Minimal resolutions of algebras

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Abstract. A method is described for constructing the minimal projective resolution of an algebra considered as a bimodule over itself. The method applies to an algebra presented as the quotient of a tensor algebra over a separable algebra by an ideal of relations which is either homogeneous or admissable (with some additional finiteness restrictions in the latter case). In particular, it applies to any finite dimensional algebra over an algebraically closed field. The method is illustrated by a number of examples, viz. truncated algebras, monomial algebras and Koszul algebras, with the aim of unifying existing treatments of these in the literature.

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1. Introduction

A projective resolution of an algebra Λ, considered as a bimodule over itself, is fundamental in governing the homological properties of the algebra. Such a resolution may be used to compute Hochschild homology and cohomology, to provide functorial projective resolutions of one-sided modules and to compute the derived functors of $\mathrm{Hom}_\Lambda(-,-)$ and $-\otimes_\Lambda -$. Any two such resolutions are homotopic, but, when Λ admits a minimal resolution, then this resolution is unique up to isomorphism and should give the most natural and efficient method for making the computations already mentioned. Of course, the minimal resolution should also be an inherently interesting invariant of Λ. It is easy to see, for example, that the length of the minimal resolution is precisely the global dimension of Λ.

The primary aim of this paper is to construct the minimal projective bimodule resolution of an algebra Λ which is presented as a quotient of a tensor algebra over a separable algebra. The principal motivating example of such a tensor algebra is the path algebra of a quiver. However, the general formalism of tensor algebras provides a broader and more natural context for our results. Hence, throughout the paper, we fix a base field k, a k-separable algebra S and an S,S-bimodule X. We let Γ be the tensor algebra of X over S and J be the augmentation ideal in Γ generated by X. Without loss of generality, all quotient algebras $\Lambda = \Gamma/I$ will be taken over ideals $I \subseteq J^2$. Almost all spaces we encounter will be at least S,S-bimodules, which form a semi-simple category, precisely because S is separable. This category also has tensor products and the symbol $\otimes$ without a subscript will always denote tensor product over S.

Eilenberg's general theory of perfect categories [**10**] provides us with two classes of quotient algebras $\Lambda = \Gamma/I$ for which minimal bimodule resolutions exist: firstly, semi-primary algebras for which the relation ideal I is 'admissable', that is, contains a power of J; and, secondly, graded algebras for which I is homogeneous, with respect to the obvious grading of Γ by the tensor powers of X. We will restrict our attention to algebras in one or other of these classes.

The individual terms in the minimal resolution are known (see for example [**12**] §1.5 when Λ is finite dimensional). Indeed, a slight refinement of Eilenberg's original theory identifies the m-th term as

$$\Lambda \otimes \mathrm{Tor}_m^{\Lambda}(S, S) \otimes \Lambda$$

where S is regarded as a Λ-module on either side via the augmentation map $\Lambda \to S$ with kernel J/I. Furthermore, we may obtain a description in terms of the ideals I and J in Γ using the formulae (cf. [**7**])

$$\mathrm{Tor}_{2n}^{\Lambda}(S, S) \cong \frac{I^n \cap JI^{n-1}J}{JI^n + I^nJ} \qquad \mathrm{Tor}_{2n+1}^{\Lambda}(S, S) \cong \frac{JI^n \cap I^nJ}{I^{n+1} + JI^nJ} \tag{1.1}$$

Our main objective is to describe the maps in the resolution in terms of data associated to Γ and in a manner compatible with (1.1).

This objective can readily be achieved at the beginning of the resolution, which is known to have the form

$$0 \to I/I^2 \xrightarrow{c} \Lambda \otimes X \otimes \Lambda \xrightarrow{d} \Lambda \otimes \Lambda \xrightarrow{e} \Lambda \to 0 \tag{1.2}$$

The maps d and e are the reductions modulo I (on both sides) of the corresponding maps d_Γ and e_Γ in the basic structure sequence for any tensor algebra (see [**9**] Prop. 2.2.6)

$$0 \to \Gamma \otimes X \otimes \Gamma \xrightarrow{d_\Gamma} \Gamma \otimes \Gamma \xrightarrow{e_\Gamma} \Gamma \to 0 \tag{1.3}$$

Explicitly, e_Γ is the multiplication map and $d_\Gamma : 1 \otimes x \otimes 1 \mapsto x \otimes 1 - 1 \otimes x$. The connecting map c in (1.2) is induced by the restriction to I of the universal derivation

$$\Delta : \Gamma \to \Gamma \otimes X \otimes \Gamma : x_1 \cdots x_r \mapsto \sum_{i=1}^{r} x_1 \cdots x_{i-1} \otimes x_i \otimes x_{i+1} \cdots x_r$$

The basic strategy of the paper involves the construction of a series of four term exact sequences

$$0 \to \frac{I^{n+1}}{I^{n+2}} \xrightarrow{c_n} \Lambda \otimes \frac{JI^n \cap I^nJ}{JI^nJ} \otimes \Lambda \xrightarrow{d_n} \Lambda \otimes \frac{I^n}{JI^n + I^nJ} \otimes \Lambda \xrightarrow{e_n} \frac{I^n}{I^{n+1}} \to 0 \tag{1.4}$$

of which (1.2) is the case $n = 0$. The sequences (1.4) can then be spliced together to form a projective resolution of Λ, which we shall refer to as the 'spliced resolution'. Comparing (1.4) to (1.1), it is clear that this resolution is not necessarily minimal, but that the terms are closely related to the required minimal terms. The excess summands occur in naturally isomorphic pairs across the splicing

$$\Lambda \otimes \frac{I^n}{I^n \cap JI^{n-1}J} \otimes \Lambda \;\cong\; \Lambda \otimes \frac{I^n + JI^{n-1}J}{JI^{n-1}J} \otimes \Lambda$$

and an important requirement thus imposed on the construction of the sequences (1.4) is that these excess summands may be readily excised, thereby yielding the minimal resolution.

The main technical idea of the paper is that the construction of (1.2) described above should be generalised to give a construction of a four term sequence

$$0 \to \frac{IL \cap LI}{ILI} \xrightarrow{c} \Lambda \otimes \frac{JL \cap LJ}{JLJ} \otimes \Lambda \xrightarrow{d} \Lambda \otimes \frac{L}{JL + LJ} \otimes \Lambda \xrightarrow{e} \frac{L}{IL + LI} \to 0 \quad (1.5)$$

defined for any Γ, Γ-bimodule L which is 'left-right projective', that is, projective both as a left Γ-module and as a right Γ-module. The case $L = \Gamma$ gives (1.2), while $L = I^n$ gives (1.4).

If we make the assumption that either J is nilpotent or L is graded, then L itself has a minimal projective resolution as a Γ, Γ-bimodule and we will show (Proposition 4.2) that this has the special form

$$0 \to \Gamma \otimes \frac{JL \cap LJ}{JLJ} \otimes \Gamma \xrightarrow{d_\Gamma} \Gamma \otimes \frac{L}{JL + LJ} \otimes \Gamma \xrightarrow{e_\Gamma} L \to 0 \qquad (1.6)$$

where e_Γ is a minimal projective cover, determined by the choice of an S, S-bimodule complement U to $JL + LJ$ in L and d_Γ is determined by the choice of an S, S-bimodule complement V to JLJ in $JL \cap LJ$, together with the choice of a left Γ-module section l_Γ of e_Γ and a right Γ-module section r_Γ of e_Γ. Explicitly, for $u \in U$ and $v \in V$,

$$\begin{aligned} e_\Gamma(1 \otimes u \otimes 1) &= u \\ d_\Gamma(1 \otimes v \otimes 1) &= l_\Gamma(v) - r_\Gamma(v) \end{aligned}$$

We will call such a sequence (1.6) a 'standard presentation' of L, determined by data $(U, V, l_\Gamma, r_\Gamma)$. Note that (1.3) is the special case $L = \Gamma$, with $U = S$ and $V = X$. We will then see (Theorem 5.2) that reducing a standard presentation (1.6) modulo I on both sides yields a sequence of the form (1.5) with certain special properties. In particular, as in (1.2), the map c is induced by a 'bimodule derivation' associated to (1.6). We encapsulate these special properties in the definition of a 'standard presentation' of $L/(IL + LI)$. These properties will, in particular, enable us to excise the excess summands from the spliced resolution to obtain the minimal resolution.

Unfortunately, the assumptions above are too strong to cover the case $L = I^n$ for an arbitrary admissable ideal I in, for example, the path algebra of a quiver with oriented cycles. Therefore, we shall need a further existence result for standard presentations of $L/(IL + LI)$ to cover cases where we do not know whether a standard presentation of L exists. The abstract existence of a sequence of the form of (1.5) will be proved (Theorem 6.2) in the degree of generality appropriate to the paper, that is, under the assumption that either I is admissable or L is graded. However, this proof is non-constructive and to actually construct (1.5) as a standard presentation we need some additional finiteness assumptions. Thus our final existence result (Theorem 6.3) is somewhat more awkward than we would like. On the other hand, it does cover the important case $L = I^n$ for any admissable ideal I in the path algebra of a finite quiver.

To appreciate the problems involved in constructing (1.6) when J is not nilpotent and L is not graded, it is sufficient to consider the case where Γ is the free associative algebra in at least two variables and L is an arbitrary admissable ideal in Γ. We do not even know, in this case, whether a map of the form e_Γ necessarily exists, that is, whether L may be generated by a complement to $JL + LJ$. It also appears to still be an open question whether, in this case, all projective bimodules are free.

The structure of the paper is as follows. Sections 2 and 3 are preparatory. In Section 2, we recall the material needed from Eilenberg's paper [**10**] and establish formulae for the terms of minimal resolutions. In Section 3, we define bimodule derivations, explain how they are associated to left-right split short exact sequences such as (1.6) and show that they may be used to induce connecting morphisms such as the map c in (1.5). Sections 4,5 & 6 contain the main technical part of the paper, that is, the definitions of and existence theorems for standard presentations, as already outlined. In Section 5 we are able to give, as an example, the minimal resolution of a truncated algebra $\Lambda = \Gamma/J^t$, because in this special case the spliced resolution coincides with the minimal resolution. Section 7 completes the main aim of the paper by showing how, in general, the excess summands may be excised from the spliced resolution to yield the minimal resolution. We formulate the final result for any left-right projective bimodule L for which $IL = LI$, although the case of most interest is $L = \Gamma$. Sections 8 & 9 discuss some aspects of the construction in detail in the case of monomial algebras and Koszul algebras, and the relationship to what is already known in these two cases. The paper ends with an Appendix containing a structure theorem, used in Section 6, for one-sided projective modules over tensor algebras. The result may not be familiar in the degree of generality stated here, although it is certainly well-known in special cases.

Additional notation and conventions.

All ideals are two-sided unless otherwise specified. An ideal in Γ denoted by I will always be contained in J^2 and the corresponding quotient algebra will always be denoted by Λ. We will not distinguish between Λ-modules and Γ-modules that are annihilated by I. Thus, if $\mathcal{L}$ is a left Λ-module, then we will write just $J\mathcal{L}$ rather than $(J/I)\mathcal{L}$, with similar conventions for right modules and bimodules. Furthermore, S will be regarded interchangeably as a subring of Γ with complement J, or as a subring of Λ with complement J/I, or as a Γ, Γ bimodule (or one sided module) via the augmentation $\Gamma \to S$ with kernel J, or as a Λ, Λ-bimodule (or one-sided module) via the augmentation $\Lambda \to S$ with kernel J/I.

When we consider Γ as a graded algebra, it will always be with respect to the obvious grading given by the tensor powers of X. Hence, by a graded Γ, Γ-bimodule L we mean one equipped with an S, S-bimodule decomposition $L = \bigoplus_{n \geq 0} L[n]$, such that $XL[n] + L[n]X \subseteq L[n+1]$. A graded morphism $f : M \to L$ is one for which $f(M[n]) \subseteq L[n]$, and a tensor product of graded modules is graded by the usual convention

$$(A \otimes B)[n] = \bigoplus_{i+j=n} A[i] \otimes B[j].$$

We will use the adjective "left-right" to describe a property that is satisfied by bimodules as one-sided modules on both sides. Thus a left-right projective bimodule is one that is projective as a left module and also projective as a right module. Similarly, a left-right split short exact sequence of bimodules is one that is split as a sequence of left modules and also as a sequence of right modules.

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2. Projective covers of bimodules

We gather some basic facts about minimal projective covers and minimal resolutions of bimodules over quotients of tensor algebras. We note first that the separability of S enables us to identify a large class of projective bimodules.

Lemma 2.1. *Let Ω_1 and Ω_2 be k-algebras. Let P be an Ω_1, S bimodule, which is projective as a left Ω_1-module, and Q an S, Ω_2-bimodule, which is projective as a right Ω_2 module. Then for any S, S-bimodule W the Ω_1, Ω_2-bimodule $P \otimes W \otimes Q$ is projective.*

Proof. This follows immediately from the equivalence of functors

$$\mathrm{Hom}_{\Omega_1,\Omega_2}(P \otimes W \otimes Q, -) \cong \mathrm{Hom}_{S,S}\Big(W, \mathrm{Hom}_{\Omega_1}\big(P, \mathrm{Hom}_{\Omega_2}(Q, -)\big)\Big)$$

and the fact that every S, S-bimodule is projective, because S is separable.

