemma 23. *Let $1 \le i \le r$. The set*

$$B_i = \{p_{xy} \circ m \ \mid \ x, y \in \mathbb{Z}^r,\ m \textit{ is a monomial in } \gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_r\},$$

is a basis for V_r.

Proof. We first demonstrate that B_i is a spanning set of V_r.

The commutation relations for V_r reduce any path from x to y in Q' to the form $p_{xy} \circ q_y$, where q_y is a monomial in $\gamma_1, \dots, \gamma_r$. The Milnor relation reduces q_y to a polynomial in $\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_r$.

We now show that B_i is linearly independent.

Let W_r be the vector space with basis

$$\{(x, y, m) \ \mid \ x, y \in \mathbb{Z}^r,\ m \text{ is a monomial in } \zeta_1, \dots, \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_r\}.$$

We define an action of V_r on W_r, via

$$f_x \circ (y, z, m) = \delta_{xy}(y, z, m),$$

$$\alpha_{x,j} \circ (y, z, m) = \begin{cases} (x, z, m), & \text{if } x + \epsilon_j = y, d(x,z) = d(y,z) + 1, \\ (x, z, \gamma_j m), & \text{if } x + \epsilon_j = y, d(x,z) = d(y,z) - 1, j \neq i, \\ -\sum_{l \neq i} (x, z, \gamma_l m), & \text{if } x + \epsilon_j = y, d(x,z) = d(y,z) - 1, j = i, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta_{x,j} \circ (y, z, m) = \begin{cases} (x, z, m), & \text{if } x - \epsilon_j = y, d(x,z) = d(y,z) + 1, \\ (x, z, \gamma_j m), & \text{if } x - \epsilon_j = y, d(x,z) = d(y,z) - 1, j \neq i, \\ -\sum_{l \neq i} (x, z, \gamma_l m), & \text{if } x + \epsilon_j = y, d(x,z) = d(y,z) - 1, j = i, \\ 0, & \text{otherwise.} \end{cases}$$

This does in fact define an action. Indeed, we defined this action precisely in such a way that the defining relations for V_r are forced to hold.

Now observe that the V_r-module W_r is generated by $\{(x, x, 1), x \in \mathbb{Z}^r\}$. In fact, the image of B_i under the map

$$V_r \to W_r : \quad v \mapsto \sum_{x \in \mathbb{Z}^r} v \circ (x, x, 1)$$

is the defining basis for W_r. Therefore, B_i is linearly independent. □

Corollary 24. $C_{V_r}(q)_{xy} = (1 - q^2)^{1-r} q^{d(x,y)}$, *for $x, y \in \mathbb{Z}^r$.* □

Lemma 25. *The actions of Λ_r on V_r are free.*

Proof. We look at the right action. For $x \in \mathbb{Z}^r$, the map

$$\Lambda_r \to f_x V_r,$$
$$v \mapsto f_x \circ v,$$

is clearly surjective, and degree preserving. To see it is an isomorphism, we observe that the Hilbert polynomials of the two sides agree. Indeed, by corollary 24, summing over all y, we see the Hilbert series of the right hand side is

$$(1-q^2)^{1-r}(1+2q+2q^2+\dots)^r = (1-q^2)^{1-r}\left(\frac{1+q}{1-q}\right)^r = (1-q^2)(1-q)^{-2r},$$

which is the Hilbert series of the left hand side. □

Let $\mathcal{X} \subset \mathbb{Z}^r$ be a Cubist subset. We now define our main objects of study.

Definition 26. *The Cubist algebras associated to $\mathcal{X}$ are*

$$U_{\mathcal{X}} = U_r / \sum_{x \in \mathbb{Z}^r \setminus \mathcal{X}} U_r e_x U_r$$

and

$$V_{\mathcal{X}} = \sum_{x,y \in \mathcal{X}} f_x V_r f_y.$$

Remark 27. $C_{V_{\mathcal{X}}}(q)_{xy} = (1-q^2)^{1-r} q^{d(x,y)}$, for $x, y \in \mathcal{X}$.

Remark 28. The algebras U_r, V_r each have an anti-involution ω, which swaps $a_{x,i}$ and $b_{x+\epsilon_i,i}$ (respectively $\alpha_{x,i}$ and $\beta_{x+\epsilon_i,i}$). These anti-involutions descend to $U_{\mathcal{X}}, V_{\mathcal{X}}$.

Remark 29. When $r = 1$, the algebras $U_{\mathcal{X}}$ and $V_{\mathcal{X}}$ are all isomorphic to the field k.

When $r = 2$, the algebras $U_{\mathcal{X}}$ are all isomorphic to the Brauer tree algebra of an infinite line, and the algebras $V_{\mathcal{X}}$ are all isomorphic to the preprojective algebra on an infinite line.

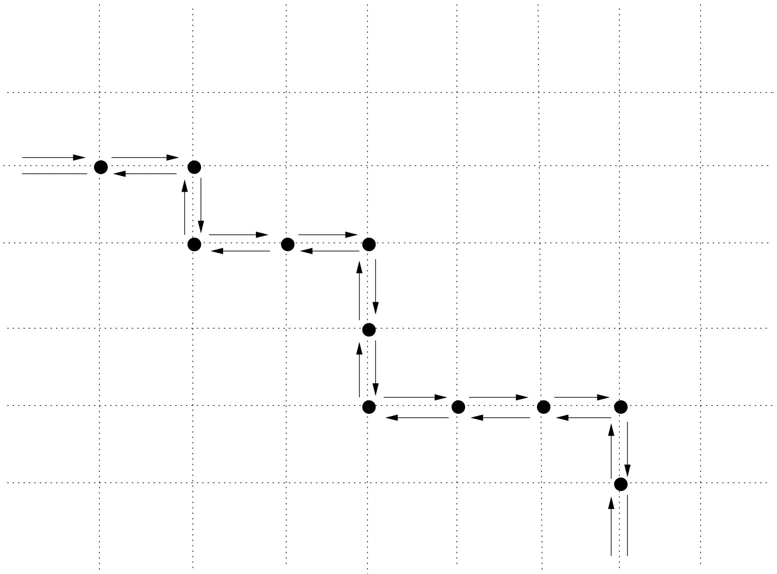

FIGURE 5. Cubist algebras when $r = 2$.

The commutation relations for the preprojective algebras come from the commutation relations for V_2 at flat elements $x \in \mathcal{X}$ and the Milnor relations at crooked elements. The commutation relations for the Brauer tree algebras come from the supercommutation relations for U_2 at flat elements $x \in \mathcal{X}$ and the Heisenberg relations at crooked elements; the square relations for the Brauer tree algebras come from the square relations for U_2 at flat elements and the supercommutation relations at crooked elements.

When $r = 3$, the algebras $U_{\mathcal{X}}$ are isomorphic to the rhombal algebras of Peach introduced in § 1.2. The rhombus tilings are obtained from the Cubist subsets $\mathcal{X}$ by projecting the cubical complex $\mathcal{C}_{\mathcal{X}}$ onto a hyperplane, as described in § 2.2. The star relations come from the Heisenberg relations for U_3, and the mirror relations from the supercommutation relations. The two rhombuses relations come in two varieties: straight paths of length two in the rhombus tiling are zero (a consequence of the square relations), and nonstraight paths of lenth two not bordering a single rhombus are zero (a consequence of the supercommutation relations).

To obtain the original presentation of the rhombal algebras given by Peach, one has to use a different presentation of U_3. This alternative choice of signs is described in remark 18.

5. Highest weight categories

In this section and the following one, we demonstrate that $U_{\mathcal{X}}$-mod, $V_{\mathcal{X}}$-mod are graded highest weight categories, in the sense of Cline, Parshall and Scott. As throughout the rest of the paper, we frequently forget the word graded, and use the term "highest weight category" as an abbreviation for "graded highest weight category".

Let $\mathcal{X} \subset \mathbb{Z}^r$ be a Cubist subset. The bijection λ between $\mathcal{X}$ and its set of facets $\mathcal{F}_{\mathcal{X}}$ established in Proposition 10 gives rise to a partial order on $\mathcal{X}$ which usually does not coincide with the restriction of the partial order on $\mathbf{Z}^r$ that we have been employing.

Proposition 30. *$\mathcal{X}$ possesses a partial order $\succeq$, generated by the relations $x \succeq y$, for $y \in \lambda x$.*

Proof. We proceed by induction on r.

When $r = 1$, the set $\mathcal{X}$ has only one element, and the lemma is trivial.

Now assume $r > 1$. Let

$$x = x^0 \succ x^1 \succ \ldots \succ x^l = x$$

be a loop in $\mathcal{X}$. To prove the lemma, we show any such loop has length zero. Without loss of generality, we may assume that $x = 0$.

Let us first observe that the loop must lie in the hyperplane $\mathbb{Z}^{r-1} \times 0 \subset \mathbb{Z}^r$. This is because by definition $x^{i+1} \in \lambda x^i + F_j$ for some j, which implies that the last coordinate of x^{i+1} is less than or equal to the last coordinate of x^i Let us write $\mathcal{Z}_0 = \mathcal{X} \cap (\mathbb{Z}^{r-1} \times 0)$. We have proved the loop lies in $\mathcal{Z}_0$.

Secondly, we prove that the loop must lie either in the subset $\mathcal{X}_0$ of $\mathcal{Z}_0$, or else in $\mathcal{Z}_0 \backslash \mathcal{X}_0$ where

$$\begin{aligned} \mathcal{X}_0 &= \{x \in \mathcal{Z}_0 \mid x - \epsilon_r \in \mathcal{X}\} \\ &= \{x \in \mathcal{Z}_0 \mid \lambda x = x + F_j, j = 1, .., r-1\}. \end{aligned}$$

Indeed, the loop cannot pass from $\mathcal{X}_0$ onto $\mathcal{Z}_0 \backslash \mathcal{X}_0$, since if $\lambda(x^i) = x^i + F_j$, and $j = 1, .., r-1$ then $x^{i+1} = x^i + h$, where $h \in F_j \cap (\mathbb{Z}^{r-1} \times 0)$, and so $h - \epsilon_r \in F_j, x^i + h - \epsilon_r \in \mathcal{X}$, which implies that $x^{i+1} \in \mathcal{X}_0$.

The relation $\succeq$ on $\mathcal{Z}_0 \backslash \mathcal{X}_0$ is precisely the restriction of the relation $\geq$ on $\mathbb{Z}^{r-1} \times 0$, so there are no loops in $\mathcal{Z}_0 \backslash \mathcal{X}_0$.

Therefore, the loop must lie in $\mathcal{X}_0$. By corollary 3, $\mathcal{X}_0$ is a Cubist subset of $\mathbb{Z}^{r-1}$. Therefore, by the inductive hypothesis, our loop has length zero. □

Remark 31. Permuting indices $1 \leq i \leq r$ with an element of the symmetric group Σ_r, we obtain an alternative partial order on $\mathcal{X}$. Indeed, conjugating $\succeq$ by elements of the symmetric group, we obtain $r!$ different partial orders on $\mathcal{X}$. The theorems of this paper hold for any such partial order.

Example 32.
(1) Consider the flat Cubist subset $\mathcal{H}_j = \{x = (x_1, \ldots, x_n) \mid x_j = 0\}$ in $\mathbb{Z}^r$. For each $x \in \mathcal{H}_j$ we have $\lambda x = x + F_j$. So for all $x, y \in \mathcal{H}_j$. $x \succeq y$ if and only if $x_i \leq y_i$ for $i < j$ and $x_i \geq y_i$ for $i > j$.

(2) The partial order on the corner configuration $\mathcal{X}_{CC}$ is more subtle. Given $x \in \mathcal{X}_{CC}$, we have $\lambda x = x + F_{m(x)}$ where $m(x) = \min\{i \mid x_i = 0\}$. We claim that $x \succeq y$ in $\mathcal{X}_{CC}$ if and only if the following hold:
   - $m(x) \geq m(y)$,
   - $x_i \leq y_i$, if $1 \leq i \leq m(y)$,
   - $x_i \geq y_i$, if $m(x) \leq i \leq r$.

   These conditions define a transitive relation on $\mathcal{X}_{CC}$ which clearly holds when $y \in \lambda x$ and therefore when $x \succeq y$. Conversely suppose that the conditions are satisfied for some $x, y \in \mathcal{X}_{CC}$. Then $x \succeq x^{(1)} \succeq \ldots \succeq x^{(m(x)-m(y))} \succeq y$, where $x^{(j)} \in \mathcal{X}_{CC}$ is defined by

$$x_i^{(j)} = \begin{cases} 0 & \text{if } m(x) - j \leq i \leq m(x), \\ x_i & \text{otherwise.} \end{cases}$$

Our proofs of the strong homological properties of the Cubist algebras, rely on the following combinatorial observation.

Proposition 33. *For $x, y \in \mathcal{X}$, the intersection $\lambda x \cap \mu y$ is a d-cube, for some dimension d.*

If $x \neq y$, and $\lambda x \cap \mu y \neq \emptyset$, then there exist some $k \in \{1, \dots, r\}, \sigma \in \{\pm 1\}$ such that $x + \sigma\epsilon_k \in \lambda x \cap \mu y$, and $d(x + \sigma\epsilon_k, y) = d(x, y) - 1$.

Proof. The intersection of $x + F_i$ with $y + C_j$ is certainly a d-cube for some d. Indeed, if it is nonempty, this subset of the $(r-1)$-cube $x + F_i$ is carved out by a number of inequalities on coordinates.

The second statement of the proposition is trivial when $r = 1$. Let us assume that the proposition is true for all Cubist subsets of $\mathbb{Z}^{r-1}$. Let $\mathcal{X}$ be a Cubist subset of $\mathbb{Z}^r$. We prove that the proposition is true for $\mathcal{X}$, and thereby for any Cubist subset, by induction. We assume that $\lambda x \cap \mu y \neq \emptyset$ for some $x \neq y$, and demonstrate the existence of some coordinate k so that the proposition is satisfied. Without loss of generality, $y = 0$.

Case 1. $\lambda y = F_r, \lambda x = x + F_r$. Therefore, $\mu y = \mathbb{Z}^{r-1}_{\leq 0} \times \mathbb{Z} \cap \mathcal{X}$. Then $x \in \mu y$, and $x + \epsilon_i \in \mu y$, for some $i < r$, and the proposition holds for $k = i$.

Case 2. $\lambda y = F_r, \lambda x = x + F_i$, for some $i < r$. Again, $\mu y = (\mathbb{Z}^{r-1}_{\leq 0} \times \mathbb{Z}) \cap \mathcal{X}$. We claim that the intersection of μy with $\mathcal{X}_0$ is empty. Indeed, suppose for a contradiction that this set is non-empty, and there exists $z \in \mathcal{X}_0 \cap \mu y \subset (\mathbb{Z}^{r-1}_{\leq 0} \times 0) \cap \mathcal{X}$. Then $z - \epsilon_r \in \mathcal{X}$. Since $z \leq 0$, we have $z - \epsilon_r \leq -\epsilon_r$. Therefore $-\epsilon_r \in \mathcal{X}^+$, and $\epsilon_1 + \dots + \epsilon_{r-1} \in \mathcal{X}^+[1]$. Thus $\epsilon_1 + \dots + \epsilon_{r-1} \notin \mathcal{X}$. However, $\lambda y = F_r$ by assumption, so $\epsilon_1 + \dots + \epsilon_{r-1} \in \mathcal{X}$, realising our contradiction.

We may now observe that the existence of a nontrivial intersection between μy and λx implies the r^{th} coordinate of x must be strictly greater than zero, and the proposition holds for $k = r$.

Case 3. $\lambda y = F_i$ for some $i < r$, $\lambda x = x + F_r$. Suppose there is a non trivial intersection $\lambda x \cap \mu y$. Then $\mu y = \mathbb{Z}^{i-1}_{\leq 0} \times \mathbb{Z} \times \mathbb{Z}^{r-i}_{\geq 0}$. If the j^{th} coordinate of x is strictly less than zero for some $j < i$, then the proposition holds for $k = j$.

If the $1^{st}, \dots, i-1^{th}$ coordinates of x are equal to zero, let us observe the i^{th} coordinate of x is strictly less than zero. Indeed, suppose for a contradiction that the $1^{st}, \dots, i-1^{th}$ coordinates of x are equal to zero, and the i^{th} coordinate of x is greater than, or equal to zero. Let $z \in \lambda x \cap \mu y$. We may assume that $z \in x + (0^i \times \{0, 1\}^{r-i})$. Thus, $z \in \mathbb{Z}^r_{\geq 0} \cap \mathcal{X}$, and $z + \epsilon_1 + \dots + \epsilon_i \in \mathcal{X}$. Thus, $\epsilon_1 + \dots + \epsilon_i \in \mathcal{X}^-$, so $-\epsilon_{i+1} - \dots - \epsilon_r \in \mathcal{X}^-[1]$, and $-\epsilon_{i+1} - \dots - \epsilon_r \notin \mathcal{X}$. However, $\lambda y = F_i$, and so $-\epsilon_{i+1} - \dots - \epsilon_r \in \mathcal{X}$, giving a contradiction.

When the $1^{st}, \dots, i-1^{th}$ coordinates of x are equal to zero, we may now observe the truth of the proposition for $k = i$.

Case 4. $\lambda y = F_i, \lambda x = x + F_j$ for some $i, j < r$. Then either the r^{th} coordinate of x is strictly greater than zero, in which case the proposition holds for $k = r$, or else $x \in \mathcal{X}_0$, in which case the induction hypothesis gives the result. □

Corollary 34. *If $x \neq y \in \mathcal{X}$, and $\lambda x \cap \mu y$ is non-empty, then there exists a $(d-1)$-cube $\mathcal{C}$, an integer $k \in \{1, \dots, r\}$, and $\sigma \in \{\pm 1\}$, such that*

$$\lambda x \cap \mu y = \mathcal{C} + \{0, \sigma\epsilon_k\},$$

and such that for all $c \in \mathcal{C}$,

$$d(x, c + \sigma\epsilon_k) = d(x, c) + 1,$$

$$d(c + \sigma\epsilon_k, y) = d(c, y) - 1.$$

Proof. By proposition 33, there exists a $(r-2)$-cube $\mathcal{C}'$ such that $\lambda x = \mathcal{C}' + \{0, \sigma\epsilon_k\}$, and for all $c \in \mathcal{C}'$, we have $d(x + \sigma\epsilon_k, y) = d(x, y) - 1$, $d(y, c + \sigma\epsilon_k) = d(y, c) - 1$, and $c \in \mu y$ exactly when $c + \sigma\epsilon_k \in \mu y$. The corollary follows upon putting $\mathcal{C} = \mathcal{C}' \cap \mu y$. □

Let $\tilde{D}_{V_\mathcal{X}}(q)$ be the $\mathcal{X} \times \mathcal{X}$ matrix, whose xy entry is $q^{d(x,y)}$, if $y \in \mu x$, and zero otherwise.

