THE SIMPLE GROUP J_4

by

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PREFACE

I declare that this dissertation is my own original work except where mention is made to the contrary in the text. All work done in collaboration with others is explicitely declared on the next page. Moreover I declare that no substantially similar work is being or has been submitted for any degree, diploma or other qualification at any other University.

I would like to thank my supervisor Professor J.G.Thompson for his help and encouragement; also Dr. J.H.Conway for many interesting conversations.

My thanks are also due to the Science Research Council, without whose financial support I could not have undertaken this research, and Trinity College, Cambridge, who bestowed upon me the privileges and emoluments of a research scholar.

Declaration of Originality

This dissertation is partly work done in collaboration with S.Norton, R.Parker, J.Conway and J.Thackray, and partly work done on my own. Attributions are as follows :

Chapter 1 : Only the last section of this is my own work. The rest is due to R.T.Curtis and J.Conway.

Chapter 2 : This is mainly due to J.Conway although Theorem 2.3 is my own.

Chapter 3 : This is mostly my own work.

Chapter 4 : The first three sections are my own work, while the last section is a description of a notation due to Conway.

Chapter 5 : This is entirely my own work.

Chapter 6 : This describes joint work of Norton, Parker, Conway, Thackray and myself culminating in the proof of the existence of the simple group J_4 .

Chapter 7 : This is my own work.

For further details see pages 4 to 6 of the introduction.

Contents

Tables and Figures

Chapter 0 Introduction

One of the most useful tools in Mathematics and Theoretical Physics is the study of an object from the point of view of its group of symmetries. Thus for example Galois in the last century proved the insolubility of the general quintic equation by radicals, by looking at the symmetry groups of field extensions; and more recently Lie Groups have played an enormous part in the theory of fundamental particles in Physics.

Groups can be broken down into basic building blocks called simple groups (under some basic assumptions satisfied by most interesting groups including all finite groups and all Lie groups) so that structure theory breaks down naturally into two parts :

1) What are the simple groups ?

2) How are they glued together to make an arbitrary group ?

There are many questions for which an answer to 1) and some induction argument gives the full answer, but 2) is in general more intractable than 1).

For compact Lie groups, the answers to both 1) and 2) were fully worked out earlier this century by Cartan, Weyl, etc.

For finite groups the situation is more complicated. Work on the classification of finite simple groups is still in progress, and a general description of the present state of the theory can be found in [1]. The second question is only tractable in restricted situations.

The currently known finite simple groups, which are thought to comprise at least most, if not all, of the possible ones, are :

(i) The Alternating Groups $\mathcal{A}_n$ for $n \geqslant 5$

(ii) The Chevalley Groups $Chev(p^n)$

(iii) The twisted Chevalley Groups ${}^rChev(p^n)$

(iv) 26 Sporadic Groups, not fitting into classes (i) - (iii)

Classes (ii) and (iii) are the finite analogues of the compact Lie groups.

Of the 26 Sporadic groups, at the beginning of 1980 two had not been proven to exist, namely F_1 the 'Monster' (so called because of its enormous size - it has $2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71$ elements) and Janko's fourth group J_4. However, in January Robert L. Griess announced that he had constructed the Monster and that his construction had been carried out entirely by hand, and in February, S.Norton, R.Parker, J.Conway, J.Thackray and I completed our proof of the existence of J_4 using a computer (see [4]). It is the latter group which I wish to discuss in this dissertation.

In the process of trying to classify finite simple groups with a 'large' extraspecial 2-subgroup (an extraspecial p-subgroup E is large in G if $E = O_p(C_G(E'))$ and $C_G(E) \leqslant E$), Janko [2] conjectured the existence of a new sporadic group J_4 . In particular, he proved that given a finite simple group G satisfying :

<u>Hypothesis A</u>

The centralizer of some involution $C_G(z) = H$ is of shape $2^{1+12}.3M_{22}.2$ with $O_2(H)$ extraspecial and containing its centralizer, and modulo $\langle z \rangle$, H splits over $O_2(H)$, with as complement the triple cover of M_{22} with the outer automorphism adjoined.

(a group is of SHAPE A.B or AB when it has a normal subgroup of shape A with quotient of shape B ; names of groups are shapes; an elementary abelian group of order p^n has shape p^n ; a special group whose centre has order p^m and index p^n has shape p^{m+n}, etc.)

then G satisfies a list of properties including :

(i) $| G | = 2^{21}.3^3.5.7.11^3.23.29.31.37.43$

(ii) A Sylow 2-subgroup of G possesses exactly one elementary abelian subgroup of order 2^{11} and the normalizer of such a subgroup is a split extension of shape $2^{11}M_{24}$ where the action of M_{24} on the elementary abelian subgroup is the same as on the even portion of the dual of the Golay Code.

(iii) G possesses exactly one conjugacy-class of self-centralizing elementary abelian subgroups of order 2^{10}, and the normalizer of such a subgroup is a split extension of shape $2^{10}L_5(2)$ where the action of $L_5(2)$ on the elementary abelian subgroup is the same as the action on the skew-square (i.e. exterior second power) of a natural 5-dimensional module.

(iv) G possesses a special 2-subgroup of shape 2^{3+12} whose normalizer is of shape $2^{3+12}(S_5 \times L_3(2))$. This group does not split over its O_2, but does contain subgroups isomorphic to S_5 and $L_3(2)$. It contains the Sylow 5-normalizer in G of shape $5.4 \times 2^3L_3(2)$ (with the $2^3L_3(2)$ non-split) and the 7-normalizer of shape $7.3 \times S_5$.

(v) For p = 23, 29, 31, 37 and 43, a Sylow p-subgroup is self-centralizing with normalizers of shape 23.22 , 29.28 , 31.10 , 37.12 and 43.14 respectively (all Frobenius groups). Sylow 3- and 11-subgroups are extraspecial with normalizers of shape $3^{1+2}.8.2$ (semidihedral 2-subgroup of order 16) and $11^{1+2}(5 \times 2S_4)$ ($2S_4$ non-split) respectively.

(vi) G possesses $PGL_2(23)$ as a subgroup.

(vii) The character table of G is known and was determined in Cambridge in 1975 by J.Conway, S.Norton, J.Thompson and D.Hunt (see Appendix A).

In January 1980 , Thompson and Norton showed that a simple group satisfying these conditions is unique, using character-theoretic methods (before the existence proof was completed) and so I shall say that any group satisfying the above conditions is 'isomorphic to J_4'.

In this dissertation, I shall develop notations for working inside J_4 , give the existence proof, and provide a presentation for J_4 by generators and relators. I shall use [2] and the character table of J_4 as my starting point.

Complete familiarity with the Mathieu groups will be assumed, although I have spent Chapter 1 developing

Curtis' MOG for M_{24} . The approach given is a recent unpublished one due to Conway.

In Chapters 2, 3, 4 and 5, I take an 'incident' set of representatives (in the sense of Smith and Ronan [11], see the CODA after Chapter 5) H, M, P and L of the four conjugacy classes of maximal 2-local subgroups of J_4 and develop notations for working with them, given that J_4 exists.

Chapter 2 gives a notation for working inside the maximal 2-local $H = C_{J_4}(z)$ of shape $2^{1+12}.3M_{22}.2$ described in Hypothesis A above. The notation is mostly due to Conway.

In Chapter 3 a particular representative M is chosen of the conjugacy class of maximal 2-locals of shape $2^{11}M_{24}$ described in (ii) above and the 'dictionary' is developed between the notations for elements of $M \cap H$ as elements of M and as elements of H . Some elementary consequences of this dictionary are then investigated, for use later on. This chapter is mostly my own work.

Chapter 4 describes a particular representative P of the class of maximal 2-locals of shape $2^{3+12}(S_5 \times L_3(2))$ described in (iv) above, and describes the notation due to Conway for elements of this. This notation is not used again but is included for the sake of completeness.

In Chapter 5 a representative L is chosen of the class of maximal 2-locals of shape $2^{10}L_5(2)$ described in (iii) above, and a particularly good complement for $O_2(L)$ in L is found. This chapter is my own work.

Chapter 6 describes the construction by Norton, Parker and Thackray of a pair of 112×112 matrices over GF(2) generating J_4 , some of the methods developed by Parker and Thackray for dealing with 2-modular representations on a computer, and the proof by Norton, Parker, Conway, Thackray and myself that the group generated by these matrices is indeed isomorphic to J_4 . The main heavy computer work in this proof is involved in showing that the skew-square of the 112-dimensional representation has an invariant subspace of dimension 4995.

In chapter 7 , further details of the geometry of the 112-dimensional representation are investigated, and a presentation for J_4 by generators and relators is proven. This presentation consists of adding two relators to the amalgamated product of a copy of M with a copy of H via their intersection $D = M \cap H$. It is conceptually easy to see what these relators are doing : one comes from the subgroup P described in chapter 4, and the other involves a subgroup $PGL_2(23)$ intersecting M in a subgroup $L_2(23)$ (c.f. (vi) above). The work of this chapter can easily be modified to give a proof that the 112-dimensional matrices described in chapter 6 generate a group isomorphic to J_4 , independent of the finding of the invariant subspace of the skew-square of the representation. This chapter is my own work.

Chapter 1 The MOG for M_{24}

Since the notations we have developed for working inside J_4 depend heavily on use of R.T.Curtis' MOG (Miracle Octad Generator) for M_{24} (see [3]), a few words about this are in order.

Let $GF(4) = \{0, 1, \omega, \bar{\omega}\}$ with the usual multiplication and addition.

Definition The HEXACODE is the self-dual code in $(GF(4))^6$ generated by the following code-words :

$(\omega\bar{\omega}\ \omega\bar{\omega}\ \omega\bar{\omega})$

$(\omega\bar{\omega}\ \bar{\omega}\omega\ \bar{\omega}\omega)$

$(\bar{\omega}\omega\ \omega\bar{\omega}\ \bar{\omega}\omega)$

and $(\bar{\omega}\omega\ \bar{\omega}\omega\ \omega\bar{\omega})$

Since these words add up to zero and clearly satisfy no other linear relations, the code generated has dimension 3.

Definition An AUTOMORPHISM of the Hexacode is a semilinear transformation of the form :

(multiplication of each coördinate by a non-zero element of GF(4)) . (permutation of the six coördinates) . (field automorphism (possibly trivial))

preserving the set of codewords.

There is a visible automorphism group $3(S_2 \wr S_3)$ given by multiplications by field elements followed by permutations preserving the given grouping of the six coördinates into three sets of two, followed by the field automorphism for odd coördinate permutations.

It is clear that every code-word is equivalent under the action of this group to one of :

$(00\ 00\ 00)$	1 word
$(01\ 01\ \omega\bar{\omega})$	36 words
$(00\ 11\ 11)$	9 words
$(\omega\bar{\omega}\ \omega\bar{\omega}\ \omega\bar{\omega})$	12 words
$(11\ \omega\omega\ \bar{\omega}\bar{\omega})$	6 words
	64 words

Adjoining another automorphism, e. g. :
(multiplication by (11 11 $\bar{\omega}\omega$)) . (permutation by (. — . . .))
. (field automorphism)
we have a non-split group $3S_6$ acting as the full automorphism group of the code.

Now we use the Hexacode to build up a binary code in $(GF(2))^{24}$ = { subsets of Ω } where Ω is a set of 24 objects arranged in a 4 X 6 array :

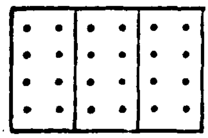
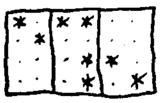


Figure 1

A subset is designated by a set of stars in the appropriate positions, e. g. :  is an 8-element subset.

Addition of subsets is given by :

$$A + B = (A \cup B) \setminus (A \cap B)$$

The six coördinates of the hexacode are put into correspondence with the six columns of this array, and elements of GF(4) are given 'interpretations' as subsets of a column as follows :

Definition The EVEN and ODD interpretations of elements of GF(4) as subsets of a column are as in the table below :

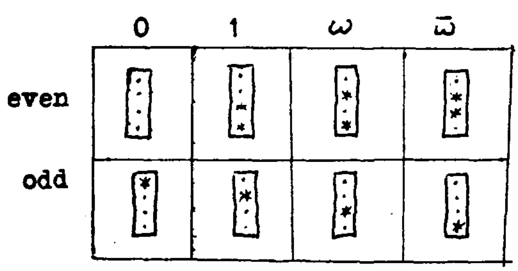
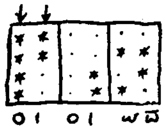
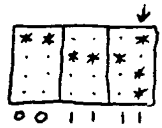


Table 1

Now let $\mathcal{C}$ be the code in { subsets of Ω } given as follows :
{ hexacode words given the EVEN interpretation in each column PLUS all elements from an even number of columns } $\cup$ { hexacode words given the ODD interpretation in each column PLUS all elements from an odd number of columns }
(PLUS in the sense of vector space addition as given above)

e. g.  and  $\in \mathcal{C}$

Then $\mathcal{C}$ has dimension 12, and it is easy to check that it is self-dual and that the words of minimal weight have weight 8. Thus $\mathcal{C}$ is the binary Golay Code and has the Mathieu group M_{24} acting on it.

<u>Definition</u> The arrangement of 24 points in the above 4 X 6 array with the code $\mathcal{C}$ defined on them and M_{24} acting on them is called the MOG (Miracle Octad Generator).

(Note that our MOG differs from Curtis' in [3] by transposition of the left-hand pair of columns)

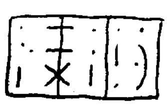
<u>Definition</u> The 8 and 12 element subsets in $\mathcal{C}$ are called (special) OCTADS and DODECADS respectively. A partitioning of Ω into six four-element subsets (tetrads) such that any two form an octad is called a SEXTET. A partitioning of Ω into three disjoint octads is called a TRIO.

The sets of eight points into which figure 1 is divided are called the BRICKS of the MOG, and they form a trio called the BRICK TRIO. The columns of the MOG form a sextet called the VERTICAL SEXTET.

<u>Some subgroups of M_{24}</u> (see [3])

<u>The Sextet Group</u>

The automorphisms in M_{24} fixing a sextet form a group of shape $2^6.3S_6$, which for the vertical sextet is generated by :

(i) Automorphisms of the hexacode lifted to its action on the MOG, e. g.  corresponds to the automorphism named at the top of the previous page.

(These automorphisms form a group of shape $3S_6$)

(ii) Codewords in the hexacode with the interpretation :

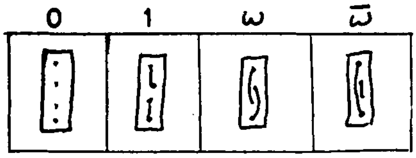
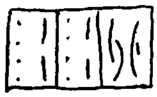


Table 2

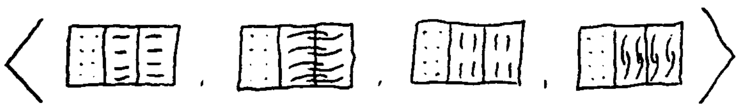
e. g.  corresponds to the codeword $(01\ \ 01\ \ \omega\bar{\omega})$

These form an elementary abelian subgroup of order 2^6 normalised by the $3S_6$ of (i).

The Octad Group

The stabilizer of an octad is a group of shape $2^4\mathcal{A}_8 = 2^4L_4(2)$ which for the left-hand octad (i.e. the left-hand brick) acts as follows :

(i) The normal 2^4 is



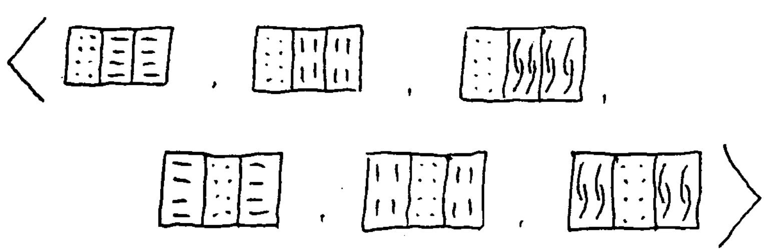
and gives the right-hand square (i. e. the complement of the left-hand brick) the structure of an affine 4-space over GF(2) on which it acts as affine translations.

(ii) Stabilizing a point in the right-hand square (we usually use the top-left point) we get a complementary $\mathcal{A}_8 \cong L_4(2)$ acting as $\mathcal{A}_8$ on the left-hand brick and as $L_4(2)$ on the right-hand square as a 4-dimensional vector space over GF(2).

The Trio Group

The stabilizer of a trio is of shape $2^6(S_3 \times L_3(2))$ acting as follows in the case of the brick trio :

(i) The normal 2^6 is



This gives each of the three bricks the structure of an affine 3-space over GF(2) on which it acts as 3 affine translations whose sum is zero.

(ii) Complementary to this there is a subgroup $S_3 \times L_2(7)$ (note that $L_3(2) \cong L_2(7)$) where the S_3 permutes the bricks "bodily" and the $L_2(7)$ acts similarly in each brick, preserving the projective line structure given by the numbering :

∞	0	∞	0	∞	0
3	2	3	2	3	2
5	1	5	1	5	1
6	4	6	4	6	4

Figure 2

There is also a subgroup $L_3(2)$ of the trio group acting as $L_2(7)$ on one of the bricks and as an $L_3(2)$ stabilizing a point in each of the other two bricks. However, this does not extend to an $S_3 \times L_3(2)$.

$L_2(23)$

In order to display the subgroup $L_2(23)$ we use a STANDARD NUMBERING of the points of the MOG diagram with the symbols ∞, 0, 1, . . . , 22 as follows :

∞	0	22	1	11	2
3	15	12	21	13	7
6	5	18	20	4	10
9	19	8	14	16	17

Figure 3

and then the $L_2(23)$ preserving the projective line structure given by this labelling is contained inside M_{24} , though note that $PGL_2(23)$ is not.

(Note that this numbering is different to that used in [3])

$M_{22}.2$

We usually take $M_{22}.2$ to be the subgroup of M_{24} fixing { 0, ∞ } setwise, and write MOG diagrams for $M_{22}.2$ with these points partitioned off thus :

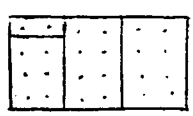


Figure 4

It is important that we observe this convention because we shall be dealing with two subgroups of J_4 of shapes $2^{1+12}.3M_{22}.2$ and $2^{11}.M_{24}$ whose elements must not be confused.

The intersection of an octad for M_{24} with $\Omega\setminus\{0, \infty\}$ is called a (special) HEXAD, HEPTAD or OCTAD for $M_{22}.2$ depending on its cardinality; similarly the intersection of a dodecad with $\Omega\setminus\{0, \infty\}$ is called a DECAD, HENDECAD or DODECAD .

Given a hexad for $M_{22}.2$, the remaining 16 points have a natural structure as a symplectic space of dimension 4 over GF(2). A division of this into four isotropic planes (2-spaces) corresponds to a pair of points in the hexad. So the stabilizer of such a HEXAD + PLANE is a group of shape $2^2(S_4 \times 2)$.

If ϑ and φ are two disjoint hexads for $M_{22}.2$, then $\mathrm{Stab}_{M_{22}.2}(\vartheta) \cap \mathrm{Stab}_{M_{22}.2}(\varphi)$ is a group isomorphic to S_6 , and the permutation actions of this group on the points of ϑ and of φ are inequivalent, and related by the outer automorphism of S_6 . Thus points in ϑ correspond to totals in φ and duads in ϑ correspond to synthemes in φ , and vice-versa (see [9]) .

Modules and cohomology for M_{24} over GF(2)

As a 12-dimensional module for M_{24}, the Golay code $\mathcal{C}$ has a unique non-trivial invariant submodule $\langle\Omega\rangle$ which has dimension 1. $\gamma\mathcal{C} = \mathcal{C}/\langle\Omega\rangle$ is an irreducible module of dimension 11 for M_{24}. $\mathcal{C}^* = \{$ subsets of $\Omega \} / \mathcal{C}$ is a 12-dimensional module dual to $\mathcal{C}$, having an irreducible 11-dimensional submodule $\mathcal{S}\mathcal{C}^* = \{$ even subsets of $\Omega \} / \mathcal{C}$.

The skew-square $\mathcal{C}^{2-}$ of $\mathcal{C}$, of dimension 66, is a uniserial module with three composition factors : $\langle\Omega\rangle_\wedge \mathcal{C}$ is an invariant 11-dimensional submodule isomorphic to $\gamma\mathcal{C}$, with quotient $\gamma\mathcal{C}^{2-} = \mathcal{C}^{2-}/(\langle\Omega\rangle_\wedge \mathcal{C}) \cong (\gamma\mathcal{C})^{2-}$, and $\mathcal{S}\mathcal{C}^{2-} = \langle C_1 \wedge C_2 : C_1, C_2 \subseteq \Omega$ and $C_1 \cap C_2 = \emptyset \rangle$ is an invariant 55-dimensional submodule containing $\langle\Omega\rangle_\wedge \mathcal{C}$. $\gamma\mathcal{S}\mathcal{C}^{2-} = \mathcal{S}\mathcal{C}^{2-}/(\langle\Omega\rangle_\wedge \mathcal{C})$ is an irreducible 44-dimensional module. The dual $\gamma\mathcal{S}(\mathcal{C}^*)^{2-}$ can be built in a similar way.

Lemma 1 A split extension J of $\mathcal{C}^*$ by M_{24} has a unique conjugacy class of complements for $O_2(J)$.

Proof Let K_1 and K_2 be two such complements. Choose an element y_1 of order 23 in $J / O_2(J)$. Then the two representatives of y_1 in K_1 and K_2 are conjugate by an element of $O_2(J)$, by Sylow's theorem, so we may suppose they are the same. Let y_2 be an element of order 11 in $J / O_2(J)$ normalizing y_1. Then the representatives of y_2 in K_1 and K_2 differ by an element of $O_2(J)$ centralizing y_1, and hence centralizing y_2. Thus since the two representatives of y_2 both have order 11, they must be equal. Now take an element y_3 of order 10 normalizing y_2 in $J / O_2(J)$. The representatives of y_3 in K_1 and K_2 differ by an element of $O_2(J)$ centralizing y_2, and have the same order, and hence are either equal or differ by the duad fixed by y_2. In the latter case, conjugating by the monad fixed by y_1 fixes the representatives of y_1 and y_2, and sends one representative of y_3 to the other. Since $J / O_2(J)$ is generated by y_1, y_2 and y_3 this means that we have found an element of $O_2(J)$ conjugating K_1 to K_2. //

Corollary 1 A split extension J' of $\mathcal{S}\mathcal{C}^*$ by M_{24} has two conjugacy classes of complements for $O_2(J')$, conjugate by an outer automorphism of J' (corresponding to an odd element of $O_2(J)$). //

Warning J_4 has a subgroup M (see chapter 3) isomorphic to the J' above, and the two classes of complements are not conjugate in J_4. They do not even have the same conjugacy class fusion maps with respect to J_4.