□

The main application of this lemma is that, for any quotients Λ_1 and Λ_2 of Γ and for any S, S-bimodule W, the induced Λ_1, Λ_2-bimodule $\Lambda_1 \otimes W \otimes \Lambda_2$ is projective. In particular, Γ itself has dimension one, in the sense of Hochschild cohomology, because the basic sequence (1.3) is a projective bimodule resolution. Hence Γ is left and right hereditary and so all ideals in Γ are left-right projective. The lemma also has the following important consequence, which shows that these ideals are also projective as Γ, S- and S, Γ-bimodules.

Corollary 2.2. *Any Ω_1, S bimodule P that is projective as a left Ω_1-module is projective as an Ω_1, S-bimodule.*

Proof. Apply Lemma 2.1 with $\Omega_2 = Q = W = S$.

□

To ensure the existence and uniqueness (up to isomorphism) of minimal resolutions of Λ, Λ-bimodules (or equivalently of left modules over the enveloping algebra $\Lambda^{ev} = \Lambda \otimes_k \Lambda^{op}$), we shall work in a perfect category, as axiomatised by Eilenberg ([**10**] §2 Axioms 1–5 & §4 Axiom 6). Strictly speaking, Eilenberg uses the term "perfect" to describe a category of modules satisfying just the first five axioms, which are sufficient only to ensure the existence and uniqueness of minimal projective covers (called 'minimal epimorphisms' in [**10**]). Axiom 6 is required to ensure that a perfect category is closed under taking kernels, so that the process of taking covers can be iterated to produce minimal resolutions. Therefore we shall require that our perfect categories also satisfy this axiom.

Using [**10**] §6 Proposition 15, we may identify two examples of perfect categories of bimodules, which are the two main examples considered in this paper. If I is admissable, and hence Λ is semi-primary, we take the category of all Λ, Λ-bimodules; while if I is homogeneous, and hence Λ is graded, we take the category of graded Λ, Λ-bimodules and graded morphisms. It is then appropriate in both cases to refer to S^{ev} as the semisimple 'top' of Λ^{ev} (because S is separable) and to

$$\operatorname{rad}(\Lambda^{ev}) = \operatorname{rad}\Lambda \otimes_k \Lambda^{op} \; + \; \Lambda \otimes_k \operatorname{rad}\Lambda^{op}$$

as the 'radical', where $\operatorname{rad}\Lambda = J/I$. The 'top' of a Λ, Λ-bimodule $\mathcal{L}$ is then taken to be the S, S-bimodule

$$T(\mathcal{L}) = \frac{\mathcal{L}}{\operatorname{rad}(\Lambda^{ev})\mathcal{L}} = \frac{\mathcal{L}}{J\mathcal{L} + \mathcal{L}J} = S \otimes_\Lambda \mathcal{L} \otimes_\Lambda S \tag{2.1}$$

Now [**10**] §3 Proposition 3 shows that every bimodule $\mathcal{L}$ has a minimal projective cover $\mathcal{P} \to \mathcal{L}$, which is unique up to isomorphism and which is characterised (amongst *all* maps from a projective to $\mathcal{L}$) by the condition that the induced map $T(\mathcal{P}) \to T(\mathcal{L})$ is an isomorphism. In our restricted context of quotients of tensor algebras, we may construct minimal projective covers in a more explicit manner than was available to Eilenberg. The key additional feature is that Λ and, consequently, Λ^{ev} split over their radicals. Therefore, for any S, S-bimodule W we have an induced Λ, Λ-bimodule

$$\mathcal{P}(W) = \Lambda \otimes W \otimes \Lambda \tag{2.2}$$

which is projective by Lemma 2.1 and clearly has top $T(\mathcal{P}(W)) = W$. We may thus reformulate Eilenberg's existence and uniqueness result as follows, writing $\pi : \mathcal{L} \to T(\mathcal{L})$ for the canonical projection.

Lemma 2.3. *If W is an S, S-bimodule and $\sigma : W \to \mathcal{L}$ an S, S-morphism such that $\pi\sigma : W \to T(\mathcal{L})$ is an isomorphism, then the induced map $\hat{\sigma} : \Lambda \otimes W \otimes \Lambda \to \mathcal{L}$ is a minimal projective cover. If, in addition, $\mathcal{L}$ is projective then $\hat{\sigma}$ is an isomorphism.*

This does indeed produce explicit minimal projective covers for any $\mathcal{L}$, because S is separable and so we may choose for W any S, S-complement of $J\mathcal{L} + \mathcal{L}J$ in $\mathcal{L}$, and thus obtain, as the required cover, the map

$$\mathcal{P}(W) \to \mathcal{L} : a \otimes w \otimes b \mapsto awb.$$

Finally, the usual iterative procedure yields a construction of a minimal projective resolution of $\mathcal{L}$ in which the mth term is $\mathcal{P}\big(T_m(\mathcal{L})\big)$, where, by **[10]** §4 Proposition 10,

$$T_m(\mathcal{L}) \cong \operatorname{Tor}_m^{\Lambda^{ev}}(S^{ev}, \mathcal{L}). \tag{2.3}$$

The Λ, Λ-bimodules of most interest in this paper are those such as $\mathcal{L} = I^n/I^{n+1}$, which have the special form $\mathcal{L} = L/LI$, where L is a left-right projective Γ, Γ-bimodule and $IL = LI$. For these special bimodules we may find other formulae for the Tor groups in (2.3), which give the formula $T_m(\Lambda) \cong \operatorname{Tor}_m^{\Lambda}(S, S)$ and the ideal quotient formulae (1.1) in the case $L = \Gamma$.

Proposition 2.4. *Let L be a left-right projective Γ, Γ-bimodule for which $IL = LI$. Then $\mathcal{L} = L/LI$ is a left-right projective Λ, Λ-bimodule and, for all $m \geq 0$,*

$$T_m(\mathcal{L}) \cong \operatorname{Tor}_m^{\Lambda}(S, \mathcal{L} \otimes_\Lambda S). \tag{2.4}$$

Furthermore, for all $n \geq 0$,

$$T_{2n}(\mathcal{L}) \cong \frac{LI^n \cap JLI^{n-1}J}{JLI^n + LI^nJ} \qquad\qquad T_{2n+1}(\mathcal{L}) \cong \frac{JLI^n \cap LI^nJ}{LI^{n+1} + JLI^nJ} \tag{2.5}$$

Proof. Since $L/IL = \Lambda \otimes_\Gamma L$, we see that $\mathcal{L}$ is left projective and, by a symmetric argument, it is also right projective. To prove (2.4), we calculate $\operatorname{Tor}_*^{\Lambda^{ev}}(S^{ev}, \mathcal{L})$ using a projective Λ, Λ-bimodule resolution $\mathcal{P}_*$ of $\mathcal{L}$. Now

$$S^{ev} \otimes_{\Lambda^{ev}} \mathcal{P}_* = S \otimes_\Lambda \left(\mathcal{P}_* \otimes_\Lambda S\right) \tag{2.6}$$

and, since $\mathcal{L}$ is right projective, $\mathcal{P}_*$ is split acyclic as a complex of right modules. Hence $\mathcal{P}_* \otimes_\Lambda S$ is a projective left Λ-module resolution of $\mathcal{L} \otimes_\Lambda S$ and the homology of the right hand side of (2.6) is precisely $\operatorname{Tor}_*^{\Lambda}(S, \mathcal{L} \otimes_\Lambda S)$.

To get from (2.4) to (2.5), we first note that

$$\mathcal{L} \otimes_\Lambda S = (L/LI) \otimes_\Gamma (\Gamma/J) = L/LJ.$$

Then we compute $\operatorname{Tor}_*^{\Lambda}(S, \mathcal{L} \otimes_\Lambda S)$ using the following projective right Λ-module resolution of S (cf. **[11]** §1 Proposition 3)

$$\cdots \to \frac{JI^n}{JI^{n+1}} \to \frac{I^n}{I^{n+1}} \to \frac{JI^{n-1}}{JI^n} \to \cdots \to \frac{J}{JI} \to \Lambda \to S \to 0 \tag{2.7}$$

Here the three general terms are in degrees $2n+1$, $2n$ and $2n-1$, and the maps are induced by the chain of inclusions

$$\cdots \subseteq JI^{n+1} \subseteq I^{n+1} \subseteq JI^n \subseteq I^n \subseteq JI^{n-1} \subseteq \cdots \subseteq I \subseteq J \subseteq \Gamma$$

Now L is flat as a left Γ-module, and so, for any ideal $H \subseteq \Gamma$, the multiplication map $H \otimes_\Gamma L \to HL$ is an isomorphism and further induces an isomorphism

$$\frac{H}{H'} \otimes_\Gamma \frac{L}{L'} \cong \frac{HL}{H'L + HL'}$$

for any ideal $H' \subseteq H$ and any submodule $L' \subseteq L$. Using this formula, we may apply the functor $- \otimes_\Gamma (L/LJ)$ to the resolution (2.7) to obtain the complex

$$\cdots \to \frac{JLI^n}{JLI^nJ} \to \frac{LI^n}{LI^nJ} \to \frac{JLI^{n-1}}{JLI^{n-1}J} \to \cdots \to \frac{JL}{JLJ} \to \frac{L}{LJ}$$

which clearly has the required homology groups.

□

Remark 2.5. The formulae (2.5), especially when $L = \Gamma$, are analogues of Gruenberg's formulae in [**15**] for group homology $H_*(G) = \mathrm{Tor}_*^{\mathbf{Z}G}(\mathbf{Z}, \mathbf{Z})$. The proof here is modelled on his proof, with (2.7) playing the role of the resolution of $\mathbf{Z}$ given in [**14**].

3. Bimodule derivations

In this paper, a key technical tool is the association of a bimodule derivation to a left-right split short exact sequence of bimodules. In this section, we explain how this association is made and how it is used.

Definition 3.1. Let Ω_1 and Ω_2 be k-algebras, and L and N be Ω_1, Ω_2-bimodules. A *bimodule derivation* $\delta : L \to N$ is a k-linear map such that

$$\delta(azb) = \delta(az)b - a\delta(z)b + a\delta(zb) \tag{3.1}$$

for all $a \in \Omega_1$, $b \in \Omega_2$ and $z \in L$. Clearly these derivations form a k-linear subspace of $\mathrm{Hom}_k(L, N)$ and include both left Ω_1-morphisms and right Ω_2-morphisms. We call δ an *inner derivation* if it may be written as the difference of a left Ω_1-morphism and a right Ω_2-morphism.

Given a left-right split short exact sequence of Ω_1, Ω_2-bimodules

$$0 \to N \xrightarrow{g} M \xrightarrow{f} L \to 0, \tag{3.2}$$

one associates to it a bimodule derivation by choosing for f a left Ω_1-splitting $l : L \to M$ and a right Ω_2-splitting $r : L \to M$. There is then a unique k-linear map $\delta : L \to N$ such that $g\delta = l - r$ and this map is a bimodule derivation. One may readily check that two derivations that are associated to the same sequence by different choices of splittings differ by an inner derivation.

Now suppose that $H_i \subseteq \Omega_i$ are two ideals. If δ is a bimodule derivation associated to (3.2), then it determines a connecting morphism

$$\widetilde{\delta} : \frac{H_1 L \cap L H_2}{H_1 L H_2} \longrightarrow \frac{N}{H_1 N + N H_2}$$

by setting $\widetilde{\delta}(w + H_1 L H_2) = \delta(w) + (H_1 N + N H_2)$. The defining property (3.1) of a derivation ensures that $\widetilde{\delta}$ is well-defined and in fact a morphism of $(\Omega_1/H_1), (\Omega_2/H_2)$-bimodules. Note that the domain is indeed such a bimodule, because

$$H_1(H_1 L \cap L H_2) + (H_1 L \cap L H_2) H_2 \subseteq H_1 L H_2.$$

Note further that adding an inner derivation to δ does not change $\widetilde{\delta}$. Hence $\widetilde{\delta}$ only depends on the left-right split sequence (3.2). The choice of the left and right splittings is only needed for explicit calculation. The use of the term "connecting morphism" is justified by the following result which plays a key role in the rest of the paper.

