SOME RECENT TRENDS IN MODULAR REPRESENTATION THEORY

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Introduction

The study of modular representation theory was in some sense started by L.E. Dickson [11] in 1902. However, it was not until R. Brauer [7] started investigating the subject that it really got off the ground. In the years between 1935 and his death in 1977, he almost single-handedly constructed the corpus of what is now regarded as the classical modular representation theory. Brauer's main motivation in studying modular representations was to obtain number theoretic restrictions on the possible behavior of ordinary character tables, and thereby find restrictions upon the structure of finite groups. His work has been indispensable in the classification of the finite simple groups. For a definitive account of modular representation theory from the Brauer viewpoint (as well as some more modern material) see Feit [13].

It was really J.A. Green who first systematically attacked the study of modular representation theory from the point of view of looking at the set of indecomposable modules, starting with his paper [14]. Green's results formed an indispensable tool in the treatment by Thompson, and then more fully by E.C. Dade [10], of blocks with cyclic defect groups. Since then, many other people have become interested in the study of the modules for their own sake.

I wish to discuss here the developments of the last five years or so in this area. I shall not discuss the progress made in the theory of blocks and representations of particular classes of groups, where a lot of progress has also been made recently. I shall concentrate on three main topics, namely almost split sequences, representation rings, and algebraic varieties associated with the cohomology of modules.

Almost split sequences

In the course of studying the representation theory of Artin algebras (of which modular group algebras are an example), M. Auslander and I. Reiten [2] developed the notion of an almost split sequence (also called an Auslander-Reiten sequence).

Definition. An almost split sequence is a short exact sequence of modules

$$0 \to A \to B \xrightarrow{\sigma} C \to 0$$

with the following properties.

i) The sequence does not split.

ii) A and C are indecomposable.

iii) Given any map $\rho: D \to C$, either ρ is a split epimorphism (i.e. C is isomorphic to a direct summand of D with projection ρ) or there is a map $\phi: D \to B$ with $\rho = \phi\sigma$.

(Thus for example σ splits on every proper submodule of C.)

Theorem (M. Auslander and I. Reiten, [2]). If Λ is an Artin algebra and C is a non-projective indecomposable Λ-module, then there exists an almost split sequence terminating in C. This sequence is unique up to isomorphism of short exact sequences.

The existence of almost split sequences has also been established for lattices over an order, so that p-adic group algebras are covered as well as modular group algebras. However, the construction is quite different, so that for a modular group algebra we have $A \cong \Omega^2 C$ while for a p-adic group algebra $A \cong \Omega C$ (Ω denotes the 'Heller operator', of taking the kernel of the projective cover; we shall also need the operator $\mho$ which sends a module to the cokernel of the injective hull).

The almost split sequences fit together to form the Auslander-Reiten quiver as follows.

If U and V are indecomposable modules, we write $\mathrm{Rad}(U,V)$ for the space of non-invertible maps from U to V (if $U \not\cong V$ then $\mathrm{Rad}(U,V)$ is just $\mathrm{Hom}(U,V)$) and $\mathrm{Rad}^2(U,V)$ for the subspace spanned by elements of the form $\alpha\beta$ for $\alpha \in \mathrm{Rad}(U,W)$ and $\beta \in \mathrm{Rad}(W,V)$. Then $\mathrm{Irr}(U,V) = \mathrm{Rad}(U,V)/\mathrm{Rad}^2(U,V)$ is an $\mathrm{End}_{kG}(U)$ - $\mathrm{End}_{kG}(V)$ bimodule, possibly zero. Let (a_{UV}, a'_{UV}) be the lengths as left $\mathrm{End}_{kG}(U)$-module and right $\mathrm{End}_{kG}(V)$-module respectively. Note that if k is algebraically closed then $a_{UV} = a'_{UV} = \dim_k \mathrm{Irr}(U,V)$. The <u>Auslander-Reiten quiver</u> is the directed graph whose vertices are the indecomposable modules, and with a directed edge from U to V labelled (a_{UV}, a'_{UV}) whenever $\mathrm{Rad}(U,V) \neq \mathrm{Rad}^2(U,V)$.