Corollary 2 There is a unique isomorphism type of uniserial module of dimension 12 for M_{24} having an 11-dimensional submodule isomorphic to $\mathcal{S}\mathcal{C}^*$. i.e. $\mathrm{Dim}\ \mathrm{Ext}^1(\mathcal{S}\mathcal{C}^*, 1) = 1$.

Lemma 2 The split extension 2^6S_6 of the natural permutation module by S_6 has a unique conjugacy class of subgroups of shape $2 \times S_6$.

Proof This follows from a similar argument to that in Lemma 1 using a Sylow 3-subgroup and a transposition mixing the two orbits of it.

Theorem Let R be the sextet group of shape $2^{12}.2^6.3S_6$ in the group J defined in Lemma 1. Then

(i) there is exactly one conjugacy class of subgroups of shape $2 \times (2^6.3S_6)$ supplementing $O_2(J)$ in R ;

(ii) any such supplement has a unique subgroup of index 2 complementing $O_2(J)$ in R and contained in some complement to $O_2(J)$ in J. Such a subgroup is contained in exactly two such complements conjugate by the involution in $Z(R)$.

Proof First, all supplements to $O_2(R)$ in R of shape $2 \times 3S_6$ are conjugate, since any such is a supplement to $O_2(S)$ in S where S is the normalizer in R of a Sylow 3-subgroup of $O_{2,3}(R)$, having shape $(2^6 \times 3).S_6$, and all such are conjugate by Lemma 2.

Now take such a supplementary subgroup R_1 of shape $2 \times 3S_6$. Then $[O_2(R), O_3(R_1)]$ is an extraspecial group of shape 2^{1+12}, and modulo the centre it decomposes as the direct sum of two different (dual) irreducible modules for R_1 so that there is a unique such decomposition. Thus such a supplement to $O_2(R)$ extends uniquely to a supplement of shape $2 \times (2^6.3S_6)$ to $O_2(J)$ in R, and all such supplements are hence conjugate, thus proving (i).

Since a complement to $O_2(J)$ in J does contain a complement to $O_2(J)$ in R, and the centralizer of the latter is the unique vector in $O_2(J)$ fixed by R, (ii) follows. //

<u>Corollary</u> Let R' be the sextet group of shape $2^{11}.2^6.3S_6$ in the group J' defined in corollary 1 to lemma 1. Then

(i) there are exactly two conjugacy classes of subgroups of shape $2 \times (2^6.3S_6)$ supplementing $O_2(J')$ in R';

(ii) any such supplement has a unique subgroup of index 2 complementing $O_2(J')$ in J'. Such a subgroup is contained in exactly two such complements conjugate by the involution in $Z(R')$. //

Under the action of R', $O_2(J')$ reduces uniserially with two proper invariant submodules :

(i) the sextet as an element of $\mathcal{S}\mathcal{C}^*$ forms an invariant 1-dimensional submodule ;

(ii) the PARITY submodule for the sextet group is defined as the collection of all elements of $\mathcal{S}\mathcal{C}^*$ intersecting each tetrad of the sextet with the same parity (i.e. all evenly or all oddly). This is well-defined since every element of $\mathcal{C}$ hits each tetrad with the same parity. This submodule has dimension 7.

Chapter 2 The Subgroup H of shape $2^{1+12}.3M_{22}.2$

In this chapter we develop a notation for working inside the subgroup $H = C_{J_4}(z)$ of shape $2^{1+12}.3M_{22}.2$ satisfying Hypothesis A on P.2 . Mostly the notation does not distinguish between an element and its product with z, so that a certain amount of information is lost, but we can usually get around this difficulty when it matters.

First, let us examine the structure of $\bar{H} = H / \langle z \rangle$. This is a split extension of $\bar{E} = E / \langle z \rangle = O_2(H) / \langle z \rangle$ by a complement $\bar{F} = F / \langle z \rangle$ of shape $3M_{22}.2$. Janko proved in [2] that $F_0 = F'$ is a proper cover $6M_{22}$. (But note that contrary to the title of [2] the full covering group of M_{22} is in fact of shape $12M_{22}$ and not $6M_{22}$, as was discovered in Summer 1979)

Looking at the 2-modular character table of $3M_{22}.2$ we see that there is a unique faithful 12-dimensional module over GF(2) and that this has the structure of a six-dimensional module over GF(4) with $O_3(\bar{F})$ acting as scalar multiplications. The outer half of $3M_{22}.2$ acts semilinearly; the inner half linearly. (†)

Let $\langle w \rangle = O_3(F)$ so that $C_H(w) = F$, and let conjugation by w in E represent multiplication by $\omega \in GF(4)$, i. e. $\omega y = y^w$.

Since $E = O_2(H)$ has an automorphism of order 3 induced by w whose centralizer in E is $\langle z \rangle$, E is of type 2_+^{1+12} (i. e. a central product of 6 copies of the Quaternion group Q_8)

Represent passage from E to $\bar{E} = E / \langle z \rangle$ by $x \to \tilde{x}$. Then the unitary structure on $\bar{E}$ preserved by $\bar{F}$ can be obtained as follows from the extraspeciality of E and the GF(4)-structure resulting from conjugation by w :

Let $\rho : \langle z \rangle \to GF(2) \subset GF(4)$

be given by $I \mapsto 0$

$z \mapsto 1$

and define $\tilde{x} \cdot \tilde{y} = \rho([x, \omega y]) + \omega\rho([x,y])$ (1)

Note that it doesn't matter which inverse images are taken for $\tilde{x}$ and $\tilde{y}$ since $\langle z \rangle$ is central .

(†) See Appendix G

<u>Lemma</u> 2.1

(1) defines a unitary structure on $\bar{E}$.

<u>Proof</u>

$$\tilde{x} \cdot \omega\tilde{y} + \bar{\omega}(\tilde{x} \cdot \tilde{y}) = \rho([x,\bar{\omega} y]) + \omega\rho([x,\omega y]) + \bar{\omega}\rho([x,\omega y]) + \rho([x,y])$$

$$= \rho([x,y] + [x,\omega y] + [x,\bar{\omega} y])$$

$$= 0$$

$$\Rightarrow \quad \tilde{x} \cdot \omega\tilde{y} = \bar{\omega}(\tilde{x} \cdot \tilde{y}) .$$

Similarly $\quad \omega\tilde{x} \cdot \tilde{y} = \omega(\tilde{x} \cdot \tilde{y})$.

$$\tilde{x} \cdot (\tilde{y} + \tilde{z}) = \rho([x, \omega(y + z)]) + \omega\rho([x, y + z])$$

$$= \rho([x, \omega y]) + \rho([x, \omega z]) + \omega\rho([x,y]) + \omega\rho([x,z])$$

$$= \tilde{x} \cdot \tilde{y} + \tilde{x} \cdot \tilde{z} .$$

Similarly $(\tilde{x} + \tilde{y}) \cdot \tilde{z} = \tilde{x} \cdot \tilde{z} + \tilde{y} \cdot \tilde{z}$. //

Note that since $\langle x, w\rangle \cong \begin{cases} Q_8 \circ 3 = 2A_4 \\ 2^3.3 = 2 \times A_4 \quad \text{or} \\ 2^2.3 = A_4 \end{cases}$

we have $\tilde{x} \cdot \tilde{x} = [x, x^w] = x^2$

so that vectors of norm 0 are involutions while those of norm 1 are elements of order 4 .

Thus the inverse image in E of an isotropic subspace in $\bar{E}$ is the same thing as an elementary abelian subgroup of E containing z and invariant under w . If X is such, then $[X, w]$ is a complement for $\langle z\rangle$ in X, natural given our choice of F .

More generally, if X is any elementary abelian subgroup of E containing z then $[X, w]^w = \{ w^2xw^2xw^2 : x \in X \}$ is such a complement for $\langle z\rangle$ in X (see also P. 26)

We express vectors in $\overline{E}$ using a 2 X 3 array of coördinates $\boxed{\begin{smallmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_4 & \lambda_5 & \lambda_6\end{smallmatrix}}$ with $\lambda_i \in GF(4)$, and inner product :

$$\boxed{\begin{smallmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \lambda_4 & \lambda_5 & \lambda_6\end{smallmatrix}} \cdot \boxed{\begin{smallmatrix}\mu_1 & \mu_2 & \mu_3\\ \mu_4 & \mu_5 & \mu_6\end{smallmatrix}} = \lambda_1\bar{\mu}_4 + \lambda_2\bar{\mu}_5 + \lambda_3\bar{\mu}_6 + \lambda_4\bar{\mu}_1 + \lambda_5\bar{\mu}_2 + \lambda_6\bar{\mu}_3 .$$

When writing down elements of $\overline{F}$ as 6 X 6 matrices over GF(4) we shall think of these vectors as row vectors $(\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6)$.

Can we find 22 objects in $\overline{E}$ which are permuted by $\overline{F}$ in the same way as the natural permutation representation of $\overline{F} / \langle \overline{w} \rangle \cong Aut(M_{22})$ on 22 points ?

To answer this question, we look for subgroups of $\overline{E}$ invariant under the action of $3M_{21} \cong SL_3(4)$, and find that the module $\overline{E}$ reduces uniserially with a unique 3-dimensional irreducible submodule with irreducible (and dual) 3-dimensional quotient. Since the induced action on this submodule is the natural action of $SL_3(4)$, it must be an isotropic subspace of $\overline{E}$.

Let W_i be the isotropic 3-space in $\overline{E}$ stabilized by $Stab_{\overline{F}}(i)$ for $i \in \Omega \setminus \{0, \infty\}$.

<u>Lemma</u> 2.2

$Dim(W_i \cap W_j) = 1$ for $i \neq j$

and $Dim(W_i \cap W_j \cap W_k) = 0$ for $i \neq j \neq k \neq i$

<u>Proof</u>

First we note that since $\overline{F}$ is 3-transitive on $\Omega \setminus \{0, \infty\}$, these numbers are independent of choice of i, j and k.

Certainly $0 \leq Dim(W_i \cap W_j) \leq 2$.

If $Dim(W_i \cap W_j) = 0$ then $Stab_{\overline{F}}(i, j) \leq Stab_{\Gamma U_6(2)}(W_i, W_j) = \{1\}$, a contradiction. If $Dim(W_i \cap W_j) = 2$ then $\langle W_i, W_j \rangle$ has dimension 4, and either $W_i \cap W_j = W_i \cap W_k \quad \forall\ i \neq j \neq k \neq i$ in which case $\bigcap_{i=1}^{22} (W_i)$ would be an invariant 2-dimensional subspace, or $W_i \cap W_j \neq W_i \cap W_k$ $\forall\ i \neq j \neq k \neq i$ in which case $\langle W_i, W_j \rangle \supseteq W_k$ by dimension counting, and so $\langle W_i : 1 \leq i \leq 22 \rangle$ is an invariant 4-dimensional subspace.

Thus $\mathrm{Dim}(W_i \cap W_j) = 1$ for $i \neq j$.

If $\mathrm{Dim}(W_i \cap W_j \cap W_k) = 1$ for $i \neq j \neq k \neq i$ then $\bigcap_{i=1}^{22} (W_i)$ would be an invariant 1-dimensional subspace .

Thus $\mathrm{Dim}(W_i \cap W_j \cap W_k) = 0$ for $i \neq j \neq k \neq i$. //

Theorem 2.3

Given a 6-dimensional unitary space over GF(4) and a set of isotropic 3-spaces of maximal size subject to the conditions :

(i) Any two intersect in a 1-dimensional space

(ii) Any three intersect trivially

then there are 22 such subspaces in the set, and the setwise stabilizer of this configuration in $\Gamma U_6(2)$ is a group isomorphic to the triple cover of M_{22} extended by the outer automorphism.

Proof

W.l.o.g. we may take the space to be $\overline{E}$.

Let $\{ X_i : 1 \leq i \leq n \}$ be the collection of subspaces and let $\{ W_i : 1 \leq i \leq 22 \}$ be the 22 subspaces defined above.

(*) Since an isotropic 3-space has only 21 isotropic 1-spaces in it, we must have $n \leq 22$, which with the lemma shows that $n = 22$. I shall find an element of $\Gamma U_6(2)$ taking $X_i \to W_i$, $1 \leq i \leq 22$, after possibly renumbering some of the X_i .

Since $\Gamma U_6(2)$ is transitive on isotropic 3-spaces, we may suppose $X_1 = W_1$.

Since $\mathrm{Stab}_{\Gamma U_6(2)}(W_1)$, of shape $2^{12}SL_3(4).S_3 = 2^{12}\Gamma L_3(4)$ acts transitively on isotropic 3-spaces intersecting W_1 in a 1-space, we may suppose $X_2 = W_2$.

Since $\mathrm{Stab}_{\Gamma U_6(2)}(W_1) \cap \mathrm{Stab}_{\Gamma U_6(2)}(W_2)$ of shape $2^6.2^4(3^2 \times SL_2(4)).2$ acts transitively on isotropic 3-spaces intersecting W_1 in a 1-space, W_2 in a 1-space and $W_1 \cap W_2$ in a 0-space, we may suppose that $X_3 = W_3$.

Now $\bigcap_{i=1,2,3} (\mathrm{Stab}_{\Gamma U_6(2)}(W_i))$, of shape $2^{2+4}.3^2.2$, has 2 orbits on subspaces X intersecting W_i in a 1-space and $W_i \cap W_j$

in a O-space for each pair $i \neq j \in \{1,2,3\}$, distinguished by whether $W_1 \cap X \leq \langle W_1 \cap W_2 , W_1 \cap W_3 \rangle$ (orbit 1)

or not (orbit 2)

(this condition is in fact symmetric in W_1 , W_2 , W_3) .

By the remark (*) on the previous page, precisely 3 of the W_i and 3 of the X_i for $i \notin \{1,2,3\}$ are in orbit 1 and 16 are in orbit 2.

By the transitivity properties of $Aut(M_{22})$ this means that $\{1,2,3\}$ together with the 3 i's for which W_i is in orbit 1 form a hexad for M_{22} which must therefore be $\{1,2,3,5,14,17\}$. Thus by applying a suitable element of $\bigcap_{i=1,2,3} (Stab_{\Gamma U_6(2)}(W_i))$ and by renumbering some X_i in orbit 1 as X_5 we may suppose that $W_5 = X_5$.

Next we see that $\bigcap_{i=1,2,3,5} (Stab_{\Gamma U_6(2)}(W_i))$ of shape $2^{1+4}.3.2$ is transitive on the elements of orbit 1 apart from W_5 , so by renumbering some X_i , $i \neq 5$ from orbit 1 as X_{14} and applying some element of this stabilizer we may take $X_{14} = W_{14}$; then after relabelling the final X_i of orbit 1 as X_{17} we automatically have $X_{17} = W_{17}$.

Finally $\bigcap_{i=1,2,3,5,14,17} (Stab_{\Gamma U_6(2)}(W_i))$ of order 2^6 is simply transitive on the 2^6 possibilities for choice of a subspace satisfying the requirements (i) and (ii) for intersections with the W_i , $i \in \{1, 2, 3, 5, 14, 17\}$ and having fixed one, the other 15 are determined. Thus we may renumber the X_i for $i \notin \{1, 2, 3, 5, 14, 17\}$ and apply a suitable element of the above stabilizer so that $X_i = W_i$, $1 \leq i \leq 22$. //

Thus by the above theorem, the following is a valid layout for the subspaces of $\bar{E}$ corresponding to the points of $\Omega \setminus \{0, \infty\}$:

∞	0	$x\ y\ z$ $0\ 0\ 0$	$S\ S\ S$ $X\ Y\ Z$	$S\ S_1S_2$ $X\ Y_1Z_2$	S_2S_1S X_2Y_1Z
$x_2y\ z$ $X_2Y\ Z$	$x_1y\ z$ $X_1Y\ Z$	$x\ 0\ 0$ $0\ y\ z$	$S\ Y\ Z$ $X\ S\ S$	$S\ Y_1Z_2$ $X\ S_1S_2$	S_2Y_1Z X_2S_1S
$x\ y_2z$ $X\ Y_2Z$	$x\ y_1z$ $X\ Y_1Z$	$0\ y\ 0$ $x\ 0\ z$	$X\ S\ Z$ $S\ Y\ S$	$X\ S_1Z_2$ $S\ Y_1S_2$	X_2S_1Z S_2Y_1S
$x\ y\ z_2$ $X\ Y\ Z_2$	$x\ y\ z_1$ $X\ Y\ Z_1$	$0\ 0\ z$ $x\ y\ 0$	$X\ Y\ S$ $S\ S\ Z$	$X\ Y_1S_2$ $S\ S_1Z_2$	X_2Y_1S S_2S_1Z

Table 3

The notation for 3-dimensional subspaces of $\bar{E}$ is as follows :

x, y and z are general elements of GF(4)

$X = y + z \qquad Y = x + z$

$Z = x + y \qquad S = x + y + z$

A subscript 1 signifies multiplication by $\omega \in GF(4)$ and a subscript 2 signifies multiplication by $\bar{\omega} \in GF(4)$

Thus for example :

$\begin{array}{|c|}\hline x\ y_2z \\ X\ Y_2Z \\ \hline\end{array}$ is the subspace spanned by $\begin{array}{|ccc|}\hline 1 & 0 & 0 \\ 0 & \bar{\omega} & 1 \\ \hline\end{array}$, $\begin{array}{|ccc|}\hline 0 & \bar{\omega} & 0 \\ 1 & 0 & 1 \\ \hline\end{array}$ and $\begin{array}{|ccc|}\hline 0 & 0 & 1 \\ 1 & \bar{\omega} & 0 \\ \hline\end{array}$

$\begin{array}{|c|}\hline X_2Y_1S \\ S_2S_1Z \\ \hline\end{array}$ is the subspace spanned by $\begin{array}{|ccc|}\hline 0 & \omega & 1 \\ \bar{\omega} & \omega & 1 \\ \hline\end{array}$, $\begin{array}{|ccc|}\hline \bar{\omega} & 0 & 1 \\ \bar{\omega} & \omega & 1 \\ \hline\end{array}$ and $\begin{array}{|ccc|}\hline \bar{\omega} & \omega & 1 \\ \bar{\omega} & \omega & 0 \\ \hline\end{array}$

We give a list of a few useful coördinate transformations on $\bar{E}$ effected by elements of $\bar{F}$ and their action on the MOG for H :

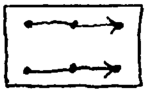
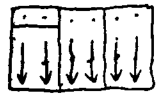
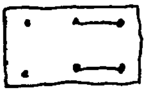
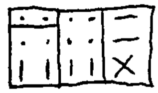
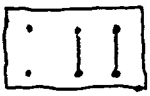
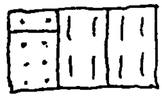
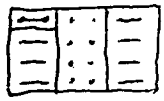
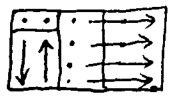
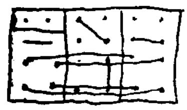
Coörd. Transformation	Effect on H's MOG
	
	
	
field aut. $\omega \leftrightarrow \bar{\omega}$	
diag $\begin{pmatrix} 1 & \omega & \bar{\omega} \\ 1 & \omega & \bar{\omega} \end{pmatrix}$	
$\begin{pmatrix} 1&0&0&0&1&1 \\ 0&1&0&1&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$	 (see also p. 56)

Table 4

Classification of vectors and isotropic subspaces in $\bar{E} = O_2(H)/\langle z \rangle$

Having obtained Table 3, it is easy enough to verify the following classifications :

Vectors

Under the action of H , E has 3 classes of involution, with z in a class of its own and $y \sim yz$ for each $y \notin \langle z \rangle$, and one class of elements of order 4, whose stabilizers in $\bar{\bar{F}} / \langle \bar{w} \rangle$ are as given in the following table :

	elts	vectors	1-spaces	norm	Stabilizer in $M_{22}.2$	J_4-class (see p.35)
(z)	1	-	-	0	$M_{22}.2$	2A
	1386	693	231	0	$2^4(S_5 \times 2)$ = 2-point stabilizer (EDGE group)	2A
	2772	1386	462	0	$2^4PGL_2(5)$ = stabilizer of hexad + total on it (TOTAL group) [9]	2B
	4032	2016	672	1	$PGL_2(11)$ = stabilizer of pair of disjoint dodecads (DUUM group)	4A
	$8191 = 2^{13}-1$					

Table 5

Isotropic 2-spaces

There are three classes of these under the action of H, which contain the following numbers of EDGE-type and TOTAL-type 1-spaces and have the following stabilizers :

number	edge 1-spaces	total 1-spaces	Stabilizer in $M_{22}.2$
462	5	0	2^4S_5 = stabilizer of hexad + included point
1155	3	2	$2^4(S_4 \times 2)$ = stabilizer of hexad + syntheme on it [9]
4620	1	4	$2^2(S_4 \times 2)$ = stabilizer of hexad + plane (see p.12)

Table 6

Isotropic 3-spaces

There are four classes of these under the action of H , which contain the following numbers of EDGE-type and TOTAL-type 1-spaces and have the following stabilizers :

number	edge 1-spaces	total 1-spaces	Stabilizer in $M_{22}.2$
22	21	0	$M_{21}.2$ = POINT stabilizer
77	15	6	2^4S_6 = HEXAD stabilizer
462	5	16	2^4S_5 = stabilizer of hexad + included point
330	7	14	$2^3(L_3(2) \times 2)$ = OCTAD stabilizer

Table 7

The containments between the isotropic subspaces are given in the following diagram :

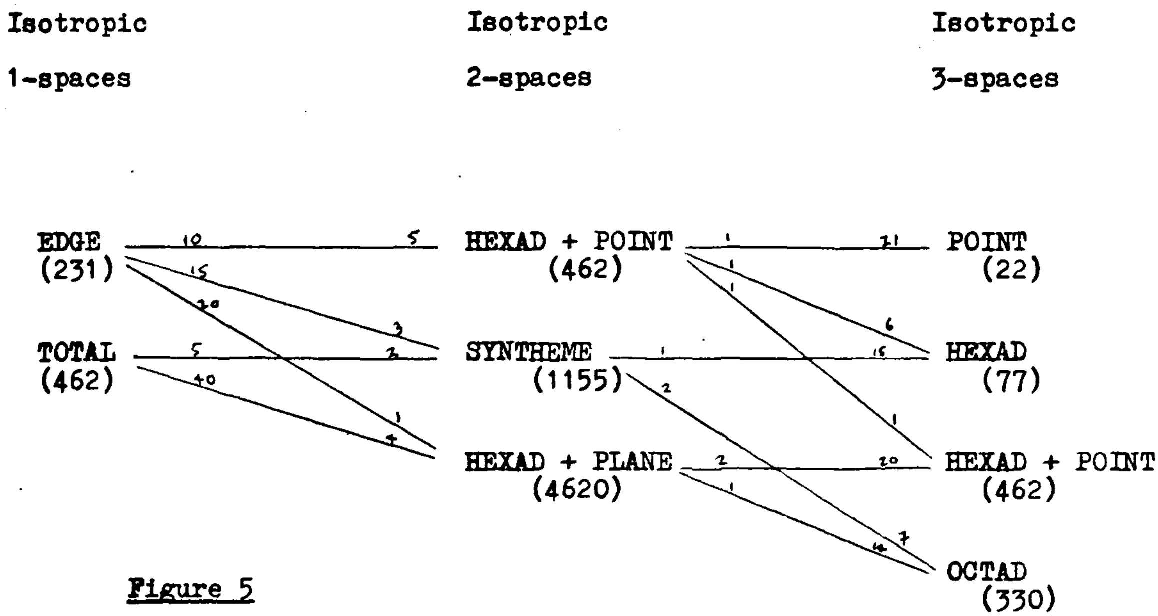


Figure 5

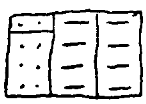
where this means, for example, that there are 330 octad type isotropic 3-spaces, each of which contains 7 syntheme type and 14 hexad + plane type isotropic 2-spaces.