Let $\tilde{D}_{U_\mathcal{X}}(q)$ be the $\mathcal{X} \times \mathcal{X}$ matrix, whose xy entry is $q^{d(x,y)}$, if $y \in \lambda x$, and zero otherwise.

(We shall eventually show that these are equal to the decomposition matrices $D_{V_\mathcal{X}}(q)$ and $D_{U_\mathcal{X}}(q)$.)

Lemma 35.

$$\tilde{D}_{U_\mathcal{X}}(q)\tilde{D}_{V_\mathcal{X}}(-q)^T = 1.$$

Proof. For $x, y \in \mathcal{X}$, the xy entry is equal to

$$\sum_{z \in \lambda x \cap \mu y} q^{d(x,z)}(-q)^{d(z,y)}.$$

If $\lambda x \cap \mu y = \emptyset$, then this sum is equal to zero.

If $x \neq y$, and $\lambda x \cap \mu y$ is non-empty, the previous corollary shows that this sum is equal to

$$\sum_{c \in \mathcal{C}} (-1)^{d(c+\sigma\epsilon_k,y)} \left(q^{d(x,c+\sigma\epsilon_k)+d(c+\sigma\epsilon_k,y)} - q^{d(x,c)+d(c,y)} \right) = 0.$$

If $x = y$, then $\lambda x \cap \mu y = \{x\}$, and the sum is equal to 1. □

Corollary 36. *Let $x, y \in \mathcal{X}$. If $x \in \mu y$, then $x \succeq y$.*

Proof. The matrix $\tilde{D}_{U_\mathcal{X}}(q)$ is lower unitriangular with respect to $\succeq$. Therefore its inverse is also lower unitriangular, with respect to $\succeq^{op}$. □

Let $1 \leq i \leq r$. Consider the subalgebra P_i of V_r generated by elements

$$\{f_x, \beta_{x,1}, \ldots, \beta_{x,i-1}, \alpha_{x,i+1}, \ldots, \alpha_{x,r} \mid x \in \mathbb{Z}^r\}.$$

Let $L(x)$ be the simple P_i-module corresponding to $x \in \mathbb{Z}^r$. Let

$$\Delta_{V,i}(x) = V_r \otimes_{P_i} L(x),$$

a V_r-module. Let $\Omega_i = k[\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_r]$ a polynomial subalgebra of Λ_r in $r-1$ variables.

Lemma 37. *The algebra P_i is free over Ω_i with basis $\{f_x \mid x \in \mathbb{Z}^r\}$. The algebra V_r is free over P_i with a basis $\{b \circ 1 \mid b \in \mathcal{B}_i\}$, where*

$$\mathcal{B}_i = \{\textit{monomials in } \alpha_1, \ldots, \alpha_i, \beta_{i+1}, \ldots, \beta_r\} \\ \cup \{\textit{monomials in } \alpha_1, \ldots, \alpha_{i-1}, \beta_i, \ldots, \beta_r\}.$$

Proof. By Lemma 25, the action of Λ_r on V_r is free, with basis $\{f_x \mid x \in \mathbb{Z}^r\}$. The commutative algebra Ω_i acts freely on Λ_r, with basis $\mathcal{B}_i$. Hence

$$V_r \;=\; \bigoplus_{x\in\mathbb{Z}^r} \Lambda_r \circ f_x \;=\; \bigoplus_{x\in\mathbb{Z}^r}\bigoplus_{b\in\mathcal{B}_i} b\Omega_i \circ f_x \;=\; \bigoplus_{b\in\mathcal{B}_i} b \circ P_i,$$

the action of Ω_i on P_i is free with basis $\{f_x\}$ and the action of P_i on V_r is free with basis $\{b \circ 1 \mid b \in \mathcal{B}_i\}$. □

Let $\mathcal{X} \subset \mathbb{Z}^r$ be a Cubist subset. Given $x \in \mathcal{X}$, we have $\lambda x = x + F_{i_x}$ for some i_x.

Corollary 38. *Let $x \in \mathcal{X}, 1 \leq i \leq r$.*

$$[\Delta_{V,i}(x) : L(y)]_q = \sum_{z\in x+C_i} q^{d(x,z)},$$
$$[f_{\mathcal{X}}\Delta_{V,i_x}(x) : L(y)]_q = \tilde{D}_{V_{\mathcal{X}}}(q)_{xy}.\square$$

Lemma 39. *Let $x \in \mathcal{X}$. Then $\Delta_{V_{\mathcal{X}}}(x)$, the standard $V_{\mathcal{X}}$-module corresponding to X with respect to $\succeq^{op}$, possesses a linear projective resolution:*

$$\ldots \to \bigoplus_{\substack{y\in\lambda x\\ d(x,y)=2}} V_{\mathcal{X}} f_y\langle 2\rangle \to \bigoplus_{\substack{y\in\lambda x\\ d(x,y)=1}} V_{\mathcal{X}} f_y\langle 1\rangle \;\to\; V_{\mathcal{X}} f_x \;\to\; \Delta_{V_{\mathcal{X}}}(x).$$

Furthermore, $\Delta_{V_{\mathcal{X}}}(x) \cong f_{\mathcal{X}}\Delta_{V,i_x}(x)$, and consequently $D_{V_{\mathcal{X}}}(q) = \tilde{D}_{V_{\mathcal{X}}}(q)$.

Proof. We first prove the existence of a linear projective resolution for $\Delta_{V_{\mathcal{X}}}(x)$, before deducing the standard property.

In fact, we first reveal a linear projective resolution of

$$\Delta_{V,i} = V_r \otimes_{\Omega_i} k,$$

for $1 \leq i \leq r$, where k is the unique graded simple Ω_i-module. Note that

$$\Delta_{V,i} \;\cong\; V_r \otimes_{P_i} \left(\bigoplus_{x\in\mathbb{Z}^r} f_x\Omega_i\right) \otimes_{\Omega_i} k \;\cong\; V_r \otimes_{P_i} \bigoplus_{x\in\mathbb{Z}^r} L(x) \;\cong\; \bigoplus_{x\in\mathbb{Z}^r} \Delta_{V,i}(x)$$

as V_r-modules, and therefore $\Delta_{V,i}(x)$ possesses a linear projective resolution. In case $i = i_x$, applying the exact functor $\oplus_{x\in\mathcal{X}} Hom(V_r f_x, -)$, we obtain a linear projective resolution of $\Delta_{V_{\mathcal{X}}}(x)$.

How to obtain the linear projective resolution of $\Delta_{V,i}$? Recall that Ω_i is a Koszul algebra, whose Koszul complex

$$k[\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_r] \otimes_k \bigvee(\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_r) \twoheadrightarrow k$$

defines a linear projective resolution of ${}_{\Omega_i}k$. Here, we write $\bigvee(W)$ for the vector space dual of $\bigwedge(W^*)$. Note that $\bigwedge(W^*)$ is Koszul dual to $S(W) \cong k[W^*]$.

Recall that P_i acts freely on V_r. Furthermore, Ω_i acts freely on P_i. Therefore, tensoring the Koszul complex for Ω_i with V_r over Ω_i, we obtain a linear projective resolution of V_r-modules,

$$V_r \bigotimes_{\Omega_i} k[\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_r] \otimes_k \bigvee(\beta_1, \ldots, \beta_{i-1}, \alpha_{i+1}, \ldots, \alpha_r)$$

$$\twoheadrightarrow V_r \bigotimes_{\Omega_i} k = \Delta_{V,i}.$$

Let $x \in \mathcal{X}$. Taking a direct summand of this complex, in case $i = i_x$, we obtain a linear projective resolution of $\Delta_{V,i_x}(x)$, whose term in differential degree d is

$$\bigoplus_{h \in F_i, |h|=d} V_r f_{x+h} \otimes k\xi_h,$$

where $\xi_h = \xi_h^1 \vee \ldots \vee \xi_h^r$, and

$$\xi_h^j = \begin{cases} \beta_j & \text{if the coefficient of } \epsilon_j \text{ in } h \text{ is } 1 \\ 1 & \text{if the coefficient of } \epsilon_j \text{ in } l \text{ is } 0 \\ \alpha_j & \text{if the coefficient of } \epsilon_j \text{ in } h \text{ is } -1 \end{cases}$$

for $h \in F_{i_x}$. Note that all the projective indecomposable terms in this complex are indexed by elements of $\mathcal{X}$. Therefore, applying the exact functor $\oplus_{x \in \mathcal{X}} Hom(V_r f_x, -)$, we obtain a projective linear resolution of the $V_{\mathcal{X}}$-module $f_{\mathcal{X}} \Delta_{V,i_x}(x)$, as described in the statement of the lemma.

Looking at the first two terms in our resolution, and observing that $x + \epsilon_j \prec x$ for $j = 1, \ldots, i_x - 1$, and $x - \epsilon_j \prec x$ for $j = i_x + 1, \ldots, r$, we perceive that $f_{\mathcal{X}} \Delta_{V,i_x}(x)$ surjects onto the standard module at x. However, we also know that $y \succeq x$, for $y \in \mu x$, and so every composition factor $L(y)$ of $f_{\mathcal{X}} \Delta_{V,i_x}(x)$ satisfies $y \succeq x$. Therefore, $f_{\mathcal{X}} \Delta_{V,i_x}(x)$ is a standard module $\Delta_{V_{CX}}(x)$ for $V_{\mathcal{X}}$. □

Theorem 40. $V_{\mathcal{X}}$-mod *is a highest weight category, with respect to* $\succeq^{op}$.

Proof. Thanks to the linear resolution of standard modules, we have the formula

$$\tilde{D}_{U_{\mathcal{X}}}(-q) C_{V_{\mathcal{X}}}(q) = D_{V_{\mathcal{X}}}(q).$$

Together with proposition 35 and the identification $D_{V_{\mathcal{X}}}(q) = \tilde{D}_{V_{\mathcal{X}}}(q)$, this implies that

$$C_{V_{\mathcal{X}}}(q) = D_{V_{\mathcal{X}}}(q)^T D_{V_{\mathcal{X}}}(q).$$

Now that this numerical manifestation of the highest weight property is evident, we may appeal to a standard argument due to Dlab [9]. Let $A = V_{\mathcal{X}}$. The existence of the (graded) highest weight structure is equivalent to the surjective multiplication map

$$\frac{Af_x}{\sum_{y \succ x} Af_y A f_x} \otimes_k \frac{f_x A}{\sum_{y \succ x} f_x A f_y A} \longrightarrow \frac{\sum_{y \succeq x} Af_y A}{\sum_{y \succ x} Af_y A}$$

being an isomorphism, for all $x \in \mathcal{X}$. Keeping in mind that we have an anitautomorphism ω of A fixing each f_x, we see that this is equivalent to the sum over $x \in \mathcal{X}$ of the Hilbert series of $f_z \Delta(x) \otimes_k f_{z'} \Delta(x)$ being equal to the sum over x of the Hilbert series of $f_z \frac{\sum_{y \succeq x} Af_y A}{\sum_{y \succ x} Af_y A} f_{z'}$, for all $z, z' \in \mathcal{X}$. This is precisely the formula,

$$C_{V_{\mathcal{X}}}(q)_{zz'} = \left(D_{V_{\mathcal{X}}}(q)^T D_{V_{\mathcal{X}}}(q)\right)_{zz'}.$$

□

Remark 41. We are using a slightly different notion of highest weight category than that introduced by Cline, Parshall and Scott. We filter projective objects by standard modules, rather than injectives by costandards. Furthermore, we do not assume the finite interval property holds with respect to our partial order $\preceq$. In other words, we do not assume that $\{z \mid x \preceq z \preceq y\}$ to be finite, for all $x, y \in \mathcal{X}$.

The following theorem can be proved by the dual of an argument given by Cline, Parshall and Scott ([8], Theorem 3.9(a)).

Theorem 42. *Suppose that $\mathcal{X}$ possesses the finite interval property. Let $\mathcal{T}_1 \subset \mathcal{X}$ be an ideal relative to $\preceq$. Let $V_{\mathcal{T}_1}$ be the quotient of $V_{\mathcal{X}}$ by the ideal generated by $f_x, x \in \mathcal{X}\backslash\mathcal{T}_1$.*

There is a full embedding of derived categories,

$$D^b(V_{\mathcal{T}_1}\text{-mod}) \hookrightarrow D^b(V_{\mathcal{X}}\text{-mod}).$$

We conclude this section by observing the finite interval property does hold for those Cubist sets which are obtained from the corner configuration $\mathcal{X}_{CC}$ by removing finitely many boxes.

Lemma 43. *Suppose that $\mathcal{X}^-$ is obtained from the corner configuration $\mathcal{X}_{CC}^-$ by removing finitely many elements. Then set $\{z \in \mathcal{X} \mid x \preceq z \preceq y\}$ is finite for all $x, y \in \mathcal{X}$.*

Proof. The finite interval property holds for $\mathcal{X}_{CC}$, by example 32(2). □

6. Standard Koszulity

Theorem 44. *U_r and V_r are Koszul dual.*

Proof. Note that Λ_r is a Koszul algebra, whose Koszul complex

$$\Lambda_r \otimes_k (\Lambda_r^!)^* \twoheadrightarrow k,$$

defines a linear projective resolution of ${}_{\Lambda_r}k$. (More generally, any commutative complete intersection with quadratic regular sequence is Koszul, see [12, §3.1].) Tensoring over with V_r over Λ_r, we obtain a linear projective resolution,

$$V_r \bigotimes_{\Lambda_r} \left(\Lambda_r \otimes_k (\Lambda_r^!)^*\right) \to V_r \bigotimes_{\Lambda_r} k = V_r^0,$$

of the degree zero part of V_r. The Koszul dual of V_r is equal to its quadratic dual, namely U_r. □

Let $\mathcal{T}$ be a finite truncation of the poset $(\mathcal{X}, \succeq^{op})$. Thus $\mathcal{T}$ is the intersection of an ideal $\mathcal{T}_1$, and a coideal $\mathcal{T}_2$ in $\mathcal{X}$, and $\mathcal{T}$ has finitely many elements. Since $V_{\mathcal{X}}$-mod has a highest weight module category, it has a finite dimensional subquotient $V_{\mathcal{T}}$, which is quasi-hereditary, and whose simple modules are indexed by $\mathcal{T}$ ([8], Theorem 3.9).

Proposition 45. *Standard modules for $V_{\mathcal{T}}$ have linear projective resolutions. $V_{\mathcal{T}}$ is Koszul.*

The Koszul dual $V_{\mathcal{T}}^!$ of $V_{\mathcal{T}}$ is quasi-hereditary, with respect to $\succeq$. Standard modules for $V_{\mathcal{T}}^!$ have linear projective resolutions.

Proof. Let $t \in \mathcal{T}$. Let $\mathcal{X}(t, N)$ be a Cubist subset of $\mathbb{Z}^r$ defined as in Lemma 15. Thus $\mathcal{X}(t, N)$ is identical to $\mathcal{X}$ in the region of radius N about t, and is obtained by removing finitely many boxes from a shift of the Corner configuration. We know by Lemma 43 that $\mathcal{X}(t, N)$ satisfies the finite interval property. Note that for $N >> 0$, the finite truncation $V_{\mathcal{T}}$ is also a finite truncation of $V_{\mathcal{X}}(t, N)$. Replacing $\mathcal{X}$ by $\mathcal{X}(t, N)$ for some $N >> 0$, if necessary, we may now assume that $\mathcal{X}$ possesses the finite interval property.

Let $\Delta(s)$ be a standard $V_{\mathcal{X}}$-module. By Lemma 39, we have a linear projective resolution,

$$\bigoplus_{t \in \lambda s} V_{\mathcal{X}} f_t \twoheadrightarrow \Delta(s),$$

of $\Delta(s)$. The term $V_{\mathcal{X}} f_t$ rests in homological degree $d(s, t)$.

We first prove that standard modules for $V_{\mathcal{T}_1}$ have linear projective resolutions. Indeed, let L be a simple $V_{\mathcal{T}_1}$-module, and assume $s \in \mathcal{T}_1$. We have, by Lemma 42,

$$Ext^*_{V_{\mathcal{T}_1}\text{-mod}}(L\langle j\rangle, \Delta(s)) \cong Ext^*_{V_{\mathcal{X}}\text{-mod}}(L\langle j\rangle, \Delta(s)).$$

Therefore, our linear projective resolution of $\Delta(s)$ in $V_{\mathcal{X}}$-mod descends to a linear projective resolution of $\Delta(s)$ in $V_{\mathcal{T}_1}$-mod.

Let $s \in \mathcal{T}$. Let $f_{\mathcal{T}} = \sum_{t\in\mathcal{T}} f_t$. A standard module for $V_{\mathcal{T}}$ is obtained by applying the functor $Hom_{V_{\mathcal{T}_1}}(V_{\mathcal{T}_1} f_{\mathcal{T}}, -)$ to the standard $V_{\mathcal{T}_1}$-module $\Delta(s)$. Applying this functor to our resolution of $\Delta(s)$, we obtain a linear resolution of $f_{\mathcal{T}}\Delta(s)$, as required. The terms in this resolution are projective, because they are sums of modules $f_{\mathcal{T}} V_{\mathcal{T}_1} f_t$ such that $t \in \lambda s \cap \mathcal{T}_1$; and therefore $t \succeq^{op} s$, and $t \in \mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T}$.