<u>Theorem</u> (Auslander and Reiten). The Auslander-Reiten quiver of an Artin algebra is a locally finite graph. If V is not projective, then there is a directed edge from U to V if and only if U is a direct summand of the middle term of the almost split sequence terminating in V. In this case a'_{UV} is its multiplicity as a direct summand. If V is projective, there is a directed edge from U to V if and only if U is a direct summand of $\mathrm{Rad}\, V$, and its multiplicity then a'_{UV}. Dually, if U is not injective, then there is a directed edge from U to V if and only if V is a direct summand of the middle term of the almost split sequence commencing with U. In this case a_{UV} is its multiplicity as a direct summand. If U is injective, there is a directed edge from U to V if and only if V is a direct summand of $U/\mathrm{Soc}\, U$, and its multiplicity is then a_{UV}.

Using techniques developed by the people who study Artin algebras, and an invariant related to the complexity of a module (see later in this article), P. Webb [17] showed that the possible shape of a connected component of the Auslander-Reiten quiver of a group algebra is quite restricted. One corollary of Webb's work is the following theorem.

<u>Theorem</u>. Suppose P is a (non-simple) projective indecomposable module for a group algebra. Then $\mathrm{Rad}\, P/\mathrm{Soc}\, P$ is a direct sum of at most four indecomposable modules.

For the group A_4 over $GF(2)$ there is a projective indecomposable module P such that $\mathrm{Rad}\, P/\mathrm{Soc}\, P$ has three direct summands, but I know of no examples with four.

<u>Representation rings</u>

The representation ring (or Green ring) of a finite group is a complex vector space $A_k(G)$ whose basis elements are in one-one correspondence with the indecomposable kG-modules. Multiplication in $A(G)$ is given by tensor products, and is well defined by the Krull-Schmidt theorem.

Green first investigated this ring in the case of a cyclic group of order p, and showed that $A(C_p)$ is semisimple. It is now known that $A(G)$ is semisimple whenever kG has finite representation type (i.e. the Sylow p-subgroups of G are cyclic, where $p = \text{char } k$), as well as a few other cases in characteristic two, while it is not semisimple in general (Zemanek, [19,20]).

In 1980, Richard Parker and I started our investigation of representation rings, the outcome of which will appear in [4]. We defined bilinear forms $(\ ,\ )$ and $\langle\ ,\ \rangle$ on $A(G)$ as follows.

$$(V,W) = \dim_k \operatorname{Hom}_{kG}(V,W)$$
$$\langle V,W\rangle = \dim_k (V,W)_1^G$$

where $(V,W)_1^G$ is the space of homomorphisms from V to W which factor through a projective module.

The second form has the advantage of being symmetric, and the forms are related as follows. Let $u = P_k - \mho(k)$ and $v = P_k - \Omega(k)$ as elements of $A(G)$, where P_k is the projective cover of the trivial module k. Then $u^* = v$ and $uv = 1$, and

$$(V,W) = \langle v\cdot V,W\rangle = \langle V,u\cdot W\rangle$$
$$\langle V,W\rangle = (u\cdot V,W) = (V,v\cdot W)$$
$$\text{(hence } (V,W) = (W,v^2\cdot V)).$$

Now if V is an indecomposable module, define

$$\tau(V) = \begin{cases} \operatorname{Soc} V & \text{if } V \text{ is projective} \\ X_{\mho V} - \Omega V - \mho V & \text{otherwise} \end{cases}$$

where $0 \to \Omega V \to X_{\mho V} \to \mho V \to 0$ is the almost split sequence terminating in $\mho V$.

$$v\cdot\tau(V) = \begin{cases} V - \operatorname{Rad} V & \text{if } V \text{ is projective} \\ V + \Omega^2 V - X_V & \text{otherwise} \end{cases}$$

where $0 \to \Omega^2 V \to X_V \to V \to 0$ is the almost split sequence terminating in V.

For general $x = \Sigma a_i V_i \in A(G)$, define $\tau(x) = \Sigma \bar{a}_i \tau(V_i)$.

Theorem (Benson and Parker [4]). If V and W are indecomposable modules then

$$\langle V,\tau(W)\rangle = (V, v\cdot\tau(W)) = \begin{cases} d_V & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

where $d_V = \dim_k(\mathrm{End}_{kG}(V)/J(\mathrm{End}_{kG}(V)))$

(= 1 if k is algebraically closed)

Thus for general $x \in A(G)$, $\langle x,\tau(x)\rangle \geq 0$ and $(x, v\cdot\tau(x)) \geq 0$ with equality if and only if $x = 0$.