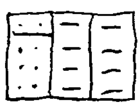
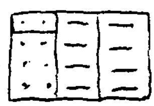
Notation for elements of H

From p. 17 we see that given an isotropic vector $\tilde{x}$ in $\bar{E}$, there is a canonical inverse image $w^2xw^2xw^2$ in E. We shall denote this inverse image with a subscript 0 and the other one with a subscript 1. For norm 1 vectors there is no good notation for distinguishing the inverse images.

Thus for example $\boxed{\begin{matrix}1&0&0\\0&0&0\end{matrix}}_0$ is a well-defined element of order 2 in E, and $\left\{ \boxed{\begin{matrix}x&y&z\\x&y&z\end{matrix}}_{x+y+z} : x+y+z \in GF(2) \right\}$ is a well-defined elementary abelian subgroup of E of order 2^5 not containing z.

For elements of F, the action on the MOG gives the element up to multiplication by an element of $Z(F) = \langle wz \rangle$. Sometimes the J_4-class of an element well-defines which representative it is, and then we append the class as a subscript. (See p.37 for a description of the J_4-classes of involutions in H)

For example,  $_{2A}$ is a well-defined element of F.

A general element of H can then be written down (in exactly two ways) as the product of an element of E and an element of F. So for example $\boxed{\begin{matrix}1&0&0\\0&0&0\end{matrix}}_0 \cdot$  $_{2A} = \boxed{\begin{matrix}1&0&0\\0&0&0\end{matrix}}_1 \cdot$  $_{2B}$

is a well-defined element of H.

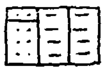
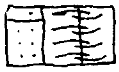
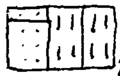
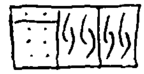
Sometimes we shall write down elements of H without worrying about which of the two representatives is concerned.

Chapter 3 The subgroup M of shape $2^{11}M_{24}$

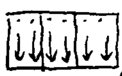
Janko [2] tells us how to find a subgroup M of J_4 of shape $2^{11}M_{24}$:

Choose a hexad for F (we choose the left-hand hexad ϑ = )

and let L_0 be its stabilizer in F , of shape 6.2^4S_6 . Let

$U_1 = O_2(L_0) = \langle$  $_2$,  $_2$,  $_2$,  $_2$, $z \rangle$,

an elementary abelian group of order 2^5 , and let $E_1 = C_E(U_1)$, the hexad-type isotropic 3-space $\begin{array}{|ccc|}x & y & z\\ x & y & z\end{array}$ for the left-hand hexad ϑ , an elementary abelian subgroup of order 2^7 . Then letting $V = E_1U_1$, V is an elementary abelian group of order 2^{11} whose normalizer $M = N_{J_4}(V)$ is a split extension of shape $2^{11}M_{24}$ where the action of M_{24} on V is the same as on $\mathcal{S}\mathcal{C}^*$.

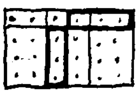
Now $D = N_H(V) = H \cap M$ is the hexad stabilizer in H of shape $2^{1+12}.3.2^4S_6$ so that D/V has shape $2^6.3S_6$, and is thus the sextet stabilizer in M/V . We are free to choose that this is the vertical sextet for M's MOG (see p. 9) , and we are free to choose that $\langle w \rangle$ acts as $\langle$  $\rangle$.

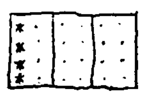
From the analysis on p.15 we know that there are two conjugacy classes of supplements of shape $2 \times (2^6.3S_6)$ to V in D , and that each has a subgroup of index two contained in exactly two complements to V in M , conjugate by z . Representatives of the two classes are :

(i) Let i be a point of H's MOG not in ϑ , so that W_i is an isotropic 3-space in $\bar{E}$ whose inverse image $\hat{W}_i$ is an elementary abelian subgroup of E of order 2^7 , of point type (see p.24). Then $\hat{W}_i\mathrm{Stab}_{L_0}(\hat{W}_i)$ is such a supplement giving 2 complementary M_{24}s of POINT type .

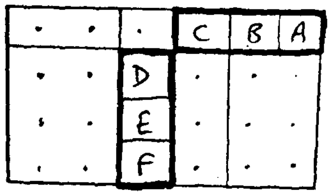
(ii) Let φ be a hexad of H's MOG disjoint from ϑ , and let $\hat{W}_\varphi$ be the inverse image in E of the isotropic 3-space W_φ in $\bar{E}$ corresponding to φ , of hexad type (see p.24). Then $\hat{W}_\varphi\mathrm{Stab}_{L_0}(\hat{W}_\varphi)$ is such a supplement giving 2 complementary M_{24}s of HEXAD type.

Any complement to V in M is then of HEXAD or POINT type depending on whether its intersection with H fixes a hexad or a point in the right-hand square of H's MOG .

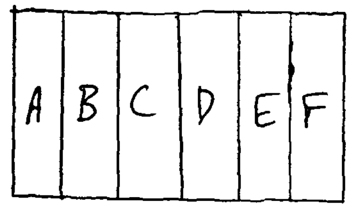
Let K be one of the two complements to V in M of hexad type extending the complement for $\langle z \rangle$ in $\hat{W}_\varphi \mathrm{Stab}_{L_0}(\hat{W}_\varphi)$ where φ =  , so that $W_\varphi = \begin{bmatrix} 0 & 0 & 0 \\ x & y & z \end{bmatrix}$.

Comparing the submodules of V that we know under the action of D / V (i. e. $\langle z \rangle$ and E_1) with the list on p. 15 we see that z is the vertical sextet  and E_1 is the PARITY submodule for the vertical sextet. Thus E_1 has 36 duad vectors and 91 sextet vectors, so that mod $\langle z \rangle$ there are 6 w-orbits of duads and 15 w-orbits of sextets. Looking at table 7 we see that a duad in E_1 mod $\langle z \rangle$ is a total type vector in $\bar{E}$ and a sextet in E_1 mod $\langle z \rangle$ is an edge type vector in $\bar{\bar{E}}$. Thus a total on the hexad θ in H's MOG corresponds to a column of the vertical sextet for M's MOG , and so the six points in θ are in duality with the six columns of the vertical sextet for M via the outer automorphism of S_6 . Thus, looking back at p. 12 we see that the points of φ are in one-one correspondence with the columns of M's MOG .

We still have the freedom of choice of this correspondence, and so for reasons which will become apparent in Chapters 4 and 5, we choose the following correspondence :



H's MOG

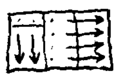


M's MOG

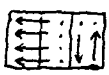
Figure 6

Having chosen this correspondence, the choice of which way round to identify $\langle w \rangle$ with  is determined,

since from table 4 we have

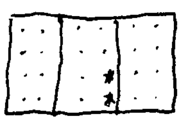
w = [ , ] (any representatives)

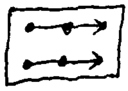
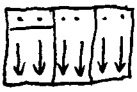
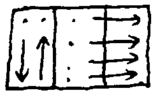
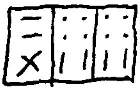
acts on M's MOG as

[ , ] =  .

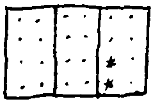
So the only freedom we have left in determining the names in H for elements of the sextet group $2^{11}.2^6.3S_6$ in M is conjugation by w .

From the definition of the correspondence in figure 6, we see that since the total type vector $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ in $\overline{E}_1$ corresponds to the point D of φ by hexad duality in $M_{22}.2$, it is a pair of disjoint duad vectors in V living in column D . Thus we may choose that

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_0$ =  (1) .

Now from table 4 we see that the element of F of order 3 acting on $\bar{\bar{E}}$ as the linear transformation  has effect  on H's MOG , and hence acts as (A)(B)(C)(DEF) on φ , and hence on the columns of M's MOG . Moreover, the element commutes with w and so it is  $.w^n$ for some n. It also commutes with the field automorphism $\omega \leftrightarrow \bar{\omega}$, which acts as (AB)(C)(D)(E)(F) on φ from table 4, and is hence  $.w^m$ for some m. Hence n = 0 .

So conjugating (1) by this we see that

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}_0 =$$ 

and $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}_0 =$$  .

Thus $\begin{bmatrix} \lambda & \mu & \nu \\ \lambda & \mu & \nu \end{bmatrix}_0$ is the element of V formed by giving

$(0\ 0\ 0\ \lambda\ \mu\ \nu)$ the even interpretation of table 1 (see p. 8).

<u>Notation</u> Every element of M has a unique expression of the form (element of V).(element of K) . We write an element as a set of stars together with a permutation on the same MOG diagram, with the understanding that the product is taken with the element of V first, followed by the element of K .

The Dictionary

In order to be able freely to translate back and forth between the names of elements of D as elements of M and as elements of H , we break it up into four parts :

(i) E_1 of shape 2^7

(ii) $O_2(K \cap D)$ of shape 2^6

(iii) $O_2(C_D(w))$ of shape 2^5

and

(iv) $C_K(w)$ of shape $3S_6$.

Every element of D has exactly two expressions as a product (element of E_1).(element of $O_2(K \cap D)$).(element of $O_2(C_D(w))$). (element of $C_K(w)$) . Notice that E is generated by (i) and (ii), V is generated by (i) and (iii), $K \cap D$ is generated by (ii) and (iv), and $F \cap D$ is generated by (iii) and (iv).

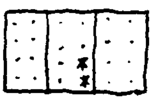
We dealt with translation of elements of E_1 in the last section, so we now deal with (ii), (iii) and (iv).

(ii) Elements of $O_2(K \cap D)$:

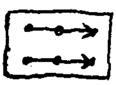
$O_2(K \cap D) = \boxed{\begin{matrix} 0 & 0 & 0 \\ x & y & z \end{matrix}}_0$; so what permutation in K is $\boxed{\begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}}_0$?

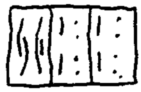
It commutes with (i. e. is orthogonal to) $\boxed{\begin{matrix} 0 & \lambda & \mu \\ 0 & \lambda & \mu \end{matrix}}$ and

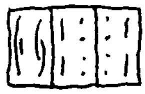
$\boxed{\begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}}_0$, which are $\mathcal{C}$ * sets in the last two columns intersecting

each evenly, and  respectively, and is in the O_2 of

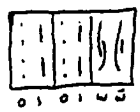
the vertical sextet group. Thus it must be .

Conjugating by the element of order 3 acting on $\bar{E}$ as the linear transformation  , as before, we get :

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_0$ = 

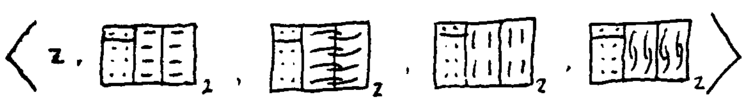
and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_0$ = 

Thus $\begin{bmatrix} 0 & 0 & 0 \\ \lambda & \mu & \nu \end{bmatrix}_0$ is the permutation in the O_2 of the vertical sextet group in K given by the hexacode word $(?\ ?\ ?\ \lambda\ \mu\ \nu)$ as described at the top of p. 10, where the question-marks are filled in in the unique possible way to make a hexacode word (see p. 7).

e. g. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & \omega & \bar{\omega} \end{bmatrix}_0$ = 

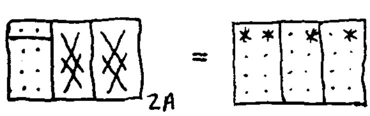
(iii) Elements of $O_2(C_D(w)) = U_1$:

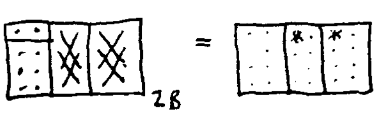
These lie in F , and their effect on H's MOG are as affine translations of the right-hand square :

i. e. 

In M these are the $\mathscr{C}^*$ sets lying in the top row of the MOG .

The correspondence is as follows :
Given such an affine translation, there is a unique cycle lying inside φ . From the correspondence in figure 6, this corresponds to a pair of columns of M's MOG . The two representatives in V of this element mod $\langle z \rangle$ =  are then the corresponding duad in the top row, and its complementary tetrad in the top row. e. g.

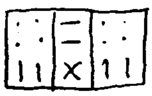


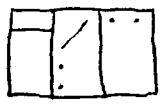
and 

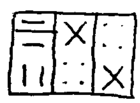
(see the next section for a description of the J_4 classes of involutions in M and H)

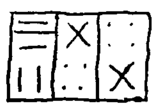
(iv) Elements of $C_K(w)$:

$C_K(w)$ has shape $3S_6$, lies in F , and is of index 2 in $Stab_F(\vartheta, \varphi)$. This means that the image of an element of $C_K(w)$ in $F / \langle wz \rangle$ depends only on the permutation effected on the six columns of M's MOG . Given this permutation, to find the image in $F / \langle wz \rangle$, fill in the corresponding permutation on the hexad φ via the correspondence given in figure 6, and fill out in the unique possible way to an element of $Stab_{F/\langle wz \rangle}(\vartheta, \varphi)$.

e. g.  $\in C_K(w)$ effects the permutation

(A)(B)(CD)(E)(F) , and hence has image  on the

hexad φ , which completes to the element  $\in F / \langle wz \rangle$,

and is hence the element  2A in F . (again see the next section)

We summarize the translation process given by this dictionary in the following table :

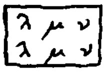
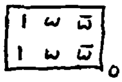
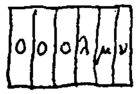
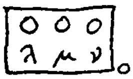
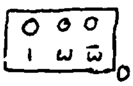
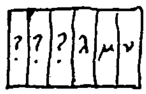
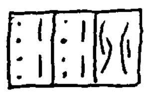
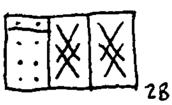
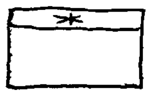
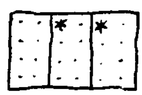
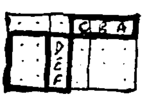
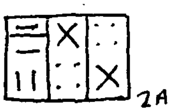
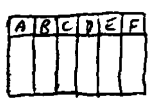
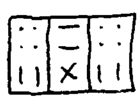
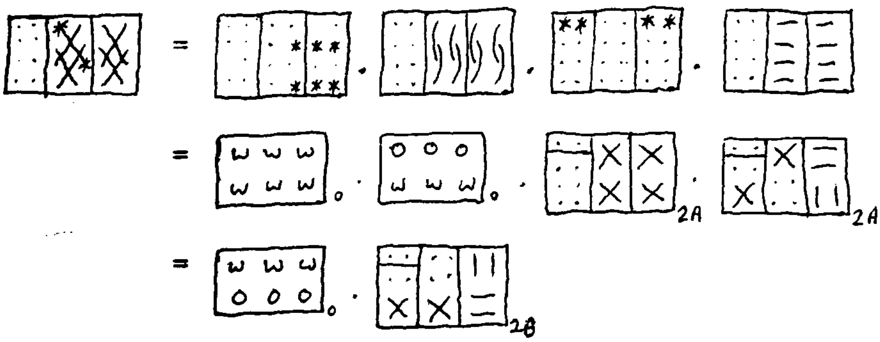
	Subgroup of H Mnemonic Example ⟷	Subgroup of M Mnemonic Example	Shape
(i)	 $\leqslant E$ PAIRED 	 $\leqslant V$ PARITY 	2^7
(ii)	 $\leqslant E$ VENTRAL 	 $\leqslant K$ VERTICAL 	2^6
(iii)	 $\leqslant F$ AFFINE 	 $\leqslant V$ AFLOAT 	2^5
(iv)	 $\leqslant F$ PERMS 	 $\leqslant K$ PERMS 	$3S_6$

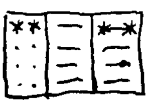
Table 8

(see also Appendix D)

Now that we have this dictionary at our disposal, we may freely translate back and forth between names for elements of D as elements of M and as elements of H . The process consists of breaking the element up as a product of elements of (i), (ii), (iii) and (iv), translating each separately via the dictionary, and then multiplying back up again.

e. g.



(Note that  has class 2B from the next section)

Conjugacy classes of involution in M and their fusion in J_4

Janko [2] gives the involution fusion pattern in J_4 . However, we shall need to know the answer more explicitely in terms of our notations for M and H .

Since z is defined to be in class 2A, all sextets in V are in class 2A . Duads in V are in class 2B. Translating a couple of these which happen to lie in E we see that edge-type elements in E are in class 2A whereas total-type elements are in class 2B. (see p. 23)

We need to examine the classes of elements in M under conjugation by K , since this corresponds to possible 'shapes' for our diagrams for elements. It turns out that every such class has a representative in E , so that the J_4-classes are as in the following table :

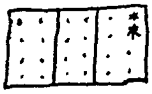
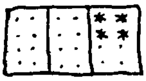
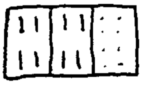
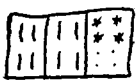
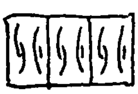
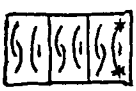
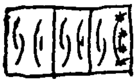
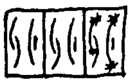
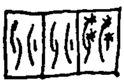
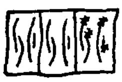
Representative of K-class in M	Name as element of E	J_4-class
$t_1 =$ 	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}_1$	2B
$t_2 =$ 	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}_1$	2A
$t_3 =$ 	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_0$	2A
$t_4 =$ 	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}_1$	2B
$t_5 =$ 	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1$	2A
$t_6 =$ 	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_0$	2A
$t_7 =$ 	$\begin{bmatrix} 0 & 1 & \omega \\ 1 & 1 & \omega \end{bmatrix}_0$	2B
$t_8 =$ 	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}_0$	2B
$t_9 =$ 	$\begin{bmatrix} 0 & 0 & 0 \\ \bar{\omega} & \omega & \bar{\omega} \end{bmatrix}_0$	2B
$t_{10} =$ 	$\begin{bmatrix} 0 & 0 & \bar{\omega} \\ \bar{\omega} & \omega & 0 \end{bmatrix}_1$	2A

Table 9

(continued on next page)

Representative of K-class in M	Name as element of E	J_4-class
t_{11} = 	$\begin{bmatrix} 0 & 0 & 0 \\ \bar{\omega} & \omega & \bar{\omega} \end{bmatrix}_1$	2B
t_{12} = 	$\begin{bmatrix} 0 & \omega & \bar{\omega} \\ \bar{\omega} & 0 & 0 \end{bmatrix}_0$	2B
t_{13} = 	$\begin{bmatrix} 0 & 1 & 1 \\ \bar{\omega} & \bar{\omega} & \omega \end{bmatrix}_0$	2A
t_{14} = 	$\begin{bmatrix} 0 & 1 & \omega \\ \bar{\omega} & \bar{\omega} & 1 \end{bmatrix}_0$	2B

<u>Table 9</u>
(continued)

Under the action of V , these fuse as follows :

$$t_3 \sim t_5 \sim t_6 \qquad t_4 \sim t_8 \qquad t_9 \sim t_{11} \sim t_{12}$$

so that M has 9 conjugacy classes of involution.

<u>Conjugacy classes of involution in H and their fusion in J_4</u>

Now we can use the dictionary to give the same information for H , choosing a representative in D from each class of involution in H .

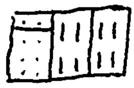
<u>Case 1</u> Elements of E :

These have already been dealt with, and we have

$s_1 = z$ has class 2A

s_2 = edge vector $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_0$ has class 2A

s_3 = total vector $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_0$ has class 2B

<u>Case 2</u> Elements of shape (element of E of norm 0) .  :

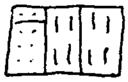
First work mod $\langle z \rangle$. The element  has effect

 on $\bar{E}$ (see table 4) so that

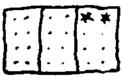
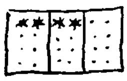
$C_{\bar{E}}$ () = $\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}$

and conjugating  by an element of E adds on a vector $\begin{bmatrix} 0 & \mu_1 & \mu_2 \\ 0 & \mu_1 & \mu_2 \end{bmatrix}$.

Thus mod $\langle z \rangle$ there are 3 classes of this shape :

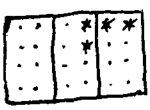
 , $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.  and $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. 

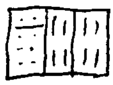
Do the involutary preimages of these in H fuse ?