We have proved that the algebra $V_{\mathcal{T}}$ is a 'Standard Koszul algebra', in the sense of Ágoston, Dlab, and Lukács [1]. In other words, $V_{\mathcal{T}}$ is a quasi-hereditary algebra, all of whose standard modules have linear projective resolutions. These authors have proved that such algebras are always Koszul, and that their Koszul duals are standard Koszul, with respect to the opposite ordering. Therefore $V_{\mathcal{T}}$ is Koszul, and $V^!_{\mathcal{T}}$ is standard Koszul with respect to $\succeq$. □

Theorem 46. *$V_{\mathcal{X}}$ is Koszul.*

Proof. Let $x \in \mathcal{X}$. Let $\mathcal{X}(x, N)$ be a Cubist subset of $\mathbb{Z}^r$ defined in Lemma 15. Thus $\mathcal{X}(x, N)$ is identical to $\mathcal{X}$ in the region of radius N about x, and is obtained by removing finitely many boxes from a shift of the Corner configuration. We know by Lemma 43 that $\mathcal{X}(x, N)$ satisfies the finite interval property. Consequently there exists a finite subset $\mathcal{T}(x, N)$ of $\mathcal{X}(x, N)$, which is the intersection of an ideal and a coideal, and contains the region of radius N about x. The algebra $V_{\mathcal{T}(x,N)}$ is therefore Koszul by the previous theorem. In particular, $V_{\mathcal{T}(x,N)}$ is a quadratic algebra, and as this is true for all x, N, the algebra $V_{\mathcal{X}}$ is quadratic. Let K be the Koszul complex associated to the quadratic algebra $V_{\mathcal{X}}$. Thus, $K = \bigoplus_{N\geq 0} K_N$ is the sum of complexes

$$K_N = \bigoplus_{i+j=N} (V_{\mathcal{X}})_i \otimes (V^!_{\mathcal{X}})^*_j.$$

To prove that $V_{\mathcal{X}}$ is Koszul, it suffices to show that $f_x K_N$ is exact for all x, and all $N \geq 1$. This is true, however, because we can identify $f_x K_N$ with the corresponding summand of the Koszul complex of $V_{\mathcal{T}(x,N)}$. □

Proposition 47. *$V_{\mathcal{X}}$ is isomorphic to the path algebra of the quiver with vertices*

$$\{f_x \mid x \in \mathcal{X}\},$$

and arrows

$$\{\alpha_{x,i} \mid x, x + \epsilon_i \in \mathcal{X}\} \cup \{\beta_{x,i} \mid x, x - \epsilon_i \in \mathcal{X}\},$$

modulo the ideal generated by quadratic relations,

$$\alpha_{x,i}\alpha_{x+\epsilon_i,j} - \alpha_{x,j}\alpha_{x+\epsilon_j,i} = 0 \ (x, x + \epsilon_i, x + \epsilon_j, x + \epsilon_i + \epsilon_j \in \mathcal{X}),$$
$$\beta_{x,i}\beta_{x-\epsilon_i,j} - \beta_{x,j}\beta_{x-\epsilon_j,i} = 0 \ (x, x - \epsilon_i, x - \epsilon_j, x - \epsilon_i - \epsilon_j \in \mathcal{X}),$$
$$\alpha_{x,i}\beta_{x+\epsilon_i,j} - \beta_{x,j}\alpha_{x-\epsilon_j,i} = 0 \ (x, x + \epsilon_i, x - \epsilon_j, x + \epsilon_i - \epsilon_j \in \mathcal{X}),$$
$$1 \leq i, j \leq r, \quad i \neq j,$$

$$\sum_i \xi_{x,i}\eta_{x,i} = 0 \ (x \in \mathcal{X}_{crooked}),$$

where

$$(\xi_{x,i}, \eta_{x,i}) = \begin{cases} (\beta_{x,i}, \alpha_{x-\epsilon_i}), & \text{if } x - \epsilon_i \in \mathcal{X} \\ (\alpha_{x,i}, \beta_{x+\epsilon_i,i}), & \text{if } x - \epsilon_i \notin \mathcal{X}. \end{cases}$$

Proof. Since $V_{\mathcal{X}}$ is Koszul, it is generated in degrees zero and one, modulo the ideal generated by quadratic relations. In degrees zero and one, by definition $V_{\mathcal{X}}$ has a basis as described in the proposition. It remains to check the quadratic relations between these generators.

The commutation relations between generators of $V_{\mathcal{X}}$ are visible as the first three families of relations given in the proposition. The Milnor relation at x is inherited from V_r if x is crooked. However, the degree two part of $f_x V_{\mathcal{X}} f_x$ has dimension $r-1$, and when x is flat, the Milnor relation need not be invoked to demonstrate that the elements $\{\beta_{x,j}\alpha_{x-\epsilon_j}, j \neq i(x)\}$ form a basis for this space. □

Corollary 48. *If $x, y \in \mathcal{X}$, then $d_{\mathcal{X}}(x,y) = d_{\mathbb{Z}^r}(x,y)$.* □

The Koszul dual of $V_{\mathcal{X}}$ is equal to the quadratic dual $V_{\mathcal{X}}^!$. The quadratic presentation of $V_{\mathcal{X}}^!$ is by the quiver with vertices,

$$\{e_x \mid x \in \mathcal{X}\}$$

arrows,

$$\{a_{x,i} \mid x, x+\epsilon_i \in \mathcal{X}\} \cup \{b_{x,i} \mid x, x-\epsilon_i \in \mathcal{X}\},$$

and relations,

$$a_{x,i}a_{x+\epsilon_i,i} = 0 \ (x \in \mathcal{X}, x+\epsilon_i \in \mathcal{X}, x+2\epsilon_i \in \mathcal{X}),$$
$$b_{x,i}b_{x-\epsilon_i,i} = 0 \ (x \in \mathcal{X}, x-\epsilon_i \in \mathcal{X}, x-2\epsilon_i \in \mathcal{X}),$$
$$1 \leq i \leq r.$$
$$a_{x,i}a_{x+\epsilon_i,j} + a_{x,j}a_{x+\epsilon_j,i} = 0 \ (x, x+\epsilon_i+\epsilon_j \in \mathcal{X}),$$
$$b_{x,i}b_{x-\epsilon_i,j} + b_{x,j}b_{x-\epsilon_j,i} = 0 \ (x, x-\epsilon_i-\epsilon_j \in \mathcal{X}),$$
$$a_{x,i}b_{x+\epsilon_i,j} + b_{x,j}a_{x-\epsilon_j,i} = 0 \ (x, x+\epsilon_i-\epsilon_j \in \mathcal{X}),$$
$$1 \leq i,j \leq r, \quad i \neq j,$$
$$b_{x,i}a_{x-\epsilon_i,i} + a_{x,i}b_{x+\epsilon_i,i} = b_{x,i+1}a_{x-\epsilon_{i+1},i+1} + a_{x,i+1}b_{x+\epsilon_{i+1},i+1},$$
$$(x \in \mathcal{X}), 1 \leq i < r.$$

Here, the term $a_{x,i}a_{x+\epsilon_i,j}$ is defined to be zero if $x+\epsilon_i$ is not an element of $\mathcal{X}$. The same convention applies to any term in the last four relations.

Lemma 49. *$V_{\mathcal{X}}^!$ is a locally finite dimensional algebra.*

Proof. The relations allow an element of degree $2r$ to be written as a sum of elements,

$$c^1_{\sigma 1} \dots c^r_{\sigma r} d^1_{\sigma 1} \dots d^r_{\sigma r}, \sigma \in \Sigma_r, \{c^i, d^i\} = \{a, b\}.$$

The term $c^1_{\sigma 1} \dots c^r_{\sigma r}$ represents a path of length r across an r-cube. All such paths are equal up to sign, by the supercommutation relations, and there exists such a path through each vertex of the r-cube. However, no r-cube is a subset of $\mathcal{X}$, so this term is equal to zero in $V_{\mathcal{X}}^!$. □

Because $V_{\mathcal{X}}^! = Ext^*_{V_{\mathcal{X}}}(V_{\mathcal{X}}^0, V_{\mathcal{X}}^0)$ is locally finite dimensional, we have the following fact:

Corollary 50. *$V_\mathcal{X}$ has finite global dimension.* □

Corollary 51. *There is a recollement of derived categories,*

$$D^b(V_r\text{-mod})_\mathcal{X} \leftrightarrows D^b(V_r\text{-mod}) \leftrightarrows D^b(V_\mathcal{X}\text{-mod}).$$

Proof. Since $V_\mathcal{X}$ has finite global dimension, a theorem of Cline, Parshall, and Scott implies that the map $D^b(V_r\text{-mod}) \to D^b(V_\mathcal{X}\text{-mod})$ extends to a recollement of derived categories ([8], Theorem 2.3). Here, $D^b(V_r\text{-mod})_\mathcal{X}$ is the category of complexes of modules, whose homology is given by simple modules outside $\mathcal{X}$. □

Theorem 52. *We have an isomorphism, $V_\mathcal{X}^! \cong U_\mathcal{X}$. In other words, $U_\mathcal{X}$ is Koszul dual to $V_\mathcal{X}$.*

Proof. The relations for $V_\mathcal{X}^!$ are precisely the relations for U_r, modulo the relation $e_\mathcal{X} = 0$. Therefore, there exists a surjection $V_\mathcal{X}^! \to U_r/U_r e_\mathcal{X} U_r = U_\mathcal{X}$ of graded algebras.

Thanks to the aforementioned recollement, there exists a surjection,

$$U_r = Ext^*_{V_r}(V_r^0, V_r^0) \to Ext^*_{V_\mathcal{X}}(V_\mathcal{X}^0, V_\mathcal{X}^0) = V_\mathcal{X}^!,$$

in the kernel of which lies $U_r e_\mathcal{X} U_r$. Thus, we have a surjection $U_\mathcal{X} \to V_\mathcal{X}^!$ of graded algebras.

We have proved the existence of graded surjections from $V_\mathcal{X}^!$ to $U_\mathcal{X}$, and back. Such maps preserve homogeneous spaces of projective indecomposable modules, which are finite dimensional. Each of these surjections is therefore an isomorphism. □

Corollary 53. *There is an equivalence of derived categories,*

$$D^b(U_\mathcal{X}\text{-mod}) \cong D^b(V_\mathcal{X}\text{-mod}).$$

Proof. By a theorem of Beilinson, Ginzburg, and Soergel, such an equivalence holds for a general pair of Koszul dual algebras, one of which is locally finite dimensional ([2], Theorem 2.12.6). □

Corollary 54. *There is a recollement of derived categories,*

$$D^b(U_\mathcal{X}\text{-mod}) \leftrightarrows D^b(U_r\text{-mod}) \leftrightarrows D^b(U_r\text{-mod})^\mathcal{X}.$$

Proof. Since $V_\mathcal{X}$ is equal to $f_\mathcal{X} V f_\mathcal{X}$, we know that $Ext^i_{U_r}(S,T) = Ext^i_{U_\mathcal{X}}(S,T)$, for all simple $U_\mathcal{X}$-modules S, T. By functorality, $Ext^i_{U_r}(M,T) = Ext^i_{U_\mathcal{X}}(M,T)$, for all finite dimensional $U_\mathcal{X}$-modules M, and all simple $U_\mathcal{X}$-modules T. Again by functorality, we find that $Ext^i_{U_r}(M,N) = Ext^i_{U_\mathcal{X}}(M,N)$, for all finite dimensional $U_\mathcal{X}$-modules M, N.

A theorem of Cline, Parshall and Scott allows us to deduce that the map $D^b(U_\mathcal{X}\text{-mod}) \to D^b(U_r\text{-mod})$ extends to a recollement ([7], Theorem 3.1). Here, $D^b(U_r\text{-mod})^\mathcal{X}$ is the quotient category of $D^b(U_r\text{-mod})$ by the thick subcategory $D^b(U_\mathcal{X}\text{-mod})$. □

Corollary 55. *$U_\mathcal{X}$-mod is a highest weight category, with respect to $\succeq$. Standard modules possess linear projective resolutions.*

Proof. For $x \in \mathcal{X}, N \geq 0$, let $\mathcal{X}(x,N)$ be a Cubist subset which can be identified with $\mathcal{X}$ in a box of diameter N around x, such that $\mathcal{X}(x,N)$ is obtained by removing boxes from a translate of the corner configuration. Such an $\mathcal{X}(x,N)$ has the finite interval property, and therefore for all finite truncations $\mathcal{T}(x,N)$, the algebra $U_{\mathcal{T}(x,N)}$ Koszul dual to $V_{\mathcal{T}(x,N)}$ is standard Koszul. Because $U_\mathcal{X}$ is locally finite dimensional, $U_\mathcal{X}$ can be identified with $U_{\mathcal{T}(x,N)}$ in a large region around x, so long as N is large enough.

For this reason, the regular $U_{\mathcal{X}}$-module possesses a Δ-filtration. As the Koszulity of $V_{\mathcal{T}(x,N)}$ implied the Koszulity of $V_{\mathcal{X}}$ in the proof of theorem 46, now the existence of linear projective resolutions for standard $U_{\mathcal{T}(x,N)}$-modules imply the existence of linear projective resolutions for standard $U_{\mathcal{X}}$-modules. □

Lemma 56. *The standard module $\Delta_{U_{\mathcal{X}}}(x)$ of $U_{\mathcal{X}}$ has a basis $\{q_y \mid y \in \lambda x\}$, with q_y in degree $d(y,x)$. If $y, y' \in \lambda x$ and $d(y',x) = d(y',y) + d(y,x)$, then $\gamma_{y'y} q_y = \pm q_{y'}$. In particular, $\Delta_{U_{\mathcal{X}}}(x)$ has simple socle $L(x^{op})\langle w \rangle$.*

Proof. Let $K^{-1} : D^b(V_{\mathcal{X}}\text{-mod}) \to D^b(U_{\mathcal{X}}\text{-mod})$ be the inverse Koszul duality functor (see [2, Theorem 1.2.6]). Then K^{-1} is a triangulated functor such that $K^{-1}(M\langle n \rangle) = K^{-1}(M)[-n]\langle -n \rangle$, $K^{-1}(V_{\mathcal{X}} e_x) = L(x)$, and $K^{-1}(L(x)) = U_{\mathcal{X}}^* e_x$. By Lemma 39, we deduce that $K^{-1}(\Delta_{V_{\mathcal{X}}}(x))$ is quasiisomorphic to a module M whose composition factors are described by the matrix $D_{U_{\mathcal{X}}}(q)$. Moreover, by Lemma 38, M has an injective resolution $U_{\mathcal{X}}^* e_x \to \oplus_{y \in \mu x, d(y,x)=1} U_{\mathcal{X}}^* e_y \langle -1 \rangle \to \ldots$. Hence M is the costandard module of $U_{\mathcal{X}}$ associated to the simple module $L(x)$, and $\Delta_{U_{\mathcal{X}}} = M^*$, the corresponding standard module, also has composition factors given by $D_{U_{\mathcal{X}}}(q)$. By comparing $D_{U_{\mathcal{X}}}(q)$ with $C_{U_{\mathcal{X}}}(q)$, we deduce that the images q_y, $y \in \lambda x$ of $\gamma_{y,x}$, $y \in \lambda x$ under a surjective homomorphism $U_{\mathcal{X}} e_x \to \Delta_{\mathcal{X}}(x)$ form a basis for $\Delta_{\mathcal{X}}(x)$, and furthermore we have $\gamma_{y'y} q_y = \pm q_y$ whenever $d(y',x) = d(y',y) + d(y,x)$. □

Corollary 57.

$$C_{U_{\mathcal{X}}}(q) = D_{U_{\mathcal{X}}}(q)^T D_{U_{\mathcal{X}}}(q).$$

Definition 58. *Let $x \in \mathcal{X}$. We call the standard $U_{\mathcal{X}}$-module $\Delta_{U_{\mathcal{X}}}(x)$ the* facetious module *corresponding to λx. It is a graded $U_{\mathcal{X}}$-module whose head is $L(x)$, and whose Hilbert series is $\sum_{y \in \lambda x} q^{d(x,y)} L(y)$.*

Theorem 59. *$U_{\mathcal{X}}$, and $V_{\mathcal{X}}$ have homogeneous cellular bases. For either algebra, there is a canonical choice of such basis, with respect to our fixed generators.*

Proof. Let $\mathcal{X} \subset \mathbb{Z}^r$ be a Cubist subset. The cellularity is immediate from the definition of S. König and C. Xi [16, Corollary 4.2]: a quasi-hereditary algebra which has a decomposition by primitive idempotents each fixed by an anti-involution is cellular. Our anti-involution is ω, which swaps $a_{x,i}$ and $b_{x+\epsilon_i,i}$ (respectively $\alpha_{x,i}$ and $\beta_{x+\epsilon_i,i}$). The grading on our algebras is compatible with the highest weight structure, and therefore with the cellular structure. Cellular bases can be canonically defined with respect to the generators of $U_{\mathcal{X}}, V_{\mathcal{X}}$, because the q-decomposition numbers are all monomials. □

7. Symmetry

Before proving an algebraic property of the Cubist algebras, we must always do some combinatorics. Let us prove some lemmas, before we deduce the symmetry of $U_{\mathcal{X}}$...

Let w_0 be the longest element of Σ_r. Let the standard partial order be the partial order explicitly written down in the paper.

Lemma 60. *Fix a partial order $\succeq$ on $\mathcal{X}$. The map $x \mapsto x^{op}$ which takes x to its opposite in λx is bijective.*

Proof. Let $\lambda : \mathcal{X} \to \mathcal{F}_X$ be the map defined by the standard partial order $\succeq$. Let $\lambda' : \mathcal{X} \to \mathcal{F}_X$ be the map defined by the partial order $\succeq^{w_0}$. Since $x^{op} = \lambda'^{-1}\lambda(x)$, and λ, λ' are bijective, the lemma holds. □

Lemma 61. *Let x, y be distinct elements of $\mathcal{X}$. Then there exists a facet F of $\mathcal{X}$ such that $x \in F$ and $y \notin F$.*

Proof. Because $\mathcal{C}_\mathcal{X}$ is homeomorphic to $\mathbb{R}^{r-1}$, any d-cube C in $\mathcal{X}$ can be characterised as the intersection of facets containing C. For this reason, $\mathcal{P}_x$ can be characterised as the largest subcomplex of $\mathcal{C}_\mathcal{X}$ whose vertices are all contained in $\mathcal{P}_x$. However, since x, y are distinct, we have $\mathcal{P}_x \neq \mathcal{P}_y$. Therefore, some vertex of $\mathcal{P}_x$ is not a vertex of $\mathcal{P}_y$. This completes the proof of the lemma. □

Definition 62. *Let $x \in \mathcal{X}$, and let ξ be a facet of $\mathcal{X}$ containing x. Let*

$$\xi_i = \begin{cases} a_i & \textit{if } x + \epsilon_i \in \xi, \\ b_i & \textit{if } x - \epsilon_i \in \xi, \\ 1 & \textit{otherwise,} \end{cases}$$

for $i = 1, \dots, r$. Let $s(\xi)$ to be the order of the set $\{\xi_i \mid \xi_i = b_i\}$.

Definition 63. *Let $\mathcal{F}_x = \{\xi \in \mathcal{F}_\mathcal{X} \mid x \in \xi\}$, for $x \in \mathcal{X}$*

Lemma 64. *If $\xi, \xi' \in \mathcal{F}_x$, then there is a sequence $\xi = \xi^0, \xi^1 \dots, \xi^l = \xi'$ in $\mathcal{F}_x$, such that $\xi^i \cap \xi^{i+1}$ is an $r-2$-cube.*

Proof. The polytope $\mathcal{P}_x$ has a 1-skeleton whose vertices correspond to elements of $\mathcal{F}_x$, and whose edges correspond to $r-2$-cubes containing x. The poset of faces of $\mathcal{P}_x$ ordered by inclusion is the opposite of the poset of cubes containing x, ordered by inculsion. The 1-skeleton of any polytope is connected. Therefore, the lemma holds. □

Lemma 65. *Suppose that $\xi, \eta \in \mathcal{F}_X$, and that $\xi \cap \eta$ is an $r-2$-cube. Then one of the following holds:*

(a) There exists $i \in [1, r]$, such that $\xi_i = a_i, \eta_i = b_i$, and $\xi_k = \eta_k$ for $k \in [1, r], k \neq i$.