Proposition 3.2. *As above, let (3.2) be a left-right split sequence of Ω_1, Ω_2-bimodules, let $\delta : L \to N$ be an associated bimodule derivation and let $H_i \subseteq \Omega_i$ be ideals. Then there is an exact sequence of $(\Omega_1/H_1), (\Omega_2/H_2)$-bimodules*

$$\begin{aligned} &\frac{H_1M \cap MH_2}{H_1MH_2} \xrightarrow{\widetilde{f}} \frac{H_1L \cap LH_2}{H_1LH_2} \xrightarrow{\widetilde{\delta}} \\ &\qquad \xrightarrow{\widetilde{\delta}} \frac{N}{H_1N + NH_2} \xrightarrow{\widetilde{g}} \frac{M}{H_1M + MH_2} \xrightarrow{\widetilde{f}} \frac{L}{H_1L + LH_2} \to 0 \end{aligned} \tag{3.3}$$

where $\widetilde{f}$ and $\widetilde{g}$ are the naturally induced morphisms. If M is a projective bimodule, then $H_1M \cap MH_2 = H_1MH_2$ and (3.3) reduces to a four-term exact sequence.

Proof. Since (3.2) is left-right split, the sequences $H_1(3.2)$ and $(3.2)H_2$ are both exact. Hence we may take the natural addition map from the direct sum of these two sequences to (3.2) and apply the Snake Lemma to obtain the following exact commutative diagram of Ω_1, Ω_2-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H_1N \cap NH_2 \arrow[r, "g"] \arrow[d] & H_1M \cap MH_2 \arrow[r, "f"] \arrow[d] & H_1L \cap LH_2 \arrow[r, "\gamma"] \arrow[d, "{(1,-1)}"] & {} \\
0 \arrow[r] & H_1N \oplus NH_2 \arrow[r, "g \oplus g"] \arrow[d] & H_1M \oplus MH_2 \arrow[r, "f \oplus f"] \arrow[d] & H_1L \oplus LH_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "g"] \arrow[d, "\pi_N"] & M \arrow[r, "f"] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
{} \arrow[r, "\gamma"] & \frac{N}{H_1N+NH_2} \arrow[r, "\widetilde{g}"] \arrow[d] & \frac{M}{H_1M+MH_2} \arrow[r, "\widetilde{f}"] \arrow[d] & \frac{L}{H_1L+LH_2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If $l : L \to M$ and $r : L \to M$ are left and right splittings of f with $g\delta = l - r$, then $l \oplus r$ is a k-linear splitting of $f \oplus f$ and so $\gamma = \pi_N \delta$. The exact sequence (3.3) is obtained immediately from the six term exact sequence that makes up the top and bottom rows of the above diagram, because $H_1LH_2 = f(H_1MH_2)$. Finally, the equality $H_1M \cap MH_2 = H_1MH_2$ clearly holds for $M = \Omega_1 \otimes_k \Omega_2$ and hence for all free bimodules and their summands, that is, all projective bimodules.

□

Remark 3.3. The ideas above can be interpreted in terms of the relative homological algebra for Ω_1, Ω_2-bimodules determined by the subfunctor $\mathrm{Ext}^1_{lr} \subseteq \mathrm{Ext}^1$ consisting of left-right split sequences. The relative projectives in this case are direct sums of summands of induced modules $A \otimes_k \Omega_2$ or $\Omega_1 \otimes_k B$, for any left Ω_1-module A or right Ω_2-module B. There is a relative projective resolution of any bimodule L which starts

$$P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} L \to 0$$

with terms

$$
\begin{aligned}
P_0 &= \big(L \otimes_k \Omega_2\big) \oplus \big(\Omega_1 \otimes_k L\big)\\
P_1 &= \Omega_1 \otimes_k L \otimes_k \Omega_2\\
P_2 &= \Omega_1 \otimes_k \Omega_1 \otimes_k L \otimes_k \Omega_2 \otimes_k \Omega_2
\end{aligned}
$$

and differentials

$$
\begin{aligned}
p_0(z_1 \otimes 1, 1 \otimes z_2) &= z_1 + z_2\\
p_1(1 \otimes z \otimes 1) &= (z \otimes 1, -1 \otimes z)\\
p_2(1 \otimes a \otimes z \otimes b \otimes 1) &= 1 \otimes azb \otimes 1 - 1 \otimes az \otimes b + a \otimes z \otimes b - a \otimes zb \otimes 1
\end{aligned}
$$

Applying $\mathrm{Hom}(-,N)$ to this resolution, one sees immediately that $\mathrm{Ext}^1_{lr}(L,N)$ is naturally isomorphic to the space of bimodule derivations $L \to N$ modulo inner derivations.

Furthermore, since the functor

$$M \mapsto \frac{H_1M \cap MH_2}{H_1MH_2}$$

is zero on relative projectives, we may interpret (3.3) as the beginning of a relative Tor sequence. If (and only if) L is left-right projective, then there is a left-right split sequence (3.2) with M projective. Hence, in this case, the relative Tor is absolute. In the special case $\Omega_1 = \Omega_2 = \Gamma$ and $H_1 = H_2 = I$, we thus obtain

$$\mathrm{Tor}_1^{\Gamma^{ev}}(\Lambda^{ev}, L) \cong \frac{IL \cap LI}{ILI} \tag{3.4}$$

Remark 3.4. If L is a left-right projective Γ,Γ-bimodule, then Lemma 2.1 shows that $L \otimes (1.3)$ is a projective bimodule resolution of L and, thus, that L has projective dimension at most one. Now, the special case $I = J$ of (3.4) is

$$\mathrm{Tor}_1^{\Gamma^{ev}}(S^{ev}, L) \cong \frac{JL \cap LJ}{JLJ}$$

and hence, if L has a minimal projective Γ,Γ-bimodule resolution, then Section 2 shows that this will have the form

$$0 \to \Gamma \otimes \frac{JL \cap LJ}{JLJ} \otimes \Gamma \longrightarrow \Gamma \otimes \frac{L}{JL + LJ} \otimes \Gamma \longrightarrow L \to 0 \tag{3.5}$$

We will see in the next section how such a minimal resolution may be constructed more explicitly.

4. Standard presentations over tensor algebras

We give the definition of a standard presentation of a left-right projective bimodule L over the tensor algebra Γ. In circumstances where Eilenberg's theory of perfect categories (cf. Section 2) implies that L has a minimal resolution, we prove that L has a standard presentation. It follows immediately from Remark 3.4 that a standard presentation is, in particular, a minimal resolution.

Definition 4.1. Let L be a left-right projective Γ, Γ-bimodule. A *standard presentation* of L is an exact sequence of Γ, Γ-bimodules

$$0 \to \Gamma \otimes V \otimes \Gamma \xrightarrow{d_\Gamma} \Gamma \otimes U \otimes \Gamma \xrightarrow{e_\Gamma} L \to 0 \tag{4.1}$$

determined by data $(U, V, l_\Gamma, r_\Gamma)$ as follows:

i) U is an S, S complement of $JL + LJ$ in L and for $u \in U$

$$e_\Gamma(1 \otimes u \otimes 1) = u \tag{4.2}$$

ii) $l_\Gamma : L \to \Gamma \otimes U \otimes \Gamma$ is a Γ, S-splitting of e_Γ,
$r_\Gamma : L \to \Gamma \otimes U \otimes \Gamma$ is an S, Γ-splitting of e_Γ,
V is an S, S-complement of JLJ in $JL \cap LJ$ and for $v \in V$

$$d_\Gamma(1 \otimes v \otimes 1) = l_\Gamma(v) - r_\Gamma(v). \tag{4.3}$$

Proposition 4.2. *If the ideal $J \subseteq \Gamma$ is nilpotent, then every left-right projective Γ, Γ-bimodule admits a standard presentation. On the other hand, for arbitrary Γ, every graded left-right projective Γ, Γ-bimodule admits a standard presentation.*

Proof. We start with the first case and suppose that L is an arbitrary left-right projective Γ, Γ-bimodule. Choose any S, S-complement U of $JL + LJ$ in L. By Lemma 2.3, the induced Γ, Γ-morphism $e_\Gamma : \Gamma \otimes U \otimes \Gamma \to L$ defined by (4.2) is surjective and provides the first part of the standard presentation. Now take the kernel of e_Γ to obtain a short exact sequence

$$0 \to N \xrightarrow{g} \Gamma \otimes U \otimes \Gamma \xrightarrow{e_\Gamma} L \to 0 \tag{4.4}$$

By Corollary 2.2, we see that L is Γ, S-projective and, similarly, S, Γ-projective. Hence for l_Γ and r_Γ we may choose any Γ, S-splitting and any S, Γ-splitting of e_Γ.

Let $\delta_\Gamma : L \to N$ be the bimodule derivation given by $g\delta_\Gamma = l_\Gamma - r_\Gamma$ and note that δ_Γ is an S, S-morphism. Applying Proposition 3.2 to (4.4) with $\Omega_i = \Gamma$, $H_i = J$ and $\delta = \delta_\Gamma$, we see that $\widetilde{\delta}_\Gamma$ is an isomorphism, because $\widetilde{e}_\Gamma$ is. Hence, if we choose any complement V of JLJ in $JL \cap LJ$, then the restriction of δ_Γ maps V isomorphically to a complement of $JN + NJ$ in N. But N is projective, because L has projective dimension at most 1 as a bimodule (Remark 3.4), and so Lemma 2.3 implies that the map $\Gamma \otimes V \otimes \Gamma \to N$ induced by δ_Γ is an isomorphism. In other words, the map $d_\Gamma : \Gamma \otimes V \otimes \Gamma \to \Gamma \otimes U \otimes \Gamma$ defined by (4.3) is an injection onto the kernel of e_Γ, as required.

This completes the proof of the first case. However, the same proof works when L is graded, without the restriction on Γ, because we may choose the data $(U, V, l_\Gamma, r_\Gamma)$ to be graded, so that δ_Γ is also graded and both applications of Lemma 2.3 remain valid. More explicitly, writing $L = \bigoplus_n L[n]$ for the graded decomposition of L, we may choose $U[n]$ to be an S,S-complement of $XL[n-1] + L[n-1]X$ in $L[n]$ and $V[n]$ to be an S,S-complement of $XL[n-2]X$ in $XL[n-1] \cap L[n-1]X$. Further note that $X \otimes L[n-1] \to L[n]$ is injective, because L is left projective (Appendix: Proposition A.1(a)). Hence, l_Γ may be defined unambiguously on $XL[n-1]$ and extended by choosing any S,S-complement to $XL[n-1]$ in $L[n]$ and any S,S-lift of this complement to $(\Gamma \otimes U \otimes \Gamma)[n]$. The choice of r_Γ is made similarly.

□

Remark 4.3. Observe that in the proof of Proposition 4.2, the data $(U, V, l_\Gamma, r_\Gamma)$, that determines the standard presentation, may be chosen arbitrarily subject to the conditions of Definition 4.1, and the additional requirement that it is graded when L is graded.

5. Standard presentations over quotient algebras I

We now generalise Definition 4.1 to bimodules over any quotient algebra $\Lambda = \Gamma/I$ and prove that a standard presentation of a Γ,Γ-bimodule L induces a standard presentation of the induced Λ,Λ-bimodule $L/(IL + LI)$. When Λ is a 'truncated' algebra, that is, $\Lambda = \Gamma/J^t$ for some $t \geq 2$, this is sufficient to enable us to describe the minimal resolution of Λ.

Definition 5.1. Let L be a left-right projective Γ,Γ-bimodule and $\mathcal{L} = L/(IL + LI)$ the induced Λ,Λ-bimodule. Let $\pi : L \to \mathcal{L}$ be the canonical projection. A *standard presentation* of $\mathcal{L}$ is an exact sequence of Λ,Λ-bimodules

$$0 \to \frac{IL \cap LI}{ILI} \xrightarrow{c} \Lambda \otimes V \otimes \Lambda \xrightarrow{d} \Lambda \otimes U \otimes \Lambda \xrightarrow{e} \frac{L}{IL + LI} \to 0 \tag{5.1}$$

determined by data (U, V, l, r, Δ) as follows:

i) U is an S,S-complement of $JL + LJ$ in L and for $u \in U$

$$e(1 \otimes u \otimes 1) = \pi(u) \tag{5.2}$$

ii) $l : L \to \Lambda \otimes U \otimes \Lambda$ is a Γ,S-morphism with $el = \pi$,
$r : L \to \Lambda \otimes U \otimes \Lambda$ is an S,Γ-morphism with $er = \pi$,
V is an S,S-complement of JLJ in $JL \cap LJ$ and for $v \in V$

$$d(1 \otimes v \otimes 1) = l(v) - r(v) \tag{5.3}$$

iii) $\Delta : L \to \Lambda \otimes V \otimes \Lambda$ is a bimodule derivation and an S,S-morphism such that $d\Delta = l - r$.
In addition, $\Delta(v) = 1 \otimes v \otimes 1$ for all $v \in V$ and for $w \in IL \cap LI$

$$c(w + ILI) = \Delta(w) \tag{5.4}$$

Note that $\Delta(ILI) = 0$, because Δ is a bimodule derivation, so that c is well-defined by (5.4). It is also easy to check that c is a Λ,Λ-morphism and that $dc = 0$.