The two involutary preimages of  are, as elements of M , s_4 =  and s_5 = 

These are respectively of class 2B and 2A .

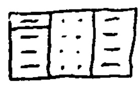
The two involutary preimages of $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.  are conjugate by $\begin{bmatrix} 0 & 0 & 0 \\ \omega & 0 & 0 \end{bmatrix}_0$ in H . Let s_6 be the preimage

 in V so that s_6 has class 2A .

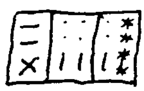
Similarly the two involutary preimages of $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.  are conjugate by $\begin{bmatrix} \omega & 0 & 0 \\ \omega & 0 & 0 \end{bmatrix}_0$ in H . Let s_7 be the preimage

 in M so that s_7 has class 2B .

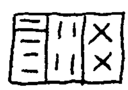
<u>Case 3</u> Elements of shape (element of E of norm 0) .  :

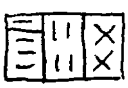
First work mod $\langle z \rangle$. This time all three elements of $\bar{F}$ with image  in F / $\langle wz \rangle$ are involutary, and they are conjugate via w . So choose the one acting on $\bar{E}$ as the field automorphism $\omega \leftrightarrow \bar{\omega}$ (see table 4) . Its centralizer in $\bar{E}$ is the set of all vectors with entries in GF(2) = {0, 1} . Conjugating by $x \in \bar{E}$ adds on a vector $x + \bar{x}$, and vectors of this form span the centralizer of our element in $\bar{E}$, so that all involutions of shape (element of E of norm 0) .  are conjugate mod $\langle z \rangle$.

The two preimages in F are, as elements of M ,

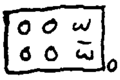
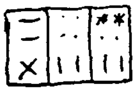
s_8 =  and s_9 =  .

s_8 is in class 2A whereas s_9 is in class 2B .

<u>Case 4</u> Elements of shape (element of E of norm 0) .  :

First work mod $\langle z \rangle$. Again all three elements of $\bar{F}$ with image  in F / $\langle wz \rangle$ are involutary and conjugate via w . So choose the one acting on $\bar{E}$ as  followed by the field automorphism $\omega \leftrightarrow \bar{\omega}$.

The centralizer of this element in $\bar{E}$ is $\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \bar{\lambda}_2 & \bar{\lambda}_3 \end{bmatrix}$ with $\lambda_1, \lambda_4 \in GF(2)$. Conjugating by an element of $\bar{E}$ adds a vector $\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_4 & \bar{\mu}_2 & \bar{\mu}_3 \end{bmatrix}$ with $\mu_1, \mu_4 \in GF(2)$ and so all such elements are conjugate in $\bar{\bar{H}}$.

The two preimages of our element in H are conjugate by  as can be checked by looking at these elements in M . Let s_{10} be the preimage which is  as an element of M , of class 2B .

Thus H has 10 conjugacy classes of involution as displayed in the following table :

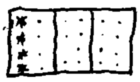
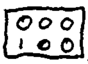
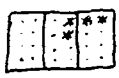
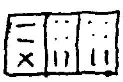
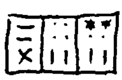
Representative of class in H	Name as element of M	J_4-class
$s_1 = z$		2A
$s_2 =$  $_0$		2A
$s_3 =$  $_0$		2B
$s_4 =$  $_{2B}$		2B
$s_5 =$  $_{2A}$		2A
$s_6 =$  $_0 \cdot$  $_{2A}$		2A
$s_7 =$  $_0 \cdot$  $_{2B}$		2B
$s_8 =$  $_{\omega \leftrightarrow \bar{\omega}}$ 2A		2A
$s_9 =$  $_{\omega \leftrightarrow \bar{\omega}}$ 2B		2B
$s_{10} \rightsquigarrow$ 		2B

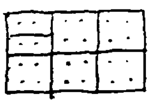
Table 10

Chapter 4 The 'Pentad' Subgroup P of shape $2^{3+12}(S_5 \times L_3(2))$

Looking again at [2] we find a recipe for obtaining in J_4 a subgroup of shape $2^{3+12}(S_5 \times L_3(2))$. In this chapter, I shall develop a notation for working inside this subgroup, and investigate the Sylow 5-normalizer and Sylow 7-normalizer in J_4, which lie inside this subgroup.

Let T be the trio group in M of shape $2^{11}.2^6(S_3 \times L_3(2))$ for the brick trio (see p.9-10). Then $Z(O_2(T)) = V_1 \leq V$ has order 2^3, and the 'Pentad' group $P = N_{J_4}(V)$ has shape $2^{3+12}(S_5 \times L_3(2))$ with $O_2(P)$ special (see p. 3).

Looking at V_1 in H, we find that it is the inverse image in E of an isotropic 1-space in $\bar{E}$, namely $\begin{array}{|ccc|} 0 & \lambda & \lambda \\ 0 & \lambda & \lambda \end{array}$ a 1-space of edge type for the edge $\{3, 15\}$ =  .

Thus $N_H(V_1)$ is the sextet group of shape $2^{1+12}.3.2^4(S_5 \times 2)$ for the square sextet  in H.

Thus

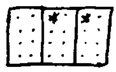
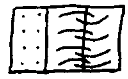
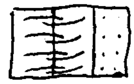
$P \cap M = N_M(V_1) = T$ has shape $\begin{cases} 2^{11}.2^6(S_3 \times L_3(2)) & \text{in } M \\ 2^{3+12}(S_4 \times L_3(2)) & \text{in } P \end{cases}$

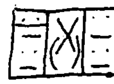
$P \cap H = N_H(V_1)$ has shape $\begin{cases} 2^{1+12}.3.2^4(S_5 \times 2) & \text{in } H \\ 2^{3+12}(S_5 \times 2^2L_2(2)) & \text{in } P \end{cases}$

and $P = \langle P \cap M, P \cap H \rangle$.

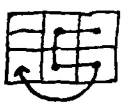
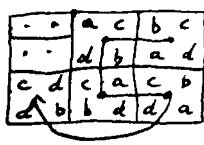
<u>The centralizer of an element of order 7</u> (c.f. p.4)

Let x_7 =  $\in M \cap P$.

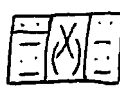
Then $C_M(x_7) = \langle x_7 \rangle \times \langle$  ,  ,  $\rangle$

$= \langle x_7 \rangle \times \langle$  2B ,  2A ,  2A $\rangle$

has shape $7 \times S_4$.

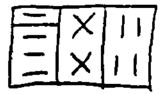
Since $C_{J_4}(x_7) = C_P(x_7)$ has shape $7 \times S_5$, and $H \cap P$ has index 7 in P , we must be able to find an element of order 5 in H commuting with x_7 . Now $N_P(x_7)$ has shape $7.3 \times S_5$ with $O_{7,3}(N_P(x_7)) = O_{7,3}(N_M(x_7)) = \langle x_7, w \rangle$. Thus the element of order 5 must commute with w , and hence lie in F . Now there is a unique element of order 5 in F which acts on the 5 tetrads of the square sextet for H 's MOG as  and makes an S_5 with the S_4 we already have, namely x_5 =  5A = (3 15)(9 13 22 20 17) (19 11 21 8 10)(6 2 1 18 4) (5 7 12 14 16)

and so we have $[x_5, x_7] = 1$.

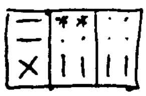
Thus $C_{J_4}(x_7) = \langle x_7, x_5,$  2A $\rangle$

Let $\Sigma = O^7(C_{J_4}(x_7)) = \langle x_5,$  2A $\rangle \cong S_5$.

The centralizer of an element of order 5

We now have enough information to find $C_{J_4}(x_5)$ of shape $5 \times 2^3L_3(2)$ (with the $2^3L_3(2)$ non-split) , since we already know that $C_P(x_5) \geqslant \langle x_5, V_1, w, x_7, Y \rangle$ where Y is the unique representative of 

commuting with x_5 (note that the $M_{22}.2$ centralizer of an element of order 5 is of order 10)

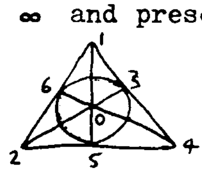
i. e. $Y =$  as an element of M .

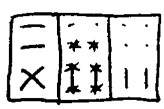
Since $\langle x_5, V_1, w, x_7, Y \rangle$ is already big enough, it must be the whole centralizer of x_5 .

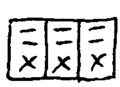
$C_M(x_5)$ is a non-split extension of shape $2^3L_3(2)$ and lies in the trio group T. Its action on the three bricks is :

Left-hand brick : $L_2(7)$ on the standard numbering on the brick .

∞	0
3	2
5	1
6	4

Right-hand brick : $L_3(2)$ fixing ∞ and preserving the projective plane structure given by  in the standard numbering on the brick .

Middle brick : Rewriting Y as  we see that the action on the middle brick is the same as the action on the right-hand brick except that a $\mathcal{C}^*$-set has been attached to each permutation on $\{0, \ldots , 6\}$ to give a non-split monomial group $2^3L_3(2)$.

Now let $\Lambda = \langle$  $, x_7 \rangle$ of shape $L_2(7)$ (i. e. the elements of $T \cap K$ acting acting on each brick, preserving the standard numbering) . Then $\Lambda \cap C_P(\Sigma) = \langle x_7, w \rangle = \mathrm{Stab}_\Lambda(\infty)$. Our notations for P will be based on the two subgroups Λ and Σ .

The structure of $O_2(P)$ and notation for elements of P

The action of $P / O_2(P) = S_5 \times L_3(2)$ on $O_2(P) / V_1$ is the same as on the tensor product of the two-dimensional module over $GF(4)$ for $S_5 \cong \Sigma L_2(4)$ written as a 4-dimensional module over $GF(2)$, with a 3-dimensional module for $L_3(2)$ dual to V_1. (We call elements of V_1 lines and of the dual V_1^* points)

Thus we should label the tetrads of the square sextet for H with the points of a projective line $PG(1,4)$. We choose the numbering :

	$\bar{\omega}$	1
∞	ω	0

Figure 7

The elements of V_1 are given names as follows :

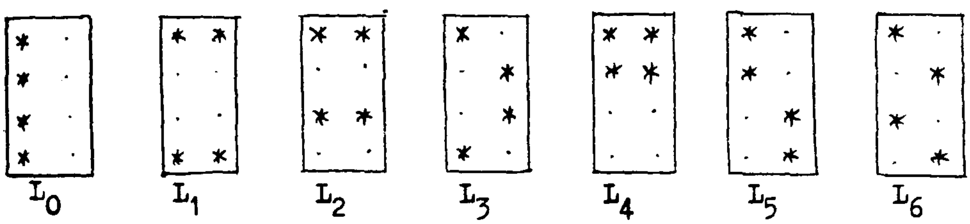
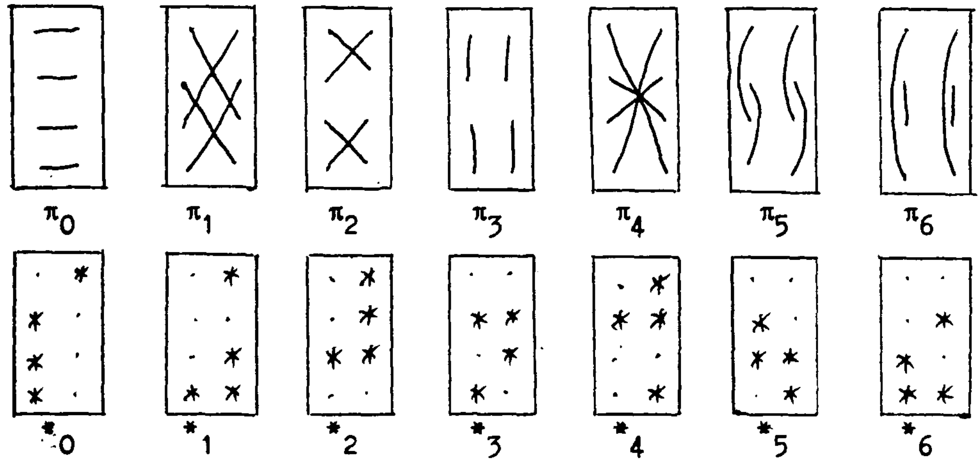


Table 11

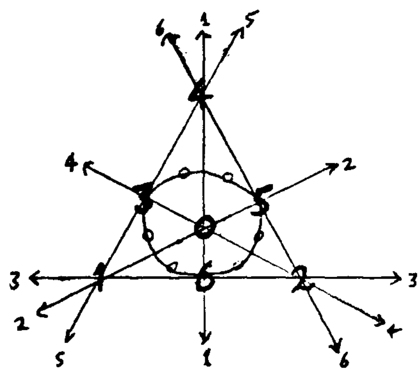
where each element of V_1 is a sextet vector in V for a sextet refining the brick trio for M, labelled above by giving the pair of tetrads comprising any brick (such sextets are similar in each brick).

We give elements of the dual of V_1 two interpretations, one as affine translations on a brick, and one as sets of four points on a brick (c. f. fig. 2) as follows :



<u>Table 12</u>

This means that lines (in V_1) and points (in V_1*) are incident as follows :



<u>Figure 8</u>

This numbering gives a 1-1 correspondence between lines and points which is respected by the group $\Lambda \cap C_P(\Sigma)$ but is NOT incidence preserving.

Thus we see that elements of $O_2(P) / V_1$ are spanned by elements of the form $(n) \otimes (x, y)$, $0 \leq n \leq 6$; $x, y \in GF(4)$. We choose a particular inverse image n_{xy} in $O_2(P)$, of $(n) \otimes (x, y)$, as follows :

$$n_{11} = \begin{array}{|c|c|c|}\hline 0 & \pi_n & \pi_n \\ \hline\end{array} \qquad n_{10} = \begin{array}{|c|c|c|}\hline 0 & *_n & *_n \\ \hline\end{array}$$

$$n_{\omega\bar{\omega}} = \begin{array}{|c|c|c|}\hline \pi_n & 0 & \pi_n \\ \hline\end{array} \qquad n_{\omega 0} = \begin{array}{|c|c|c|}\hline *_n & 0 & *_n \\ \hline\end{array} \qquad \in M$$

$$n_{\bar{\omega}\omega} = \begin{array}{|c|c|c|}\hline \pi_n & \pi_n & 0 \\ \hline\end{array} \qquad n_{\bar{\omega} 0} = \begin{array}{|c|c|c|}\hline *_n & *_n & 0 \\ \hline\end{array}$$

and then linearly :

$$n_{x+x'\ y+y'} = n_{x\ y} \cdot n_{x'\ y'}$$

Let $tr(x) = x + \bar{x} \in \{0, 1\}$, and let

$$n^*_{x\,y} = n_{xy} \circ L_n^{tr(xy)}$$

Then elements of P can be written in the form :

(element of V_1)$\circ$(product of n_{xy}'s)$\cdot$(element of Σ)$\cdot$(element of Λ)

The multiplication rules are :

(i) $Z(O_2(P))$:

$$L_a^{\;2} = 1$$

$$L_a \circ L_b = L_c$$

where L_a, L_b, L_c are three lines intersecting in a point

(ii) $O_2(P)$:

$$n_{x\ y} \cdot n_{x'\ y'} = n_{x+x'\ y+y'}$$

$$m_{x\ y} \cdot n_{x\ y} = r_{x\ y} \cdot L_a^{tr(xy)}$$

where (m, n, r) are 3 points lying on line a

$$[\, m_{x\ y} , n_{x'\ y'} \,] = L_a^{tr\begin{vmatrix} x, & y \\ x' & y' \end{vmatrix}}$$

where a is the line joining m and n

(iii) Action of Λ on $O_2(P)$:

Let $\lambda \in \Lambda$. Then

$(L_a)^{\lambda} = L_b$ where $\lambda : a \mapsto b$ as lines

$(n_{x\,y})^{\lambda} = m_{x\,y}$ where $\lambda : n \mapsto m$ as points

(iv) Action of Σ on $O_2(P)$:

Let $\sigma \in \Sigma$. Then

$(L_a)^{\sigma} = L_a$

$(n^*_{x\,y})^{\sigma} = n^*_{x'\,y'}$

where $\sigma : (x, y) \to (x', y')$

as an element of $\Sigma L_2(4)$

(v) $[\Sigma, \Lambda]$:

$[\sigma, \lambda] = n_{\overline{ac}\;\overline{bd}}$

where $\lambda : \infty \mapsto n$

and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $(1, 0) \to (a, b)$, $(0, 1) \to (c, d)$

as an element of $\Sigma L_2(4)$

Remark

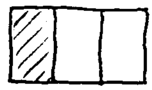
Let $M \ast_D H$ be the free amalgamated product of groups isomorphic to M and H with the vertical sextet group in M identified with the left-hand hexad group in H via the dictionary of Chapter 2 (i.e. D as a subgroup of each identified). Let $\dfrac{M \ast_D H}{\langle [x_5, x_7] \rangle}$ be the quotient of this by the normal closure of the element $[x_5, x_7]$. Then there is a surjective map

$$\frac{M \ast_D H}{\langle [x_5, x_7] \rangle} \rightarrow J_4$$

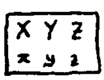
given in the obvious way. We shall be investigating the kernel of this map in Chapter 8 , but for the moment, let us remark that $\langle M \cap P , H \cap P \rangle$, as a subgroup of $\dfrac{M \ast_D H}{\langle [x_5, x_7] \rangle}$ is isomorphic to P ; i.e. that all relations in P follow from the dictionary of p. 31 and the fact that x_5 commutes with x_7 . This follows easily from the fact that $O_2(P) \leqslant D$.

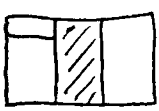
Chapter 5 The subgroup L of shape $2^{10}L_5(2)$

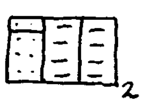
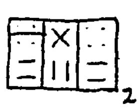
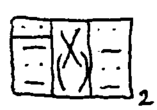
Looking once more at [2] we find a recipe for obtaining in J_4 a subgroup of shape $2^{10}L_5(2)$. In this chapter I shall investigate this subgroup and its intersections with M , H and P .

Let $A_0 \leq V$ be the subgroup of even $\mathcal{C}$*-sets with a representative contained in the left-hand octad  ,

so that $|A_0| = 2^6$. Then $N_M(A_0)$ is the octad group in M of shape $2^{11}.2^4L_4(2)$ (see p. 10) and $C_K(A_0)$ has order 2^4 and consists of affine translations of the right-hand square . Let $A = A_0C_K(A_0)$ so that A is elementary abelian of order 2^{10} , and let $L = N_{J_4}(A)$. Then L is of shape $2^{10}L_5(2)$, L splits over $O_2(L)$, and the action of $L / O_2(L)$ on $O_2(L)$ is irreducible and is the same as the action on the skew-square of a natural 5-dimensional irreducible module .

Looking at A in H , we find that $A \cap E$ has order 2^7 , and is the inverse image in E of the isotropic 3-space  in $\bar{E}$ of octad type for the 'middle' octad

 of H's MOG .

$A \cap F = \langle z,$  ,  ,  $\rangle$

has order 2^4 , and $A = \langle A \cap E , A \cap F \rangle$.

Thus $N_H(A)$ is the octad group of shape $2^{1+12}.3.2^3(L_3(2) \times 2)$ for the middle octad of H's MOG .

As a subgroup of P , A is

$\langle V_1, \{n_{x\,y} : x, y \in \{0, 1\}\}, (\omega \leftrightarrow \bar{\omega}) \in \Sigma \rangle$

so that $N_P(A)$ is the inverse image in P of the centralizer in $P / O_2(P)$ of the field automorphism $(\omega \leftrightarrow \bar{\omega})$ and has shape $2^{3+12}((S_3 \times 2) \times L_3(2))$ *in P*.

So we have :

$L \cap M = N_M(A)$ has shape $\begin{cases} 2^{10}.2^4L_4(2) & \text{in } M \\ 2^{10}.2^4L_4(2) & \text{in } L \end{cases}$

(but note that the two subgroups A and $L \cap V$ of size 2^{10} are not the same)

$L \cap H = N_H(A)$ has shape $\begin{cases} 2^{1+12}.3.2^3(L_3(2) \times 2) & \text{in } H \\ 2^{10}.2^6(L_2(2) \times L_3(2)) & \text{in } L \end{cases}$

$L \cap P = N_P(A)$ has shape $\begin{cases} 2^{3+12}((2 \times S_3) \times L_3(2)) & \text{in } P \\ 2^{10}.2^6(L_3(2) \times L_2(2)) & \text{in } L \end{cases}$

We shall find a nice basis with respect to which we shall write elements of $L / O_2(L)$ as 5×5 matrices in such a way that the intersections with M, H and P are three of the four maximal parabolics defined by the upper triangular matrices with respect to this basis. Then we shall find a particular complement to $O_2(L)$ in L , in order to be able to write elements of L in the form

(sum of wedge-products) . 5×5 matrix .

Looking at A_0 as a module for $N_M(A) / A$ we see that it is isomorphic to the skew-square of $C_K(A_0)$. The rule for finding the wedge-product of two different involutions λ and μ of $C_K(A_0)$ as an element of A_0 is as follows :

The orbits of $\langle \lambda , \mu \rangle$ of length four are congruent mod $\mathcal{C}$, and define a $\mathcal{C}^*$-set $\lambda \circ \mu$ in A_0 (we denote this wedge product by a circle to distinguish it from the wedge-product $\wedge$ from $W \times W$ to A to be defined later)

To which conjugacy class in J_4 do elements of A belong ?