(b) There exists $i \in [1, r]$, such that $\xi_i = b_i, \eta_i = a_i$, and $\xi_k = \eta_k$ for $k \in [1, r], k \neq i$.

(c) There exist distinct $i, j \in [1, r]$, such that $(\xi_i, \xi_j) = (a_i, 1)$, $(\eta_i, \eta_j) = (1, a_j)$, and $\xi_k = \eta_k$ for $k \in [1, r], k \neq i, j$.

(d) There exist distinct $i, j \in [1, r]$, such that $(\xi_i, \xi_j) = (b_i, 1)$, $(\eta_i, \eta_j) = (1, b_j)$, and $\xi_k = \eta_k$ for $k \in [1, r], k \neq i, j$.

Proof. We only need eliminate a couple of possibilities. The first is the existence of distinct $i, j \in [1, r]$, such that $(\xi_i, \xi_j) = (a_i, 1)$, $(\eta_i, \eta_j) = (1, b_j)$, and $\xi_k = \eta_k$ for $k \in [1, r], k \neq i, j$. However, such an arrangement implies that $x_\xi = x + \sum_{k, x+\epsilon_k \in \xi} \epsilon_k \in \xi \subset \mathcal{X}$, as well as $x_\eta = x - \sum_{k, x-\epsilon_k \in \eta} \epsilon_k \in \eta \subset \mathcal{X}$. Thus $x_\eta, x_\eta[1] = x_\xi \in \mathcal{X}$, which cannot happen.

The remaining possibility is the existence of distinct $i, j \in [1, r]$, such that $(\xi_i, \xi_j) = (a_i, 1)$, $(\eta_i, \eta_j) = (1, b_j)$, and $\xi_k = \eta_k$ for $k \in [1, r], k \neq i, j$. This we can eliminate for identical reasons. □

Lemma 66. *Let $x \in \mathcal{X}$, and let ξ, η be facets of $\mathcal{X}$ containing x. Then*

$$(-1)^{s(\xi)} e_x \circ \xi_1 \dots \xi_{r-1} \xi_{r-1}^\omega \dots \xi_1^\omega = (-1)^{s(\eta)} e_x \circ \eta_1 \dots \eta_{r-1} \eta_{r-1}^\omega \dots \eta_1^\omega,$$

and these are non-zero elements of $U_\mathcal{X}$.

Proof. The written elements of $U_\mathcal{X}$ are non-zero, by the cellularity of $U_\mathcal{X}$. By lemma 64, we may assume that $\xi \cap \eta$ is an $r-2$-cube. By lemma 65, we should check cases (a)-(d).

Case (a): By the supercommutation relations, the left hand side is equal to

$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r a_i b_i \xi_r^\omega \dots \xi_{i+1}^\omega \xi_i^\omega \dots \xi_1^\omega,$$

whilst the right hand side is equal to

$$(-1)^{s(\xi)+1}e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r b_i a_i \xi_r^\omega \dots \xi_{i+1}^\omega \xi_{i-1}^\omega \dots \xi_1^\omega.$$

Let $j \neq i$ be that number such that $\xi_j = \eta_j = 1$. The difference of the left and right hand side is

$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r (a_i b_i + b_i a_i) \xi_r^\omega \dots \xi_{i+1}^\omega \xi_i^\omega \dots \xi_1^\omega,$$

which is equal to

$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r (a_j b_j + b_j a_j) \xi_r^\omega \dots \xi_{i+1}^\omega \xi_i^\omega \dots \xi_1^\omega,$$

by the Heisenberg relation. Note that $x - \sum_{k, x-\epsilon_k \in \xi} \epsilon_k \in \mathcal{X}$, and so its shift, $x + \epsilon_j + \sum_{k, x+\epsilon_i \in \xi} \epsilon_k \notin \mathcal{X}$, and thus

$$e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r a_j = \pm e_x \circ a_j \prod_{k, x+\epsilon_k \in \xi} \xi_k \prod_{k, x-\epsilon_k \in \xi} \xi_k = 0.$$

Similarly, $e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r b_j = 0$, and we have proved the difference of left and right hand side is zero.

Case (b) is proved identically to case (a).

Case (c): The left hand side is equal to

$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r a_i b_i \xi_r^\omega \dots \xi_{i+1}^\omega \xi_i^\omega \dots \xi_1^\omega,$$

whilst the right hand side is equal to

$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{j-1}\xi_{j+1} \dots \xi_r a_j b_j \xi_r^\omega \dots \xi_{j+1}^\omega \xi_j^\omega \dots \xi_1^\omega.$$

By the Heisenberg relations, we have

$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r (a_i b_i + b_i a_i) \xi_r^\omega \dots \xi_{i+1}^\omega \xi_i^\omega \dots \xi_1^\omega =$$
$$(-1)^{s(\xi)}e_x \circ \xi_1 \dots \xi_{j-1}\xi_{j+1} \dots \xi_r (a_j b_j + b_j a_j) \xi_r^\omega \dots \xi_{j+1}^\omega \xi_j^\omega \dots \xi_1^\omega.$$

However, $x - \epsilon_j = \sum_{k, x-\epsilon_k \in \xi} \epsilon_k \in \mathcal{X}$, and so its shift, $x + \sum_{k, x+\epsilon_i \in \xi} \epsilon_k \notin \mathcal{X}$, and thus

$$e_x \circ \xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_r b_i = \pm e_x \circ b_i \prod_{k, x-\epsilon_k \in \xi} \xi_k \prod_{k, x+\epsilon_k \in \xi} \xi_k = 0.$$

Similarly, $e_x \circ \xi_1 \dots \xi_{j-1}\xi_{j+1} \dots \xi_r b_j = 0$, and we have proved the equality of left and right hand side.

Case (d) is proved identically to case (c). □

Lemma 67. *Let F be a facet of a cubist set $\mathcal{X}$, containing an element x. Then there exists $\sigma \in \Sigma_r$, such that $F = x + F_i^\sigma$.*

Proof. We have

$$F = \left\{ x + \sum_{j \in S} a_j \epsilon_j + \sum_{j \notin S \cup \{i\}} a_j \epsilon_j, \quad a_j = 0, 1 \right\}$$

for some $S \subseteq \{1, \dots, r\}$ and $i \notin S$. Let σ be some element of Σ_r, such that $\sigma(\{1, \dots, |S|\}) = S$. □

Theorem 68. *$U_\mathcal{X}$ is symmetric.*

Proof. Let us define a bilinear form on $U_{\mathcal{X}}$ by the formula

$$(u_1, u_2) = \sum_{x \in \mathcal{X}} c(u_1.u_2),$$

where $c(u)$ the coefficient of the element $(-1)^{s(\xi)}\xi_1 \dots \xi_1 \xi_{r-1}^\omega \dots \xi_1^\omega$ of lemma 66 in u. The form is clearly associative. Let us prove its non-degeneracy.

By lemma 66, we know that the the degree $2r-2$ part of $U_{\mathcal{X}}$ is isomorphic to $k\mathcal{X}$ as a $U_{\mathcal{X}}$-$U_{\mathcal{X}}$-bimodule, and that $U_{\mathcal{X}}$ vanishes in degree $2r-1$. Therefore, let $0 \neq u \in e_x U_{\mathcal{X}} e_y$ be homogeneous of degree $i < 2r-2$. We are required to show that $U_{\mathcal{X}}^{>0}.u \neq 0$. Suppose not. Then the socle of $U_{\mathcal{X}} e_y$ contains a summand isomorphic to $L(x)\langle i \rangle$. Hence the same is true of the socle of one of the factors $\Delta_{U_{\mathcal{X}}}(z)\langle d(z,y)\rangle$, $y \in \lambda z$ in a standard filtration of $U_{\mathcal{X}} e_y$. By Lemma 60 there is a unique $z \in \mathcal{X}$ such that $z^{op} = x$, and by Lemma 56 we must have $y \in \lambda z$ and $i = d(z,y) + r - 1$. In particular we have $y \neq x$.

Now the above argument remains valid for any conjugate of the partial order on $\mathcal{X}$. By Lemma 61 there is a facet F of $\mathcal{X}$ such that $x \in F$ and $y \notin F$. Let z' be the vertex of F opposite x. By lemma 67, we may choose a partial order $\succeq'$ on $\mathcal{X}$, with respect to which $\lambda' z' = F$. Then $z'^{op} = x$ with respect to λ', but $y \notin \lambda' z'$, which is a contradiction.

Let us finally observe that $(,)$ is symmetric. We need to see that the Nakayama automorphism N of $U_{\mathcal{X}}$, defined by

$$(x, y) = (N(y), x),$$

is trivial. The Nakayama automorphism is a graded homomorphism, so it is enough to know that $N(x) = x$, for arrows x, since these generate $U_{\mathcal{X}}$. The explicit formula of lemma 66 makes this clear. □

Corollary 69. *Every principal indecomposable $U_{\mathcal{X}}$-module has radical length $2r-1$. The global dimension of $V_{\mathcal{X}}$ is $2r-1$.*

Corollary 70. *$U_{\mathcal{X}}$ is Ringel self-dual.*

Proof. Projective modules are also injective, and therefore tilting. □

R. Martínez-Villa has characterised Koszul self-injective algebras by a noncommutative Gorenstein property [18], providing a corollary to theorem 68:

Corollary 71. *There is an isomorphism of graded $k\mathcal{X}$-$k\mathcal{X}$-bimodules,*

$$Ext^*_{V_{\mathcal{X}}\text{-mod}}(k\mathcal{X}, V_{\mathcal{X}}) \cong k\mathcal{X}\langle 2r-2 \rangle . \square$$

Let A be a $\mathbb{Z}_+$-graded algebra. Let A-nil be the Serre subcategory of A-mod, of modules on which $A_{>0}$ acts nilpotently. Let

$$A\text{-Qmod} = A\text{-mod} / A\text{-nil},$$

the non-commutative analogue of coherent sheaves on a projective scheme $Proj(A)$ associated to a commutative graded algebra A. Let A-qmod be the subcategory of A-Qmod of graded $V_{\mathcal{X}}$-modules with projective resolutions whose terms are all finitely generated. A theorem of R. Martínez-Villa and A. Martsinkovsky [19] provides a second corollary to theorem 68:

Corollary 72. *(Serre duality) For graded $V_{\mathcal{X}}$-modules M, N, we have,*

$$Ext^i_{V_{\mathcal{X}}\text{-qmod}}(M,N) \cong Ext^{2r-3-i}_{V_{\mathcal{X}}\text{-qmod}}(N, M\langle -2r+2\rangle)^*.\square$$

Note that in the above duality, we do not twist the module N by an automorphism of $V_{\mathcal{X}}$. The absence of such an automorphism (canonical bundle) is known to some as the *Calabi-Yau* property. The Calabi-Yau dimension of $V_{\mathcal{X}}$-qmod is $2r-3$. In particular, when $r = 3$, these categories have Calabi-Yau dimension 3. String theorists are said to be interested in such things.

To end this section, let us give a few definitions concerning the algebra $U_{\mathcal{X}}$ which will be of some application in the sequel.

Definition 73. *Let $\omega : kQ \longrightarrow kQ$ be the graded antiautomorphism, defined by $\omega(e_x) = e_x$, $\omega(a_{x,i}) = b_{x+\epsilon_i,i}$, $\omega(b_{x,i}) = a_{x-\epsilon_i,i}$. We have $\omega(R) = R$, so ω defines antiautomorphisms on U_r and $U_{\mathcal{X}}$ which we still call ω.*

Given a graded $U_{\mathcal{X}}$-module $V = \oplus V_n$, we make the dual $V^* = \oplus(V^*)_n = \oplus V^*_{-n}$ a graded $U_{\mathcal{X}}$-module via $(u\phi)(v) = \phi(\omega(u)v)$.

Definition 74. *Let κ be the automorphism of $\mathbb{Z}^r$ defined by $\kappa(x) = -x$.*

Then $G = \mathbb{Z}^r \rtimes \langle\kappa\rangle$ acts as a group of automorphisms on $\mathbb{Z}^r$ (the $\mathbb{Z}^r$ by translations) and thus as a group of automorphisms of kQ, with $g(e_x) = e_{g(x)}$; $g(a_{x,i}) = a_{g(x),i}$ and $g(b_{x,i}) = b_{g(x),i}$ if $g \in \mathbb{Z}^r$; and $\kappa(a_{x,i}) = b_{g(x),i}$ and $\kappa(b_{x,i}) = a_{g(x),i}$ if $g \notin \mathbb{Z}^r$.

Let $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r) \in \mathbb{Z}^r$. All paths in Q from x to y of shortest length have the same image in U_r, up to sign, by virtue of the anticommutation relations. In order to be precise, we make the following

Definition 75. *Let $\gamma_{xy} \in Q$ be the path of shortest length starting at x, visiting $(y_1, x_2, \ldots, x_n)$, $(y_1, y_2, x_3, \ldots, x_n)$, $\ldots$, $(y_1, \ldots, y_{n-1}, x_n)$ in succession and ending at y.*

8. Derived equivalences

In this section we show how certain mutations of Cubist subsets correspond to derived equivalences of Cubist algebras. These 'flips' play an important role in the study of rhombus tilings (see, e.g., [17, §2]).

Lemma 76. *Let $\mathcal{X}$ be a Cubist subset of $\mathbb{Z}^r$ and let $z \in \mathcal{X}$. Then the following statements are equivalent.*

1. *z is a maximal element of $\mathcal{X}$ with respect to the partial order $\leq$.*
2. *For all $x \in \mathbb{Z}^r$, if $z[-1] < x \leq z$, then $x \in \mathcal{X}$.*
3. *The subset $\mathcal{X}'$ of $\mathbb{Z}^r$ obtained from $\mathcal{X}$ be replacing z by $z[-1]$ is a Cubist subset.*
4. *The polytope $\mathcal{P}_z$ is an $r-1$-simplex.*

For most of the remainder of this section we fix a Cubist subset $\mathcal{X} \subset \mathbb{Z}^r$ and an element $z \in \mathcal{X}$ satisfying the conditions of the Lemma. Our aim is the following result

Theorem 77. *There exists an equivalence of triangulated categories*

$$F : D^b(U_{\mathcal{X}'}\text{-mod}) \xrightarrow{\sim} D^b(U_{\mathcal{X}}\text{-mod})$$

such that $F(X\langle n\rangle) \cong F(X)\langle n\rangle$ *for* $X \in D^b(U_{\mathcal{X}'}\text{-mod})$ *and* $n \in \mathbb{Z}$. *Moreover, for all* $x \in \mathbb{Z}^r$, *we have* $F(U'_{\mathcal{X}} e_x) \cong \Gamma_x$, *where* Γ_x *is a complex explicitly described in* §8.2.

8.1. **Structure around a flip.** We begin by describing the structure of $U_{\mathcal{X}}$ near z. Given $n \in \mathbb{Z}$, let

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}]$$

be the associated 'quantum integer'. Note that $[-n] = -[n]$.

Lemma 78. (1) *The element* $\zeta_z = e_z b_{z,i} a_{z-\epsilon_i,i} e_z$ *in* $U_{\mathcal{X}}$ *is independent of* i.

(2) *The elements*

$$\gamma_{x,z}\zeta_z^s, \qquad z[-1] < x \leq z, \quad 0 \leq s \leq r - 1 - d(z, x)$$

form a basis of $U_{\mathcal{X}} e_z$, *and the elements*

$$\zeta_z^s \gamma_{z,x}, \qquad z[-1] < x \leq z, \quad 0 \leq s \leq r - 1 - d(z, x)$$

form a basis of $e_z U_{\mathcal{X}}$.

(3) *We have* $C_{U_{\mathcal{X}}}(q)_{zy} = q^{-d(z,x)} + q^{-d(z,x)+2} + \ldots + q^{d(z,x)} = q^{r-1}[r - d(z, x)]_q$.

(4) *Suppose* $z[-1] < x \leq z$ *and let* $\rho(\gamma_{z,x}) : U_{\mathcal{X}} e_z\langle d(z, x)\rangle \to U_{\mathcal{X}} e_x$ *be the homomorphism defined by right multiplication by* $\gamma_{z,x}$. *Then* $\mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}} e_z\langle n\rangle, \mathrm{coker}(\rho(\gamma_{z,x}))) = 0$ *for all* $n \in \mathbb{Z}$.

Proof. Part (1) is a consequence of the Heisenberg relations because $z + \epsilon_i \notin \mathcal{X}$ for all i. Define a kQ-module W with basis

$$\{w_{x,s} \mid z[-1] < x \leq z, \quad 0 \leq s \leq r - 1 - d(z, x)\}$$

with $\deg(w_{x,s}) = d(z, x) + 2s$ and action

$$e_y w_{x,s} = \begin{cases} w_{x,s} & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

$$a_{y,i} w_{x,s} = \begin{cases} (-1)^\sigma w_{x+\epsilon_i,s+1} & \text{if } y = x \text{ and } x + \epsilon_i \in \mathcal{X}, \\ 0 & \text{otherwise,} \end{cases}$$

$$b_{y,i} w_{x,s} = \begin{cases} (-1)^\sigma w_{x-\epsilon_i,s} & \text{if } y = x,\ x - \epsilon_i \in \mathcal{X}, \text{ and } s \neq r - 1 - d(z, x), \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma = \#\{j \in \{1, \ldots, i - 1\} \mid y_j = x_j - 1\}$. It is easy to check that the defining relations of U_r hold, and that W is annihilated by e_x for all $x \notin \mathcal{X}$. Hence W is a $U_{\mathcal{X}}$-module. There is a unique homomorphism $\psi : U_{\mathcal{X}} e_z \longrightarrow W$ such that $\psi(e_z) = w_{z,0}$. We have $\psi(\gamma_{x,z}\zeta_z^s) = w_{x,s}$, so ψ is surjective. The dimension of W is

$$\sum_{z[-1]<x\leq z} (r - d(z, x)) = \sum_{j=1}^{r} \binom{r}{j} j = r \sum_{j=1}^{r} \binom{r-1}{j-1} = r2^{r-1}.$$

On the other hand the dimension of $U_{\mathcal{X}} e_z$ is $\sum D_{U_{\mathcal{X}}}(1)_{yz} D_{U_{\mathcal{X}}}(1)_{yx}$, where the sum is over all x, y such that $z, y \in \lambda x$. This is also $r2^{r-1}$, since z is contained in exactly r facets of $\mathcal{X}$, by Lemma 76. We deduce that ψ is an isomorphism and that the first half of part (2) is true. The second half is obtained by applying the antiautomorphism ω. Part (3) is follows immediately. The last part is also a consequence of the second, because

$$\mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}}e_z\langle n\rangle, \mathrm{coker}(\rho(\gamma_{z,x}))) \cong \left(\frac{e_zU_{\mathcal{X}}e_x}{e_zU_{\mathcal{X}}\gamma_{z,x}}\right)_n.$$

□

8.2. **The tilting complex.** For each $x \in \mathbb{Z}^r$ we define

$$\Gamma_x = \begin{cases} \mathrm{cone}(U_{\mathcal{X}}e_z\langle d(z,x)\rangle \xrightarrow{\rho(\gamma_{z,x})} U_{\mathcal{X}}e_x) & \text{if } z[-1] \le x \le z, \\ U_{\mathcal{X}}e_x & \text{otherwise.} \end{cases}$$

So each Γ_x is a complex of projective $U_{\mathcal{X}}$-modules concentrated in homological degrees -1 and 0. Observe that Γ_z is contractible and that $\Gamma_{z[-1]}$ is isomorphic to $U_{\mathcal{X}}e_z\langle r\rangle[1]$. Hence Γ_z is nonzero as an object of $D^b(U_{\mathcal{X}}\text{-mod})$ if and only if $x \in \mathcal{X}'$.