Theorem 5.2. *Let L be a left-right projective Γ,Γ-bimodule and I an ideal in Γ. Then a standard presentation of L canonically determines a standard presentation of $L/(IL+LI)$.*

Proof. Let Δ_Γ be the bimodule derivation determined by $d_\Gamma\Delta_\Gamma = l_\Gamma - r_\Gamma$. Applying Proposition 3.2 to the standard presentation (4.1), with $\Omega_i = \Gamma$, $H_i = I$ and $\delta = \Delta_\Gamma$, we obtain the exact sequence (5.1) as the sequence (3.3). The spaces U and V are unchanged, as they do not depend on I. The maps l, r and Δ are the composites of l_Γ, r_Γ and Δ_Γ with the canonical projections $\pi_U : \Gamma\otimes U\otimes\Gamma \to \Lambda\otimes U\otimes\Lambda$ and $\pi_V : \Gamma\otimes V\otimes\Gamma \to \Lambda\otimes V\otimes\Lambda$.

□

Example 5.3. For any $s \geq 0$, there is a short exact sequence

$$0 \to \Gamma\otimes X^{s+1}\otimes\Gamma \xrightarrow{d_\Gamma} \Gamma\otimes X^s\otimes\Gamma \xrightarrow{e_\Gamma} J^s \to 0 \tag{5.5}$$

with

$$d_\Gamma : 1\otimes x_0x_1\cdots x_s\otimes 1 \mapsto x_0\otimes x_1\cdots x_s\otimes 1 \;-\; 1\otimes x_0x_1\cdots x_{s-1}\otimes x_s$$

which generalises (1.3). The exactness of (5.5) is easily verified by restricting attention to each component of fixed total degree. This sequence is the standard presentation of $L = J^s$ with $U = X^s$, $V = X^{s+1}$ and left and right splittings

$$\begin{aligned} l_\Gamma &: x_1\cdots x_n \mapsto x_1\cdots x_{n-s}\otimes x_{n-s+1}\cdots x_n\otimes 1 \\ r_\Gamma &: x_1\cdots x_n \mapsto 1\otimes x_1\cdots x_s\otimes x_{s+1}\cdots x_n \end{aligned}$$

The associated bimodule derivation Δ_Γ is then given by

$$\begin{aligned} \Delta_\Gamma(x_1\ldots x_n) &= \sum_{i=1}^{n-s} x_1\ldots x_{i-1}\otimes x_i\ldots x_{i+s}\otimes x_{i+s+1}\ldots x_n \\ &= \sum p\otimes v\otimes q \end{aligned}$$

where p, v, q run over all triples such that $x_1\ldots x_n = pvq$ with $v\in V$. In particular, $\Delta_\Gamma(u) = 0$ for $u\in U$ and $\Delta_\Gamma(v) = 1\otimes v\otimes 1$ for $v\in V$.

As already remarked, the case $s = 0$ of (5.5) is the exact sequence (1.3), which is a standard presentation of Γ itself. Applying Theorem 5.2 with $L = \Gamma$, we then obtain (1.2) as the standard presentation of $\Lambda = \Gamma/I$.

As a second application, we find an explicit minimal resolution of any truncated algebra $\Lambda = \Gamma/J^t$ for any $t \geq 2$. For each $n \geq 0$, apply Theorem 5.2 with $I = J^t$ and $L = J^{nt}$, with standard presentation given by (5.5), to obtain standard presentations

$$0 \to \frac{J^{(n+1)t}}{J^{(n+2)t}} \xrightarrow{c_n} \Lambda\otimes X^{nt+1}\otimes\Lambda \xrightarrow{d_n} \Lambda\otimes X^{nt}\otimes\Lambda \xrightarrow{e_n} \frac{J^{nt}}{J^{(n+1)t}} \to 0$$

We may splice these together to obtain a resolution

$$\cdots \longrightarrow \Lambda\otimes X^{nt+t}\otimes\Lambda \xrightarrow{\partial_{2n+2}} \Lambda\otimes X^{nt+1}\otimes\Lambda \xrightarrow{\partial_{2n+1}} \Lambda\otimes X^{nt}\otimes\Lambda \longrightarrow \cdots$$
$$\cdots \xrightarrow{\partial_1} \Lambda\otimes\Lambda \xrightarrow{e_0} \Lambda \to 0$$

with differentials $\partial_{2n+1} = d_n$ and $\partial_{2n+2} = c_n e_{n+1}$ or, more explicitly,

$$\partial_{2n+1}(1\otimes x_1\cdots x_{nt+1}\otimes 1) = x_1\otimes x_2\cdots x_{nt+1}\otimes 1 - 1\otimes x_1\cdots x_{nt}\otimes x_{nt+1}$$
$$\partial_{2n+2}(1\otimes x_1\cdots x_{nt+t}\otimes 1) = \sum_{i=1}^{t} x_1\cdots x_{i-1}\otimes x_i\cdots x_{nt+i}\otimes x_{nt+i+1}\cdots x_{nt+t}$$

Now, one easily calculates that

$$\frac{I^n\cap JI^{n-1}J}{JI^n+I^nJ}\cong X^{nt} \qquad \frac{JI^n\cap I^nJ}{I^{n+1}+JI^nJ}\cong X^{nt+1}$$

and so, by Proposition 2.4 and its preceding discussion, the above resolution is the minimal resolution of Λ.

Liu & Zhang [**16**] found a special case of this resolution when Γ is the path algebra over k of a quiver consisting of a single oriented cycle with r vertices. In this case, it is clear that the $2n$th syzygy I^n/I^{n+1} is isomorphic to Λ if and only if r divides nt. Hence the minimal resolution is periodic with period $2\lambda/t$, where $\lambda = \mathrm{l.c.m}(r,t)$.

6. Standard presentations over quotient algebras II

We construct standard presentations of $L/(IL+LI)$ in the case when L does not have one itself. Thus throughout this section I is an admissable ideal in Γ. We first establish the existence of an exact sequence with the terms of (1.5). We will make use of the following technical lemma.

Lemma 6.1. *Let $\sigma : Q\to P$ be a quasi-invertible morphism in an abelian category, that is, there is a morphism $\tau : P\to Q$ such that $\sigma\tau\sigma=\sigma$ and $\tau\sigma\tau=\tau$. If d is a second morphism $Q\to P$ such that $d-\sigma$ is in the radical of the category, then there is an exact sequence of the form*

$$0\to \operatorname{Ker} d\to \operatorname{Ker}\sigma\to \operatorname{Coker}\sigma\to \operatorname{Coker} d\to 0.$$

Proof. With respect to the decompositions $Q = \operatorname{Im}\tau\oplus\operatorname{Ker}\sigma$ and $P=\operatorname{Im}\sigma\oplus\operatorname{Ker}\tau$ we may write

$$\sigma=\begin{pmatrix} s & 0\\ 0 & 0\end{pmatrix} \qquad \tau=\begin{pmatrix} t & 0\\ 0 & 0\end{pmatrix} \qquad d=\begin{pmatrix} \theta & \phi\\ \psi & \chi\end{pmatrix}.$$

where s and t are mutually inverse isomorphisms between $\operatorname{Im}\tau$ and $\operatorname{Im}\sigma$. Then, since $d-\sigma$ is in the radical, the map

$$1_Q+\tau(d-\sigma)=\begin{pmatrix} t\theta & t\phi\\ 0 & 1\end{pmatrix}$$

is an automorphism and hence θ is an isomorphism. This implies that the kernel and cokernel of d are isomorphic to those of $\chi-\psi\theta^{-1}\phi : \operatorname{Ker}\sigma\to\operatorname{Ker}\tau$, and thus the proof is completed by observing that $\operatorname{Ker}\tau\cong\operatorname{Coker}\sigma$.

□

Theorem 6.2. *Let L be a left-right projective Γ,Γ-bimodule and I an admissable ideal in Γ. Then there is a presentation of $L/(IL+LI)$ of the form*

$$0 \to \frac{IL\cap LI}{ILI} \longrightarrow \Lambda\otimes\frac{JL\cap LJ}{JLJ}\otimes\Lambda \longrightarrow \Lambda\otimes\frac{L}{JL+LJ}\otimes\Lambda \longrightarrow \frac{L}{IL+LI}\to 0 \tag{6.1}$$

Proof. Since the exact sequence (1.3) is left split, it remains exact after applying the functor $L\otimes_\Gamma -$. Since L is right projective, Proposition A.1(a) in the Appendix (transposed for right modules) implies that the multiplication map $L\otimes X\to LJ$ is an isomorphism. Hence, we may write $L\otimes_\Gamma (1.3)$ as

$$0\to LJ\otimes\Gamma \xrightarrow{d_\Gamma} L\otimes\Gamma \xrightarrow{e_\Gamma} L\to 0 \tag{6.2}$$

where $d_\Gamma : zx\otimes 1\mapsto zx\otimes 1 - z\otimes x$ for $z\in L$ and $x\in X$. Reducing this map modulo J on both sides, the term $z\otimes x$ does not contribute and we obtain the map

$$\sigma_S : \frac{LJ}{JLJ} \longrightarrow \frac{L}{JL}$$

induced by the inclusion of LJ in L. Choosing isomorphisms $LJ\cong\Gamma\otimes\frac{LJ}{JLJ}$ and $L\cong\Gamma\otimes\frac{L}{JL}$ as in Theorem A.4 in the Appendix, we write (6.2) as

$$0\to\Gamma\otimes\frac{LJ}{JLJ}\otimes\Gamma \xrightarrow{d_\Gamma} \Gamma\otimes\frac{L}{JL}\otimes\Gamma \xrightarrow{e_\Gamma} L\to 0 \tag{6.3}$$

and note that the image of $d_\Gamma - 1\otimes\sigma_S\otimes 1$ is contained in $J\otimes\frac{L}{JL}\otimes\Gamma+\Gamma\otimes\frac{L}{JL}\otimes J$.

Lemma 2.1 implies that (6.3) is a projective bimodule resolution of L and, since L is left-right projective, we may apply Proposition 3.2 to obtain

$$0\to\frac{IL\cap LI}{ILI} \longrightarrow \Lambda\otimes\frac{LJ}{JLJ}\otimes\Lambda \xrightarrow{d} \Lambda\otimes\frac{L}{JL}\otimes\Lambda \longrightarrow \frac{L}{IL+LI}\to 0 \tag{6.4}$$

Writing $\sigma = 1\otimes\sigma_S\otimes 1$, we see that

$$\operatorname{Im}(d-\sigma)\subseteq \operatorname{rad}\Lambda\otimes\frac{L}{JL}\otimes\Lambda+\Lambda\otimes\frac{L}{JL}\otimes\operatorname{rad}\Lambda$$

and so $d-\sigma$ is in the radical of the category of Λ,Λ-bimodules. The required result then follows from Lemma 6.1 and the fact that

$$\operatorname{Ker}\sigma\cong\Lambda\otimes\frac{JL\cap LJ}{JLJ}\otimes\Lambda,\qquad \operatorname{Coker}\sigma\cong\Lambda\otimes\frac{L}{JL+LJ}\otimes\Lambda$$

and σ is quasi-invertible as a Λ,Λ-morphism, because S is separable and so σ_S is quasi-invertible as an S,S-morphism.

□

We now show that, with some finiteness restrictions, the sequence (6.1) may be constructed as a standard presentation.

Theorem 6.3. *Let Γ be a finitely-generated tensor algebra over S, that is, X is finitely-generated as an S,S-bimodule. Let L be a finitely-generated left-right projective Γ,Γ-bimodule and let I be an admissable ideal in Γ. Suppose further that there exist admissable ideals $I_1', I_2' \subseteq I$ such that*

$$I_1'L = LI_2' \subseteq ILI. \tag{6.5}$$

Then there is a standard presentation of $L/(IL+LI)$, in which U and V may be chosen arbitrarily.

Remark 6.4. Suppose that L is an ideal in Γ which contains a power of J. Then $J^m \subseteq ILI \subseteq I$ for some $m \geq 2$, and $I_1' = (LJ)^m$ and $I_2' = (JL)^m$ satisfy the condition (6.5). Thus the theorem applies to all such ideals in the path algebra of a finite quiver. On the other hand, under the condition $IL = LI$, which is of most importance in this paper, we may simply choose $I_1' = I_2' = I^2$.

Proof. As usual we define the quotient algebras $\Lambda = \Gamma/I$, $\Lambda_1' = \Gamma/I_1'$ and $\Lambda_2' = \Gamma/I_2'$, and introduce the notation

$$\mathcal{L} = \frac{L}{IL+LI} \qquad \mathcal{L}' = \frac{L}{I_1'L} = \frac{L}{LI_2'}$$

for the reductions of L to a Λ,Λ-bimodule and a Λ_1',Λ_2'-bimodule respectively, and $\pi : L \to \mathcal{L}$ and $\pi' : L \to \mathcal{L}'$ for the canonical projections. A crucial observation is that $\mathcal{L}'$, being isomorphic to both $\Lambda_1' \otimes_\Gamma L$ and $L \otimes_\Gamma \Lambda_2'$, is a left-right projective Λ_1',Λ_2'-bimodule. It is this which will enable us to choose the maps l, r and Δ required by Definition 5.1.