From the analysis on p. 35-37 we see that :

(i) elements of $C_K(A_0)$ are in class 2A

(ii) duads in A_0 are in class 2B

(iii) sextets in A_0 are in class 2A

(iv) elements of the form

sextet s in A_0 . element λ of $C_K(A_0)$

are in class $\begin{cases} \text{2A} & \text{if } \exists\, \mu \in C_K(A_0) \text{ s. t. } s = \lambda \circ \mu \\ \text{2B} & \text{otherwise} \end{cases}$

(v) elements of the form

duad in A_0 . element of $C_K(A_0)$ are in class 2B

Thus if we let $W = \langle a, b, c, d, e \rangle$ be an abstract 5-dimensional space over GF(2) so that W^{2-}, the skew-square of W , is ten-dimensional (we write wedge-products in W^{2-} with a $\wedge$) then we can identify W^{2-} with A via :

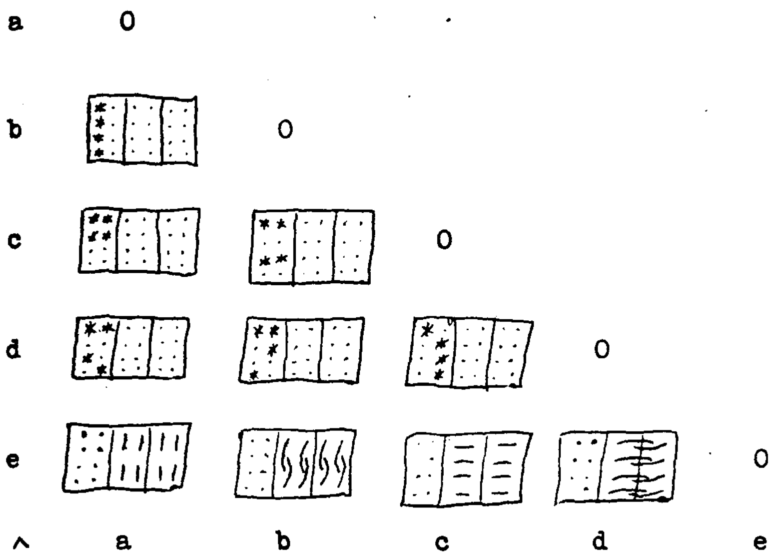


Table 13

and with this identification we find that for

$\lambda, \mu \in \langle a, b, c, d \rangle \ , \quad \lambda \wedge \mu = (\lambda \wedge e) \circ (\mu \wedge e)$

and so an element of A is in

class 2A if it is a simple wedge-product

class 2B otherwise.

Lemma 5.1

If X is a vector-space over a field having all square roots, and $\dim(X) \neq 4$, then every automorphism of X^{2-} preserving the set of wedge-products is the skew-square of an element of $\Gamma L(X)$.

Proof We examine the subspaces of X^{2-} such that every element is a wedge-product of elements of X . Such a subspace corresponds to a collection of 2-spaces in X such that if X_0 and X_1 are in the collection then $\dim(X_0 \cap X_1) = 1$, and every subspace Y with $X_0 \cap X_1 < Y < \langle X_0, X_1 \rangle$ is also a member of the collection.

The possible structures of such collections of subspaces, for given dimension of subspace of X^{2-}, are by induction as follows :

dimension 1 : a single 2-space X_0

dimension 2 : $\{Y : X_0 < Y < X_1\}$ for a 1-space X_0 and a 3-space X_1

dimension 3 : There are now two possibilities :

(i) $\{Y : Y < X_0 \text{ and } \dim Y = 2\}$ for a 3-space X_0

(ii) $\{Y : X_0 < Y < X_1 \text{ and } \dim Y = 2\}$ for a 1-space X_0 and a 4-space X_1

dimension 4 : (i) above cannot be extended to a 4-space and so the only possibility is $\{Y : \dim Y = 2 \text{ and } X_0 < Y < X_1\}$ for a 1-space X_0 and a 5-space X_1

dimension $n > 4$: The only possibility here is $\{Y : \dim Y = 2 \text{ and } X_0 < Y < X_1\}$ for a 1-space X_0 and an $(n+1)$-space X_1

Let $\mathcal{M}(X)$ be the collection of subspaces of X^{2-} of maximal dimension such that every element is a wedge-product of elements of X, so that we have shown that if $\dim X \geqslant 5$ then every element of $\mathcal{M}(X)$ has dimension $\dim X - 1$ and is of the form $X_0 \wedge X$ for a uniquely determined 1- space X_0.

Assume now that $\dim X \geqslant 5$.

Three 1-spaces X_0, X_1 and X_2 lie in a 2-space iff

$$\dim((X_0 \wedge X) \cap (X_1 \wedge X) \cap (X_2 \wedge X)) = 1$$

Thus am automorphism ρ of X^{2-} preserving the collection of wedge-products determines a permutation of the 1-spaces preserving whether three such lie in a 2-space. Hence there is an element of $P\Gamma L(X)$ determining the same permutation. Choose a preimage ε for this in $\Gamma L(X)$ and look at $\rho^{-1}\varepsilon^{2-}$.

This fixes each element of $\mathcal{M}(X)$, and hence fixes each intersection of a pair of elements of $\mathcal{M}(X)$. But these are precisely the 1-spaces of wedge-products.

This is enough to show that $\rho^{-1}\varepsilon^{2-}$ is a scalar transformation λI . Then $\rho = (\sqrt{\lambda}\ \varepsilon)^{2-}$.

Thus the result is proven for dim $X \geqslant 5$. The result is clear for dim $X \leqslant 3$, and is false for dim $X = 4$. (take the skew-square of a duality for example) //

Thus we have a well-defined action of $L / O_2(L)$ on W , and can hence write elements of $L / O_2(L)$ as 5×5 matrices acting on W as the space of row vectors. The groups $(M \cap L)/ O_2(L)$, $(H \cap L)/ O_2(L)$ and $(P \cap L)/ O_2(L)$ are as follows :

$(M \cap L)/ O_2(L)$ $$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} a\ b\ c\ d\ \ e \\ \left(\begin{array}{c|c} * & 0 \\ \hline * & 1 \end{array}\right) \end{array}$$

$(H \cap L)/ O_2(L)$ $$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} a\ b\ \ c\ d\ e \\ \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right) \end{array}$$

$(P \cap L)/ O_2(L)$ $$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} a\ b\ c\ \ d\ e \\ \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right) \end{array}$$

Figure 9

We now attack the problem of a complement for $O_2(L)$ in L . We notice that $O_2(L)$ has a natural conjugacy class of complements in $L \cap M$, a representative of which is $C_K(A_0)^{\{\infty,22\}}$ where $\{\infty,22\}$ is a duad in V . Can we extend this to a complement for $O_2(L)$ in L ?

Lemma 5.2

$L_5(2)$ has the following presentation by generators and relations :

$$\left\langle a_i,\ b_i,\ 1 \leqslant i \leqslant 4 \ \middle|\ \begin{array}{l} \left.\begin{array}{l}(a_ia_j)^n = 1 \\ (b_ib_j)^n = 1\end{array}\right\} \ n = \begin{cases} 1 \text{ if } |i-j| = 0 \\ 4 \text{ if } |i-j| = 1 \\ 2 \text{ if } |i-j| \geqslant 2 \end{cases} \\ (a_ib_j)^n = 1 \quad n = \begin{cases} 3 \text{ if } i = j \\ 2 \text{ if } i \neq j \end{cases} \\ (a_ib_ia_{i+1})^3 = (b_{i+1}b_ia_{i+1})^3 = (b_{i+1}a_ia_{i+1})^3 \\ = 1 \quad \text{for each} \quad 1 \leqslant i \leqslant 3 \end{array} \right\rangle$$

Proof These relations are satisfied by

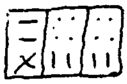
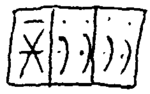
$$a_1 = \begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \quad a_3 = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \end{pmatrix} \quad a_4 = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&1&1 \end{pmatrix}$$

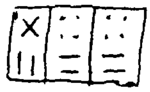
$$b_1 = \begin{pmatrix} 1&1&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \quad b_2 = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&1&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \quad b_3 = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} \quad b_4 = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&0&1 \end{pmatrix}$$

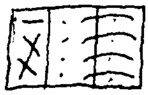
and successive coset enumerations of the $2^{r+1}-1$ cosets of $\langle a_i,\ b_i,\ 1 \leqslant i \leqslant r-1\ ,\ a_r \rangle$ in $\langle a_i,\ b_i,\ 1 \leqslant i \leqslant r \rangle$ and the 2^{r+1} cosets of $\langle a_i,\ b_i,\ 1 \leqslant i \leqslant r \rangle$ in $\langle a_i,\ b_i,\ 1 \leqslant i \leqslant r\ ,\ a_{r+1} \rangle$ show that these relations are sufficient. ♦

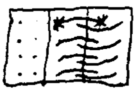
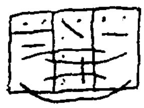
Theorem 5.3

The following elements of M and H generate a complement L_1 for $O_2(L)$ in L :

$\begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$ =  $\in M \cap H$ $\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$ =  $\in M \cap H$

$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$ =  $\in M \cap H$ $\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$ =  $\in M$

$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&1&0\\0&0&0&0&1\end{pmatrix}$ =  $\in M \cap H$ $\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$ =  $\in M \cap H$

$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&1\end{pmatrix}$ =  $\in M \cap H$ $\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}$ =  $_{2A}$ $= \alpha \in H$

Proof

Taking these as the a_i and b_i of the lemma, we see that each of the given relations is either satisfied in M or H , or is the relation :

$[\ \alpha\ ,$  $] = 1$

This relation holds in the pentad group, as is easily checked. //

Remark

The proof of the theorem shows that in the group $\frac{M \ast_D H}{\langle [x_5, x_7] \rangle}$ described on p. 48 the subgroup $\langle M \cap L , H \cap L \rangle$ is isomorphic to the group L described in this chapter; i.e. all relations in L follow from the dictionary of p. 31 and the fact that x_5 commutes with x_7 .

Having the complement L_1 , we now have a notation for elements of L as

(element of $O_2(L)$).(element of L_1)

i. e. as

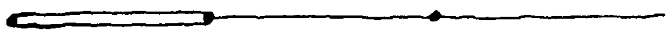
(sum of wedge-products).(5 X 5 matrix)

The reader is now referred to Appendix D where many useful elements of J_4 are written in the notations for those of M, H, P and L in which they lie.

CODA Larghetto

The Smith-Ronan Diagram for J_4

In [11] , Smith and Ronan investigate diagrams for groups 'of GF(2) type' . These are supposed to be an extension of the Dynkin Diagram notation for Chevalley Groups over GF(2) . The nodes of the diagram represent an 'incident' set of maximal 2-local subgroups in such a way that suppressing a particular node and all the edges leading from it leaves the diagram for the quotient of that maximal 2-local by its O_2 . For example, the following is their diagram for M_{24} :



$2^4L_4(2)$ OCTAD $2^6(S_3 \times L_3(2))$ TRIO $2^63Sp_4(2) = 2^63S_6$ SEXTET

Figure 10

In their notation, a missing node is one which cannot be suppressed, and which magically reappears when certain of the other nodes have been suppressed. Thus it might be better to re-draw the above diagram as

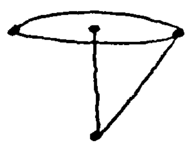


Figure 11

with the rule that a trapped node cannot be suppressed.

Then the diagram for $3S_6$ would be

Figure 12

In this notation the diagram for $M_{22}.2$ is

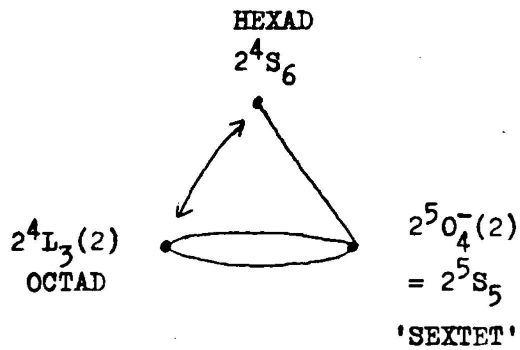


Figure 13

that for $3M_{22}.2$ is

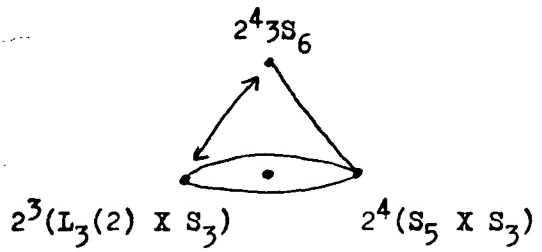


Figure 14

and finally the diagram for J_4 is

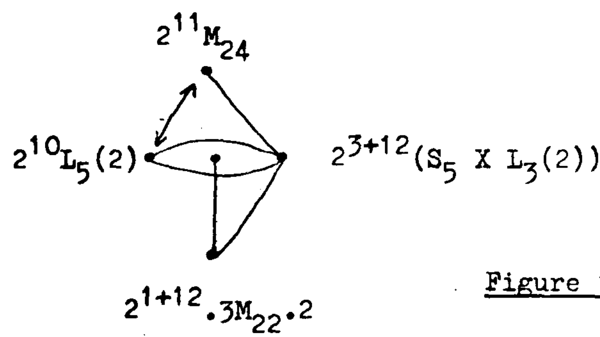


Figure 15

The notions of incidence involved in these diagrams were used as a guide to the choice of notations for H, M, P and L; in H we choose an incident hexad, sextet and octad to define $H \cap M$, $H \cap P$ and $H \cap L$, and in M we choose an incident sextet, trio and octad to define $M \cap H$, $M \cap P$ and $M \cap L$. Then the notations for P

and L follow naturally.

In the Chevalley groups, not only can we choose an incident set of maximal 2-locals, but we can also choose an 'incident' set of Levi Complements in these maximal 2-locals. Corresponding to this, our 'complements' $F < H$, $K < M$, Λ and $\Sigma < P$ and $L_1 < L$ seem to be incident in a similar way.

(Other examples of trapped nodes are :

$U_4(3).2^2$

$3^2U_4(3).2^2$

$Fi_{24}.2$

$3Fi_{24}.2$

and perhaps a little more far-fetched are :

$U_4(2).2$

$Sz.2$

$3Sz.2$)

Chapter 6 The construction of J_4 via the 112-dimensional 2-modular representation Δ

In this chapter, I shall describe the construction by Norton, Parker and Thackray of a pair of 112 X 112 matrices over GF(2) generating J_4 (see [6]), some of the methods developed by Parker and Thackray (see [7]) for dealing with 2-modular representations on a computer, and the proof by Norton, Parker, Conway, Thackray and myself that the group generated by these matrices is indeed isomorphic to J_4. In the next chapter I shall develop more of the internal structure of the module.

When the ordinary character table of J_4 had been produced in Cambridge in 1975, it was seen that the smallest ordinary character degree was 1333, and that even that was irrational, so that it would be quite difficult to construct the group via this ordinary representation. Hence Thompson decided to examine the possibilities for a modular representation of small degree. By restricting to various subgroups, he showed that for $p \neq 2$, there could be no non-trivial p-modular representation of degree less than 1333, and that for $p = 2$, the smallest possibility was a (self-dual) 112-dimensional representation over GF(2). Thus he made :

Conjecture 1

J_4 has a representation of degree 112 over GF(2).

The problem was then to try to construct a pair of 112 X 112 matrices over GF(2) generating this representation of J_4. To do this, it was necessary to find a moderately large subgroup of odd characteristic. Looking again at the ordinary character table of J_4, it was seen that $|U_3(11)|$

divides $|J_4|$, and that a set of apparently consistent character restrictions from J_4 to $U_3(11)$ could be written down . Thus we have :

<u>Conjecture 2</u>

J_4 has a subgroup U isomorphic to $U_3(11)$.

Suppose J_4 exists and satisfies conjectures 1 and 2. Under the action of U , the 112-dimensional module $\triangle$ would have to be uniserial 1+110+1 , with a unique invariant 1-dimensional submodule $\triangle_1^U$ and a unique invariant 111-dimensional submodule $\triangle_{111}^U$; $\triangle_{111}^U / \triangle_1^U$ lifts to characteristic 0 .

(Notation : when $\triangle$ has a unique invariant submodule of dimension n under the action of a group X , we denote this submodule by $\triangle_n^X$)

An element of order 11 in the centre of a Sylow 11-subgroup acts fixed point freely on $\triangle_{111}^U / \triangle_1^U$, and hence has a unique fixed point on $\triangle / \triangle_1^U$. Thus this module is a quotient of the permutation module on the cosets of a Sylow 11-normalizer (i.e. on isotropic vectors). It turns out that the kernel is generated by fixed points on even elements of the module. Thus $\triangle / \triangle_1^U$ can easily be built as matrices. Moreover, $\triangle_{111}^U$ is dual to this module. Since $\triangle_{111}^U / \triangle_1^U$ is self-dual, it is then possible to choose bases so that the matrices for these two modules agree in 110 rows and columns. Gluing them together leaves only one bit of the resulting matrices unresolved :

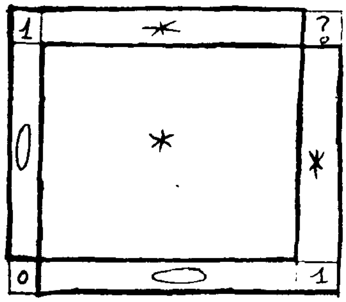


Figure 16

Since the group of all matrices obtained by filling in this bit in both possible ways is a group of shape $U_3(11) \times 2$, choosing an odd order generating set resolves the ambiguity.

Now we need an extra element to complete this to a representation of J_4 . This was done by enlarging the Sylow 11-normalizer :

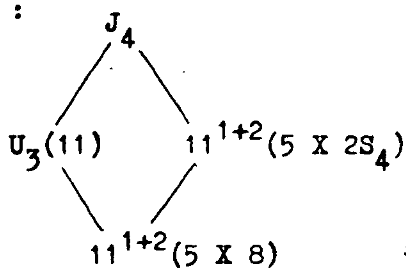


Figure 17

Under the action of the J_4 Sylow 11-normalizer, $\triangle$ decomposes as a direct sum of a 110-dimensional module and a 2-dimensional module (with S_3 action on the latter). Restricting to the subgroup of shape $11^{1+2}.5$, the 110-space becomes a 55-space over GF(4) . Thus we can find a matrix of order 3 normalizing this subgroup in the required way, by the methods described in the next section, and then all such matrices can be obtained from this one by multiplication by a GF(4)-scalar matrix.

Thus very few possibilities were found for a set of generating matrices for J_4 . All but one of these possibilities was rejected immediately by taking random products of the generating matrices and finding an element whose order is not the order of an element of J_4 .

Thus we were left with a particular set of generating matrices for a group which we believed to be isomorphic to J_4 . The problem was then to prove that this was indeed the case. Before I give this proof, I shall discuss the computer techniques used to verify certain facts about the module needed in the proof.

Computer Techniques for Modular Representations of Finite Groups

This section is a brief description of a body of techniques developed by Richard Parker and Jon Thackray for dealing with modular representations of finite groups on a computer (see [7] for further details). These work in principle for any finite field, but have so far been implemented only over GF(2), where the techniques are most efficient.

A group representation is stored as a set of non-singular matrices generating the group. Since we mostly deal with groups which can be generated by two matrices, the programmes have been written to store a group representation as a pair of matrices. All vectors considered are row vectors acted on the right by matrices.

The basic operations defined on matrices and vectors are the following :

(i) Rank
(ii) Addition
(iii) Multiplication
(iv) Inversion
(v) Transposition
(vi) Tensor Product
(vii) Exterior Powers Etc.
(viii) Null Space
(ix) 'Invariant Subspace'
(x) 'Standard Base'
(xi) Top Left
(xii) Split

and a few more technical operations.

Only the last five of these need explanation.

Null space :

This takes as input a singular matrix say of nullity n , and gives as output a non-singular matrix the first n rows of which give a basis for the null space, and the rest of which complete this to a basis for the whole space.

Invariant Subspace :

This takes as input a pair of matrices and a vector. It then finds the submodule generated by the vector under the action of the matrices, and gives as output the dimension n of this space, and a matrix the first n rows of which give a basis for the subspace and the rest of which complete this to a basis for the whole space.

Standard Base :

This takes as input a pair of matrices and a vector not in any proper invariant subspace. The output is a matrix whose rows form a basis obtained in a standard way from the input. (i.e. conjugating the input by a matrix will have the effect of conjugating the output by the same matrix.)

Top Left :

This takes as input a matrix and an integer n , and gives as output the top-left n x n portion of the matrix.

Split :

This takes as input two matrices and a vector. It finds the submodule generated by the vector under the action of the matrices, and outputs the two group elements in the two new representations : submodule and quotient module.

With the above tools, the complete lattice of submodules for a representation may be found as follows :

(i) The nullity trick :

The basic observation here is that a 'random' element of the group algebra as a matrix in a given representation over a finite field is quite likely to have small non-zero nullity. This is at its best over GF(2) and gets worse for bigger fields. (For example for an absolutely irreducible representation over GF(2) the probability of a random matrix having nullity 1 is about 0.56 , over GF(3) about 0.42 , and over GF(5) about 0.24 . For a reducible representation the probabilities are not much different, unless the representation is really a representation over a larger field of the same characteristic, written over the small field. Then the nullities are all divisible by the degree of the field extension.)

Now suppose we are given a reducible representation and we wish to find an invariant submodule. The first step is to take random elements of the group algebra in the representation until one is found of small non-zero nullity. The next step is to take the non-zero null vectors of this element one by one and find the invariant subspace generated by each under the action of the group. Since it is quite likely that the element has nullity in a proper submodule if there is one, this means that a proper invariant subspace will be found after a few tries like this. We may then extract the submodule and quotient module as new matrix representations of the group.

(ii) The irreducibility test :

Having applied the nullity trick to a representation and not found an invariant subspace, we may wish to try to prove that the representation is irreducible. To do this, again we take random elements of the group algebra in the representation until we find one of small non-zero nullity (preferably nullity 1), and then we take all its non-zero null vectors and find the submodule generated by each under the action of the group. If in each case the whole space is found to be the answer, we repeat the procedure with the transpose of the matrix and the transpose inverse of the generators of the group. If the whole space is again found to be the answer in each case, we know that the representation is irreducible.

(iii) Isomorphism types; fingerprints :

Having obtained the composition factors of the module, we wish to know which are isomorphic. The first and obvious remark is that modules of different dimensions are non-isomorphic. Secondly, a quick and easy test for non-isomorphism is to find an element of the group algebra with different nullities in the two representations. Thus we try to find a short list of test-elements of the group algebra, so that the 'fingerprint' of a module - i.e. the set of nullities of the test-elements - is enough to ascertain the isomorphism type.

The method for proving that two modules are isomorphic is as follows :

First, we find an element of the group algebra of small non-zero nullity (again preferably nullity 1) in the representations (the nullities had better be the same !) . Then we take a particular null vector in the first representation and find the standard base (see above) with respect to this vector. Doing the same for each null vector in the other representation, we find either that :

(a) there is a null vector such that expressing the matrices for each representation with respect to the standard bases for these null vectors, the two representations have identical matrices, in which case the representations are isomorphic, or

(b) no such null vector has this property, in which case the representations are non-isomorphic.