Proposition 79. *The complexes Γ_z satisfy the hypotheses of Theorem 16.*

Proof. The generation condition clearly holds, because $U_{\mathcal{X}}e_z \cong \Gamma_{z[-1]}\langle -r\rangle[-1]$ and, for all $x \neq z$, $U_{\mathcal{X}}e_x$ is isomorphic to Γ_x or to the cone of a morphism from Γ_x to $U_{\mathcal{X}}e_z\langle d(z,x)\rangle[-1]$. So it remains to prove that for $x, x' \in \mathcal{X}$ and $m, n \in \mathbb{Z}$ with $m \neq 0$,

$$\mathrm{Hom}_{D^b(U_{\mathcal{X}}\text{-mod})}(\Gamma_x\langle n\rangle, \Gamma_{x'}[m]) = 0.$$

This is clear unless $m = 1$ or $m = -1$, since Γ_x and $\Gamma_{x'}$ are complexes concentrated in degree -1 and 0. Thus it suffices to show that $\mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}}e_z\langle n\rangle, H^0(\Gamma_{x'})) = 0$ and $\mathrm{Hom}_{U_{\mathcal{X}}}(H^0(\Gamma_x)\langle n\rangle, U_{\mathcal{X}}e_z) = 0$ for all $n \in \mathbb{Z}$. The first is true by Lemma 78 and the second follows because $U_{\mathcal{X}}$ is a symmetric algebra (Theorem 68). □

We form the graded endomorphism ring

$$E := \oplus_{n\in\mathbb{Z}} E_n, \qquad E_n = \oplus_{x,y\in\mathbb{Z}^r} \mathrm{Hom}_{D^b(U_{\mathcal{X}}\text{-mod})}(\Gamma_x\langle n\rangle, \Gamma_y),$$

and put $e'_x = id_{\Gamma_x}$ for each $x \in \mathbb{Z}^r$. By Theorem 16 there exists an equivalence

$$F : D^b(\mathrm{mod}(E^{op})) \xrightarrow{\sim} D^b(\mathrm{mod}(U_{\mathcal{X}}))$$

such that $F(E^{op}e'_x) \cong \Gamma_x$ for all $x \in \mathbb{Z}^r$, and $F(X\langle n\rangle) \cong F(X)\langle n\rangle$ for $X \in D^b(E^{op}\text{-mod})$ and $n \in \mathbb{Z}$. To complete the proof of Theorem 77 we shall construct an isomorphism between $U_{\mathcal{X}'}$ and E^{op}.

8.3. **Identification of the endomorphism ring.** We define a graded homomorphism

$$\Phi : kQ \longrightarrow E^{\mathrm{op}}$$

as follows:

$$\Phi(e_x) : \Gamma_x \to \Gamma_x = \mathrm{id}_{\Gamma_x}$$

$$\Phi(a_{x,i}) : \Gamma_x\langle 1\rangle \to \Gamma_{x+\epsilon_i} = \begin{cases} ((-1)^{\sigma_{x,i}}\rho(\zeta_z), \rho(a_{x,i})) & \text{if } z[-1] \le x, x+\epsilon_i \le z, \\ (0, \rho(a_{x,i})) & \text{otherwise,} \end{cases}$$

$$\Phi(b_{x,i}) : \Gamma_x\langle 1\rangle \to \Gamma_{x-\epsilon_i} = \begin{cases} ((-1)^{\sigma_{x,i}}\,\mathrm{id}, \rho(b_{x,i})) & \text{if } z[-1] \le x, x-\epsilon_i \le z, \\ (0, \rho(b_{x,i})) & \text{otherwise,} \end{cases}$$

where $\sigma_{x,i} = \#\{j < i \mid y_j \neq x_j\}$ and chain maps (f_{-1}, f_0) are specified by their components f_{-1}, f_0 in degrees -1 and 0. Using Lemma 78 it is straightforward to check that these are indeed chain maps.

Proposition 80. *Φ is surjective.*

Proof. Let $f : \Gamma_x\langle n\rangle \to \Gamma_y$ be a chain map. We want to show that the image of Φ contains f. It certainly contains a chain map whose degree 0 component agrees with f, so by taking their difference and scaling we may assume that $z[-1] \leq x, y \leq z$, and that $f = (\rho(\zeta_z^s), 0)$ where $2s = n + d(z,x) - d(z,y)$. Since f is a chain map, we have $\zeta_z^s \gamma_{z,y} = 0$ which implies by Lemma 78 that $s \geq r - d(z,y)$. Hence $n \geq 2r - d(z,x) - d(z,y) \geq 0$.

If $n = 0$ then $f = \Phi(e_{z[-1]})$. If $n = 1$ then $f = \Phi(a_{z[-1],i})$ or $f = \Phi(b_{z[-1]+\epsilon_i,i})$ for some i. Now we assume $n \geq 2$ and argue by induction on n. Suppose that $y = z[-1]$. Since Γ_z is contractible we may assume that $x \neq z$ and choose i such that $z[-1] \leq x + \epsilon_i \leq z$. Then f is the composition of $\Phi(a_{x,i})\langle n-1\rangle$ and $(\rho(\zeta_z^s), 0) : \Gamma_{x+\epsilon_i}\langle n-1\rangle \to \Gamma_{z[-1]}$.

Suppose on the other hand that $y \neq z[-1]$. Choose i such that $z[-1] \leq y - \epsilon_i \leq z$. Then f is the composition of $(\rho(\zeta_z^{s-1}), 0) : \Gamma_x\langle n\rangle \to \Gamma_{y-\epsilon_i}\langle 1\rangle$ and $\Phi(b_{y-\epsilon_i,i})$. The former is a chain map because $s - 1 \geq r - d(z, y - \epsilon_i)$. □

Proposition 81. *Φ factors through the natural homomorphism $kQ \to U_{\mathcal{X}'}$.*

Proof. Since $\Phi(e_x) = id_{\Gamma_x} = 0$ for $x \notin \mathcal{X}'$, it suffices to show that Φ kills the defining relations of U_r. We shall show that in fact the image of any relation under Φ is the zero chain map (not merely nullhomotopic); note that this is clear in homological degree 0.

- Square relations (U0): For any $x \in \mathbb{Z}^r$, at least one of x, $x + \epsilon_i$, and $x + 2\epsilon_i$ is not in $z - \{0,1\}^r$ and hence $\Phi(a_{x,i}a_{x+\epsilon_i,i}) = 0$. A similar argument applies to $\Phi(b_{x,i}b_{x-\epsilon_i,i})$.
- Supercommutation relations (U1): Consider $\Phi(a_{x,i}a_{x+\epsilon_i,j} + a_{x,j}a_{x+\epsilon_j,i})$. We may assume that both Γ_x and $\Gamma_{x+\epsilon_i+\epsilon_j}$ are nonzero in degree -1, and therefore that $x, x+\epsilon_i, x+\epsilon_j, x+\epsilon_i+\epsilon_j \in z - \{0,1\}^r$. Then the component of $\Phi(a_{x,i}a_{x+\epsilon_i,j} + a_{x,j}a_{x+\epsilon_j,i}) = 0$ in degree -1 is right multiplication by
$$\left((-1)^{\sigma_{x,i}+\sigma_{x+\epsilon_i,j}} + (-1)^{\sigma_{x,j}+\sigma_{x+\epsilon_j,i}}\right)\zeta_z^2 = 0.$$
The argument that $\Phi(b_{x,i}b_{x+\epsilon_i,i} + b_{x,j}b_{x+\epsilon_j,i}) = 0$ and $\Phi(a_{x,i}b_{x+\epsilon_i,j} + b_{x,j}a_{x-\epsilon_j,i}) = 0$ is similar.
- Heisenberg relations (U2): If $x \notin z - \{0,1\}^r$, then the degree -1 component of $\Phi(a_{x,i}b_{x+\epsilon_i,i} + b_{x,i}a_{x-\epsilon_i,i})$ is 0. If $x \in z - \{0,1\}^r$ then exactly one of $x + \epsilon_i$ and $x - \epsilon_i$ is in $z - \{0,1\}^r$, and therefore the degree -1 component of $\Phi(a_{x,i}b_{x+\epsilon_i,i} + b_{x,i}a_{x-\epsilon_i,i})$ is $\rho(\zeta_z)$, which does not depend on i.

□

By virtue of these two propositions we have a surjective homomorphism $U_{\mathcal{X}'} \to E^{\mathrm{op}}$. We now show that this is actually an isomorphism, by demonstrating that $C_{U_{\mathcal{X}'}}(q) = C_{E^{\mathrm{op}}}(q)$.

Lemma 82. *We have*
$$C_{E^{\mathrm{op}}}(q)_{x,y} = \begin{cases} C_{U_{\mathcal{X}}}(q)_{x,y} - q^{r-1}[r - d(z,x) - d(z,y)]_q & \text{if } z[-1] \leq x \leq z, \\ C_{U_{\mathcal{X}}}(q)_{x,y} & \text{otherwise.} \end{cases}$$

Proof. By Lemma 78,

$$C_{U_{\mathcal{X}}}(q)_{z,x} = \begin{cases} q^{r-1}[r-d(z,x)]_q & \text{if } z[-1] \leq x \leq z, \\ 0 & \text{otherwise,} \end{cases}$$

and since $\omega(e_x U_{\mathcal{X}} e_z) = e_z U_{\mathcal{X}} e_x$, we have $C_{U_{\mathcal{X}}}(q)_{x,z} = C_{U_{\mathcal{X}}}(q)_{z,x}$.

Hence

$$\begin{aligned} C_{E^{\mathrm{op}}}(q)_{x,y} &= \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}_{D^b(U_{\mathcal{X}}\text{-mod})}(\Gamma_x\langle n\rangle, \Gamma_y)\ q^n \\ &= \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}}e_x\langle n\rangle, U_{\mathcal{X}}e_y)\ q^n \\ &\quad + \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}}e_z\langle d(z,x)+n\rangle, U_{\mathcal{X}}e_z\langle d(z,y)\rangle)\ q^n \\ &\quad - \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}}e_x\langle n\rangle, U_{\mathcal{X}}e_z\langle d(z,y)\rangle)\ q^n \\ &\quad - \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}_{U_{\mathcal{X}}}(U_{\mathcal{X}}e_z\langle d(z,x)+n\rangle, U_{\mathcal{X}}e_y)\ q^n. \end{aligned}$$

So if $z[-1] \leq x, y \leq z$, then

$$\begin{aligned} C_{E^{\mathrm{op}}}(q)_{x,y} &= C_{U_{\mathcal{X}}}(q)_{x,y} + q^{r-1+d(z,y)-d(z,x)}[r]_q \\ &\quad - q^{r-1+d(z,y)}[r-d(z,x)]_q - q^{r-1-d(z,x)}[r-d(z,y)]_q \\ &= C_{U_{\mathcal{X}}}(q)_{x,y} - q^{r-1}[r-d(z,x)-d(z,y)]_q, \end{aligned}$$

and otherwise $C_{E^{\mathrm{op}}}(q)_{x,y} = C_{U_{\mathcal{X}}}(q)_{x,y}$. □

By using the automorphism κ we see that the results of this section apply to a dual situation in which we specify an element of a Cubist subset minimal with respect to $\leq$. Taking in particular the Cubist subset $\mathcal{X}'$ and the minimal element $z[-1]$, we obtain a graded endomorphism ring E' together with an epimorphism $U_{\mathcal{X}} \to (E')^{\mathrm{op}}$, and the formula

$$C_{(E')^{\mathrm{op}}}(q)_{x,y} = \begin{cases} C_{U_{\mathcal{X}'}}(q)_{x,y} - q^{r-1}[r-d(z[-1],x)-d(z[-1],y)]_q & \text{if } z[-1] \leq x, y \leq z, \\ C_{U_{\mathcal{X}'}}(q)_{x,y} & \text{otherwise.} \end{cases}$$

There is an equivalence of categories,

$$F' : D^b((E')^{op}\text{-mod})) \xrightarrow{\sim} D^b(U_{\mathcal{X}'}\text{-mod}).$$

Note that $r - d(z[-1],x) - d(z[-1],y) = d(z,x) + d(z,y) - r$.

We are now ready to show that $C_{U_{\mathcal{X}'}}(q)_{xy} = C_{E^{\mathrm{op}}}(q)_{xy}$ for all $x, y \in \mathbb{Z}^r$, and thus complete the proof of Theorem 77. Because Γ_z is contractible we may assume that $x \neq z$ and $y \neq z$. If $z[-1] \leq x, y \leq z$, then

$$\begin{aligned} C_{E^{\mathrm{op}}}(q)_{x,y} &= C_{U_{\mathcal{X}}}(q)_{x,y} + q^{r-1}[r-d(z,x)-d(z,y)]_q \\ &\geq C_{(E')^{\mathrm{op}}}(q)_{x,y} + q^{r-1}[r-d(z,x)-d(z,y)]_q \\ &= C_{U_{\mathcal{X}'}}(q)_{x,y} \\ &\geq C_{E^{\mathrm{op}}}(q)_{x,y}, \end{aligned}$$

where $\geq$ means an inequality holds for each pair of corresponding coefficients. We deduce that the inequalities are actually equalities.

Remark 83. By Koszul duality, the derived categories of $U_{\mathcal{X}}$-mod, $V_{\mathcal{X}}$-mod are equivalent. Therefore, there also exists an equivalence of triangulated categories

$$D^b(V_{\mathcal{X}'}\text{-mod}) \xrightarrow{\sim} D^b(V_{\mathcal{X}}\text{-mod}).$$

Remark 84. The simple $V_{\mathcal{X}}$-module $L(z)$ has extension algebra $e_z U_{\mathcal{X}} e_z \cong k[\zeta_z]/\zeta_z^r \cong H^*(\mathbb{P}^{r-1})$ in the category of ungraded $V_{\mathcal{X}}$-modules. Objects whose extension algebras are isomorphic to $H^*(\mathbb{P}^n)$ are called $\mathbb{P}^n$-objects by D. Huybrechts and R. Thomas [13], and give rise to self-equivalences whenever they appear in the derived category of a smooth projective variety.

In our setting, self-equivalences of $D^b(U_{\mathcal{X}}\text{-mod})$ can be obtained by composing equivalences,

$$F \circ F' : D^b(U_{\mathcal{X}}\text{-mod}) \xrightarrow{\sim} D^b(U_{\mathcal{X}'}\text{-mod}) \xrightarrow{\sim} D^b(U_{\mathcal{X}}\text{-mod}).$$

8.4. **Another formula for the entries of the graded Cartan matrix of $U_{\mathcal{X}}$.** We can now derive a different formula for the entries of $C_{U_{\mathcal{X}}}(q)$, one which isn't tied to a choice of quasi-hereditary structure on $U_{\mathcal{X}}$.

For any $x \in \mathcal{X}$ we define

$$I_{\mathcal{X}}(x) = \{z \in \mathcal{X} \mid x \leq z \leq x[1]\}.$$

Lemma 85. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be Cubist subsets containing x, and suppose that $I_{\mathcal{X}_1}(x) = I_{\mathcal{X}_2}(x)$. Then for all $y \in \mathbb{Z}^r$, we have $C_{U_{\mathcal{X}_1}}(q)_{xy} = C_{U_{\mathcal{X}_2}}(q)_{xy}$.*

Proof. Let $\mathcal{X} \subset \mathbb{Z}^r$ be a Cubist subset. Using the fact that $U_{\mathcal{X}}$ is quasihereditary (Corollary 55) and the formula for its decomposition matrix given in Lemma 56, we have

$$C_{U_{\mathcal{X}}}(q)_{xy} = \sum_{\substack{z \in \mathcal{X} \\ x,y \in \lambda z}} q^{d(x,z)+d(y,z)} = \sum_{\substack{F \in \mathcal{F}_{\mathcal{X}} \\ x,y \in F}} q^{d(x,\lambda^{-1}(F))+d(y,\lambda^{-1}(F))}.$$

Hence it suffices to show that whether or not a facet $F \in \mathcal{F}$ containing x is contained in $\mathcal{X}$ depends only on $I_{\mathcal{X}}(x)$. For some $S \subseteq \{1, \dots, r\}$ and $i \notin S$, the facet F consists of all $x' \in \mathbb{Z}^r$ such that $x' \leq x + \sum_{j \in S} a_j \epsilon_j$ and $x' \geq x - \sum_{j \notin S \cup \{i\}} a_j \epsilon_j$. Hence F is contained in $\mathcal{X}$ if and only if $x + \sum_{j \in S} \epsilon_j \in \mathcal{X}$ and $x + \sum_{j \in S \cup \{i\}} \epsilon_j \notin \mathcal{X}$. □

Proposition 86. *Let $\mathcal{X}$ be a Cubist subset. Then for all $x, y \in \mathcal{X}$, we have*

$$C_{U_{\mathcal{X}}}(q)_{xy} = \sum_{z \in I_{\mathcal{X}}(x) \cap I_{\mathcal{X}}(y)} q^{r-1}[r - d(z,x) - d(z,y)]_q.$$

Proof. We induct on $|I_{\mathcal{X}}(x)|$. If $I_{\mathcal{X}}(x) = \{x\}$, then the sum on the right hand side of the desired equality contains a single term $q^{r-1}[r - d(x,y)]_q$ if $y \leq x \leq y[1]$ and is zero otherwise. This is in agreement with part (3) of Lemma 78.