We start by choosing arbitrary S,S complements U of $JL + LJ$ in L and V of JLJ in $JL \cap LJ$ and setting

$$\mathcal{P}(U) = \Lambda \otimes U \otimes \Lambda \qquad \mathcal{P}'(U) = \Lambda_1' \otimes U \otimes \Lambda_2'$$

and $\mathcal{P}(V)$ and $\mathcal{P}'(V)$ similarly. In addition, let $\pi_U : \mathcal{P}'(U) \to \mathcal{P}(U)$ be the reduction map modulo I/I_1' on the left and I/I_2' on the right, and π_V similarly. We may also identify U with a complement of $J\mathcal{L} + \mathcal{L}J$ in $\mathcal{L}$ and of $J\mathcal{L}' + \mathcal{L}'J$ in $\mathcal{L}'$, and hence, by Lemma 2.3, obtain minimal projective covers $e : \mathcal{P}(U) \to \mathcal{L}$ and $e' : \mathcal{P}'(U) \to \mathcal{L}'$.

Because $\mathcal{L}'$ is left-right projective, we may choose for e' a Λ_1', S-section $l' : \mathcal{L}' \to \mathcal{P}'(U)$ and an S, Λ_2'-section $r' : \mathcal{L}' \to \mathcal{P}'(U)$. We define $l, r : L \to \mathcal{P}(U)$ to be the composite maps

$$l = \pi_U l' \pi' \qquad r = \pi_U r' \pi'$$

and also define

$$\begin{aligned} d' &: \mathcal{P}'(V) \to \mathcal{P}'(U) : 1 \otimes v \otimes 1 \mapsto l'\pi'(v) - r'\pi'(v) \\ d &: \mathcal{P}(V) \to \mathcal{P}(U) : 1 \otimes v \otimes 1 \mapsto l(v) - r(v) \end{aligned}$$

so that $\pi_U d' = d\pi_V$, $e'd' = 0$, $ed = 0$ and $el = \pi = er$.

Let $\mathcal{K}' = \mathrm{Ker}(e')$, so that we have a short exact sequence

$$0 \to \mathcal{K}' \xrightarrow{i'} \mathcal{P}'(U) \xrightarrow{e'} \mathcal{L}' \to 0 \tag{6.6}$$

and let $\Delta' : \mathcal{L}' \to \mathcal{K}'$ be the unique map such that $i'\Delta' = l' - r'$. Note that $d' = i'q'$ where

$$q' : \mathcal{P}'(V) \to \mathcal{K}' : 1 \otimes v \otimes 1 \mapsto \Delta'\pi'(v)$$

We show first that q' is onto by applying Proposition 3.2 to (6.6) with $\Omega_i = \Lambda'_i$, $H_i = J/I'_i$, $\delta = \Delta'$. For in this case $\widetilde{e}' : U \to \mathcal{L}'/(J\mathcal{L}' + \mathcal{L}'J)$ is an isomorphism, so we may deduce that

$$\widetilde{\Delta}' : \frac{J\mathcal{L}' \cap \mathcal{L}'J}{J\mathcal{L}'J} \to \frac{\mathcal{K}'}{J\mathcal{K}' + \mathcal{K}'J}$$

is an isomorphism. Since

$$\widetilde{\pi}' : \frac{JL \cap LJ}{JLJ} \to \frac{J\mathcal{L}' \cap \mathcal{L}'J}{J\mathcal{L}'J}$$

is also an isomorphism, we see that $\Delta'\pi'$ maps V isomorphically to a complement of $J\mathcal{K}' + \mathcal{K}'J$ in $\mathcal{K}'$ and hence, by Lemma 2.3, q' is a minimal projective cover.

Now let

$$\pi_{\mathcal{K}} : \mathcal{K}' \to \mathcal{K} = \frac{\mathcal{K}'}{I\mathcal{K}' + \mathcal{K}'I}$$

be the canonical projection from $\mathcal{K}'$ to its Λ, Λ-bimodule reduction $\mathcal{K}$, and let $q : \mathcal{P}(V) \to \mathcal{K}$ and $i : \mathcal{K} \to \mathcal{P}(U)$ be the reductions of q' and i' respectively, so that $iq = d$, $\pi_U i' = i\pi_{\mathcal{K}}$ and $\pi_{\mathcal{K}} q' = q\pi_V$. Note that q is still epi, but i need not be mono and $\mathcal{K}$ need not be $\mathrm{Ker}(e)$.

Applying Proposition 3.2 to (6.6) with $\Omega_i = \Lambda'_i$, $H_i = I/I'_i$, $\delta = \Delta'$, we obtain the four term exact sequence

$$0 \to \frac{I\mathcal{L}' \cap \mathcal{L}'I}{I\mathcal{L}'I} \xrightarrow{\widetilde{\Delta}'} \mathcal{K} \xrightarrow{i} \mathcal{P}(U) \xrightarrow{e} \mathcal{L} \to 0 \tag{6.7}$$

and we further observe, from (6.5), that π' induces an isomorphism

$$\widetilde{\pi}' : \frac{IL \cap LI}{ILI} \to \frac{I\mathcal{L}' \cap \mathcal{L}'I}{I\mathcal{L}'I}.$$

Comparing (6.7) with (6.1) we see that $\mathcal{K}$ and $\mathcal{P}(V)$ have the same finite composition length as S, S-bimodules and hence $q : \mathcal{P}(V) \to \mathcal{K}$, which is already known to be epi, is actually an isomorphism. Thus we obtain an isomorphism of exact sequences

$$\begin{array}{ccccccccc}
0 \to & \frac{IL\cap LI}{ILI} & \xrightarrow{c} & \mathcal{P}(V) & \xrightarrow{d} & \mathcal{P}(U) & \xrightarrow{e} & \mathcal{L} & \to 0 \\
 & \downarrow \widetilde{\pi}' & & \downarrow q & & \| & & \| & \\
0 \to & \frac{I\mathcal{L}'\cap\mathcal{L}'I}{I\mathcal{L}'I} & \xrightarrow{\widetilde{\Delta}'} & \mathcal{K} & \xrightarrow{i} & \mathcal{P}(U) & \xrightarrow{e} & \mathcal{L} & \to 0
\end{array}$$

where c is induced by the map $\Delta : L \to \mathcal{P}(V)$ defined by

$$\Delta = q^{-1}\pi_{\mathcal{K}}\Delta'\pi'.$$

We finally show that Δ satisfies the requirements of Definition 5.1, namely that

$$\begin{aligned} d\Delta &= i\pi_K\Delta'\pi' \\ &= \pi_U i'\Delta'\pi' \\ &= \pi_U(l' - r')\pi' \\ &= l - r \end{aligned}$$

and that, for $v \in V$,

$$\begin{aligned} \Delta(v) &= q^{-1}\pi_{\mathcal{K}}\Delta'\pi'(v) \\ &= q^{-1}\pi_{\mathcal{K}}q'(1 \otimes v \otimes 1) \\ &= \pi_V(1 \otimes v \otimes 1) \\ &= 1 \otimes v \otimes 1 \end{aligned}$$

This completes the proof of Theorem 6.3.

□

7. Minimal resolutions

We now complete the main objective of the paper by showing how standard presentations are used to construct minimal resolutions. Therefore we must work with sufficient conditions on the tensor algebra Γ, the ideal I and the left-right projective Γ, Γ-bimodule L to ensure that $\mathcal{L} = L/(IL + LI)$ does admit a standard presentation and a minimal resolution. To be precise, we assume that either (i) $J \subseteq \Gamma$ is nilpotent and I and L are arbitrary, or (ii) Γ is arbitrary, I is homogeneous and L is graded, or (iii) Γ and L are finitely generated, I is admissable and condition (6.5) holds. Hence we may apply Theorem 5.2 or Theorem 6.3 and, writing $\mathcal{N} = (IL \cap LI)/ILI$, obtain a standard presentation

$$0 \to \mathcal{N} \xrightarrow{c} \mathcal{P}(V) \xrightarrow{d} \mathcal{P}(U) \xrightarrow{e} \mathcal{L} \to 0 \tag{7.1}$$

determined by data (U, V, l, r, Δ) as in Definition 5.1. Furthermore, writing $L_+ = IL \cap LI$, we may choose a complement U_+ of $JL_+ + L_+J + ILI$ in L_+ to obtain, by Lemma 2.3, a minimal projective cover

$$e_+ : \mathcal{P}(U_+) \to \mathcal{N} : 1 \otimes u \otimes 1 \mapsto u + ILI.$$

and thus a three term projective resolution of $\mathcal{L}$

$$\mathcal{P}(U_+) \xrightarrow{ce_+} \mathcal{P}(V) \xrightarrow{d} \mathcal{P}(U) \xrightarrow{e} \mathcal{L} \to 0 \tag{7.2}$$

By construction, $e : \mathcal{P}(U) \to \mathcal{L}$ is a minimal projective cover, but, since $d(1 \otimes v \otimes 1) = l(v) - r(v)$, it is easy to see that the summand $\mathcal{P}(V \cap L_+)$ of $\mathcal{P}(V)$

is contained in $\mathrm{Ker}(d)$, so that d is not necessarily a minimal projective cover of $\mathrm{Ker}(e)$. This summand must also occur in $\mathcal{N}$ and $\mathcal{P}(U_+)$, from where it is mapped isomorphically by c and e_+. Hence it is possible to 'excise' this summand from the map ce_+ and reduce (7.2) to a smaller resolution. We will now show that there is an optimal choice of V such that this excision leaves us with the first three terms of a minimal resolution.

To optimise the choice of V, we consider the following lattice of submodules of L.

$$\begin{tikzcd}
 & & JLJ \arrow[dl] & & \\
JL \cap LJ & L_+ + JLJ \arrow[l] & & L_+ \cap JLJ \arrow[ul] \arrow[dl] & JL_+ + L_+J + ILI \arrow[l] \\
 & & L_+ \arrow[ul] & &
\end{tikzcd} \tag{7.3}$$

We then choose S, S-complements

$$\begin{aligned}
&Z \text{ of } L_+ \cap JLJ \text{ in } L_+, \\
&T_1 \text{ of } L_+ + JLJ \text{ in } JL \cap LJ, \\
&T_2 \text{ of } JL_+ + L_+J + ILI \text{ in } L_+ \cap JLJ.
\end{aligned} \tag{7.4}$$

Note that Z is also a complement of JLJ in $L_+ + JLJ$, and so, in constructing (7.2), we may choose

$$V = Z \oplus T_1 \qquad U_+ = Z \oplus T_2. \tag{7.5}$$

It is important to note at this stage that, under the assumptions (i), (ii) or (iii) above, there is no restriction on the choices of complements U and V that go into the standard presentation (see Remark 4.3 or Theorem 6.3), and, similarly, no restriction on the choice of complement U_+ that determines the cover of $\mathcal{N}$. Hence, in particular, there is no obstruction to making the choices in (7.5).

The restriction of ce_+ to the summand $\mathcal{P}(Z) \subseteq \mathcal{P}(U_+)$ is given by

$$ce_+(1 \otimes z \otimes 1) = \Delta(z) = 1 \otimes z \otimes 1$$

for $z \in Z \subseteq V$, where the second equality is part of Definition 5.1(iii). Hence, with respect to the direct sum decompositions induced by (7.5),

$$ce_+ = \begin{pmatrix} 1 & * \\ 0 & \partial_2 \end{pmatrix} \qquad d = \begin{pmatrix} 0 & \partial_1 \end{pmatrix}.$$

We may now excise the common summand $\mathcal{P}(Z)$ from $\mathcal{P}(U_+)$ and $\mathcal{P}(V)$ and obtain

Proposition 7.1. *The complex*

$$\mathcal{P}(T_2) \xrightarrow{\partial_2} \mathcal{P}(T_1) \xrightarrow{\partial_1} \mathcal{P}(U) \xrightarrow{e} \mathcal{L} \to 0 \tag{7.6}$$

is the beginning of a minimal resolution of $\mathcal{L}$. The maps in (7.6) are given by

$$\begin{aligned} e &: 1 \otimes u \otimes 1 \mapsto u + (IL + LI) \\ \partial_1 &: 1 \otimes t \otimes 1 \mapsto l(t) - r(t) \\ \partial_2 &: 1 \otimes t \otimes 1 \mapsto (1 \otimes p_1 \otimes 1)\Delta(t) \end{aligned} \tag{7.7}$$

where $p_1 : V \to T_1$ is the canonical projection associated to the decomposition (7.5).

Proof. It is already clear that the sequence (7.6) is exact. To show that it is a minimal resolution, we need to show, by (2.3), that $T_i \cong \operatorname{Tor}_i^{\Lambda^{ev}}\big(S^{ev}, \mathcal{L}\big)$ for $i = 1, 2$.