Corollary. Suppose V and W are kG-modules with the property that for any kG-module U, $\dim_k \mathrm{Hom}_{kG}(V,U) = \dim_k \mathrm{Hom}_{kG}(W,U)$. Then $V \cong W$ (and similarly if $\dim_k \mathrm{Hom}_{kG}(U,V) = \dim_k \mathrm{Hom}_{kG}(U,W)$).

We also investigated certain direct sum decompositions of A(G) associated with subgroups of G, and used these to investigate the notions of species, vertex and origin, which I shall not define here. As an example of such a decomposition, let $i_{H,G}$ and $r_{G,H}$ denote the induction and restriction maps.

Theorem (Benson and Parker [4]). Let $H \leq G$. Then

$$A(G) = \mathrm{Ker}(r_{G,H}) \oplus \mathrm{Im}(i_{H,G})$$

as a direct sum of ideals, and

$$A(H) = \mathrm{Im}(r_{G,H}) \oplus \mathrm{Ker}(i_{H,G})$$

as a direct sum of a subring and a subspace.

Corollary. Let $H \leq G$.

i) Suppose V_1 and V_2 are kG-modules such that $V_1{\downarrow_H}{\uparrow^G} \cong V_2{\downarrow_H}{\uparrow^G}$. Then $V_1{\downarrow_H} \cong V_2{\downarrow_H}$.

ii) Suppose W_1 and W_2 are kH-modules such that $W_1{\uparrow^G}{\downarrow_H} \cong W_2{\uparrow^G}{\downarrow_H}$. Then $W_1{\uparrow^G} \cong W_2{\uparrow^G}$.

We then went on to apply the nondegeneracy of the inner products, and the notions of species, vertex and origin, to investigate finite dimensional summands of A(G) satisfying certain natural properties. We showed how to set up, for each such summand, a pair of dual tables resembling Brauer's modular irreducible and projective indecomposable character tables. Our analogues for the centralizer orders are certain real algebraic numbers which need be neither positive nor rational! P. Webb [18] has also investigated some numerical properties of these tables.

In [5], I went on to investigate a certain special lambda-ring structure on A(G), and to relate this to the concepts introduced in [4].

Complexity and cohomology varieties

This subject started off with some work of Quillen [15,16] describing the structure of the set of prime ideals of the (even) equivariant cohomology ring of a compact Lie group $H_G^{ev}(X)$ with coefficients in a permutation representation X. His main results, when interpreted for finite groups, give a description of Spec $H^{ev}(G,\mathbf{Z}/p\mathbf{Z})$ in terms of the elementary abelian p-subgroups and their normalizers (the Quillen stratification theorem). In particular he showed that dim Spec $H^{ev}(G,\mathbf{Z}/p\mathbf{Z})$ is equal to the maximal rank of an elementary abelian p-subgroup of G.

The next move was made by Chouinard [9], who showed that an arbitrary module V in characteristic p is projective if and only if $V{\downarrow_E}$ is projective for all elementary abelian p-subgroups E.

Since then, the subject has progressed quite far, and the following is a summary of the present state of the subject.

Definitions

Let k be a field of characteristic p, and V a kG-module. Let $\alpha_1,\dots,\alpha_r$ be the degrees of homogeneous generators for $H^*(G,k)$ (which is known to be finitely generated). If

$$\cdots \to P_1 \to P_0 \to V \to 0$$

is a minimal projective resolution of V then the Poincaré series $\eta_V(t) = \Sigma t^i \dim(P_i)$ is a rational function of the form $p(t)/\pi(1-t^{\alpha_i})$, with p a polynomial with integer coefficients. The order of the pole of $\eta_V(t)$ at $t = 1$ is called the complexity of V, written $cx_G(V)$, and measures the rate of growth of $\dim(P_i)$.

Remark. The invariant used by Webb in his analysis of the Auslander-Reiten quiver was the value of the analytic function $(\pi\alpha_i)(1-t)^{cx_G(V)}\eta_V(t)$ at $t = 1$. (I believe this is related to the cardinality of the generic fibre of the natural map $\mathrm{Proj}(Z(\mathrm{Ext}_G^{ev}(V,V))) \to \mathrm{Proj}(\mathrm{Ext}_G^{ev}(k,k))$, but I have not had time to check this yet.)