Now suppose $|I_{\mathcal{X}}(x)| > 1$. By Lemma 85 we may assume that $\mathcal{X} = \mathcal{X}^- \setminus \mathcal{X}^-[-1]$, where $\mathcal{X}^-$ is the ideal in $\mathbb{Z}^r$ generated by $I_{\mathcal{X}}(x)$. Choose an element $v \in I_{\mathcal{X}}(x)$ maximal with respect to $\leq$. Then v is also a maximal element of $\mathcal{X}$ with respect to $\leq$. By Lemma 76 the subset $\mathcal{X}'$ of $\mathbb{Z}^r$ obtained from $\mathcal{X}$

by replacing v by $v[-1]$ is Cubist. We have $I_{\mathcal{X}'}(x) = I_{\mathcal{X}}(z) \setminus \{v\}$, so by induction the stated formula holds for $C_{U_{\mathcal{X}'}}(q)_{xy}$. Hence for all $y \in \mathcal{X}$ we have

$$\begin{aligned} C_{U_{\mathcal{X}}}(q)_{xy} &= \begin{cases} C_{U_{\mathcal{X}'}}(q)_{xy} + q^{r-1}[r - d(v,x) - d(v,y)]_q & \text{if } v \in I_{\mathcal{X}}(x), \\ C_{U_{\mathcal{X}'}}(q)_{xy} & \text{otherwise} \end{cases} \\ &= \sum_{z \in I_{\mathcal{X}}(x) \cap I_{\mathcal{X}}(y)} q^{r-1}[r - d(z,x) - d(z,y)]_q, \end{aligned}$$

where the first equality is by Lemma 82, and the second by induction. □

9. Formulae

We assemble nine elegant formulae, which combine to give purely combinatorial relations. It seems difficult to imagine how such expressions could have been conceived, without the Cubist algebras.

Note that the formulae involving decomposition matrices hold for any of the $r!$ possible highest weight structures, but depend on the given highest weight structure. The formulae which do not involve decomposition matrices are independent of highest weight structure.

Theorem 87. *Combinatorial formulae for decomposition matrices:*

$$D_{U_{\mathcal{X}}}(q)_{xy} = \sum_{z \in \lambda x} \delta_{zy} q^{d(z,x)},$$

$$D_{V_{\mathcal{X}}}(q)_{xy} = \sum_{z \in \mu x} \delta_{zy} q^{d(z,x)}.$$

Combinatorial formulae for Cartan matrices:

$$C_{U_{\mathcal{X}}}(q)_{xy} = \sum_{z \in I_{\mathcal{X}}(x) \cap I_{\mathcal{X}}(y)} q^{r-1}[r - d(z,x) - d(z,y)]_q.$$

$$C_{V_{\mathcal{X}}}(q)_{xy} = (1 - q^2)^{1-r} q^{d(x,y)},$$

Brauer formulae for Cartan matrices:

$$C_{U_{\mathcal{X}}}(q) = D_{U_{\mathcal{X}}}(q)^T D_{U_{\mathcal{X}}}(q),$$

$$C_{V_{\mathcal{X}}}(q) = D_{V_{\mathcal{X}}}(q)^T D_{V_{\mathcal{X}}}(q),$$

Transpose formulae:

$$C_{U_{\mathcal{X}}}(q) = C_{U_{\mathcal{X}}}(q)^T.$$

$$C_{V_{\mathcal{X}}}(q) = C_{V_{\mathcal{X}}}(q)^T.$$

Inverse formulae:

$$D_{U_{\mathcal{X}}}(q)^T . D_{V_{\mathcal{X}}}(-q) = 1.$$

$$C_{U_{\mathcal{X}}}(q) . C_{V_{\mathcal{X}}}(-q) = 1.$$

Symmetry formula:

$$C_{U_{\mathcal{X}}}(q^{-1}) = q^{2-2r} C_{U_{\mathcal{X}}}(q).$$

Proof. The combinatorial formulae for the decomposition matrices were proved during our study of highest weight categories (corollary 38, lemma 56). The combinatorial formula for $C_{U_{\mathcal{X}}}(q)$ was proved in the last section (proposition 86). The combinatorial formula for $C_{V_{\mathcal{X}}}(q)$ is recorded as remark 27.

The Brauer formulae for Cartan matrices in terms are abstract consequences of $U_{\mathcal{X}}$-mod, $V_{\mathcal{X}}$-mod being highest weight categories with duality [8, Theorem 3.1.11]. The transpose formulae follow immediately.

The inverse formula relating the decomposition matrices of $U_{\mathcal{X}}, V_{\mathcal{X}}$ was proven as lemma 35. The inverse formula relating the Cartan matrices of $U_{\mathcal{X}}, V_{\mathcal{X}}$ is an abstract consequence of $U_{\mathcal{X}}, V_{\mathcal{X}}$ being Koszul dual ([2], Theorem 2.11.1).

The symmetry formula holds because $U_{\mathcal{X}}$ is symmetric. □

10. Interpretation

The algebras $U_{\mathcal{X}}$ look like blocks of finite group algebras. Let us detail this metaphor.

Consider the diagram,

$$\begin{array}{ccc} \text{Blocks of finite groups} & \longrightarrow & \text{Abelian Categories} \\ & \searrow \quad \swarrow & \\ & \text{Triangulated categories .} & \end{array}$$

Here the horizontal arrow describes a functor Φ, which takes B to its module category B-mod. The southwest pointing functor Ψ carries an abelian category to its derived category. The southeast pointing functor Υ takes a block to its derived category. We have the following vague conjectures:

Conjecture 88. *1. The image of Φ is small.*
2. The image of Υ is very small.

We should be more precise. Let k have characteristic p. Let P be a p-group, and B_P be the set of blocks with defect group P. Let b be a block of some group in which P is normal, and let B_b be the set of blocks whose Brauer correspondent is Morita equivalent to b.

Conjecture 89. *1. (P. Donovan) For any P, $|im(\Phi_{B_P})| < \infty$.*
2. (M. Broué) For abelian P, $|im(\Upsilon_{B_b})| = 1$.

Specialising to symmetric groups, both parts of conjecture 89 are theorems. Let $B_{\Sigma,w}$ be the set of blocks of symmetric groups of weight w.

Theorem 90. *1. (J. Scopes) $|im(\Phi_{B_{\Sigma,w}})| < \infty$.*
2. (J. Chuang, R. Rouquier) $|im(\Upsilon_{B_{\Sigma,w}})| = 1$.

Various investigations into blocks of symmetric groups suggest the following parallel to theorem 90:

Conjecture 91. *Let $w < p$.*

$$im(\Phi_{B_{\Sigma,w}}) \subset \left\{ \begin{array}{c} e\mathcal{A}e\text{-mod}, \ \ \mathcal{A} \textit{ a standard Koszul, symmetric algebra,} \\ \textit{graded in degrees } 0,1,\dots,2w, \\ e \textit{ an idempotent in } \mathcal{A} \end{array} \right\}.$$

The relation between the algebras $U_{\mathcal{X}}$ and blocks of symmetric groups should now be clear. We have proved the $U_{\mathcal{X}}$'s possess all of the strong properties of the algebras $\mathcal{A}$ of the above conjecture, as well as various refinements of these properties. We have also revealed a multitude of derived equivalences between $U_{\mathcal{X}}$'s, as one expects to find between blocks of finite groups.

In fact, the similarity between blocks of symmetric groups and Cubist algebras appears to be more than merely formal. In the following section, we make a precise connection between the algebras $U_{\mathcal{X}}$ in case $r = 3$, and symmetric group blocks of defect 2.

It would be interesting if there were generalisations of conjecture 91 to other families of finite groups. The stated conjecture already appears to be quite deep.

Remark 92. Let G be a finite group. Let $\underline{\chi}$ be the submatrix of the ordinary character table of G, whose columns are indexed by p-regular elements. The matrix $\underline{\mathcal{B}}$ of p-Brauer characters of G, is related to $\underline{\chi}$ by the formula,

$$\underline{\chi} = D.\underline{\mathcal{B}},$$

where D is the p-decomposition matrix of G. Therefore, to compute $\underline{\mathcal{B}}$ from $\underline{\chi}$, one uses the formula $\underline{\mathcal{B}} = D^{-1}\underline{\chi}$, where D^{-1} is a left inverse of D.

It is always easier to find the ordinary character table of G than the table of Brauer characters. Therefore, from a computational point of view, D^{-1} is more important than D itself.

For the algebra $U_{\mathcal{X}}$, we know exactly what the inverse of the decomposition matrix is: it is the q-decomposition matrix of $V_{\mathcal{X}}$, evaluated at $q = -1$.

For the algebras $\mathcal{A}$ which we expect to control blocks of symmetric groups of abelian defect, the same situation ought to arise. The Koszul dual of such an algebra will have a q-decomposition matrix. It is the evaluation of this matrix at $q = -1$ which should allow for the direct computation of the Brauer character table $\underline{\mathcal{B}}$ of a related block from the ordinary character table $\underline{\chi}$ of the relevant symmetric group.

11. Symmetric group blocks and rhombal algebras

11.1. **Overview.** Here we establish a direct connection between rhombal algebras and some blocks of symmetric groups, making more precise and complete the observations of Michael Peach [22, §4]. Let B be a weight 2 block of a symmetric group in characteristic $p \neq 2$. Then B has $\frac{1}{2}(p-1)(p+2)$ simple modules. In this section we will prove the following result.

Theorem 93. *There exists a Cubist subset $\mathcal{X} \subset \mathbb{Z}^3$, an idempotent $e \in U_{\mathcal{X}}$ and an idempotent $f \in B$ such that $eU_{\mathcal{X}}e$ and fBf are isomorphic as algebras, each having $\frac{1}{2}(p-1)p$ simple modules.*

Actually we shall obtain a more precise result, in which $\mathcal{X}$ is described explicitly in terms of the combinatorics associated to B. The strategy of the proof is to first construct an isomorphism directly for a special class of blocks, the Rouquier blocks, which are known to have a description in terms of wreath products [4]. Then the result is extended to all blocks using the derived equivalences between Cubist algebras in §8, together with known equivalences between blocks of symmetric groups: the Morita equivalences of Scopes [26] and the derived equivalences of Rickard [25].

11.2. **Blocks of symmetric groups.** We begin by sketching the combinatorics of the block theory of the symmetric groups, referring the reader to the standard references [14] and [20] for more details. Let $\mathfrak{S}_n$ be the symmetric group of degree n and let k be a field of characteristic p. The simple modules D^λ of $k\mathfrak{S}_n$ are parametrized by p-regular partitions λ of n.

We describe a method due to Gordon James [15] for representing partitions which is useful in this context. We consider an abacus with p vertical half-infinite runners, with positions labelled $0, 1, \ldots$ from left to right and then top to bottom. Thus the positions on the i-th runner are labelled $i, i+p, i+2p, \ldots$. Given any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, an abacus representation of λ is obtained by placing N beads in positions $\lambda_1 + N - 1,\ \lambda_2 + N - 2,\ \ldots,\ \lambda_N$, where N is any integer at least as big as the number of parts of λ. By moving all the beads as far up their runners as possible one obtains an abacus representation of a partition of $n - wp$ for some $w \geq 0$. This partition is the *core* of λ and w is the *weight* of λ. The simple modules D^λ and D^μ belong to the same block of $k\mathfrak{S}_n$ if and only if λ and μ have the same core (and therefore the same weight). This statement, known as Nakayama's Conjecture, allows us to assign a core κ and weight w to each block B of $k\mathfrak{S}_n$. Any two blocks of the same weight have the same number of simple modules. We denote by Λ_B the set of all partitions (of n) with core κ and weight w. Then the simple modules D^λ of B are indexed by the p-regular partitions in Λ_B.

We choose mutually orthogonal idempotents $f_\lambda \in B$ such that Bf_λ is a projective cover of D^λ. Then $(\sum f_\lambda)B(\sum f_\lambda)$ is a basic algebra Morita equivalent to B.

11.3. **Weight 2 blocks.** We now assume that $p > 2$ and restrict our attention to blocks of weight 2. This class of blocks has been well studied. Peach and our work on rhombal algebras has been partly inspired by the general results of Scopes [27] and Tan [29], the determination of decomposition numbers by Richards [23] and the calculation of quivers and relations by Erdmann and Martin [11] and by Nebe [21].

We shall describe a natural parametrization of the simple modules in any block of weight 2 by the set

$$\mathcal{S} = \{(u, v) \in \mathbb{Z}^2 \mid 0 \leq u \leq v \leq p-1,\ (u, v) \neq (0, 0)\}.$$

The simple modules corresponding to the subset

$$\mathcal{P} = \{(u, v) \in \mathbb{Z}^2 \mid 0 \leq u < v \leq p-1\}$$

will survive in a truncation of the block which will be shown to be isomorphic to a truncation of a rhombal algebra.

Let B be a block of $k\mathfrak{S}_n$ of weight 2. Consider an abacus representation of the associated p-core partition κ. Let $q_0, \ldots, q_{p-1}$ be the first unoccupied positions in each of the p runners, relabelled so that $q_0 < \ldots < q_{p-1}$, and define the *pyramid* of B to be

$$\mathcal{P}_B = \{(u, v) \in \mathcal{P} \mid q_v - q_u < p\}.$$

This is a corruption of a notion introduced by Matthew Richards [23]; his pyramid, defined for any weight, contains the same information in weight 2 as ours. Richards proves that if $(u, v) \in \mathcal{P}_B$, then $(u, w), (w, v) \in \mathcal{P}_B$ whenever $u < w < v$, and that any subset of $\mathcal{P}$ with this property is equal to $\mathcal{P}_B$

for some block B of weight 2. We also define

$$\begin{aligned}\mathcal{S}_B &= \{(u,v) \in \mathcal{S} \mid q_v - q_u < p\}. \\ &= \mathcal{P}_B \cup \{(u,u) \in \mathbb{Z}^2 \mid 1 \leq u \leq p-1\}.\end{aligned}$$

Example 94. Let k be a field of characteristic 7, and let B be the block of $k\mathfrak{S}_{42}$ of weight 2 corresponding to the 7-core partition $\kappa = (12,6,6,1,1,1,1)$. Choosing $N = 7$ we get place beads on the abacus in positions $1,2,3,4,10,12,18$. The first unoccupied positions on each of the runners are, from left to right, $0,8,9,17,25,5,6$. Hence

$$(q_0,q_1,q_2,q_3,q_4,q_5,q_6) = (0,5,6,8,9,17,25),$$

$$\mathcal{P}_B = \{(0,1),(0,2),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\},$$

$$\mathcal{S}_B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(0,1),(0,2),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}.$$

We shall use a variation of a shorthand, due to Scopes [26, 27], for labelling the partitions in Λ_B:

- $\langle u,v\rangle$ for the partition (whose abacus display is) obtained (from the abacus display of κ) by moving the beads at positions $q_u - p$ and $q_v - p$ down one position, i.e. to positions q_u and q_v. Here $u \neq v$.
- $\langle u\rangle$ for the partition obtained by moving the bead at $q_u - p$ to $q_u + p$.
- $\langle u,u\rangle$ for the partition obtained by moving the bead at $q_u - 2p$ to $q_u - p$, and the beas at $q_u - p$ to q_u.

The same set of shorthands labels is used for all blocks of weight 2. However the subset of labels that corresponds to p-regular partitions in Λ_B depends on B.

Scopes [26] considers pairs of blocks related to each other by 'swapping adjacent runners'. Even though her results are valid for blocks of arbitrary weight, here we just describe the weight 2 case. Suppose that there exists an abacus display of κ and $0 \leq s < t \leq p-1$ such that $q_t - q_s = mp + 1$, where $m > 0$. Then by moving the beads from positions $q_t - p,\ q_t - 2p, \ldots,\ q_t - mp$, to the unoccupied positions $q_t - p - 1,\ q_t - 2p - 1, \ldots,\ q_t - mp - 1$ we obtain the abacus display of a p-core partition $\bar{\kappa}$. Then B and the block $\bar{B}$ of weight 2 with p-core $\bar{\kappa}$ are said to form a $[2:m]$ pair. It is easy to describe the relationship between the pyramids of B and $\bar{B}$: if $m \geq 2$ then $\mathcal{P}_{\bar{B}} = \mathcal{P}_B$, and if $m = 1$ then $\mathcal{P}_{\bar{B}}$ is the disjoint union of $\mathcal{P}_B$ and $\{(s,t)\}$. For an arbitrary block B of weight 2, there exists a sequence $B_0, \ldots, B_l$ of blocks of weight 2 such that $\mathcal{P}_{B_0} = \emptyset$, $B_l = B$ and for $i = 1, \ldots, l$, the blocks B_{i-1} and B_i form a $[2:m]$ pair for some m.

By Scopes [27], there exists a bijection

$$\Phi = \Phi_{B,\bar{B}} : \Lambda_B \xrightarrow{\sim} \Lambda_{\bar{B}}$$

such that

- $\Phi(\lambda)$ is p-regular if and only if λ is p-regular,
- $\Phi(\lambda)$ and λ have the same shorthand notation, except in the following cases when $m = 1$:

$$\begin{aligned}\Phi(\langle t,t\rangle) &= \langle s\rangle, \\ \Phi(\langle s,t\rangle) &= \langle t,t\rangle, \\ \Phi(\langle s\rangle) &= \langle s,t\rangle.\end{aligned}$$

In case $m = 1$, we are extending Scopes' Definition 3.4 in [27] by taking, in her notation, $\Phi(\alpha) = \bar{\alpha}$, $\Phi(\beta) = \bar{\gamma}$, and $\Phi(\gamma) = \bar{\beta}$.

We now produce the promised parametrization of simple modules in blocks of weight 2.

Proposition 95. (1) *Let B be a block of weight 2. Then the map*

$$\lambda_B : \mathcal{S} \to \Lambda_B$$

defined by

$$\lambda_B(u,v) = \begin{cases} \langle u+1, v\rangle & \text{if } (u,v) \notin \mathcal{S}_B \text{ and } (u+1,v) \notin \mathcal{S}_B \\ \langle v, v\rangle & \text{if } (u,v) \notin \mathcal{S}_B \text{ and } (u+1,v) \in \mathcal{S}_B \\ \langle u, v+1\rangle & \text{if } (u,v) \in \mathcal{S}_B \text{ and } (u,v+1) \in \mathcal{S}_B \\ \langle u\rangle & \text{if } (u,v) \in \mathcal{S}_B \text{ and } (u,v+1) \notin \mathcal{S}_B \end{cases}$$

is an bijection of $\mathcal{S}$ onto the set of p-regular partitions in Λ_B.