Let $\mathcal{M} = \operatorname{Ker}(e)$ and split (7.1) into the two short exact sequences

$$\begin{aligned} 0 \to \mathcal{N} \xrightarrow{c} \mathcal{P}(V) \longrightarrow \mathcal{M} \to 0 \\ 0 \to \mathcal{M} \longrightarrow \mathcal{P}(U) \xrightarrow{e} \mathcal{L} \to 0 \end{aligned}$$

Reducing these modulo J on both sides and replacing $\mathcal{N}/(J\mathcal{N} + \mathcal{N}J)$ by U_+ using the isomorphism $\widetilde{e}_+$ induced by e_+, we obtain the exact sequences

$$0 \to \operatorname{Tor}_2^{\Lambda^{ev}}\big(S^{ev}, \mathcal{L}\big) \longrightarrow U_+ \xrightarrow{\widetilde{ce_+}} V \longrightarrow \frac{\mathcal{M}}{J\mathcal{M} + \mathcal{M}J} \to 0 \tag{7.8a}$$

$$0 \to \operatorname{Tor}_1^{\Lambda^{ev}}\big(S^{ev}, \mathcal{L}\big) \longrightarrow \frac{\mathcal{M}}{J\mathcal{M} + \mathcal{M}J} \longrightarrow U \xrightarrow{\widetilde{e}} \frac{\mathcal{L}}{J\mathcal{L} + \mathcal{L}J} \to 0 \tag{7.8b}$$

Now for $t \in T_2 \subseteq JLJ$

$$ce_+(1 \otimes t_2 \otimes 1) = \Delta(t_2) \in J\mathcal{P}(V) + \mathcal{P}(V)J$$

and so, with respect to the decompositions (7.5),

$$\widetilde{ce_+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The exactness of (7.8a) implies that $T_2 \cong \operatorname{Tor}_2^{\Lambda^{ev}}\big(S^{ev}, \mathcal{L}\big)$ and that ∂_1 induces an isomorphism of T_1 with the top of $\mathcal{M}$. Since $\widetilde{e}$ is an isomorphism, the exactness of (7.8b) then shows that $T_1 \cong \operatorname{Tor}_1^{\Lambda^{ev}}\big(S^{ev}, \mathcal{L}\big)$.

□

From Proposition 7.1 we may obtain the following formulae for the Tor-groups $T_i(\mathcal{L}) = \operatorname{Tor}_i^{\Lambda^{ev}}(S^{ev}, \mathcal{L})$, for general $\mathcal{L} = L/(IL + LI)$.

$$\begin{aligned} T_1(\mathcal{L}) &= \frac{JL \cap LJ}{(IL \cap LI) + JLJ} \\ T_2(\mathcal{L}) &= \frac{IL \cap LI \cap JLJ}{J(IL \cap LI) + (IL \cap LI)J + ILI} \end{aligned}$$

From the growing complexity of these formulae, we are rather pessimistic about the possibility of explicitly extending (7.6) to a full resolution for general $\mathcal{L}$. However, if we add to the assumptions (i), (ii) or (iii) the condition $IL = LI$, then the situation improves considerably. To start with we may omit (6.5) from (iii) because this new condition implies (6.5) (see Remark 6.4). Furthermore, Proposition 2.4 provides nice formulae for all the Tor groups that induce the terms in the minimal resolution. More importantly, one of the assumptions (i), (ii) or (iii) will also apply with L replaced by LI^n for all $n \geq 0$, and thus there will exist standard presentations

$$0 \to \frac{LI^{n+1}}{LI^{n+2}} \xrightarrow{c_n} \Lambda \otimes V_n \otimes \Lambda \xrightarrow{d_n} \Lambda \otimes U_n \otimes \Lambda \xrightarrow{e_n} \frac{LI^n}{LI^{n+1}} \to 0 \tag{7.9}$$

determined by data $(U_n, V_n, l_n, r_n, \Delta_n)$, as in Definition 5.1, in which the complements U_n and V_n may be chosen as in (7.5) so that

$$U_n = Z_n \oplus T_{2n} \qquad V_n = Z_{n+1} \oplus T_{2n+1} \tag{7.10}$$

where

$$\begin{aligned}
&Z_n \text{ is an } S,S\text{-complement of } LI^n \cap JLI^{n-1}J \text{ in } LI^n,\\
&T_{2n} \text{ is an } S,S\text{-complement of } JLI^n + LI^nJ \text{ in } LI^n \cap JLI^{n-1}J,\\
&T_{2n+1} \text{ is an } S,S\text{-complement of } LI^{n+1} + JLI^nJ \text{ in } JLI^n \cap LI^nJ.
\end{aligned} \tag{7.11}$$

Hence we may splice together the sequences (7.9) to obtain a projective Λ,Λ-bimodule resolution

$$\cdots \xrightarrow{d_{n+1}} \mathcal{P}(U_{n+1}) \xrightarrow{c_n e_{n+1}} \mathcal{P}(V_n) \xrightarrow{d_n} \cdots \xrightarrow{d_0} \mathcal{P}(U_0) \xrightarrow{e_0} L/LI \to 0 \tag{7.12}$$

As in the discussion leading to Proposition 7.1, each projective summand $\mathcal{P}(Z_n)$ may be excised from the map $c_n e_{n+1}$ and we thereby obtain

Theorem 7.2. *The standard presentations (7.9) together with their associated data determine a minimal resolution of L/LI*

$$\cdots \longrightarrow \mathcal{P}(T_m) \xrightarrow{\partial_m} \cdots \longrightarrow \mathcal{P}(T_1) \xrightarrow{\partial_1} \mathcal{P}(T_0) \xrightarrow{\varepsilon} L/LI \to 0 \tag{7.13}$$

The maps in (7.13) are given by

$$\begin{aligned}
\partial_{2n}(1 \otimes t \otimes 1) &= (1 \otimes p_{2n-1} \otimes 1)\Delta_{n-1}(t)\\
\partial_{2n+1}(1 \otimes t \otimes 1) &= (1 \otimes p_{2n} \otimes 1)(l_n - r_n)(t)
\end{aligned} \tag{7.14}$$

where $p_{2n-1} : V_{n-1} \to T_{2n-1}$ and $p_{2n} : U_n \to T_{2n}$ are the canonical projections associated to the decompositions (7.10).

Proof. As with Proposition 7.1, it is already clear that (7.13) is exact. The fact that it is a minimal resolution follows by comparing (7.11) with Proposition 2.4. $\square$

As a special case, we have achieved the aim of the paper.

Corollary 7.3. *Let $\Lambda = \Gamma/I$ and suppose that either (i) the augmentation ideal $J \subseteq \Gamma$ is nilpotent, or (ii) I is homogeneous, or (iii) Γ is finitely-generated and I is admissable. Then Λ has a minimal projective Λ, Λ-bimodule resolution*

$$\cdots \longrightarrow \mathcal{P}(T_m) \xrightarrow{\partial_m} \cdots \longrightarrow \mathcal{P}(T_1) \xrightarrow{\partial_1} \mathcal{P}(T_0) \xrightarrow{\varepsilon} \Lambda \to 0$$

where the spaces T_m and maps ∂_m are as described above in (7.9)–(7.14) with $L = \Gamma$.

8. Monomial algebras

In this section, we restrict to the case where Γ is the path algebra of a quiver over a field k. Hence Γ has a canonical basis consisting of all paths in the quiver, which we shall call *monomials*. Furthermore, we shall work entirely with *monomial subspaces* of Γ, that is, subspaces spanned by monomials. Under sums and intersection, such subspaces form a complemented lattice, which is also closed under products. The set of monomials which form the basis for a monomial subspace U will be denoted $U^\flat$. For any monomial u, its length (as a path in the quiver) is denoted $|u|$. As usual, the idempotents in Γ corresponding to the vertices of the quiver are monomials of length zero.

We shall show that a monomial ideal L has a canonical standard presentation and, if $I \subseteq J^2$ is another monomial ideal such that $IL = LI$, then the minimal projective bimodule resolution of L/LI over the monomial algebra $\Lambda = \Gamma/I$ may be described uniquely in terms of the monomial bases of its terms. In the specific case $L = \Gamma$, we obtain an explicit minimal resolution of Λ itself.

We begin by introducing some additional terminology. A monomial subspace U is called *elementary* if it does not contain a pair of monomials u, v such that u is a proper factor of v. The main way in which these will arise is as follows: if $B \subseteq A$ are monomial ideals and $JA + AJ \subseteq B$, then the monomial subspace complement to B in A, that is, the subspace spanned by $A^\flat - B^\flat$, is elementary.

Let U be an elementary monomial subspace of Γ. Then to each monomial $w \in \Gamma^\flat$ we may associate its *U-factorisation*, namely, the finite ordered list of its distinct factorisations

$$w = p_1 u_1 q_1 = p_2 u_2 q_2 = \cdots = p_r u_r q_r \tag{8.1}$$

with $u_i \in U^\flat$ and $p_i, q_i \in \Gamma^\flat$. The ordering is determined by the condition that $|p_1| < |p_2| < \cdots < |p_r|$ or, equivalently, that $|q_1| > |q_2| > \cdots > |q_r|$. Clearly, this list will be empty precisely when w is not in the monomial ideal generated by U.

Let L be a monomial ideal. Since it is graded by the length function, Proposition 4.2 implies that it has a standard presentation

$$0 \to \Gamma \otimes V \otimes \Gamma \xrightarrow{d_\Gamma} \Gamma \otimes U \otimes \Gamma \xrightarrow{e_\Gamma} L \to 0 \tag{8.2}$$

We may — and shall — choose U and V to be the elementary monomial subspace complements of the monomial ideals $JL+LJ$ in L and JLJ in $JL\cap LJ$ respectively. Observe that $U^\flat$ is the minimal set of monomial generators of L as a two-sided ideal, and $V^\flat$ consists of all monomials in $L^\flat$ whose U-factorisation has the form $v = u_1 q_1 = p_2 u_2$. In addition, there are canonical choices for the splittings l_Γ and r_Γ of e_Γ, which then determine d_Γ and Δ_Γ. To be precise, if $w \in L^\flat$ has U-factorisation as in (8.1), then

$$\begin{aligned} l_\Gamma(w) &= p_r \otimes u_r \otimes q_r \\ r_\Gamma(w) &= p_1 \otimes u_1 \otimes q_1 \end{aligned} \tag{8.3}$$

Thus, for $v \in V^\flat$,

$$d_\Gamma(1 \otimes v \otimes 1) = l_\Gamma(v) - r_\Gamma(v) = p_2 \otimes u_2 \otimes 1 - 1 \otimes u_1 \otimes q_2. \tag{8.4}$$

For those $w \in L^\flat$ whose U-factorisation contains precisely one term, we have $l_\Gamma(w) = r_\Gamma(w)$ and so $\Delta_\Gamma(w) = 0$. Otherwise, the U-factorisation (8.1) of w determines a non-empty V-factorisation

$$w = p_1 v_1 q_2 = p_2 v_2 q_3 = \cdots = p_{r-1} v_{r-1} q_r, \tag{8.5}$$

with $v_i \in V^\flat$, and it may easily be verified that setting

$$\Delta_\Gamma(w) = \sum_{i=1}^{r-1} p_i \otimes v_i \otimes q_{i+1}, \tag{8.6}$$

gives the required equalities $d_\Gamma \Delta_\Gamma = l_\Gamma - r_\Gamma$, and $\Delta_\Gamma(v) = 1 \otimes v \otimes 1$ for $v \in V$.

We now introduce a *monomial algebra* $\Lambda = \Gamma/I$, where $I \subseteq J^2$ is a monomial ideal with $IL = LI$. By Theorem 5.2, the standard presentation (8.2), with additional data $(l_\Gamma, r_\Gamma, \Delta_\Gamma)$ given by (8.3) and (8.6), determines a standard presentation of $\mathcal{L} = L/LI$ with additional data (l, r, Δ). Replacing L by $L_+ = LI$, we may also obtain a standard presentation of LI/LI^2 and splice these two together to give

$$\mathcal{P}(V_+) \xrightarrow{d_+} \mathcal{P}(U_+) \xrightarrow{ce_+} \mathcal{P}(V) \xrightarrow{d} \mathcal{P}(U) \xrightarrow{e} \mathcal{L} \to 0 \tag{8.7}$$

Following the excision procedure of Section 7, we will make the direct sum decompositions

$$V = Z \oplus T_1 \qquad U_+ = Z \oplus T_2 \qquad V_+ = Z_+ \oplus T_3 \tag{8.8}$$

where all the summands are elementary monomial subspaces, because all the ideals in the lattice (7.3) are monomial. We will now prove, amongst other things, that the maps in (8.7) are diagonal with respect to these decompositions.