Theorem (Properties of complexity). Let $H \leq G$, let V be a kG-module and W a kH-module.

i) $cx_G(V) = cx_G(V^*) = cx_G(V \otimes V^*) = cx_G(\Omega V)$

ii) $cx_G(V) \geq cx_H(V\downarrow_H)$

iii) $cx_G(W\uparrow^G) = cx_H(W)$

iv) If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of kG-modules then $cx_G(V_i) \leq \max(cx_G(V_j), cx_G(V_k))$, $\{i,j,k\} = \{1,2,3\}$. In particular, the two largest complexities are equal.

v) $cx_G(V \oplus V') = \max(cx_G(V), cx_G(V'))$

vi) If D is a vertex of V then $cx_G(V) = cx_D(V\downarrow_D)$

vii) $cx_G(V) + cx_G(V') - p\text{-rank}(G) \leq cx_G(V \otimes V') \leq \min(cx_G(V), cx_G(V'))$

viii) $cx_G(V) \leq cx_G(k) = p$-rank of G, where k denotes the trivial kG-module (c.f. (xii)).

ix) $cx_G(V) = 0$ if and only if V is projective.

x) (Eisenbud [12]) $cx_G(V) = 1$ if and only if V is periodic.

xi) If $|G:H| = p^n \cdot r$ with $(p,r) = 1$ then

$$cx_H(V\downarrow_H) \leq cx_G(V) \leq cx_H(V\downarrow_H) + n$$

xii) (Alperin, Evens [1]) $cx_G(V) = \max_E cx_E(V\downarrow_E)$ as E ranges over the elementary abelian p-subgroups of G.

xiii) If V and V' are in the same connected component of the Auslander-Reiten quiver of kG-modules and neither V nor V' is projective then $cx_G(V) = cx_G(V')$.

Note that (xii) is a generalization of Chouinard's result.

Definitions. Suppose k is algebraically closed. Denote by X_G the affine variety $\mathrm{Max}(H^{ev}(G,k))$ of maximal ideals of the even cohomology ring, with the Zariski topology. Then X_G is a union of lines through the origin, so we may form a projective variety $\bar{X}_G = \mathrm{Proj}(H^{ev}(G,k))$ of one smaller dimension.

Denote by $\mathrm{Ann}_G(V)$ the ideal of $H^{ev}(G,k)$ consisting of those elements annihilating $H^*(G,V)$. The support of a module V, written $X_G(V)$, is the set of all maximal ideals $M \in X_G$ which contain $\mathrm{Ann}_G(V \otimes S)$ for some module S. Denote by $I_G(V)$ the ideal of $H^{ev}(G,k)$ consisting of those elements x such that for all modules S, there exists a positive integer j with $H^*(G, V\otimes S)\cdot x^j = 0$ (cup product). Then $X_G(V) = \mathrm{Max}(H^{ev}(G,k)/I_G(V))$ is a homogeneous subvariety of X_G, and $\bar{X}_G(V) = \mathrm{Max}(H^{ev}(G,k)/I_G(V))$ is a projective (closed) subvariety of $\bar{X}_G$.

For H a subgroup of G, denote by $t_{H,G}$ the map from X_H to X_G induced by $\mathrm{res}_{G,H}: H^{ev}(G,k) \to H^{ev}(H,k)$.

Theorem (Properties of cohomology varieties). Let $H \leq G$, let V be a kG-module and W a kH-module.

i) $\dim(X_G(V)) = cx_G(V)$

ii) $X_G(V) = X_G(V^*) = X_G(V\otimes V^*) = X_G(\Omega V)$

iii) $X_H(V\downarrow_H) = t^{-1}_{H,G}(X_G(V))$

iv) $X_G(W\uparrow^G) = t_{H,G}(X_H(W))$

v) If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of kG-modules then $X_G(V_i) \subseteq X_G(V_j) \cup X_G(V_k)$, $\{i,j,k\} = \{1,2,3\}$

vi) $X_G(V\oplus V') = X_G(V) \cup X_G(V')$

vii) (Avrunin, Scott [3]) $X_G(V \otimes V') = X_G(V) \cap X_G(V')$

viii) $X_G(V) = \{0\}$ if and only if V is projective

ix) $X_G(V) = \bigcup_E t_{E,G}(X_E(V\downarrow_E))$ as E ranges over the elementary abelian p-subgroups of G.

x) If V and V' are indecomposable modules in the same connected component of the Auslander-Reiten quiver of kG-modules and neither V nor V' is projective then $X_G(V) = X_G(V')$.

xi) Given a closed homogeneous subvariety $X \subseteq X_G$ there is a module V with $X_G(V) = X$.

xii) If $X_G(V) \cap X_G(W) = \{0\}$ then $\mathrm{Ext}^i_G(V,W) = 0$ for all $i > 0$.