(2) *If B and $\bar{B}$ form a $[2:m]$ pair, then*

$$\Phi_{B,\bar{B}} \circ \lambda_B = \lambda_{\bar{B}}.$$

Proof. First suppose that $\mathcal{P}_B = \emptyset$. Then $\lambda_B(u,u) = \langle u\rangle$ and $\lambda_B(u,v) = \langle u+1, v\rangle$ if $u < v$. Thus λ_B is a bijection onto the set of partitions in Λ_B whose shorthand labels do not involve 0, which are precisely the p-regular ones. This proves statement (1) in this special case.

For an arbitrary block B, there is a sequence of blocks starting at one with empty pyramid and ending at B such that each successive pair of blocks forms a $[2:m]$ pair for some m. Hence in order to prove both statements in general it suffices to show that statement (2) holds for a fixed $[2:m]$-pair of blocks B and $\bar{B}$ under the assumption that statement (1) holds for B. This is clearly true if $m \geq 2$, because then $\mathcal{S}_B = \mathcal{S}_{\bar{B}}$ and Φ preserves shorthand labels. So let us suppose that B and $\bar{B}$ form a $[2:1]$ pair. We have $\mathcal{P}_{\bar{B}} = \mathcal{P}_B \cup \{(s,t)\}$ for some $0 \leq s < t \leq p-1$. Note that $(s+1,t), (s,t-1) \in \mathcal{S}_B$ and $(s-1,t), (s,t+1) \notin \mathcal{S}_B$. Hence

$$\begin{aligned} \Phi(\lambda_B(s,t)) &= \Phi(\langle t,t\rangle) = \langle s\rangle = \lambda_{\bar{B}}(s,t), \\ \Phi(\lambda_B(s-1,t)) &= \Phi(\langle s,t\rangle) = \langle t,t\rangle = \lambda_{\bar{B}}(s-1,t), \\ \Phi(\lambda_B(s,t-1)) &= \Phi(\langle s\rangle) = \langle s,t\rangle = \lambda_{\bar{B}}(s,t-1). \end{aligned}$$

Remembering our assumption that statement (1) holds for B, we also see that for any $(u,v) \in \mathcal{S} \setminus \{(s,t), (s-1,t), (s,t-1)\}$, the shorthand labels for $\Phi(\lambda_B(u,v))$ and $\lambda_B(u,v)$ are the same and therefore that $\Phi(\lambda_B(u,v)) = \lambda_{\bar{B}}(u,v)$. ☐

Example 96. We take $p = 7$ and $\kappa = (12,6,6,1,1,1,1)$ as in Example 94. The graph in Figure 6 records the bijection of Proposition 95. Its vertices are in bijection with the p-regular partitions in Λ_B, each of which has a shorthand label as well as a label by an element of $\mathcal{S}$, via λ_B. The shorthand labels are placed to the right of the vertices, and the $\mathcal{S}$-labels to the left, in boldface. The subset $\mathcal{P}$ is indicated by black vertices and the pyramid $\mathcal{P}_B$ by square black vertices. Two vertices are connected by an edge if an only if there exists a nonsplit extension of one of the corresponding simple B-modules by the other; in fact, by replacing each edge by a pair of directed edges in opposite directions one obtains the 'extension quiver' of B.

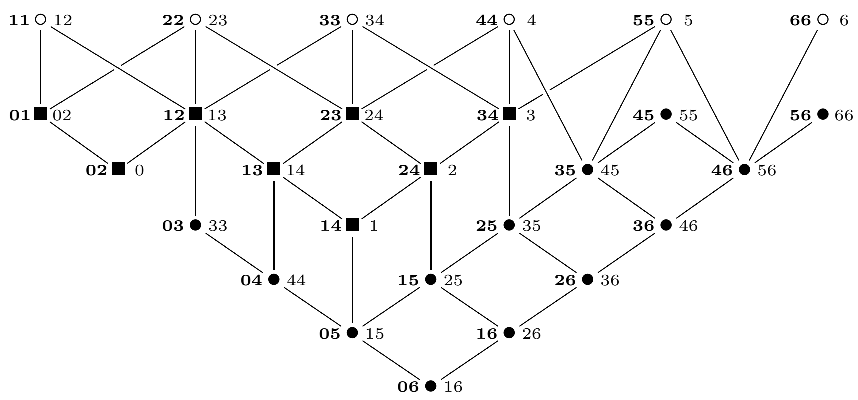

FIGURE 6. Extension quiver of B when $p = 7$ and $\kappa = (12, 6, 6, 1, 1, 1, 1)$

11.4. **Gradings.** We expect that all blocks of weight $w < p$ should have gradings compatible with radical filtrations. This is easy to verify when $w < 2$, and has been proved by Peach for $w = 2$.

Theorem 97 (Peach [22])**.** *Let B be a block of weight 2. Then there exists a grading $B = \oplus_{i=0}^{4} B_i$ such that* $\text{rad}^j(B) = \oplus_{i=j}^{4} B_i$.

We may assume that the idempotents f_λ introduced in §11.2 are in B_0.

11.5. **Morita and derived equivalences.** There is a strong relationship between the module categories of blocks in a Scopes pair. Suppose that B and $\bar{B}$ form a $[2 : m]$ pair of blocks. Scopes [26] proves that, if $m \geq 2$, there is an equivalence

$$F' : B\text{-mod} \xrightarrow{\sim} \bar{B}\text{-mod}$$

such that

$$F'(Bf_\lambda) \cong \bar{B} f_{\Phi(\lambda)}$$

for all p-regular $\lambda \in \Lambda_B$. Her results are true in greater generality, for blocks of arbitrary weight. Rickard [25] (see also [5]) built on the ideas of Scopes, proving the existence of some derived equivalences between blocks. A special case of Rickard's result (see [3] and [22, §5]) states that if B and $\bar{B}$ form a $[2 : 1]$ pair, then there exists an equivalence

$$F' : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\bar{B}\text{-mod}),$$

such that, in the notation of the analysis of $[2 : 1]$ pairs in §11.3,

$$F'(Bf_{\langle t,t\rangle}\langle 3\rangle[1]) \cong \bar{B} f_{\Phi(\langle t,t\rangle)}$$

and

$$F'(\text{cone}(P'_\lambda \xrightarrow{\zeta'_\lambda} Bf_\lambda)) \cong Bf_{\Phi(\lambda)}$$

for all p-regular $\lambda \in \Lambda_B$ apart from $\langle t, t\rangle$, where ζ'_λ is a projective cover of the smallest submodule M of Bf_λ such that $\text{Hom}_B(Bf_{\langle t,t\rangle}\langle n\rangle, Bf_\lambda/M) = 0$ for all $n \in \mathbb{Z}$.

We will only make use of truncated versions of these equivalences. Let

$$f = \sum_{(u,v)\in\mathcal{P}} f_{\lambda_B(u,v)} \in B \quad \text{and} \quad \bar{f} = \sum_{(u,v)\in\mathcal{P}} f_{\lambda_{\bar{B}}(u,v)} \in \bar{B}.$$

If $m \geq 2$, then by Proposition 95 the Morita equivalence F' above induces an equivalence

$$F = F_{B,\bar{B}} : fBf\text{-mod} \xrightarrow{\sim} \bar{f}\bar{B}\bar{f}\text{-mod}$$

such that

$$F(fBf_{\lambda_B(u,v)}) \cong \bar{f}\bar{B}f_{\lambda_{\bar{B}(u,v)}}$$

for all $(u,v) \in \mathcal{P}$. If $m = 1$, then $\langle t,t\rangle = \lambda_B(s,t)$ and, by Proposition 95, we have $\lambda \in \lambda_B(\mathcal{P})$ if and only if $\Phi(\lambda) \in \lambda_{\bar{B}}(\mathcal{P})$. Hence the equivalence $F' : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\bar{B}\text{-mod})$ induces an equivalence

$$F = F_{B,\bar{B}} : D^b(fBf\text{-mod}) \xrightarrow{\sim} D^b(\bar{f}\bar{B}\bar{f}\text{-mod})$$

such that

$$F(fBf_{\lambda_B(s,t)}\langle 3\rangle[1]) \cong \bar{f}\bar{B}f_{\lambda_{\bar{B}(s,t)}}$$

and

$$F(\text{cone}(P_{(u,v)} \xrightarrow{\zeta_{(u,v)}} fBf_{\lambda_B(u,v)})) \cong \bar{f}\bar{B}f_{\lambda_{\bar{B}}(u,v)}$$

for $(u,v) \in \mathcal{P} \setminus \{(s,t)\}$, where $\zeta_{(u,v)}$ is a projective cover of the smallest submodule M of $fBf_{\lambda_B(u,v)}$ such that $\mathrm{Hom}_{fBf}(fBf_{\lambda_B(s,t)}\langle n\rangle, fBf_{\lambda_B(u,v)}/M) = 0$ for all $n \in \mathbb{Z}$.

11.6. **Main result.** We are now ready to state and prove our result linking Cubist algebras and blocks of symmetric groups. Let B be a block of weight 2. Define

$$x_B : \big\{(u,v) \in \mathbb{Z}^2 \mid u < v\big\} \longrightarrow \mathbb{Z}^3$$

by

$$x_B(u,v) = \begin{cases} (-u-1, v, 0) & \text{if } (u,v) \in \mathbb{Z}^2 \setminus \mathcal{P}_B, \\ (-u, 1+v, 1) & \text{if } (u,v) \in \mathcal{P}_B. \end{cases}$$

Then

$$\mathcal{X}_B = \mathrm{Im}(x_B) \cup \{(i,j,1) \in \mathbb{Z}^3 \mid i+j \leq 1\}$$

is a Cubist subset of $\mathbb{Z}^3$. Indeed,

$$\mathcal{X}_B^- = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{\leq 0} \,\cup\, \{(i,j,1) \in \mathbb{Z}^3 \mid i+j \leq 1 \text{ or } (-i, j-1) \in \mathcal{P}\}$$

is an ideal in $\mathbb{Z}^3$ such that $\mathcal{X}_B = \mathcal{X}_B^- \setminus \mathcal{X}_B^-[-1]$.

Example 98. As in earlier examples, we take $p = 7$ and $\kappa = (12,6,6,1,1,1,1)$. Figure 7 shows part of $\mathcal{X}_B$ realised in the plane as a rhombus tiling. The image of $\mathcal{P}$ under x_B is indicated by black vertices and that of $\mathcal{P}_B$ by square black vertices. Compare with Figure 6.

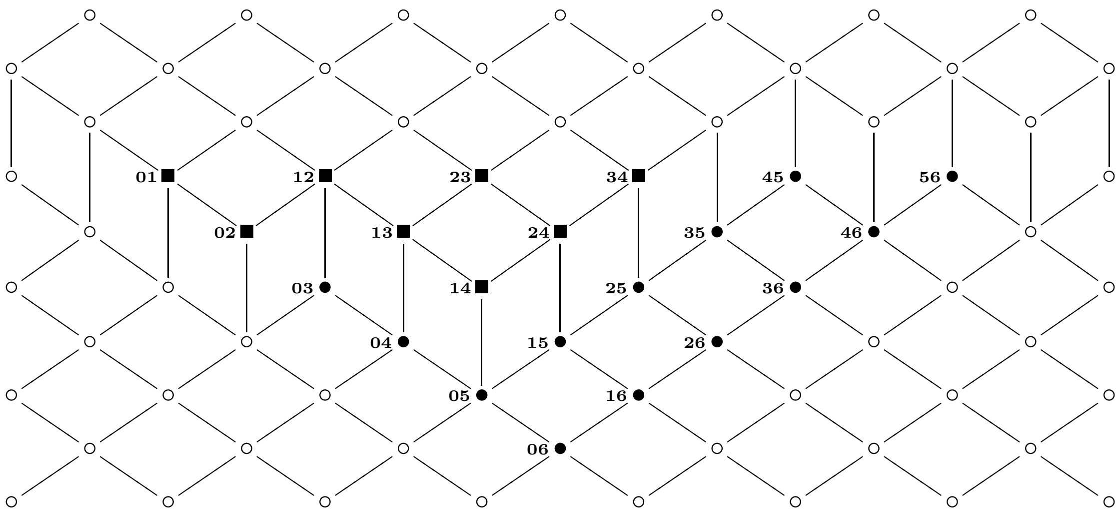

FIGURE 7. Part of the Cubist subset $\mathcal{X}_B$ when $p = 7$ and $\kappa = (12, 6, 6, 1, 1, 1, 1)$

Let $U_{\mathcal{X}_B}$ be the Cubist algebra corresponding to $\mathcal{X}_B$ and put $e = \sum_{x \in x_B(\mathcal{P})} e_x \in U_{\mathcal{X}_B}$.

Theorem 99. *We have an equivalence*

$$F_B : eU_{\mathcal{X}_B}e\text{-mod} \xrightarrow{\sim} fBf\text{-mod}$$

such that for all $(u, v) \in \mathcal{P}$,

$$F_B(eU_{\mathcal{X}_B}e_{x_B(u,v)}) \cong fBf_{\lambda_B(u,v)}.$$

Proof. The case $\mathcal{P}_B = \emptyset$ is handled in §11.7.

Now suppose that B and $\bar{B}$ form a $[2 : m]$ pair and assume that the theorem is known to hold for the block B. We need to prove that it holds for $\bar{B}$.

Suppose that $m \geq 2$. Then $\mathcal{P}_B = \mathcal{P}_{\bar{B}}$, and hence $x_B = x_{\bar{B}}$ and $\mathcal{X}_B = \mathcal{X}_{\bar{B}}$. So we simply define $F_{\bar{B}} = F_{B,\bar{B}} \circ F_B$.

Suppose that $m = 1$. We have $\mathcal{P}_{\bar{B}} = \mathcal{P}_B \cup \{(s, t)\}$, where $0 \leq s < t \leq p-1$. Thus $x_B(u, v) = x_{\bar{B}}(u, v)$ for $(u, v) \in \mathcal{P} \setminus \{s, t\}$, and $x_{\bar{B}}(s, t) = x_B(s, t)[1]$, and $\mathcal{X}_{\bar{B}} = (\mathcal{X}_B \setminus \{(x_B(s, t)\}) \cup \{x_B\,(s, t)\,[1]\}$. Let $z = x_B(s, t)$. By Theorem 77 and Lemma 78 we have an equivalence

$$G' : D^b\,(U_{\mathcal{X}_B}\text{-mod}) \xrightarrow{\sim} D^b\left(U_{\mathcal{X}_{\bar{B}}}\text{-mod}\right)$$

such that

$$G'(U_{\mathcal{X}_B}e_z\langle 3\rangle[1]) \cong U_{\mathcal{X}_{\bar{B}}}e_{z[1]}.$$

and

$$G'(\text{cone}(Q'_x \xrightarrow{\xi'_x} U_{\mathcal{X}_B}e_x)) \cong U_{\mathcal{X}_{\bar{B}}}e_x$$

for $x \in \mathcal{X}_B \setminus \{z\}$, where ξ'_x is a projective cover of the smallest submodule M of $U_{\mathcal{X}_B}e_x$ such that $\text{Hom}_{U_{\mathcal{X}_B}}(U_{\mathcal{X}_B}e_z\langle n\rangle, U_{\mathcal{X}_B}e_x/M) = 0$ for all $n \in \mathbb{Z}$. We know that Q'_x is either 0 or isomorphic to $U_{\mathcal{X}_B}e_z\langle n\rangle$ for some $n \in \mathbb{Z}$.

Since $(s,t) \in \mathcal{P}$, we have $ee_z \neq 0$, which implies the existence of an equivalence

$$G : D^b\left(eU_{\mathcal{X}_B}e\text{-mod}\right) \xrightarrow{\sim} D^b\left(\bar{e}U_{\mathcal{X}_{\bar{B}}}\bar{e}\text{-mod}\right)$$

such that

$$G(eU_{\mathcal{X}_B}e_z\langle 3\rangle[1]) \cong \bar{e}U_{\mathcal{X}_{\bar{B}}}e_{z[1]}.$$

and

$$G(\text{cone}(Q_x \xrightarrow{\xi_x} eU_{\mathcal{X}_B}e_x)) \cong \bar{e}U_{\mathcal{X}_{\bar{B}}}e_x$$

for $x \in x_B(\mathcal{P}) \setminus \{z\}$, where ξ_x is a projective cover of the smallest submodule M of $eU_{\mathcal{X}_B}e_x$ such that $\text{Hom}_{eU_{\mathcal{X}_B}}(eU_{\mathcal{X}_B}e_z\langle n\rangle, eU_{\mathcal{X}_B}e_x/M) = 0$ for all $n \in \mathbb{Z}$. Put

$$F_{\bar{B}} = F_{B,\bar{B}} \circ RF_B \circ H : D^b(\bar{e}U_{\mathcal{X}_{\bar{B}}}\bar{e}) \xrightarrow{\sim} D^b(\bar{f}\bar{B}\bar{f}),$$

where H is a quasi-inverse to G. Then we have

$$\begin{aligned} F_{\bar{B}}(\bar{e}U_{\mathcal{X}_{\bar{B}}}e_{x_{\bar{B}}(s,t)}) &\cong F_{B,\bar{B}}(RF_B(eU_{\mathcal{X}_B}e_{x_B(s,t)}[1]) \\ &\cong F_{B,\bar{B}}(fBf_{\lambda_B(s,t)}[1]) \\ &\cong \bar{f}\bar{B}\bar{f}_{\lambda_B(s,t)}, \end{aligned}$$

and for $(u,v) \in \mathcal{P} \setminus \{(s,t)\}$,

$$\begin{aligned} F_{\bar{B}}(\bar{e}U_{\mathcal{X}_{\bar{B}}}e_{x_{\bar{B}}(u,v)}) &\cong F_{B,\bar{B}}(RF_B(\text{cone}(Q_x \xrightarrow{\xi_x} U_{\mathcal{X}_B}e_{x_{B(u,v)}}))) \\ &\cong F_{B,\bar{B}}(\text{cone}(F_B(Q_x) \xrightarrow{F_B(\xi_x)} Be_{\lambda_B(u,v)})) \\ &\cong \bar{f}\bar{B}\bar{f}_{\lambda_B(u,v)}. \end{aligned}$$

It follows that $F_{\bar{B}}$ restricts to an equivalence $\bar{e}U_{\mathcal{X}_{\bar{B}}}\bar{e}\text{-mod} \xrightarrow{\sim} \bar{f}\bar{B}\bar{f}\text{-mod}$. □

11.7. **Rouquier blocks.** Now we assume that $\mathcal{P}_B = \emptyset$. In this case B is known as a *Rouquier block* of weight 2 and is particularly well understood. We shall prove that there is an isomorphism $eU_{\mathcal{X}_B}e \xrightarrow{\sim} fBf$ sending $e_{x_B(u,v)}$ to $f_{\lambda_B(u,v)}$ for all $(u,v) \in \mathcal{P}$, and thus complete the proof of Theorem 99.