(iv) Non-existence of composition factors :

In order to show that an irreducible module A is not a composition factor of a module B , it is sufficient to find an element of the group algebra having greater nullity on A than on B .

(v) The Lattice of Submodules :

We now know the isomorphism types of all the composition factors, and wish to know the lattice of submodules. For this it is enough to be able to tell what are the bottom constituents of a module (i.e. the socle) and then to 'peel off' bottom constituents one at a time. To find all submodules of our representation B isomorphic to a particular irreducible module A , we find an element of the group algebra having non-zero nullity on A and small nullity on B . We then take each of the null vectors of this element on the representation B , and for each one we find the invariant submodule generated by it under the action of the group. The number of times the submodule generated is isomorphic to A tells us the number of copies of A there are in the socle of B , if we know the correct field of definition of A (i.e. if we know the centralizer ring of A) .

There are also many short-cuts for decreasing the amount of work needed, and so far these have made the methods usable for representations of dimensions up to about 10000 .

The Group Game; the Centralizer of an Involution

In its strictest sense, the group game is the following :

There are two players, A and B . A thinks of a group and gives B a list of symbols for generators. B may then ask A for the symbol for any word in A's generators. The only restriction on A is that if he names the same element twice he must use the same symbol. B's task is to find out what group A is thinking of.

Example :

A gives B the list a_1 , a_2 .

B's question	A's answer
a_1a_1	a_3
a_3a_3	a_3
a_2a_2	a_3
a_1a_2	a_4
$a_4a_4a_4a_4a_4a_4$	a_4

At this point B deduces that A is thinking of the dihedral group of order 10 .

Our problem with J_4 was very similar, with the computer as player A and us as player B , except that we could obtain information inadmissible to the group game player. For example, if 1+x has different rank from 1+y , then x is not conjugate to y . (For each conjugacy class of element x in J_4 , the rank of 1+x in the representation Δ is given in Appendix C , after the existence has been proven)

Our tasks were to show :

(i) The centralizer of some involution z in the group generated by our matrices has the form given in Hypothesis A on p. 3 .

(ii) The group generated by our matrices is simple.

Then by the characterization given in Janko [2] the group generated would be proven to be J_4 .

Inside the group game, there is a quick and easy method for finding a subgroup of the centralizer of an involution x , which is probably the whole of $O^{2'}(C(x))$, as follows :

First, randomly multiply elements together until an element of even order is obtained, and then take a suitable power of it in order to obtain an involution x . Repeat the process to obtain another involution y , preferably not conjugate to x . Then $\langle x , y \rangle$ is a dihedral group, and if xy has even order $2n$ (which it does if $x \not\sim y$) then $(xy)^n$ is an involution commuting with x . Repeating this process with conjugates of y or with other random involutions, soon the whole of $O^{2'}(C(x))$ will be obtained. However, there is never a guarantee that it has all been found.

In our group of matrices, we found an involution z with $\text{Rank}(1+z) = 50$, and found by the above method a group H centralizing it. It was thus suspected that H was of the form given in Hypothesis A on p. 3 . The action of H on a suitable composition factor of Δ under this action was used to identify $H / O_{2,3}(H)$ (†) with $\text{Aut}(M_{22})$. A supplement F of shape $6M_{22}.2$ to $O_2(H)$ in H waw then found by the following method, which is again a group game technique :

Take an element $w \in O_{2,3}(H)$ of order 3 (i.e. an element of order 3 acting trivially on the above composition factor) and define $F = N_H(w)$. Given an involution t in $H \setminus H'$, either $w^t = w^{-1}$, in which case $t \in F$, or $w^t = xw^{-1}$ with $x \in O_2(H)$. In the

(†) i.e. a quotient of H which will later turn out to be $H/O_{2,3}(H)$. The same comment applies to the rest of this and the next paragraph.

latter case $\langle t , w , z\rangle / \langle z\rangle \cong S_4$ with t acting as $(01)(2)(3)$ and w as $(0)(123)$. Thus $twtw^2t$ is an element of F equivalent to t modulo $O_{2,3}(H)$. F is generated by elements of this form.

Next, Todd's presentation of M_{22} given in [10] was used to prove by generators and relators that H really is as in Hypothesis A . Thus the notation developed in Chapter 2 can be used to describe elements of H .

As the next step, two involutions of total type (see p. 23) lying in the subgroup E_1 of $O_2(H)$ (see p. 27) were taken, and again by the above method, subgroups of their centralizers were found. By means of looking at the action of the group M generated by these on an invariant submodule of Δ of dimension 12 under this action, this group M was identified with the group of shape $2^{11}M_{24}$ described in (ii) on p. 3 , and again Todd's presentation of M_{24} was used to prove by generators and relators that M really is isomorphic to the group described there. Notation for M was chosen in such a way that the dictionary of p. 31 held, and this dictionary was explicitely verified on the computer. Thus the rest of the analysis of Chapter 3 holds in the group of matrices $G = \langle M , H\rangle$.

Next, the relation $[x_5,x_7] = 1$ (see p. 42) was verified, so that our group G is a quotient of the group $\dfrac{M \ast_D H}{\langle [x_5,x_7]\rangle}$ defined on p. 48 . Thus by the remarks on p. 48 and p. 57 there are subgroups P and L of the shapes described in chapters 4 and 5 , intersecting M and H in the ways described there.

In fact we also checked on the computer directly by generators and relators using the presentation given in [12] that the elements given on p. 56 do indeed generate a subgroup isomorphic to $L_5(2)$, since this fact is used in our proof that G is isomorphic to J_4.

The Actions of H , M , P and L on Δ ; Sacred Vectors

The methods of p. 65 – 70 were used to show that under the action of each of H , M , P and L , Δ reduces almost uniserially, as illustrated in the following diagrams :

H	M	P	L
$\overline{10}$	1	$4_1 \otimes 1$	$1 \oplus 5$
30	$\overline{11}\ (S\mathcal{C}^*)$	$4_2 \otimes 3_p$	$\overline{10}$
$\left.\begin{matrix}10\\ \overline{10}\end{matrix}\right\} \oplus 12$	$\overline{44}\ (\gamma S(\mathcal{C}^*)^{2-})$	$(1 \otimes X) \oplus (4_1 \otimes 3_1)$	$\overline{40}$
	$44\ (\gamma S\mathcal{C}^{2-})$	$4_2 \otimes 8$	40
30	$11\ (P\mathcal{C})$	$(1 \otimes X) \oplus (4_1 \otimes 3_p)$	10
10	1	$4_2 \otimes 3_1$	$1 \oplus 5$
		$4_1 \otimes 1$	

Table 14

These diagrams indicate all submodules, and the numbers are dimensions of composition factors. Bars indicate duality, so that $\overline{10}$ denotes a module dual to 10 . For example, this means that under the action of H , there are invariant submodules of dimensions 10 , 40 , 50 , 52 , 60 , 62 , 72 and 102 (exactly one of each). See p. 62 for the notation for these submodules.

For P , any irreducible module is the tensor product of one for S_5 and one for $L_3(2)$. For S_5 , the module 4_1 is the deleted permutation module on 5 points, and 4_2 is the doubly deleted permutation module on 6 points. For $L_3(2)$, 3_1 (lines) is a module isomorphic to the module $Z(O_2(P))$, and 3_p (points) is the dual of this module (see p. 45).

X is a self-dual uniserial module for $L_3(2)$ with diagram :

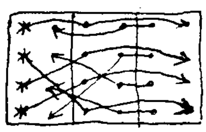
$$\begin{matrix} 3_p \\ 3_1 \\ 3_p \\ 3_1 \end{matrix}$$

Under the action of M , we see that there is a unique non-zero fixed vector v_∞ . We shall call the images of v_∞ under the group G the 'sacred vectors' .

Under the action of K , $\Delta_{12}^{M} \cong \mathcal{C}$, so that we may identify vectors in this with octads, dodecads, etc.

<u>Two Generators for $G = \langle M , H \rangle$</u>

<u>Lemma</u> $G = \langle g_1 , g_2 \rangle$, where

$g_1 =$  $\in M$, and $g_2 = x_5^{-1} x_7$.

<u>Proof</u> Certainly $\langle M , H \rangle \supseteq \langle g_1 , g_2 \rangle$, so we must prove that $\langle M , H \rangle \subseteq \langle g_1 , g_2 \rangle$. We prove this in five stages :

(i) $K = \langle g_1^{\,2} , g_2^{\,5} \rangle$:

$g_1^{\,2} \in K$, and since $[x_5 , x_7] = 1$, $g_2^{\,5} = x_7^{\,5} \in K$. We examine the set of maximal subgroups of M_{24} which are transitive and have order divisible by 7 . There are the trio group of shape $2^6(S_3 \times L_3(2))$ and the octern group of shape $L_3(2)$. We see firstly that x_7 is in only one trio group , namely that for the brick trio , and that $g_1^{\,2}$ is not in this. Secondly, in the octern group the set of fixed points of an element of order 7 is a set of imprimitivity whereas this does not hold for the fixed points of x_7 .

in $\langle g_1^{2}, g_2^{5}\rangle$.

(ii) $F' = \langle g_1 , g_2^{7}\rangle$:

$g_1 \in F'$, and since $[x_5,x_7] = 1$, $g_2^{7} = x_5^{3} \in F'$.
$\langle g_1 , g_2^{7}\rangle$ is transitive on the 22 points , whereas all proper subgroups of M_{22} are intransitive.

(iii) $M = \langle g_1 , g_2^{5}\rangle$:
$g_1^{3} = z \in V$, which is an irreducible module for K .

(iv) $E \leqslant \langle g_1 , g_2\rangle$ since $E \leqslant M$.

(v) $H \leqslant \langle g_1 , g_2\rangle$ since M contains elements in the outer half of F . //

These generators g_1 and g_2 are shown in appendix B .

<u>Splitting the skew-square of Δ ; the neighbourhood of a vector</u>

The most important step in the proof that G is isomorphic to J_4 was the finding of an invariant submodule of the skew-square Δ^{2-} of Δ under the action of the two generators g_1 and g_2 . This took the computer about 100 min. of central processing unit time, and produced an invariant submodule Δ^{2-}_{4995} of dimension 4995 (Δ^{2-} has dimension 6216) generated by a vector in $\Delta^M_1 \wedge \Delta^M_{12}$.

<u>Definition</u> Given a subspace X of Δ , we define the NEIGHBOURHOOD of X by

$$\mathcal{N}(X) = \{ w \in \Delta : \forall v \in X , \ w \wedge v \in \Delta^{2-}_{4995} \}$$

and the neighbourhood of a vector is the neighbourhood of the 1-dimensional space spanned by it.

The neighbourhood of a subspace X is clearly invariant under $\mathrm{Stab}_G(X)$.

The following facts needed in the proof that G is isomorphic to J_4 were verified by computer :

(i) $\mathcal{N}(\Delta^M_1) = \Delta^M_{12}$

(ii) The non-zero vector in Δ^L_1 is an octad vector in Δ^M_{12} (for the left - hand octad)

(iii) $\mathcal{N}(\Delta^L_1) = \Delta^L_6$

(iv) If v is a dodecad vector in Δ^M_{12} then $\mathcal{N}(v)$ has dimension 2 .

(v) $\Delta^L_{16} \leq \Delta^M_{56}$

(vi) If w_1 , w_2 and w_3 are three octad vectors in Δ^M_{12} for disjoint octads then $\mathcal{N}(w_1) \cap \mathcal{N}(w_2) \cap \mathcal{N}(w_3)$ has dimension 2 , and consists of v_∞ , another sacred vector in $\Delta^M_{56} \setminus \Delta^M_{12}$ called the TRIO vector $f_1(w_1, w_2, w_3)$ (see p. 93) and a non-sacred vector.

(vii) An orthogonal form on Δ was found which is invariant under the action of g_1 and g_2 .

The structure of these facts will become clearer in the next chapter.

As an example of how these facts were proven, we showed (iii) to be true by showing that $\mathcal{N}(\Delta^L_1)$ contains some element of $\Delta^L_6 \setminus \Delta^L_1$ and does not contain some element of $\Delta^L_{16} \setminus \Delta^L_6$. Then since $\mathcal{N}(\Delta^L_1)$ is L-invariant , it must be Δ^L_6 .

The proof that G is isomorphic to J_4

Stage 1 $C_G(v_\infty) = M$:

First, $C_G(v_\infty)$ fixes Δ^M_{12} setwise by (i). If v and v' are respectively an octad vector and a dodecad vector in Δ^M_{12}, facts (ii), (iii) and (iv) above show that

$$\dim \mathcal{N}(v) \neq \dim \mathcal{N}(v')$$

Thus octad vectors in Δ^M_{12} are really different from dodecad vectors under the action of $C_G(v_\infty)$, and hence this acts as M_{24} on $\Delta^M_{12} / \Delta^M_1 \cong \mathcal{PC}$. Hence the action on Δ^M_{12} is at most $2^{11}M_{24}$, and so it is sufficient to show that $C_G(\Delta^M_{12})$ is trivial.

First we observe that $Fix_\Delta(C_G(\Delta^M_{12}))$ is a subspace of Δ invariant under M. Now $C_G(\Delta^M_{12})$ stabilizes $\mathcal{N}(w_1) \cap \mathcal{N}(w_2) \cap \mathcal{N}(w_3)$ where w_1, w_2 and w_3 are three octad vectors in Δ^M_{12} for disjoint octads, and hence by (vi) above, it stabilizes the trio vector $f_1(w_1 , w_2 , w_3)$ so that $C_G(\Delta^M_{12}) \leqslant C_G(\Delta^M_{56}) \leqslant C_G(\Delta^L_{16})$ by (v).

If $y \in C_G(\Delta^L_{16})$ then y stabilizes v_∞ and so $\exists\ x \in M$ such that the action of x on Δ^M_{12} is the same as that of y. But then $yx^{-1} \in C_G(\Delta^M_{12}) \leqslant C_G(\Delta^L_{16})$ so that $x \in C_M(\Delta^L_{16}) = \{1\}$. Hence $y \in C_G(\Delta^M_{12})$.

Thus we have $C_G(\Delta^M_{12}) = C_G(\Delta^L_{16})$, so that $Fix_\Delta(C_G(\Delta^M_{12}))$ is invariant under both M and L. But looking on p. 75 we see that this implies that it is the whole of Δ. Thus $C_G(\Delta^M_{12}) = \{1\}$ as required.

Stage 2 Since Fix $(O_2(M))$ is invariant under M, it must be Δ_1^M. Thus the orbits of $O_2(M)$ on sacred vectors are all of even length except $\{ v_\infty \}$ and so there are oddly many sacred vectors. Hence a Sylow 2-subgroup of M is a Sylow 2-subgroup of G.

Stage 3 $C_G(z) = H$:
(recall that $\langle z \rangle = Z(H)$)

$(1 + z)^2 = 0$, so $\mathrm{Ker}(1 + z) / \mathrm{Im}(1 + z)$ is invariant under $C_G(z)$. Since $\mathrm{Rank}(1 + z) = 50$,

$$\dim(\mathrm{Ker}(1 + z) / \mathrm{Im}(1 + z)) = 12$$

(c.f. p. 75)

and the action of H on this is as $3M_{22}.2$. But

For every X with $3M_{22}.2 < X \leqslant Sp_{12}(2)$, the order of a Sylow 2-subgroup of X is greater than the order of a Sylow 2-subgroup of $3M_{22}.2$ (*) (see footnote (†) on next page)

so that the action of $C_G(z)$ on it is just $3M_{22}.2$.

Now consider a minimal normal subgroup of the kernel of this action, mod $\langle z \rangle$. If it is a direct product of isomorphic non-abelian simple groups, then the Sylow 2-subgroup is elementary abelian of order 2^{12} and the automorphism group contains $3M_{22}.2$, which is clearly absurd. If it is an elementary abelian group of odd order then Δ decomposes as a direct sum of at least two eigenspaces under the action of this group, each of which is invariant under the action of H, again absurd. Thus $O_2(H / \langle z \rangle)$ is the unique minimal normal subgroup.

Thus any odd order chief factor of $C_G(z) / \langle z \rangle$ acts on $O_2(H / \langle z \rangle)$ non-trivially, so that by (*) we have $H = C_G(z)$ as required.

<u>Stage 4</u> G is simple :

Suppose N is a minimal normal subgroup of G . Since there are no normal subgroups of M and H intersecting $M \cap H$ in the same way, either $N \geq \langle M , H \rangle = G$, or $N \cap M = N \cap H = \{1\}$. But then N has odd order, and is hence elementary abelian, and again Δ decomposes as a direct sum of at least two eigenspaces of N , a contradiction, since $O_2(M)$ has a 1-dimensional fixed space.

Thus by the characterization of Janko [2], G is isomorphic to J_4 .

//

p. 81 (†): The representation of $C_G(z)$ on $Ker_\Delta(1+z)/Im_\Delta(1+z)$ is symplectic for the following reason:

fact (vii) gives an isomorphism $\Delta \cong \Delta^*$, and hence $Ker_\Delta(1+z)/Im_\Delta(1+z) \cong Ker_{\Delta^*}(1+z)/Im_{\Delta^*}(1+z)$

$$\cong (Ker_\Delta(1+z)/Im_\Delta(1+z))^*$$

There are many ways of proving the fact (*). For example, one could look at the list of groups with Sylow 2-subgroups of size 2^8 and containing $3M_{22}.2$, and having a 12-dimensional 2-modular representation.

Chapter 7

The Geometry of Δ ; a Presentation for J_4 by Generators and Relators

Since the stabilizer of v_∞ is exactly M , the sacred vectors are in one-one correspondence with the right cosets of M in G (throughout this chapter the word coset will automatically mean right coset) so that there are exactly 173 067 389 of them. In this chapter I shall investigate further properties of the module Δ from the point of view of the geometry of the set of sacred vectors at the same time as making an abstract coset enumeration for the cosets of M in a group $\hat{G}$ defined below on p. 89 by generators and relators , with the aim of proving that $\hat{G} \cong G$. Thus many of the arguments in this chapter have two simultaneous contexts. This is merely a device for saving having to write down the same arguments twice, and I hope this does not cause too much confusion.

For the purpose of the detailed analysis of stabilizers needed in this chapter I need a technical definition :

Definition

For X_1 , $X_2 \leqslant X$ finite groups , define

$$\mathcal{M}(X_1 , X_2) = \{ N \leqslant X_1 \mid N < N' \leqslant X_1 \Rightarrow N \cap X_2 < N' \cap X_2 \}$$

Lemma 7.1

(i) $\mathcal{M}(X_1 , X_2) = \{ N \leqslant X_1 \} \Leftrightarrow X_1 \leqslant X_2$

(ii) $\mathcal{M}(X_1 , X_2) = \{ X_1 \} \Leftrightarrow X_1 \cap X_2 = \{1\}$

(iii) $X_2 \leq X_3 \Rightarrow \mathcal{M}(X_1, X_2) \subseteq \mathcal{M}(X_1, X_3)$

(iv) $X_1 \leq X_3 \Rightarrow \mathcal{M}(X_1, X_2) = \{ N \cap X_1 \mid N \in \mathcal{M}(X_3, X_2)\}$

(v) If $X_3 \trianglelefteq \langle X_1, X_2 \rangle$ then

$$N / X_3 \in \mathcal{M}(X_1 / X_3, X_2 / X_3) \Rightarrow N \in \mathcal{M}(X_1, X_2)$$

(vi) If G acts on a set S, and $x \in S$, suppose

$Y \subseteq \mathrm{Stab}_{X_1}(x)$ and

$$\langle Y, \mathrm{Stab}_{X_1 \cap X_2}(x) \rangle \in \mathcal{M}(X_1, X_2) .$$

Then

$$\langle Y, \mathrm{Stab}_{X_1 \cap X_2}(x) \rangle = \mathrm{Stab}_{X_1}(x) .$$

<u>Proof</u>

Clear //

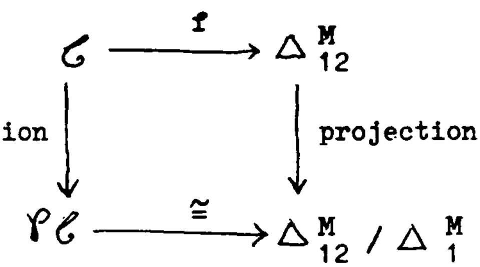
Part (vi) of this lemma will be used repeatedly.

<u>One-functions and Nought-functions</u>

Under the action of $M \,/\, O_2(M)$, $\Delta^M_{12} / \Delta^M_1 \cong \mathcal{PC}$ (see p. 13). Let $\psi : \mathcal{PC} \times \mathcal{SC}^* \to GF(2)$ be the bilinear map establishing duality between $\mathcal{PC}$ and $\mathcal{SC}^*$; i.e. $\psi(x,y) = |\hat{x} \cap \hat{y}| \pmod 2$ where $\hat{x}$ and $\hat{y}$ are subsets of Ω representing x and y . Then $O_2(M)$ acts on Δ^M_{12} by

$$x : y \mapsto y + \psi(x,y).v_\infty \quad .$$

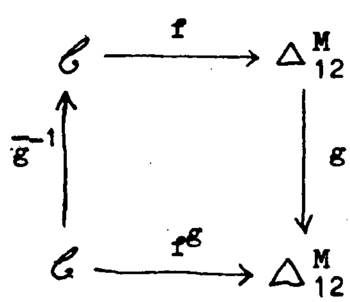
Now let $f : \mathcal{C} \to \Delta^M_{12}$ be a linear function such that



commutes.

There are 2^{12} such functions, since the difference between two is a linear function from $\mathcal{C}$ to Δ^M_1 , i.e. an element of $\mathcal{C}^*$. For 2^{11} of these, $f(\Omega) = v_\infty$, and these are called the one-functions. For the remaining 2^{11} , $f(\Omega) = 0$, and these are called the nought-functions.

$g \in M$ with image $\bar{g} \in M_{24}$ acts on the set of such f's via



and the stabilizer of a particular such function f is a complement to $O_2(M)$ in M. The two types of complement given in this way (c.f. p. 14) are called one-type complements and nought-type complements. It turns out that a hexad-type complement (see p. 28) is the same as a one-type complement and a point-type is the same as a nought-type. Thus our complement K corresponds to a particular one-function f_1. The other f's are then of the form f_1^x where $x \in \Omega$ and $|x|$ is odd for a nought-function and even for a one-function.