Proposition 8.1.

a) *If $v \in V^\flat$ has U-factorisation $v = u_1q_1 = p_2u_2$, then either $q_1, p_2 \in I^\flat$ and $v \in Z^\flat$ or $q_1, p_2 \in J^\flat - I^\flat$ and $v \in T_1^\flat$.*

b) *Let $t_2 \in T_2^\flat$ have U-factorisation $t_2 = p_1u_1q_1 = \cdots = p_ru_rq_r$, which determines its V-factorisation $t_2 = p_1v_1q_2 = \cdots = p_{r-1}v_{r-1}q_r$. Then $r \geq 3$,*

$$\begin{array}{lll} p_1 \in S^\flat & p_2, \ldots, p_{r-1} \in J^\flat - I^\flat & p_r \in I^\flat - JI^\flat \\ q_1 \in I^\flat - IJ^\flat & q_2, \ldots, q_{r-1} \in J^\flat - I^\flat & q_r \in S^\flat \end{array}$$

and $v_1, \ldots, v_{r-1} \in T_1^\flat$. Hence ce_+ is diagonal.

c) *In the U_+-factorisation $t_3 = u_1q_1 = p_1u_2$ of $t_3 \in T_3^\flat$, the U_+-factors u_1, u_2 are both in $T_2^\flat$ and hence the map d_+ is diagonal.*

Proof. For (a), suppose that $q_1 \in I$ so that $v \in LI = IL$. Then $v = p_2u_2$ and $u_2 \notin JL$, so $p_2 \in I$. Hence, the only possibilities are $q_1, p_2 \in I^\flat$ or $q_1, p_2 \in J^\flat - I^\flat$. Since by definition $Z \subseteq LI$, the first occurs when $v \in Z^\flat$ and the second otherwise.

For (b), note first that $t_2 \in IL^\flat - (JIL + ILJ)^\flat$, hence $p_r \in I^\flat - JI^\flat$ and $q_r \in S^\flat$. Similarly, $q_1 \in I^\flat - IJ^\flat$ and $p_1 \in S^\flat$. But now $t_2 \in JLJ$ implies that $r \geq 3$, and $t_2 \notin (JLI + ILJ)^\flat$ implies that $p_i, q_i \in J^\flat - I^\flat$ for $1 < i < r$. Finally, since $t_2 \notin JLI + LIJ$ we see that $v_i \notin LI$, and hence $v_i \in T_1$ (as in (a)).

For (c), observe from (a) that $q_1 \notin I$, but $u_1q_1 = p_2u_2 \in JLI$ and hence $u_1 \in JLJ \cap U_+ = T_2$. Similarly, $p_2 \notin I$ implies that $u_2 \in T_2$.

□

As in Section 7, the partial resolution (8.7) may be extended to the full 'spliced resolution' (7.12), which in this case is uniquely determined by the assumption that all choices respect the monomial bases. Hence the superfluous summands may be excised to give the minimal projective resolution

$$\cdots \longrightarrow \mathcal{P}(T_m) \xrightarrow{\partial_m} \cdots \longrightarrow \mathcal{P}(T_1) \xrightarrow{\partial_1} \mathcal{P}(T_0) \xrightarrow{\varepsilon} L/LI \to 0 \qquad (8.9)$$

where the T_m's are the unique elementary monomial subspaces satisfying (7.11). In fact, this resolution may be constructed in the following algorithmic way, involving no reference to the spliced resolution.

Theorem 8.2. *The terms T_m in the resolution (8.9) may be constructed inductively by letting $T_0^\flat$ be the elementary set of monomial generators of L and then, for $n \geq 0$, setting*

$$\begin{aligned} T_{2n+1}^\flat &= \Big[T_{2n}^\flat(J^\flat - I^\flat) \quad \cap \quad (J^\flat - I^\flat)T_{2n}^\flat\Big] \quad - \quad \Big[(J^\flat - I^\flat)T_{2n}^\flat(J^\flat - I^\flat)\Big] \\ T_{2n+2}^\flat &= \Big[(J^\flat - I^\flat)T_{2n}^\flat(J^\flat - I^\flat) \quad \cap \quad T_{2n}^\flat(I^\flat - IJ^\flat) \quad \cap \quad (I^\flat - JI^\flat)T_{2n}^\flat\Big] \\ &\quad - \quad \Big[(J^\flat - I^\flat)T_{2n}^\flat(I^\flat - IJ^\flat) \quad \cup \quad (I^\flat - JI^\flat)T_{2n}^\flat(J^\flat - I^\flat)\Big] \end{aligned} \qquad (8.10)$$

The maps $\partial_m : \Lambda \otimes T_m \otimes \Lambda \to \Lambda \otimes T_{m-1} \otimes \Lambda$ are determined as follows:

$$\partial_{2n+1}(1 \otimes v \otimes 1) = p_2 \otimes u_2 \otimes 1 - 1 \otimes u_1 \otimes q_1$$

where $v = u_1q_1 = p_2u_2$ is the T_{2n}-factorisation of $v \in T_{2n+1}$, and

$$\partial_{2n+2}(1 \otimes w \otimes 1) = \sum_{i=1}^{r-1} p_i \otimes v_i \otimes q_{i+1}$$

where $w = p_1v_1q_2 = \cdots = p_{r-1}v_{r-1}q_r$ is the T_{2n+1}-factorisation of $w \in T_{2n+2}$.

Recall that, in the last formula of the theorem, the T_{2n+1}-factorisation is determined by the T_{2n}-factorisation (8.1) and has the special properties stated in Proposition 8.1(b).

Proof. Parts (a) and (b) of Proposition 8.1 with L replaced by LI^n show that the two formulae in (8.10) are true with T_{2n} replaced by $U_n = Z_n \oplus T_{2n}$. In the case $n = 0$, we actually have $T_0 = U_0$ and thus the formulae are true. For $n \geq 1$, Proposition 8.1(c) with L replaced by LI^{n-1} shows that the U_n-factors of $w \in T^\flat_{2n+1}$ are all in T_{2n}. Hence the LHS of the first formula is contained in the RHS, and it only remains to prove that

$$T^\flat_{2n}\big(J^\flat - I^\flat\big) \,\cap\, \big(J^\flat - I^\flat\big)T^\flat_{2n} \,\cap\, \big(J^\flat - I^\flat\big)Z^\flat_n\big(J^\flat - I^\flat\big) \;=\; \emptyset$$

In fact, the argument used in the proof of Proposition 8.1(c) shows that

$$\big(J^\flat - I^\flat\big)U^\flat_n \,\cap\, \big(J^\flat - I^\flat\big)Z^\flat_n\big(J^\flat - I^\flat\big) \;=\; \emptyset.$$

Similarly, Proposition 8.1(b) shows that the LHS of the second formula is contained in the RHS, and equality follows from the fact that

$$\begin{aligned} U^\flat_n\big(I^\flat - IJ^\flat\big) \,\cap\, \big(J^\flat - I^\flat\big)Z^\flat_n\big(I^\flat - IJ^\flat\big) \;&=\; \emptyset \\ \big(I^\flat - JI^\flat\big)U^\flat_n \,\cap\, \big(I^\flat - JI^\flat\big)Z^\flat_n\big(J^\flat - I^\flat\big) \;&=\; \emptyset \end{aligned}$$

The formulae for the maps ∂_m come from (7.14),(8.4) and (8.6) together with the fact from Proposition 8.1 that the maps ce_+ and d_+ are diagonal, so the extra projections in (7.14) are not required.

$\square$

Remark 8.3. In the case $L = \Gamma$, that is, $L/LI = \Lambda$, the monomials in $T^\flat_m$ are well-known in the literature under the name '$(m-1)$-chains' or 'associated paths' ([**1**], [**2**], [**3**], [**13**]). As such, they are defined by a recursion which is somewhat simpler than (8.10), but which is apparently one sided. They were originally used to construct the minimal resolution of S as a one-sided Λ-module, but the two-sided resolution of Λ, with maps exactly as in Theorem 8.2, has now been constructed directly by Bardzell [**4**], who shows in particular (loc. cit. Lemma 3.1) that the two one-sided recursions are equivalent.

Example 8.4. Let Λ be a monomial algebra in which the relation ideal I is generated by a single monomial. In most cases Λ has global dimension two, because $T^\flat_3$ is empty. However, this fails precisely when the generator of I has the form

$(yz)^t y$ with $t \geq 1$, where y is a monomial of length ≥ 1 and z is a monomial of length ≥ 0 such that yz is a basic cycle, that is, a path with the same initial and final vertex which is not a power of a shorter cycle. In this case, Λ has infinite global dimension and the projectives $\mathcal{P}(T_m)$ in the minimal resolution are all isomorphic for $m \geq 2$, although they have different natural gradings. More precisely,

$$T_{2n}^\flat = \left\{(yz)^{nt+n-1}y\right\}$$
$$T_{2n+1}^\flat = \left\{(yz)^{nt+n}y\right\}$$

so that T_m is isomorphic to the S, S-bimodule generated by y. Using this isomorphism, so that $\mathcal{P}(T_m) = \Lambda \otimes \langle y \rangle \otimes \Lambda$, the minimal resolution is periodic after $\mathcal{P}(T_2)$ of period two, with differentials

$$\partial_{2n+1} : 1 \otimes y \otimes 1 \mapsto yz \otimes y \otimes 1 - 1 \otimes y \otimes zy$$
$$\partial_{2n+2} : 1 \otimes y \otimes 1 \mapsto \sum_{p+q=t} (yz)^p \otimes y \otimes (zy)^q$$

Example 8.5. One way of building up more complicated examples is to take coproducts. More explicitly, let Λ be a monomial algebra and suppose that the arrow set Q_1 of the underlying quiver is the disjoint union of two subsets Q_1' and Q_1'' in such a way that each minimal monomial relation is a path consisting entirely of arrows that lie in one or other of these two subsets. Then Λ is the coproduct over S of the monomial algebras Λ' and Λ'' which are the obvious quotients of the path algebras of the smaller quivers with arrow sets Q_1' and Q_1'' respectively. If we denote by T_m, T_m' and T_m'' the S, S-bimodules that induce the mth term in the minimal projective resolution of Λ, Λ' and Λ'' respectively, then it is easy to see that

$$T_m = T_m' \oplus T_m'' \tag{8.11}$$

for $m \geq 1$. This is true by construction for $m = 1, 2$ and follows inductively from the observation that, for $m \geq 3$, any element $t \in T_m^\flat$ has a pair of factorisations $t = t_1 q = p t_2$ with $t_1, t_2 \in T_{m-1}^\flat$, and $p, q \in J^\flat - I^\flat$ so that t_1 and t_2 must have at least one arrow in common. Furthermore, the differentials in the resolution (8.9) for Λ are the direct sums of the corresponding differentials for Λ' and Λ''.

9. Koszul algebras.

In this final section, we discuss one class of algebras, Koszul algebras, for which minimal projective bimodule resolutions are essentially already in the extensive literature. For an up-to-date and detailed treatment of the theory, we refer the reader to [**5**] §2. We recall the definition and some basic properties of Koszul algebras and discuss their relationship to the point of view of this paper.

To be Koszul, an algebra $\Lambda = \Gamma/I$ should be graded and the left Λ-module S should have a projective resolution whose mth term is generated in degree m, that is, by its component in degree m. Since such a resolution must necessarily be minimal, this is equivalent to requiring that $\mathrm{Tor}_m^\Lambda(S, S)$ should be purely of degree m, and thus also equivalent to requiring that Λ should have a (necessarily minimal) projective bimodule resolution whose mth term is generated in degree m.

Now, if Λ is Koszul, then it is necessarily quadratic, that is, the ideal I is generated by an S, S-submodule $R \subseteq X^2 = X \otimes X$. We will show that, in this more general case, it is possible to identify the degree m component of $\mathrm{Tor}_m^\Lambda(S, S)$, and also to write down explicitly a complex, which will give the minimal resolution of Λ, when Λ is Koszul. Define, as in [**5**] §2.6, the sequence of S, S-submodules $K_m \subset X^m$ by $K_0 = S$, $K_1 = X$ and for $m \geq 2$

$$K_m = \bigcap_{p+q=m-2} X^p R X^q \tag{9.1}$$

Proposition 9.1. *The natural graded isomorphisms (1.1) identify K_m with the m-th graded part of $\mathrm{Tor}_m^\Lambda(S, S)$.*

Proof. Suppose first that $m = 2n+2$. Since $JI^{n+1} + I^{n+1}J$ starts in degree $m+1$, the degree m part of $\mathrm{Tor}_m^\Lambda(S, S)$ is identified by (1.1) with the degree m part of $I^{n+1} \cap JI^nJ$. Clearly the degree m parts of I^{n+1} and JI^nJ are R^{n+1} and XR^nX respectively. Thus what we need to show is that

$$K_{2n+2} = R^{n+1} \cap XR^nX.$$

A similar argument shows that the case $m = 2n+1$ reduces to

$$K_{2n+1} = XR^n \cap R^nX.$$

Both of these formulae follow immediately from the fact that for all $n \geq 0$

$$R^{n+1} = \bigcap_{p+q=n} X^{2p} R X^{2q} \tag{9.2}$$

which we prove by induction, starting from the trivial case $n = 0$. Given the formula for R^n, we may write the right-hand-side of (9.2) as $X^{2n}R \cap R^nX^2$. Choosing a direct sum decomposition $X^2 = R \oplus R'$, we have $R^nX^2 = R^{n+1} \oplus R^nR'$. The result then follows because $R^{n+1} \subseteq X^{2n}R$, while

$$X^{2n}R \cap R^nR' \subseteq X^{2n}(R \cap R') = 0.$$

Note that in this proof we have made repeated use of Proposition A.1(b). $\square$

It is then clear that Koszul algebras are precisely those quadratic algebras for which $\mathrm{Tor}_m^\Lambda(S,S) = K_m$, that is, those which have a resolution whose mth term is

$$\mathcal{P}(K_m) = \Lambda \otimes K_m \otimes \Lambda.$$

In fact, we can generalise [**5**] Theorem 2.6.1 to show that, whenever Λ is quadratic, there is a 'bimodule Koszul complex' $\big(\mathcal{P}(K_*), d\big)$ which is exact precisely when Λ is Koszul. The differential d is made by combining the differentials from the left and right module Koszul complexes, whose terms are $\Lambda \otimes K_*$ and $K_* \otimes \Lambda$ respectively. More precisely, the inclusions

$$\begin{aligned} i_l &: K_m \hookrightarrow X \otimes K_{m-1} \\ i_r &: K_m \hookrightarrow K_{m-1} \otimes X \end{aligned}$$

which give the differentials in the left and right module Koszul complexes, may be used to define

$$\begin{aligned} d_l &: \mathcal{P}(K_m) \to \mathcal{P}(K_{m-1}) : 1 \otimes z \otimes 1 \mapsto i_l(z) \otimes 1 \\ d_r &: \mathcal{P}(K_m) \to \mathcal{P}(K_{m-1}) : 1 \otimes z \otimes 1 \mapsto (-1)^m \otimes i_r(z) \end{aligned}$$

We then define $d = d_l + d_r$ and observe that d is a differential, because d_l and d_r are differentials and $d_l d_r + d_r d_l = 0$.