We have

$$\mathcal{X} = \mathcal{X}_B = \mathcal{X}_0 \cup \mathcal{X}_1,$$

where

$$\mathcal{X}_0 = \text{Im}(x_B) = \{(i,j,0) \mid i+j \geq 0\}$$

and

$$\mathcal{X}_1 = \{(i,j,1) \mid i+j \leq 1\}.$$

We shall in fact prove that the algebra $\sum_{x,x' \in \mathcal{X}_0} e_x U_{\mathcal{X}} e_{x'}$ is isomorphic to a truncation of an infinite-dimensional wreath product. Then a further truncation will yield the desired isomorphism.

11.7.1. *Reformulation in terms of wreath products.* Let A be the path algebra of the infinite quiver

$$\cdots \underset{\delta_{-1}}{\overset{\gamma_{-2}}{\rightleftarrows}} \cdot_{g_{-1}} \underset{\delta_{0}}{\overset{\gamma_{-1}}{\rightleftarrows}} \cdot_{g_0} \underset{\delta_{1}}{\overset{\gamma_{0}}{\rightleftarrows}} \cdot_{g_1} \underset{\delta_{2}}{\overset{\gamma_{1}}{\rightleftarrows}} \cdots$$

modulo the relations $\gamma_i\gamma_{i+1} = 0,\ \delta_i\delta_{i-1} = 0,\ \gamma_i\delta_{i+1} + \delta_i\gamma_{i-1} = 0,$ for $i \in \mathbb{Z}$; this is a graded algebra with each arrow in degree 1. (We saw in Remark 29 that this algebra is isomorphic to $U_{\mathcal{Y}}$, for any Cubist subset $\mathcal{Y} \subset \mathbb{Z}^2$.)

We now form the wreath product $A \wr \mathfrak{S}_2 = A \otimes A \otimes k\mathfrak{S}_2$, graded with $k\mathfrak{S}_2$ in degree 0. Let σ be the nonidentity element of $\mathfrak{S}_2$. The following elements of $A \wr \mathfrak{S}_2$ are a complete set of orthogonal idempotents:

$$g_{ij} = g_i \otimes g_j \otimes 1, \quad (i, j \in \mathbb{Z}, i < j),$$
$$g_{ii}^{\pm} = g_i \otimes g_i \otimes (1 \pm \sigma)/2, \quad (i \in \mathbb{Z}).$$

Now observe that $(g_1 + \ldots + g_p)A(g_1 + \ldots + g_p)$ is a Brauer tree algebra of a line with $p-1$ simple modules, so is Morita equivalent to the principal block of $k\mathfrak{S}_p$. It follows that $g'(A \wr \mathfrak{S}_2)g'$ is Morita equivalent to the principal block of $k\mathfrak{S}_p \wr \mathfrak{S}_2$, where $g' = \sum_{1 \le i < j \le p-1} g_{ij} + \sum_{1 \le i \le p-1} g_{ii}^{+} + \sum_{1 \le i \le p-1} g_{ii}^{-}$.

On the other hand there is a Morita equivalence between the block B and the principal block of $k\mathfrak{S}_p \wr \mathfrak{S}_2$ (see [3] and [4]), where the correspondence between simple modules is known explicitly. Passing to truncated algebras, we obtain an isomorphism $fBf \xrightarrow{\sim} g(A \wr \mathfrak{S}_2)g$ that sends $f_{\langle i,j \rangle}$ to g_{ij} for $1 \le i \le j \le p-1$, where g_{ii} is defined to be g_{ii}^{+} if i is even and g_{ii}^{-} if i is odd, and $g = \sum_{1 \le i \le j \le p-1} g_{ij}$.

For $(u, v) \in \mathcal{P}$, we have $\lambda_B(u, v) = \langle u+1, v \rangle$ and $x_B(u, v) = (-u-1, v, 0)$. So to obtain the desired isomorphism between $eU_{\mathcal{X}}e$ and fBf, it suffices to get an isomorphism $eU_{\mathcal{X}}e \xrightarrow{\sim} g(A \wr \mathfrak{S}_2)g$ sending $e_{(-i,j,0)}$ to g_{ij} for $1 \le i \le j \le p-1$.

It will be more convenient to prove a stronger statement. Let

$$\tilde{e}U_{\mathcal{X}}\tilde{e} := \bigoplus_{x,x' \in \mathcal{X}_0} e_x U_{\mathcal{X}} e_{x'} \qquad \text{and} \qquad \tilde{g}(A \wr \mathfrak{S}_2)\tilde{g} := \bigoplus_{i,j,i',j' \in \mathbb{Z},\ i \le j,\ i' \le j'} g_{ij}(A \wr \mathfrak{S}_2)g_{i'j'}.$$

We will show that there is an isomorphism $\tilde{e}U_{\mathcal{X}}\tilde{e} \xrightarrow{\sim} \tilde{g}(A \wr \mathfrak{S}_2)\tilde{g}$ sending $e_{(-i,j,0)}$ to g_{ij} for $i, j \in \mathbb{Z}$, $i \le j$.

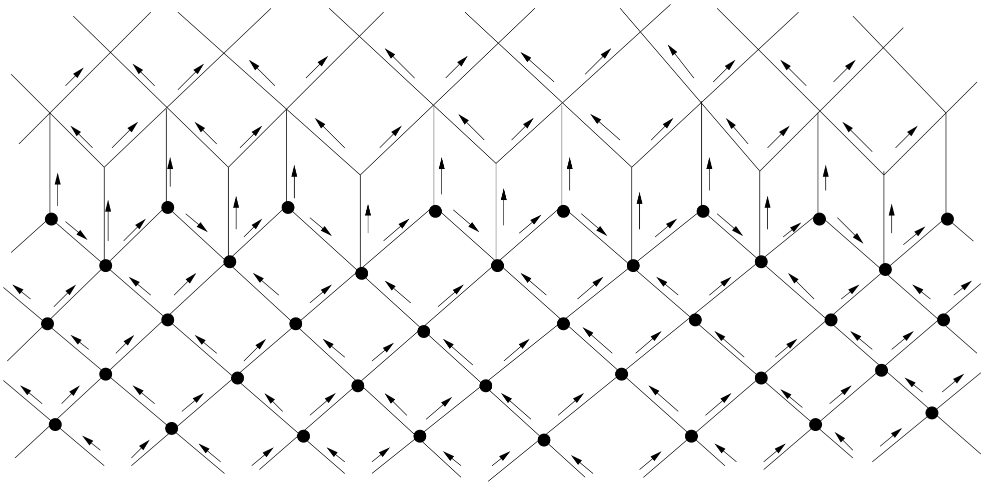

FIGURE 8. $\mathcal{X}_B$ in case $\mathcal{P}_B = \emptyset$.

11.7.2. *A presentation for $\tilde{e}U_{\mathcal{X}}\tilde{e}$.* We first observe that $\tilde{e}U_{\mathcal{X}}\tilde{e}$ is a Koszul algebra, since $\mathcal{X}$ is a Cubist subset and $\mathcal{X}_0$ is an ideal in $(\mathcal{X}, \preceq)$, as in Figure 8. In particular it is quadratic algebra.

So using the alternative presentation of U_3 given in Remark 18 we deduce a presentation of $\tilde{e}U_{\mathcal{X}}\tilde{e}$ by quiver and relations. The quiver has vertices

$$\{e_{i,j} \mid i, j \in \mathbb{Z},\ i + j \geq 0\},$$

and arrows

$$\{\alpha_{i,j,1}, \alpha_{i,j,2} \mid i + j \geq 0\},$$
$$\{\beta_{i,j,1}, \beta_{i,j,2} \mid i + j \geq 1\}.$$

The arrows $\alpha_{i,j,1}$ and $\alpha_{i,j,2}$ are directed from $e_{i,j}$ to $e_{i+1,j}$ and from $e_{i,j}$ to $e_{i,j+1}$, and the arrows $\beta_{i,j,1}$ and $\beta_{i,j,2}$ are directed from $e_{i,j}$ to $e_{i-1,j}$ and from $e_{i,j}$ to $e_{i,j-1}$. Then $\tilde{e}U_{\mathcal{X}}\tilde{e}$ is isomorphic to the path algebra of this quiver, modulo relations

$$(0)\qquad \begin{aligned}
\alpha_{i,j,1}\alpha_{i+1,j,1} &= 0 \quad (i+j \geq 0),\\
\alpha_{i,j,2}\alpha_{i,j+1,2} &= 0 \quad (i+j \geq 0),\\
\beta_{i,j,1}\beta_{i-1,j,1} &= 0 \quad (i+j \geq 2),\\
\beta_{i,j,2}\beta_{i,j-1,2} &= 0 \quad (i+j \geq 2);
\end{aligned}$$

$$(1)\qquad \begin{aligned}
\alpha_{i,j,1}\alpha_{i+1,j,2} &= \alpha_{i,j,2}\alpha_{i,j+1,1} \quad (i+j \geq 0),\\
\beta_{i,j,1}\beta_{i-1,j,2} &= \beta_{i,j,2}\beta_{i,j-1,1} \quad (i+j \geq 2),\\
\alpha_{i,j,1}\beta_{i+1,j,2} &= \beta_{i,j,2}\alpha_{i,j-1,1} \quad (i+j \geq 1),\\
\alpha_{i,j,2}\beta_{i,j+1,1} &= \beta_{i,j,1}\alpha_{i-1,j,2} \quad (i+j \geq 1),\\
\alpha_{-j,j,1}\beta_{-j+1,j,2} &= 0 \quad (j \in \mathbb{Z}),\\
\alpha_{-j,j,2}\beta_{-j,j+1,1} &= 0 \quad (j \in \mathbb{Z});
\end{aligned}$$

$$(2)\qquad \begin{aligned}
\alpha_{i,j,1}\beta_{i+1,j,1} &= \beta_{i,j,1}\alpha_{i-1,j,1} \quad (i+j \geq 2),\\
\alpha_{i,j,2}\beta_{i,j+1,2} &= \beta_{i,j,2}\alpha_{i,j-1,2} \quad (i+j \geq 2),\\
\alpha_{-j,j+1,1}\beta_{-j+1,j+1,1} - \beta_{-j,j+1,1}\alpha_{-j-1,j+1,1} &= \beta_{-j,j+1,2}\alpha_{-j,j,2} - \alpha_{-j,j+1,2}\beta_{-j,j+2,2} \quad (j \in \mathbb{Z}),\\
\alpha_{-j,j,1}\beta_{-j+1,j,1} &= \alpha_{-j,j,2}\beta_{-j,j+1,1} \quad (j \in \mathbb{Z}).
\end{aligned}$$

The three groups of relations come from (U0), (U1') and (U2'), respectively.

11.7.3. *Construction of the isomorphism.* We define a homomorphism

$$\tilde{e}U_{\mathcal{X}}\tilde{e} \to \tilde{g}(A \wr \mathfrak{S}_2)\tilde{g}$$

by

$$\begin{aligned}
e_{i,j} &\mapsto g_{-i,j},\\
\alpha_{i,j,1} &\mapsto \begin{cases} \delta_{-i} \otimes g_j \otimes 1 & \text{if } i+j \geq 1,\\ \delta_j \otimes g_j \otimes 1 + (-1)^j g_j \otimes \delta_j \otimes \sigma & \text{if } i+j = 0,\end{cases}\\
\alpha_{i,j,2} &\mapsto \begin{cases} g_{-i} \otimes \gamma_j \otimes 1 & \text{if } i+j \geq 1,\\ g_j \otimes \gamma_j \otimes 1 + (-1)^j \gamma_j \otimes g_j \otimes \sigma & \text{if } i+j = 0,\end{cases}\\
\beta_{i,j,1} &\mapsto \begin{cases} \gamma_{-i} \otimes g_j \otimes 1 & \text{if } i+j \geq 2,\\ \frac{1}{2}\left(\gamma_{j-1} \otimes g_j \otimes 1 + (-1)^j \gamma_{j-1} \otimes g_j \otimes \sigma\right) & \text{if } i+j = 1,\end{cases}\\
\beta_{i,j,2} &\mapsto \begin{cases} g_{-i} \otimes \delta_j \otimes 1 & \text{if } i+j \geq 2,\\ \frac{1}{2}\left(g_{j-1} \otimes \delta_j \otimes 1 + (-1)^{j-1} g_{j-1} \otimes \delta_j \otimes \sigma\right) & \text{if } i+j = 1.\end{cases}
\end{aligned}$$

This yields a homomorphism: that the images of the generators satisfy the relations stated in §11.7.2 is a straightforward calculation in $A \wr \mathfrak{S}_2$.

The graded Cartan matrix $C(q)$ for $\tilde{e}U_{\mathcal{X}}\tilde{e}$ is just a submatrix of that of $U_{\mathcal{X}}$, which can be calculated using Proposition 86. We get

$$C(q)_{(i,j),(i',j')} = \begin{cases} 1+q^2+q^4 & \text{if } (i,j)=(i',j') \text{ and } i+j=0, \\ 1+3q^2+q^4 & \text{if } (i,j)=(i',j') \text{ and } i+j=1, \\ 1+2q^2+q^4 & \text{if } (i,j)=(i',j') \text{ and } i+j\geq 2, \\ q+q^3 & \text{if } |i-i'|+|j-j'|=1, \\ q^2 & \text{if } |i-i'|=1 \text{ and } |j-j'|=1, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing (i,j) by $(-i,j)$ and (i',j') by $(-i',j')$, we obtain the same answers for the graded Cartan matrix of $\tilde{g}(A \wr \mathfrak{S}_2)\tilde{g}$, by direct calculation (see, e.g., Proposition 7.1 of [6]). So to show that our homomorphism is an isomorphism as desired, it suffices to demonstrate that it is surjective. This can be done by making the following observations:

- The degree 0 and degree 1 components of $\tilde{g}(A \wr \mathfrak{S}_2)\tilde{g}$ are contained in the image of our homomorphism, so it is enough to show that they generate $\tilde{g}(A \wr \mathfrak{S}_2)\tilde{g}$ as an algebra.
- The algebra $A \wr \mathfrak{S}_2$ is generated by its degree 0 and 1 components.
- Define h_{ii} to be g_{ii}^- if i is odd and g_{ii}^+ if i is even. Then $h_{ii}(A \wr \mathfrak{S}_2)_1 g_{i'j'} \neq 0$ if and only if $g_{i'j'}(A \wr \mathfrak{S}_2)_1 h_{ii} \neq 0$ if and only if $(i',j')=(i-1,i)$ or $(i',j')=(i,i+1)$.
- The product maps

$$g_{i-1,i}\tilde{g}\,(A \wr \mathfrak{S}_2)_1\,\tilde{g} \otimes \tilde{g}\,(A \wr \mathfrak{S}_2)_1\,\tilde{g}g_{i-1,i} \to g_{i-1,i}\,(A \wr \mathfrak{S}_2)_2\,g_{i-1,i}$$

and

$$g_{i-1,i}\tilde{g}\,(A \wr \mathfrak{S}_2)_1\,\tilde{g} \otimes \tilde{g}\,(A \wr \mathfrak{S}_2)_1\,\tilde{g}g_{i,i+1} \to g_{i-1,i}\,(A \wr \mathfrak{S}_2)_2\,g_{i,i+1}$$

and

$$g_{i,i+1}\tilde{g}\,(A \wr \mathfrak{S}_2)_1\,\tilde{g} \otimes \tilde{g}\,(A \wr \mathfrak{S}_2)_1\,\tilde{g}g_{i-1,i} \to g_{i,i+1}\,(A \wr \mathfrak{S}_2)_2\,g_{i-1,i}$$

are surjective.

12. Open questions

Let $r=2$. The affine Lie algebra gl_∞ is associated to the infinite Brauer line $U_{\mathcal{X}}$, via the Cartan matrix $C_{U_{\mathcal{X}}}(-1)$. Via a construction of Ringel-Hall type, the positive part of gl_∞ can be obtained as a Lie algebra of constructible functions on the set of indecomposable $V_{\mathcal{X}}$-modules on which arrows act nilpotently (G. Lusztig, H. Nakajima). D. Joyce has shown analogously how to associate a Lie algebra $\mathcal{L}(\mathcal{A})$ to any abelian category $\mathcal{A}$. Are the examples $\mathcal{L}(V_{\mathcal{X}}\text{-nil})$ of any distinguished interest, when $r>2$? Can one define analogues of the full Lie algebra gl_∞ in this setting, and not only its positive part ?

Can one deform the Cubist algebras in an interesting way ? In case $r=2$, PBW deformations of $V_{\mathcal{X}}$ have been introduced by Crawley-Boevey ("deformed preprojective algebras"), and deformations of $V_{\mathcal{X}} \wr \Sigma_n$ by Etingof, and Ginzburg ("symplectic reflection algebras").

Let p be a prime number. Let $(K,\mathcal{O},k)$ be a p-modular system. In case $r=2$, there is a polynomial deformation $\tilde{U}_{\mathcal{X}}=\mathcal{O}[z]\otimes U_{\mathcal{X}}$ of $U_{\mathcal{X}}$ defined over $\mathcal{O}$, such that $\tilde{U}_{\mathcal{X}}/(z-p)$ is the Green order associated to an infinite line, defined over $\mathcal{O}$. Can one make analogous constructions for $r>2$?

Is it possible to deform the Cubist algebras, whilst preserving all their homological structure (including decomposition matrices, etc.) ?

Classify all symmetric algebras with highest weight module categories, whose Loewy length is ≤ 5.

In case $r = 2$, the preprojective algebra $V_{\mathcal{X}}$ is closely related to the hereditary algebra kA_∞ with linear quiver of type A_∞. There are analogues of this algebra, and its Koszul dual, in case $r > 2$. Indeed, the algebras

$$\mathcal{U}_X = \bigoplus_{z,x\in\mathcal{X},i\in\mathbb{Z}} Ext^i_{V_{\mathcal{X}}\text{-mod}}(\Delta_{V_{\mathcal{X}}}(z), \Delta_{V_{\mathcal{X}}}(x) < i >),$$

$$\mathcal{V}_X = \bigoplus_{z,x\in\mathcal{X},i\in\mathbb{Z}} Ext^i_{U_{\mathcal{X}}\text{-mod}}(\Delta_{U_{\mathcal{X}}}(z), \Delta_{U_{\mathcal{X}}}(x) < i >),$$

are Koszul dual to each other, with Cartan matrices $D_{U_{\mathcal{X}}}(q)$, $D_{V_{\mathcal{X}}}(q)$. One can associate such a pair of algebras to any dual pair of standard Koszul algebras. Are these of any interest ?

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