The last relator ; the group $\hat{G}$

Let x_{23} be the element of order 23 in K acting as the permutation $x \mapsto x+1$ on M's MOG , and let x_{11} be the element of order 11 in K acting as $x \mapsto 2x$ and hence normalizing x_{23} . Let f be the element of G of order 2 normalizing x_{23} and centralizing x_{11} (the Sylow 23-normalizer in J_4 is Frobenius of shape 23.22). Then Janko [2] has shown that

$$\mathcal{J} = (M \cap M^f)\langle f\rangle \cong PGL_2(23)$$

with $\mathcal{J} \cap M \cong L_2(23)$. (f is the element $x \mapsto -x$)

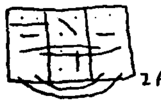
Thus v_∞^f is a sacred vector with $Stab_M(v_\infty^f) = \mathcal{J} \cap M$. So $\mathcal{J} \cap M$ has a two-dimensional fixed space, and v_∞^f lies outside Δ_{111}^M . The image of v_∞^f in Δ / Δ_{100}^M is an odd $\mathcal{C}$*-set stabilized by $\mathcal{J} \cap M$, i.e. $\{\infty\}$, and hence $\mathcal{J} \cap M$ lies in the nought-type complement corresponding to this $\mathcal{C}$*-set.

The element f was constructed by computer, and in particular it was shown that

$$f = m_1 h_1 m_2 h_2 m_3 h_3 m_4$$

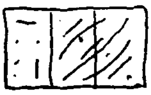
where :

$m_1 =$  $\cdot(\infty\ 6\ 22\ 12\ 1\ 7\ 3\ 21\ 4\ 11\ 10\ 8\ 9\ 19)$
$(0\ 14\ 16\ 17\ 20\ 13\ 2)(5\ 18)(15)$ $\in M$

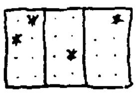
$h_1 = \alpha =$  $\in F$

$m_2 = (\infty\ 15\ 14\ 8\ 2\ 17\ 11\ 21\ 4\ 22)(1\ 7)$
$(0\ 18\ 6\ 10\ 20\ 12\ 13\ 3\ 16\ 5)(9\ 19)$ $\in K$

$h_2 = x_5{}^3 \in F$

$m_3 =$  $\in K$

$h_3 = [\ (\infty)(1\ 9\ 11\ 3\ 21\ 10\ 18\ 5\ 14\ 17\ 7)$
$(0)(2\ 15\ 22\ 16\ 13\ 12\ 6\ 20\ 4\ 8\ 19)\ ]_{22A} \in F$

$m_4 =$  $\cdot(\infty\ 0\ 11\ 21\ 2\ 10\ 15\ 17\ 22\ 7\ 5\ 8\ 16)$
$(3\ 14\ 18\ 20\ 9\ 13\ 16)(4\ 19)(12)$ $\in M$

and f is an involution in class 2B .

Letting f be this element of the group $M \ast_D H$ defined on p. 48, we see that G is a quotient of the group

$$\hat{G} = \frac{M \ast_D H}{\langle [x_5, x_7], f^{-1} x_{23} f x_{23} \rangle}$$

In the rest of this chapter I shall show that this is in fact a presentation for J_4 ; i.e. that

<u>Theorem 7.2</u>

The surjection $\hat{G} \to G$ is an isomorphism.

If required, Todd's presentation [10] for the Mathieu Groups may be used to make this into an explicit presentation by generators and relators, but I see no point in writing down the details.

I shall use the same symbols for cosets of M in $\hat{G}$ as for sacred vectors, so that for example v_∞ is the identity coset of M. Each coset/sacred vector will have two names, one as an element of an M-orbit and one as an element of an H-orbit. My arguments will be phrased in terms of cosets, but will apply both to cosets and sacred vectors. The relationship between the set of sacred vectors and the linear structure of Δ is discussed in square brackets after the relevant cosets have been investigated abstractly. Thus these parts may be omitted without hindering the analysis of $\hat{G}$ and the proof of theorem 7.2 .

By the remarks on p. 48 and p. 57 , $\hat{G}$ has subgroups P and L of the shapes given in chapters 4 and 5 intersecting M and H in the way described there.

It is clear from the analysis on p. 35 - 40 that M and H contain the fusion of involutions in M and H so that in $\hat{G}$, there are at least two classes 2A and 2B of involutions such that every involution of M and of H is in one of these classes in the way shown in Tables 9 and 10.

We may use the language of nought-functions and one-functions for the subgroup M of $\hat{G}$, since M still has a faithful representation of dimension 12 isomorphic to the module $\triangle^{M}_{12}$ for M as a subgroup of G .

The idea of the argument for proving that $\hat{G} \cong G$ is as follows :

The map $\hat{G} \to G$ induces a map from the set $\hat{\mathcal{S}}$ of cosets of M in $\hat{G}$ to the set $\mathcal{S}$ of sacred vectors. It is sufficient to prove that this map is injective.

A character-theoretic calculation shows that under the actions of H and M , the set $\mathcal{S}$ falls into respectively 9 and 7 orbits. The intersection of an M-orbit with an H-orbit decomposes as a union of orbits of $D = M \cap H$, of which there are 36 altogether. Appendix E , due to S.Norton [6] , shows how these intersections of orbits decompose, in each case as a union of at most two D-orbits.

What I shall do is to find a set of 7 particular elements of $\mathcal{J}$ representing the 7 M-orbits on $\mathcal{J}$, and show that their $\hat{G}$-stabilizer is at least as big as the G-stabilizer of their image in $\mathcal{J}$, and likewise for the 9 H-orbits. This will be by a process of going backwards and forewards between M- and H-orbits. For example, we may take an element of an M-orbit, and find its H-stabilizer by the process of taking the $M \cap H$-stabilizer and using one of the extra relations if necessary to complete this to the whole H-stabilizer. If the resulting group is in $\mathcal{M}(H, M)$ then we may conclude by Lemma 7.1 (vi) that the whole of the H-stabilizer of this coset has been found, and that it is the same in $\hat{G}$ as it is in G. The method will become clearer when you see it in practice. This produces for us two sets of 173 067 389 cosets of M in $\hat{G}$, the first closed under multiplication by elements of M and the second closed under multiplication by elements of H. Thus all that remains is to identify the two sets with each other, so that $|\hat{G} : M| = 173\,067\,389$. This will complete the proof that $\hat{G} \cong G$.

To make the identifications, we proceed as follows. Each of the two collections of cosets falls into 36 orbits under D. If we can show that one element of a D-orbit in the one set is the same as one element of a D-orbit in the other set, the whole orbits are automatically identified. Thus we must check 36 identifications. Some of these will follow from the definition of the coset, and the rest will follow from the relation $[x_5, x_7] = 1$, as will be seen (c.f. p.104 where the first example of an

'x_5-x_7 square' occurs). Thus the tables in Appendices E and F will be verified.

To convert this into an abstract proof of the existence of J_4 , the following extra verifications are necessary :

(i) The subgroup D of M really is isomorphic to the subgroup D of H . This is not clear from the analysis of Chapter 3 , but shouldn't be too difficult to prove.

(ii) The D-stabilizers of a representative of each of the 36 D-orbits are the same in the two sets of cosets. For some of these this follows from definition, but for the rest an explicit check is necessary.

Then J_4 will have been constructed as a group of permutations of 173 067 389 objects.

The Pentad ; the M-orbit T of Trio cosets

Since $|P : P \cap M| = 5$, v_∞ has 5 images under P , which are permuted like the 5 tetrads in figure 7 (p. 44). Thus we label these cosets v_∞ , v_0 , v_1 , v_ω and $v_{\bar\omega}$, and they form a 'Pentad' on which the Pentad group P acts.

[The space spanned by these vectors is invariant under P , and hence from p. 75 we see that it is Δ^P_4 . Thus

$$v_\infty + v_0 + v_1 + v_\omega + v_{\bar\omega} = 0 \qquad - \quad (1) \qquad]$$

Let α be the element of P with action $(\infty\ 0)(1)(\omega)(\bar\omega)$ on PG(1,4) defined on p. 56 . Then $v_\infty\alpha = v_0$, and so $Stab_{M\cap P}(v_0) = M \cap (M \cap P)^\alpha$ is a group of shape $2^9.2^6(S_3 \times L_3(2))$ generated by the even $\mathcal{C}$*-sets in V hitting each octad of the brick trio evenly, and the trio group in K for the brick trio.

Since $Stab_{M\cap P}(v_0) \in \mathcal{M}(M , P)$, we see by lemma 7.1 (vi) that $Stab_M(v_0) = Stab_{M\cap P}(v_0)$.

Thus the M-orbit of v_0 consists of $2^2.3795$ cosets called the TRIO cosets (c.f. p. 79) and that such a coset is determined by a trio on M's MOG and a one-function f , where two one functions determine the same coset iff they are conjugate by a $\mathcal{C}$*-set in V hitting each of the octads of the trio evenly.

We write f(a,b,c) for the trio coset defined by the trio { a,b,c } and the one-function f . We denote by $f^{a,b}(a,b,c)$ the trio coset determined by the same trio

$\{ a,b,c \}$ but by a one-function conjugate to f by a $\mathcal{C}^*$-set hitting the octads a and b oddly and c evenly. Thus while $f^{a,b}$ is not a well-defined one-function, $f^{a,b}(a,b,c)$ is a well-defined trio vector. So for example labelling the brick trio as $\boxed{1 \,|\, \bar{\omega} \,|\, \omega}$ we have

$$\begin{aligned} v_0 &= f_1(\text{brick trio}) = f_1(1,\bar{\omega},\omega) \\ v_1 &= f_1^{\bar{\omega},\omega}(1,\bar{\omega},\omega) \\ v_\omega &= f_1^{1,\bar{\omega}}(1,\bar{\omega},\omega) \\ v_{\bar{\omega}} &= f_1^{1,\omega}(1,\bar{\omega},\omega) \end{aligned}$$

[On the computer it was checked that $v_0 \in \Delta^M_{56}$, so that all trio vectors are in Δ^M_{56}. In particular, $\Delta^M_1 \leqslant \Delta^P_4 \leqslant \Delta^M_{56}$.

Since the vectors $f(a,b,c)$ and $f^{a,b}(a,b,c)$ are conjugate by an element of $O_2(M)$, they must have the same image in $\Delta^M_{56} / \Delta^M_{12}$, and hence

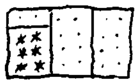
$$f(a,b,c) + f^{a,b}(a,b,c) \in \Delta^M_{12} .$$

Its image in $\Delta^M_{12} / \Delta^M_1$ is stabilized by $\mathrm{Stab}_M(c)$, and so since it cannot be v_∞ (or else $f^{a,b}(a,b,c) = f^{b,c}(a,b,c)$) it must be $f(c)$ or $f(c) + v_\infty$. If it were $f(c) + v_\infty$, then $f^{b,c}(a,b,c) + f^{a,c}(a,b,c) = f(a) + f(b)$ $= f(c) + v_\infty$, which implies that $v_0 + v_1 + v_\omega + v_{\bar{\omega}} = 0$, contradicting the known structure of Δ^P_4. Hence we have

$$f(a,b,c) + f^{a,b}(a,b,c) = f(c) \qquad - \quad (2) \qquad]$$

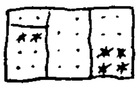
The H-orbit $\textcircled{H}$ of Hexad cosets

Since $|H : H \cap M| = 77$, v_∞ has 77 images under H . The stabilizer of such a coset is the stabilizer of a hexad ρ for H's MOG , and we write this coset as $\textcircled{H}(\rho)$. Thus for example

$v_\infty = \textcircled{H}($  $) = \textcircled{H}(\vartheta)$ - (*)

(see p. 27)

Since $\alpha \in H$, we see that

$v_0 = \textcircled{H}($  $)$

$\textcircled{H} \cap T$:

Thus the 60 hexad cosets for hexads intersecting the hexad ϑ in two points (these hexads clearly form a D-orbit) are the same as the 60 trio cosets for the 15 trios incident with (i.e. made up of pairs of tetrads of) the vertical sextet for M's MOG . The rule for translation is as follows :

Rule $\textcircled{H}$ - T :

The hexad intersects ϑ in a duad, which determines a syntheme on φ (see pp. 12 , 27 and 28) and hence a trio refining the vertical sextet for M's MOG , via the correspondence in figure 6 on p. 29 . The other four points of the hexad either miss φ , in which case the trio coset is f_1(trio) , or intersects it in one of the duads of the syntheme, in which case if the other two duads determine octads a and b of the trio, the coset is $f_1^{a,b}$(trio) .

There is one such Rule for every D-orbit (Rule I - Ⓗ is statement (*) above ; the M-orbit I is $\{ v_\infty \}$). I shall give these rules only when they take a fairly simple form, as these are the only cases of interest. For the rest, I shall simply give one identification; acting on this by D gives the 'rule'.

[It was checked by computer that $v_\infty \in \Delta^H_{10}$ so that all hexad vectors are in Δ^H_{10}.

The module Δ^H_{10} has $O_{2,3}(H)$ in its kernel, and as a module for $Aut(M_{22})$ it is isomorphic to the module of even $\mathcal{C}$-sets including both or none of $\{ \infty, 0 \}$, modulo $\langle \Omega \rangle$. Hexads as such $\mathcal{C}$-sets are the same as hexads as sacred vectors.]

The M-orbit S of Sextet cosets

Let g be an element of H with $\theta^{g} = \varphi$. Then $Stab_D(\Theta(\varphi)) = M \cap D^g$, which is a subgroup of M of shape $2^7.2^6.3S_6$ and is generated by the even $\mathcal{C}$*-sets in V hitting each tetrad of the vertical sextet with the same parity (i.e. the PARITY subgroup E_1 (see p. 28)), and the sextet group in K for the vertical sextet.

Since $Stab_D(\Theta(\varphi)) \in \mathcal{M}(M, H)$, lemma 7.1 (vi) shows that $Stab_M(\Theta(\varphi)) = Stab_D(\Theta(\varphi))$.

Thus the M-orbit of $\Theta(\varphi)$ consists of $2^4.1771$ cosets called the SEXTET cosets, and each is determined by a sextet on M's MOG and a one-function f , where two one-functions determine the same coset iff they are conjugate by a $\mathcal{C}$*-set in V hitting each of the tetrads of the sextet evenly.

As on p. 93 , we write f(u,v,w,x,y,z) for the sextet coset defined by the sextet {u,v,w,x,y,z} and the one-function f , and similarly define $f^{u,v}(u,v,w,x,y,z)$ to be the coset defined by the same sextet and a one-function conjugate to f by a $\mathcal{C}$*-set hitting u and v oddly and w, x, y and z evenly. Thus for example

$$\Theta(\varphi) = f_1(\text{vertical sextet}) .$$

$\underline{\Theta \cap S}$:

So we see that the 16 hexad cosets for hexads disjoint from θ are the same as the 16 sextet cosets for the vertical sextet, the correspondence being given by:

<u>Rule Θ - S</u> :

Let the hexad be φ^x, with x an element of U_1 (i.e. an affine translation on the right-hand square of H's MOG - see p. 27). If $x \neq 1$, then x determines a pair of columns u, v of M's MOG as on p. 32 .

The vertical sextet coset is then

f_1(vertical sextet) if $x = 1$

$f_1^{u,v}$(vertical sextet) if $x \neq 1$.

[On the computer it was checked that $\Theta(\varphi)$, and hence all sextet vectors, are in Δ^M_{56}, so that we have $\Delta^P_4 \leqslant \Delta^H_{10} \leqslant \Delta^M_{56}$. Thus as before we have

$$f(u,v,w,x,y,z) + f^{u,v}(u,v,w,x,y,z) \in \Delta^M_{12},$$

and must equal either $f(u+v)$ or $f(u+v) + v_\infty$. Thus

$$f^{w,x}(u,v,w,x,y,z) + f^{y,z}(u,v,w,x,y,z)$$
$$= f(w+x) + f(y+z)$$
$$= f(u+v) + v_\infty .$$

Conjugating by a $\mathcal{C}^*$-set in V hitting w and x oddly and u, v, y and z evenly, this gives

$$f(u,v,w,x,y,z) + f^{u,v}(u,v,w,x,y,z)$$
$$= f(u+v) + v_\infty \quad - \quad (3)$$

Now conjugating (1) by an element of H taking the duad {3, 15} to the duad {2, 11} on H's MOG , we see that

$$f_1(A,B,C,D,E,F) + f_1^{A,B}(A,B,C,D,E,F) + f_1^{CD,EF}(AB,CD,EF)$$
$$+ f_1^{CE,DF}(AB,CE,DF) + f_1^{CF,DE}(AB,CF,DE) = 0 ,$$

where $\{A,B,C,D,E,F\}$ is the vertical sextet for M's MOG (c.f. figure 6 , p. 29).

Combining this with (2) and (3) , we get

$$f_1(AB,CD,EF) + f_1(AB,CE,DF) + f_1(AB,CF,DE) = v_\infty$$

and hence for any sextet $\{u,v,w,x,y,z\}$ and any one-function f ,

$$f(uv,wx,yz) + f(uv,wy,xz) + f(uv,wz,xy) = v_\infty \qquad - \; (4)$$

Now these four vectors are hexad vectors for hexads satisfying :

(i) Any two intersect in two points

(ii) Any three intersect trivially .

Since $Aut(M_{22})$ is transitive on such configurations, this means that if ϑ_1, ϑ_2, ϑ_3 and ϑ_4 are any four hexads satisfying (i) and (ii) above, then

$$\Theta(\vartheta_1) + \Theta(\vartheta_2) + \Theta(\vartheta_3) + \Theta(\vartheta_4) = 0 \qquad - \; (5)$$

In particular,

$$f_1^{BE,CF}(AD,BE,CF) + f_1(AF,BD,CE) + f_1(AF,BD,CE)$$
$$+ f_1^{A,D}(A,B,C,D,E,F) = 0 \quad ,$$

and hence using (2) and (3) and acting by elements of M , we see that for any sextet $\{u,v,w,x,y,z\}$ and any one-function f ,

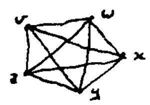
$$f(ux,vy,wz) + f(uy,vz,wx) + f(uz,vx,wy)$$
$$= f(u,v,w,x,y,z) + v_\infty \qquad - \; (6) \qquad]$$

[<u>Note added in proof</u>

It has now been shown on the computer that the modules $\mathcal{S}\,(\Delta^M_{12})^{2-}$ and Δ^M_{56}/Δ^M_1 are conjugate by an outer automorphism of M (c.f. p. 13 and cor. 1 on p. 14)

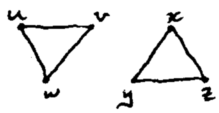
Thus if f is a one-function, thinking of an odd $\mathcal{C}$ *-set as an outer automorphism of M , and hence as an isomorphism between Δ^M_{56}/Δ^M_1 and $\mathcal{S}\,(\Delta^M_{12})^{2-}$ we have :

$$f^u(u,v,w,x,y,z) = f(v)_\wedge f(w) + f(v)_\wedge f(x) + f(v)_\wedge f(y) + f(v)_\wedge f(z) + f(w)_\wedge f(x) + f(w)_\wedge f(y) + f(w)_\wedge f(z) + f(x)_\wedge f(y) + f(x)_\wedge f(z) + f(y)_\wedge f(z)$$

(graphically u. 

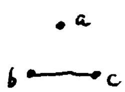
)

$$f^{u,v,w}(u,v,w,x,y,z) = f(u)_\wedge f(v) + f(u)_\wedge f(w) + f(v)_\wedge f(w) + f(x)_\wedge f(y) + f(x)_\wedge f(z) + f(y)_\wedge f(z)$$

(

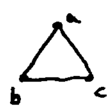
)

$$f^a(a,b,c) = f(b)_\wedge f(c)$$

(

)

$$f^{a,b,c}(a,b,c) = f(a)_\wedge f(b) + f(a)_\wedge f(c) + f(b)_\wedge f(c)$$

(

)

Then relations (2), (3), (4) and (6) above follow immediately modulo Δ^M_1 . Maybe an extension of this construction could be used as an abstract construction for the module Δ under the action of M and hence give an easy naming system for the sacred vectors. Adjoining some further automorphism would then provide an abstract geometric construction for J_4 .]

The action of L ; the H-orbit Φ of Big-Octad Cosets

Since $|M : M \cap L| = 31$, v_∞ has 31 images under L corresponding to the non-zero points of W^* , the dual of W . Since $\alpha \in L$ (see p. 56) , $v_\infty \alpha = v_0 = f_1$(brick trio) is one of these. This has 30 images under $M \cap L$, namely $f_1^{\,x}(\mathcal{O},b,c)$ for the 15 trios $\{\mathcal{O},b,c\}$ containing the left-hand octad $\mathcal{O}$ for M's MOG , and $x \in L \cap V$, i.e. $f_1(\mathcal{O},b,c)$ and $f_1^{\,b,c}(\mathcal{O},b,c)$.

[From the relations (2) and (4) on p. 94 and p. 99 we see that the set

$\{ 0 , f_1(\mathcal{O}) , v_\infty , f_1(\mathcal{O})+v_\infty , f_1(\mathcal{O},b,c) , f_1^{\,b,c}(\mathcal{O},b,c) , f_1(\mathcal{O},b,c) + v_\infty , f_1^{\,b,c}(\mathcal{O},b,c) + v_\infty \mid \{\mathcal{O},b,c\}$ is a trio containing the octad $\mathcal{O} \}$

is closed under addition and invariant under L and is hence the module Δ^L_6 . This decomposes as a direct sum of $\Delta^L_1 = \langle f_1(\mathcal{O}) + v_\infty \rangle$ and $\Delta^L_5 = \{ 0 , f_1(\mathcal{O}) , f_1(\mathcal{O},b,c) + v_\infty \} \cong W^*$,

as in the following diagram :

	$f_1(\mathcal{O}) + v_\infty$	v_∞	$f_1^{\,b,c}(\mathcal{O},b,c)$, $f_1(\mathcal{O},b,c)$
$\Delta^L_5 \rightarrow$	0	$f_1(\mathcal{O})$	$f_1(\mathcal{O},b,c) + v_\infty$, $f_1^{\,b,c}(\mathcal{O},b,c) + v_\infty$
	$\uparrow$ Δ^L_1		

Table 15

As L-modules, $\Delta^L_{16} / \Delta^L_6 \cong (\Delta^L_5)^{2-} = (O_2(L))^*$, the dual of the module $O_2(L)$. It was checked by computer that the vector f_1(vertical sextet) is in Δ^L_{16} . The L-orbit of this vector is easily seen to be the following set :

280 sextet vectors for the 35 sextets in which $\mathcal{O}$ is a union of tetrads, 8 vectors for each sextet as follows :

$f_1(u,v,w,x,y,z)$ 1
$f_1^{u,v}(u,v,w,x,y,z)$ 1
$f_1^{w,x}(u,v,w,x,y,z)$ 6
(where $u + v = \sigma$)

and 30 trio vectors for the 15 trios containing σ, 2 vectors $f_1^{\sigma,a}(\sigma,a,b)$ and $f_1^{\sigma,b}(\sigma,a,b)$ for each such trio $\{\sigma,a,b\}$.