Now, it is clear that applying the functor $- \otimes_\Lambda S$ to either of the complexes $\big(\mathcal{P}(K_*), d\big)$ or $\big(\mathcal{P}(K_*), d_l\big)$ gives the left module Koszul complex, which provides a resolution of S when Λ is Koszul, and further that

$$\big(\mathcal{P}(K_*), d\big) \otimes_\Lambda S \otimes \Lambda \cong \big(\mathcal{P}(K_*), d_l\big). \tag{9.3}$$

There are, of course, analogous statements for right modules.

Theorem 9.2. *The quadratic algebra Λ is a Koszul algebra if and only if its bimodule Koszul complex defined above is exact.*

Proof. The first differential $d : \mathcal{P}(K_1) \to \mathcal{P}(K_0)$ is precisely the map d occurring in (1.2), whose cokernel is Λ. Hence, if the bimodule Koszul complex is exact, that is, exact at all $\mathcal{P}(K_m)$ with $m \geq 1$, then it is a resolution of Λ whose mth term is generated in degree m. Hence, by the remarks at the beginning of this section, Λ is a Koszul algebra. To see this another way, if the bimodule Koszul complex is exact, then it is split exact as a complex of right modules and so, applying the functor $- \otimes_\Lambda S$, we deduce directly that the left module Koszul complex is exact.

Conversely, if Λ is Koszul, then the left module Koszul complex is exact by [**5**] Theorem 2.6.1 and hence, by (9.3), the complex $\big(\mathcal{P}(K_*), d_l\big)$ is exact. To deduce that the complex $\big(\mathcal{P}(K_*), d\big)$ is exact, we first observe that the differentials d_l and d_r, as well as d, all preserve the grading of $\mathcal{P}(K_*)$ by total degree. Hence it is sufficient to prove the exactness of each finite subcomplex of fixed total degree N, whose mth term is

$$\mathcal{P}(K_m)[N] = \bigoplus_{m+p+q=N} \Lambda[p] \otimes K_m \otimes \Lambda[q]$$

where $\Lambda[p]$ denotes the pth graded component of Λ. Now, each such subcomplex is the total complex of a finite double complex

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
\cdots & \Lambda[p]\otimes K_m\otimes\Lambda[q] & \xrightarrow{d_l} & \Lambda[p+1]\otimes K_{m-1}\otimes\Lambda[q] & \cdots \\
 & \downarrow d_r & & \downarrow d_r & \\
\cdots & \Lambda[p]\otimes K_{m-1}\otimes\Lambda[q+1] & \xrightarrow{d_l} & \Lambda[p+1]\otimes K_{m-2}\otimes\Lambda[q+1] & \cdots \\
 & \vdots & & \vdots &
\end{array}$$

The rows of this double complex are exact at every term $\Lambda[p]\otimes K_m\otimes\Lambda[q]$ with $m\geq 1$ and hence, by a familiar argument (see for example the proof of [**17**] Lemma 2.7.3) the total complex is exact at every term with $m\geq 1$.

□

Remark 9.3. If Λ is the coproduct over S of two Koszul algebras Λ' and Λ'', then the explicit construction of T_m as K_m enables us to prove (8.11) in this case as well, and deduce that Λ is also Koszul. We would expect that (8.11) holds for coproducts over S of any quotients of tensor algebras considered in this paper and further that the maps in the minimal resolution are also direct sums as in Example 8.5.

Appendix

The main purpose of this appendix is to prove the structure theorem (Theorem A.4) for projective Γ, S-bimodules that was used in the proof of Theorem 6.2. However, this result is a simple modification of a one-sided structure theorem (Theorem A.3) for projective modules over a tensor algebra over any semisimple artinian ring, and this appears to be the natural degree of generality for the result. Therefore, up to and including the proof of Theorem A.3, we will assume only that S is a semisimple artinian ring. As usual, X is any S,S-bimodule, Γ is the tensor algebra of X over S, and $\otimes$ denotes tensor product over S.

We will show that the projective left Γ-modules are precisely the induced modules, that is those of the form $F=\Gamma\otimes Y$, where Y is an S-module. Since S is semisimple any such Y is projective over S, and hence any such F is certainly projective over Γ. On the other hand, any projective Γ-module P is isomorphic to a summand of an induced module: for example, by choosing a splitting of the epimorphism

$$\Gamma\otimes P\to P:\ a\otimes p\mapsto ap$$

Therefore, it will be more than sufficient to show that any submodule of an induced module is induced.

The fact that projective modules are induced is well-known in special cases. There are several proofs that, if Γ is a free associative algebra over a field, then projective Γ-modules are free. The elegant proof in [**8**] Theorem 11.5.1 uses Schreier sets, as in one of the well-known proofs that subgroups of free groups are free.

This proof is readily adapted to show that projective modules over the path algebra of a quiver with finite vertex set are induced. Alternatively, this quiver algebra case may be obtained, albeit rather artifically, as an application of the coproduct theory in [**6**]. The proof we give below uses what appears to be a new method, which applies to tensor algebras in general.

We start with two fundamental properties of an induced module F.

Proposition A.1.

a) For any S-submodule $U \subseteq F$, the multiplication map

$$\mu_U : X \otimes U \to XU : x \otimes u \mapsto xu$$

is an isomorphism.

b) For any pair of S-submodules $U, V \subseteq F$,

$$X(U \cap V) = XU \cap XV.$$

Proof. (a) Observe first that $\mu_F : X \otimes F \to XF$ is an isomorphism by the definition of multiplication in the tensor algebra. Since S is semisimple, the functor $X \otimes -$ is exact. Hence, applying this functor to the inclusion $U \subseteq F$ yields an inclusion $X \otimes U \subseteq X \otimes F$ and the result follows.

(b) Consider the diagram of inclusions

$$\begin{array}{ccc} U \cap V & \overset{j_U}{\longrightarrow} & U \\ \downarrow j_V & & \downarrow i_U \\ V & \overset{i_V}{\longrightarrow} & F \end{array}$$

and the associated exact sequence

$$0 \to U \cap V \overset{(j_U, j_V)}{\longrightarrow} U \oplus V \overset{(i_U, -i_V)}{\longrightarrow} F \qquad (A.1)$$

Applying the exact functor $X \otimes -$, we obtain

$$0 \to X \otimes (U \cap V) \longrightarrow X \otimes U \oplus X \otimes V \longrightarrow X \otimes F$$

Now, by (a), the multiplication maps provide an isomorphism of this exact sequence with

$$0 \to X(U \cap V) \longrightarrow XU \oplus XV \longrightarrow XF$$

in which the components of the maps are inclusions, as in (A.1). The exactness of this sequence at $XU \oplus XV$ is the required result.

$\square$

Corollary A.2. *Let L be any Γ-submodule of F. If Z is an S-submodule of L such that $Z \cap XL = 0$, then the multiplication map $\Gamma \otimes Z \to L$ is injective.*

Proof. By iterating Proposition A.1(b), we see that $X^n Z \cap X^{n+1} L = 0$ for all n. Hence the sum $\sum_{n \geq 0} X^n Z$ is direct. By iterating Proposition A.1(a), we see that the multiplication maps $X^{\otimes n} \otimes Z \to X^n Z$ are isomorphisms, which completes the proof.

$\square$

Theorem A.3. *Let L be any Γ-submodule of F. Then there is an S-complement Z of XL in L such that the Γ-morphism*

$$\pi : \Gamma \otimes Z \to L : a \otimes z \mapsto az$$

is an isomorphism. Thus, submodules of induced modules are induced.

Proof. Corollary A.2 shows that π is injective for any S-complement Z. The problem is to show that there is a complement for which π is surjective. It is easy to see that π is not surjective for all complements: for example, $L = \Gamma = k[x]$ and $Z = k\langle 1 + x\rangle$.

Consider the 'degree' filtration of F with terms

$$F(n) = \bigoplus_{p \leq n} X^{\otimes p} \otimes Y$$

and let $L(n) = F(n) \cap L$ be the induced filtration of L. Clearly, $L(n) \subseteq L(n+1)$, because $F(n) \subseteq F(n+1)$, for all $n \geq 0$, and L is the union of the $L(n)$'s, because F is the union of the $F(n)$'s. The key point is that this filtration of L inherits a further property from the degree filtration of F: namely, for all $n \geq 0$

$$L(n) \cap XL = XL(n-1). \tag{A.2}$$

To see this, we make the S-module decomposition $F = F(n) \oplus X^{n+1}F$ and let $p_n : F \to X^{n+1}F$ be the canonical projection, so that $L(n) = \operatorname{Ker}\big(p_n|_L\big)$. Now consider the commutative diagram

$$\begin{array}{ccccc}
X \otimes L & \xrightarrow{1 \otimes i_L} & X \otimes F & \xrightarrow{1 \otimes p_{n-1}} & X \otimes X^n F \\
\downarrow \mu_L & & \downarrow \mu_F & & \downarrow \mu_{X^n F} \\
XL & \xrightarrow{i_{XL}} & XF & \xrightarrow{p_n|_{XF}} & X^{n+1}F
\end{array}$$

in which i_L and i_{XL} are inclusion maps, and the three vertical maps are multiplication maps and hence isomorphisms by Proposition A.1(a). The commutativity of the second square is clear, because F is induced. Using, once more, the fact that $X \otimes -$ is an exact functor, the kernel of the composite of the top row is $X \otimes L(n-1)$, and so its image under the multiplication map is $XL(n-1)$. On the other hand, the kernel of the composite of the bottom row is $L(n) \cap XL$ and so we have proved (A.2).

Now let $Z = \sum_{n \geq 0} Z[n]$, where $Z[n]$ is an S-complement of $L(n-1) + XL(n-1)$ in $L(n)$. An easy induction shows that $L(n) = \sum_{p+q \leq n} X^p Z[q]$. Hence $\pi : \Gamma \otimes Z \to L$ is surjective, as required. It remains to show that Z is a complement to XL, that is,that $Z \cap XL = 0$, since the surjectivity of π already implies that $Z + XL = L$.

Let $z = z_n + z_{n-1} + \cdots + z_0 \in Z$ with $z_i \in Z[i]$. If also $z \in XL$, then $z \in L(n) \cap XL = XL(n-1)$, by (A.2). But $z - z_n \in L(n-1)$, so $z_n \in Z[n] \cap \big(L(n-1) + XL(n-1)\big)$ and hence $z_n = 0$. Repeating the argument, we find that $z_{n-1} = \cdots = z_0 = 0$, and so $z = 0$ as required.

□

In the proof of Theorem 6.2, we actually need the following two sided result, for which we must bring back the assumption that S is separable.

Theorem A.4. *Let L be a left projective Γ, S-bimodule. Then there exists a Γ, S-isomorphism*

$$\Gamma \otimes \frac{L}{JL} \to L$$

whose reduction modulo J on the left is the identity on L/JL.

Proof. By Corollary 2.2 and the observation at the beginning of the Appendix, L may be identified with a Γ, S-submodule of an induced module $F = \Gamma \otimes Y$, where Y is an S, S-bimodule. One may easily see that the complement Z in Theorem A.3 may be chosen to be an S, S-submodule of L and so π is a Γ, S-isomorphism as required.

□

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