Definitions. If E is an elementary abelian p-group of rank r, then X_E is a vector space of dimension r. We define

$$X_E^+ = X_E \setminus \bigcup_{E'<E} t_{E',E}(X_{E'})$$

$$X_{G,E} = t_{E,G}(X_E) \qquad X_{G,E}^+ = t_{E,G}(X_E^+)$$

$$X_E^+(V) = X_E(V) \setminus \bigcup_{E'<E} t_{E',E}(X_{E'}(V))$$

$$X_{G,E}(V) = t_{E,G}(X_E(V)) \qquad X_{G,E}^+(V) = t_{E,G}(X_E^+(V))$$

Thus X_E^+ is the space X_E with all the hyperplanes defined over $\mathbb{Z}/p\mathbb{Z}$ removed.

Theorem (Quillen stratification for modules, Avrunin, Scott [3]). $X_G(V)$ is a disjoint union of the locally closed subvarieties $X_{G,E}^+(V)$ as E runs over a set of representatives of conjugacy classes of elementary abelian p-subgroups of G. The group $W_G(E) = N_G(E)/C_G(E)$ acts freely on $X_E^+(V)$, and $t_{E,G}$ induces a finite homeomorphism

$$X_E^+(V)/W_G(E) \to X_{G,E}^+(V)$$

(i.e. homeomorphism in the Zariski topology; Quillen calls this map an 'inseparable isogeny').

The natural map

$$\varinjlim_{E} X_E(V) \to X_G(V)$$

is a bijective finite morphism.

Finally, the following recent result of Carlson seems to be very important, since it is practically our only tool apart from Mackey decomposition for showing that a module decomposes as a direct sum.

Theorem (Carlson, [8]). If $X_G(V) \subseteq X_1 \cup X_2$, where X_1 and X_2 are closed homogeneous subvarieties of X_G with $X_1 \cap X_2 = \{0\}$, then we may write $V = V_1 \oplus V_2$ with $X_G(V_1) \subseteq X_1$ and $X_G(V_2) \subseteq X_2$. In particular if V is indecomposable then $\bar{X}_G(V)$ is topologically connected.

We may express these results in terms of $A(G)$ as follows. If X is a subset of $\bar{X}_G$, denote by $A(G,X)$ the linear span in $A(G)$ of the modules V for which $\bar{X}_G(V) \subseteq X$.

Theorem (properties of $A(G,X)$). Let $H \leq G$, X a subset of $\bar{X}_G$ and X' a subset of $\bar{X}_H$.

i) $A(G,X)$ is an ideal in $A(G)$

ii) $A(H,t^{-1}_{H,G}(X)) \supseteq r_{G,H}(A(G,X))$

iii) $A(G,t_{H,G}(X')) \supseteq i_{H,G}(A(H,X'))$

iv) $A(G,X)$ is closed under Ω, under taking dual modules, under forming extensions of modules, and under taking direct summands of modules.

v) $A(G,\phi)$ is the linear span of the projective modules

vi) $A(G,X_1 \cap X_2) = A(G,X_1) \cap A(G,X_2)$

vii) $A(G,X_1 \cup X_2) \supseteq A(G,X_1) + A(G,X_2)$ with equality if $X_1 \cap X_2 = \phi$.

viii) If $X_1 \subsetneqq X_2$ then $A(G,X_1) \subsetneqq A(G,X_2)$

ix) The indecomposable modules in $A(G,X)$ form a union of connected components of the Auslander-Reiten quiver of kG-modules.

x) The bilinear forms $(\ ,\ )$ and $\langle\ ,\ \rangle$ are non-singular on $A(G,X)$.

xi) Let ψ^n and λ^n denote the operations on $A(G)$ introduced in [5]. Then $\psi^n(A(G,X)) \subseteq A(G,X)$ for n coprime to p, while $\psi^p(A(G,X)) \subseteq A(G,X^{(p)})$, where $^{(p)}$ is the Frobenius map on varieties. Thus if $X = X^{(p)}$, $\lambda^n(A(G,X)) \subseteq A(G,X)$ for all n.

Remark. Most of the topics discussed here are investigated at greater length in my forthcoming book [6].

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