It can be seen that $O_2(L)$ fuses these vectors in pairs as follows :

$$f_1(u,v,w,x,y,z) \sim f_1^{u,v}(u,v,w,x,y,z)$$
$$f_1^{w,x}(u,v,w,x,y,z) \sim f_1^{y,z}(u,v,w,x,y,z)$$
$$f_1^{\sigma,a}(\sigma,a,b) \sim f_1^{\sigma,b}(\sigma,a,b)$$

so that the image of this orbit in $\Delta^L_{16} / \Delta^L_6$ is an L-invariant set of 155 vectors. L has two orbits on the non-zero vectors in this 10-dimensional module, namely 155 pure wedge-products and 868 sums of two wedge-products, and so these sacred vectors have as image the pure wedge-products in this space.

From p. 75 we see that $\Delta^L_{16} / \Delta^L_5$ is a uniserial module of shape $\frac{10}{1}$, i.e. having unique invariant submodule Δ^L_6 / Δ^L_5. The 310 vectors above have distinct images in this space, and we can choose one vector from each pair as follows in such a way that the resulting set is invariant under our complement L_1, and span a 10-dimensional subspace of $\Delta^L_{16} / \Delta^L_5$:

$f_1(u,v,w,x,y,z)$

$f_1^{w,x}(u,v,w,x,y,z)$

$f_1^{\sigma,a}(\sigma,a,b)$

where the point 22 of M's MOG is in the tetrad w and the octad a.

So under the action of L_1, $\Delta^L_{16} / \Delta^L_5$ decomposes as a direct sum $1 + 10$, and complements conjugate to L_1 are in one-one correspondence with hyperplanes in $\Delta^L_{16} / \Delta^L_5$ not containing Δ^L_6 / Δ^L_5.]

Now consider the trio coset $c_1 = f_1^{a,b}\left(\begin{array}{|c|c|} \hline & a \\ \sigma & \\ \cline{2-2} & b \\ \hline \end{array} \right)$

Then $\mathrm{Stab}_L(c_1)$ contains $A = O_2(L)$ and has image

$$\begin{pmatrix} * & 0 & * & * & * \\ * & 1 & * & * & * \\ * & 0 & * & * & * \\ * & 0 & * & * & * \\ * & 0 & * & * & * \end{pmatrix} \text{ in } L / A .$$

Thus $\mathrm{Stab}_{L \cap H}(c_1)$ is the subgroup of H of shape $2^{1+9}.1.2^4L_3(2)$ generated by the centralizer in F of the element $\omega \leftrightarrow \bar{\omega}($  $)$, of shape $2^1.2^4L_3(2)$ and the subgroup

$$E(c_1) = \left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{bmatrix}_i \;\middle|\; \lambda_1+\lambda_5+\lambda_6,\ \lambda_2+\lambda_4+\lambda_6,\ \lambda_3+\lambda_4+\lambda_5 \text{ are in } GF(2);\ i = 0 \text{ or } 1 \right\}$$

of E of shape 2^{1+9}.

Since $\mathrm{Stab}_{L \cap H}(c_1) \in \mathcal{M}(H, L)$, lemma 7.1 (vi) shows that $\mathrm{Stab}_L(c_1) = \mathrm{Stab}_{L \cap H}(c_1)$. Thus the orbit of c_1 under H consists of $2^3.3.330$ cosets called the BIG-OCTAD cosets (to distinguish them from the Little-Octad cosets defined on p. 106).

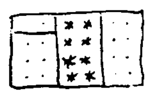
Notation for Big-Octad cosets :

Suppose x is another Big-Octad coset. Then for some $h \in H$, $x = c_1 h$, and $\mathrm{Stab}_H(x) = \mathrm{Stab}_H(c_1)^h$. Let $h = y_1 y_2$ with $y_1 \in E$ and $y_2 \in F$. Since $C_F(E_{(c_1)}) = \mathrm{Stab}_F(c_1)$, the coset x is determined by $E_{(c_1)}{}^h$ and the coset $E_{(c_1)}{}^h \cdot y_2$ in E , and hence by the pair

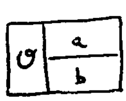
$$((\overline{E}_{(c_1)}{}^h)^\perp , \langle \overline{E}_{(c_1)}{}^h , \overline{y}_2 \rangle^\perp) ,$$

where perps are taken in the GF(2)-orthogonal structure on $\overline{E}$.

Now $\overline{E}_{(c_1)}{}^\perp = \left\{ \begin{array}{|ccc|} \hline X & Y & Z \\ x & y & z \\ \hline \end{array} : x, y, z \in GF(2) \right\}$

is the set of fixed points of the involution $(\omega \leftrightarrow \bar{\omega})$ in $\overline{F}$ (acting as  on H's MOG) on the octad-type 3-space in $\overline{E}$ for the middle octad  .

Thus the Big-Octad cosets are in one-one correspondence with pairs (Φ_1 , Φ_2) where Φ_1 is the set of fixed points in an octad type 3-space of an involution in $\overline{F}$ stabilizing the octad pointwise, and Φ_2 is either Φ_1 or a GF(2)-hyperplane in Φ_1 . We write $\Phi (\Phi_1 , \Phi_2)$ for the corresponding coset. If $\Phi_1 = \Phi_2$, we abbreviate $\Phi (\Phi_1 , \Phi_2)$ to $\Phi (\Phi_1)$. Thus for example

$f_1^{a,b}($  $) = \Phi (\overline{E}_{(c_1)})$.

$\Phi \cap T$:

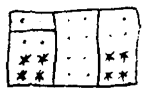
So we see that the $2^3.3.30$ Big-Octad cosets for octads disjoint from $\mathcal{G}$ are the same as the $2^2.180$ trio cosets for trios meeting the vertical sextet $(4^2.0^4)(2^4.0^2)^2$ (the notation here has the obvious meaning, and is the same as the notation used by R.T.Curtis [3] (p.41-43) in his analysis of orbits of maximal subgroups of M_{24} on other maximal subgroups).

Now we produce our first example of an 'x_5-x_7 square' (c.f. p. 91) :

Since $[x_5 , x_7] = 1$, $x_5{}^3x_7{}^3x_5{}^2x_7{}^4 = 1$.

Let $c_2 = c_1x_5{}^2 = \Phi(\bar{E}_{(c_2)}{}^{\perp})$ where

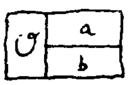
$$\bar{E}_{(c_2)}{}^{\perp} = \left\{ \begin{array}{|ccc|} 0 & x & x \\ y & z & z \end{array} : x, y, z \in GF(2) \right\}$$

so that c_2 is a Big-Octad coset for the octad  .

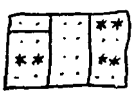
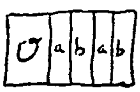
Then we have

$\Phi(\bar{E}_{(c_2)}{}^{\perp})$

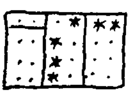
$\downarrow x_5{}^3$

$\Phi(\bar{E}_{(c_1)}{}^{\perp}) = f_1{}^{a,b}($  $)$

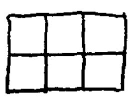
$\downarrow x_7{}^3$

$\Theta($  $) = f_1{}^{a,b}($  $)$

$\downarrow x_5{}^2$

$\Theta($  $) = f_1$(vertical sextet)

$\downarrow x_7{}^4$

f_1(square sextet )

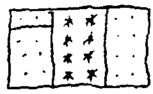
and hence $\overline{\Phi}(\overline{E}_{(c_2)}{}^{\perp}) = f_1$(square sextet) .

<u>$\Phi \cap S$</u> :

Thus we see that the $2^3.3.60$ Big-Octad cosets for octads meeting ϑ in 4 points are the same as the $2^4.90$ sextet cosets for sextets meeting the vertical sextet $(2^2.0^4)^6$.

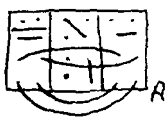
The H-orbit Υ of Little-Octad cosets

Let c_3 be the sextet coset $f_1\left(\begin{array}{|cc|cc|cc|} u & v & w & w & w & w \\ v & u & x & x & x & x \\ v & u & y & y & y & y \\ v & u & z & z & z & z \end{array} \right)$.

Then $\mathrm{Stab}_D(c_3)$ is the subgroup of H of shape $2^{0+6}.3.2^4S_4$ generated by the octad-type subgroup $E_{(c_3)} = \begin{array}{|ccc|} X & Y & Z \\ x & y & z \end{array}_0$ of E of shape 2^{0+6}, for the middle octad  for H's MOG, and a subgroup of F of shape $3 \circ 2^4S_4$ stabilizing the middle octad and the left-hand hexad. A set of generators for this group is shown in Appendix F on p. 152, taking all but the last of the elements shown there.

Let $c_4 = c_3x_7^{\,3} = f_1\left(\begin{array}{|cc|cc|cc|} u & u & w & x & w & x \\ v & v & w & x & w & x \\ u & v & y & z & y & z \\ v & u & y & z & y & z \end{array} \right)$

$= \left(\begin{array}{|ccc|} x & 0 & x \\ z & y & z \end{array} \; : \; x, y, z \in GF(2) \right)$

Then c_4 is stabilized by $\alpha =$ . From Chapter 4 we know that $[\,\alpha\,,\,x_7\,] = 1$ and hence c_3 is also stabilized by α.

Now $\langle \mathrm{Stab}_D(c_3)\,,\,\alpha \rangle$ has shape $2^{0+6}.3.2^4L_3(2)$ and is in $\mathcal{M}(H\,,\,M)$, and hence by lemma 7.1 (vi) $\mathrm{Stab}_H(c_3) = \langle \mathrm{Stab}_D(c_3)\,,\,\alpha \rangle$.

Thus there are $2^7.330$ cosets in the H-orbit of c_3, called the LITTLE-OCTAD cosets.

Notation for Little-Octad cosets :

Since $N_H(N_H(Stab_H(c_3))) = N_H(Stab_H(c_3)) = Stab_H(c_3) \times \langle z \rangle$ there are two Little-Octad cosets stabilized by $Stab_H(c_3)$, namely c_3 and $c_3.z$, and this pair of cosets is stabilized setwise by $N_H(Stab_H(c_3))$. Similarly to the Big-Octad notation, such a pair of Little-Octad cosets is specified by a pair (Ψ_1 , Ψ_2) where Ψ_1 is an octad-type isotropic 3-space in $\bar{E}$ and Ψ_2 is either Ψ_1 or a GF(2)-hyperplane in Ψ_1 . We label this pair of cosets as $\Psi(\Psi_1 , \Psi_2)$ and abbreviate this to $\Psi(\Psi_1)$ when $\Psi_1 = \Psi_2$.

Thus for example

$$\{ c_3 , c_3.z \} = \Psi(\bar{E}_{(c_3)}) .$$

$\underline{\Psi \cap S}$:

Thus we see that the $2^7.30$ Little-Octad cosets for octads disjoint from the hexad $\mathcal{G}$ are the same as the $2^4.240$ sextet cosets for sextets hitting the vertical sextet $(3.1.0^4)^2(1^4.0^2)^4$.

<u>The M-orbit Z of Sextet-line cosets</u>

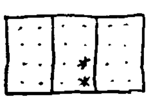
Let $c_5 = c_4 x_5^{\,2} = c_3 x_7^{\,3} x_5^{\,2} = \Phi\left(\begin{array}{|ccc|} 0 & x & x \\ x_2+z & x_1+y & x_2+y \end{array} : x,\ y,\ z \in GF(2)\right)$

a Big-Octad coset for the octad  . Then

$Stab_D(c_5)$ is a subgroup of M of shape $2^4.2^6(S_3 \times D_8)$.
A set of generators for this group is given in Appendix F on p. 146 , taking all but the last of the elements shown there.

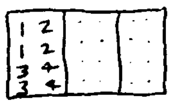
Now $w^{x_7^{\,3}}$ fixes c_4 , and from Chapter 4 we know that $\langle w , x_7 \rangle \leqslant C_P(x_5)$ so that $w^{x_7^{\,3}}$ fixes $c_4 x_5^{\,2} = c_5$.
Moreover, $\langle Stab_D(c_5) , w^{x_7^{\,3}} \rangle$, of shape $2^4.2^6(S_3 \times S_4)$, is in $\mathcal{M}(M , H)$, and hence by lemma 7.1 (vi)
$Stab_M(c_5) = \langle Stab_D(c_5) , w^{x_7^{\,3}} \rangle$.

Thus the M-orbit of c_5 consists of $2^7.26565$ cosets called the SEXTET-LINE cosets (since the stabilizer has as image in M / V the stabilizer of a line of sextets refining a trio - such sextets form a projective plane PG(2,2)) .

Since $N_M(N_M(Stab_M(c_5))) = N_M(Stab_M(c_5)) = \langle Stab_M(c_5) , t \rangle$ where t =  , the stabilizer of c_5 stabilizes exactly two such cosets c_5 and $c_5.t$, and the setwise stabilizer of this pair of cosets is $N_M(Stab_M(c_5))$.

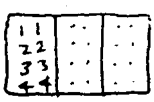
Notation for Sextet-line cosets :

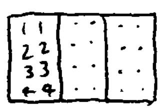
A pair of sextet-line cosets is thus determined by a line of sextets s_1 , s_2 , s_3 refining a trio and a one-function f , and two such one-functions determine the same sextet-line coset iff they are conjugate by an element of the appropriate sextet-line type 5-space in V (i.e. by the appropriate conjugate of $\langle \mathrm{Stab}_V(c_5)\ ,\ t \rangle$). We shall write this pair of sextet-line cosets as $f(s_1,s_2,s_3)$. We draw a sextet-line as a set of four pairs of points whose union is an octad, in such a way that the sum of any two pairs is a tetrad determining a sextet in the line. Thus for example

$f_1($  $) = \{\ c_5\ ,\ c_5 \circ t\ \}$

$\Phi \cap Z$:

Thus the $2^3.3.240$ Big-Octad cosets for octads meeting θ in two points are the same as the $2^7.45$ sextet-line cosets for the trios incident with the vertical sextet and such that the vertical sextet is contained in the line.

Applying the relation $x_7{}^3 x_5{}^3 x_7{}^4 x_5{}^2 = 1$ to $f_1($  $)$, we see that

$f_1($  $) = \Psi\left(\begin{array}{ccc} 0 & x & x \\ y & z & z \end{array} \right)$

as a pair of cosets. Thus :

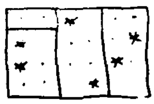
$\underline{\Psi \cap Z}$:

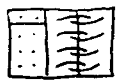
The $2^7.60$ Little-Octad cosets for octads meeting θ in four points are the same as the $2^7.60$ sextet-line cosets for trios incident with the vertical sextet, and such that the vertical sextet is not contained in the line. The pairing as Little-Octad cosets is the same as the pairing as sextet-line cosets.

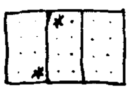
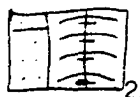
The H-orbit $\triangle$ of Duad cosets

Let $c_6 = f_1\left(\begin{array}{cc|cc|cc} 1&1&2&2&3&3\\ 1&1&2&2&3&3\\ 3&3&1&1&2&2\\ 3&3&1&1&2&2 \end{array}\right)$, a trio coset.

Then $\mathrm{Stab}_D(c_6)$ is a subgroup of H of shape $2^{1+6}.1.2^5S_4$. A set of generators for this group is given in Appendix F on p. 153, taking all but the last of the elements shown there.

Now $c_6^{\,x_7^{\,3}} = f_1\left(\begin{array}{cc|cc|cc} 1&3&2&1&3&2\\ 1&3&2&1&3&2\\ 1&3&2&1&3&2\\ 1&3&2&1&3&2 \end{array}\right) = \ominus($  $)$

is invariant under $x_5 \circ$  $_{2A}$, and so c_6 is invariant under the image of this under $x_7^{\,4}$, i.e. it is invariant under

$x_5 \cdot$  $= \begin{array}{|ccc|} 0 & 0 & \omega \\ \bar\omega & 0 & \bar\omega \end{array}_{\,1} \cdot x_5 \cdot$  $_{2A}$

and hence under $\begin{array}{|ccc|} \omega & 0 & \bar\omega \\ 1 & \omega & 1 \end{array}_{\,0} \cdot x_5$.

But $\left\langle \mathrm{Stab}_D(c_6) , \begin{array}{|ccc|} \omega & 0 & \bar\omega \\ 1 & \omega & 1 \end{array}_{\,0} \cdot x_5 \right\rangle$ is a subgroup of H of shape $2^{1+6}.1.2^5S_5$ and lies in $\mathcal{M}(H , M)$, and hence by lemma 7.1 (vi)

$$\mathrm{Stab}_H(c_6) = \left\langle \mathrm{Stab}_D(c_6) , \begin{array}{|ccc|} \omega & 0 & \bar\omega \\ 1 & \omega & 1 \end{array}_{\,0} \cdot x_5 \right\rangle .$$

This has as image in $Aut(M_{22})$ the duad stabilizer for the duad  .

So the H-orbit of c_6 consists of $2^6.3.231$ cosets called the DUAD cosets, and we have :

$\underline{\Delta \cap T}$:

The $2^6.3.15$ duad cosets for duads in the hexad ϑ are the same as the $2^2.720$ trio cosets for trios hitting the vertical sextet $(2^4.0^2)^3$.

Now let $c_7 = f_1\left(\begin{array}{|cc|cc|cc|} 1 & 1 & 3 & 2 & 3 & 2 \\ 2 & 2 & 3 & 2 & 3 & 2 \\ 1 & 2 & 1 & 3 & 1 & 3 \\ 2 & 1 & 1 & 3 & 1 & 3 \end{array} \right)$, a trio coset. By equating $c_7\alpha$ with $c_7x_7{}^4\alpha x_7{}^3$ we see that :

$\underline{\Delta \cap Z}$:

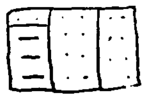
The $2^6.3.120$ duad cosets for duads not intersecting ϑ are the same as the $2^7.180$ sextet-line cosets for trios meeting the vertical sextet $(4^2.0^4)(2^4.0^2)^2$ and the sextet-line in this consisting of those sextets incident with the trio which hit the vertical sextet $(2^2.0^4)^6$.

The H-orbit Ξ of Big-Syntheme cosets

Let $c_8 = f_1($

1	2	3	3	3	3
2	1	3	3	3	3
2	1	2	2	2	2
2	1	1	1	1	1

$)$, a trio coset.

Then $\mathrm{Stab}_D(c_8)$ is a subgroup of H of shape $2^{0+5}.1\circ 2^4(2^3S_3)$. A set of generators for this group is given in Appendix F on p.154 . The image in $\mathrm{Aut}(M_{22})$ of $\mathrm{Stab}_D(c_8)$ is the stabilizer of the syntheme  in the hexad ϑ .

Since every proper subgroup of M_{22} is intransitive, $\mathrm{Stab}_D(c_8) \in \mathcal{M}(H , M)$, so lemma 7.1 (vi) shows that $\mathrm{Stab}_H(c_8) = \mathrm{Stab}_D(c_8)$.
Thus the H-orbit of c_8 consists of $2^8.3.1155$ cosets called the BIG-SYNTHEME cosets, and we have :

$\Xi \cap T$:

The $2^8.3.15$ Big-Syntheme cosets for the synthemes in the hexad ϑ are the same as the $2^2.2880$ trio cosets for trios cutting the vertical sextet $(2^4.0^2)(3.1^5)^2$.

Equating $c_8\alpha$ with $c_8x_7{}^3\alpha x_7{}^4$ we see that :

$\Xi \cap Z$:

The $2^8.3.180$ Big-Syntheme cosets for hexads meeting ϑ in two points comprising a duad of the syntheme are the same as the $2^7.1080$ Sextet-line cosets for trios meeting the vertical sextet $(4^2.0^4)(2^4.0^2)^2$ and the sextet-lines in this other than the $((2^2.0^4)^6)^3$ one .

The remaining H-orbits Σ, Γ, X and Λ, and M-orbits F, N and L.

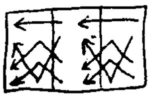
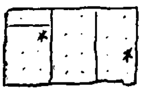
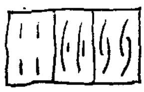
Continuing in the same way, we find the following :

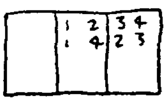
Let $c_9 = f_1($

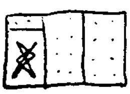
u	u	u	v	w	x
v	v	u	v	w	x
w	x	z	y	y	y
x	w	y	z	z	z

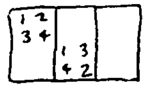
$)$, a sextet coset.

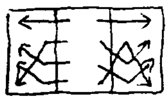
Then $\mathrm{Stab}_H(c_9) = \mathrm{Stab}_D(c_9) \in \mathcal{M}(H, M)$ is a subgroup of shape $2^{0+4}.1.2^4(2 \times S_4)$ generated by the elements shown on p. 155. Again the image in $\mathrm{Aut}(M_{22})$ is the stabilizer of a syntheme in $\mathcal{G}$, and the H-orbit of c_9 consists of $2^9.3.1155$ cosets called the LITTLE-SYNTHEME cosets, forming the H-orbit Σ.

Let $c_{10} = c_6.$  $.\alpha$, a dual coset for the duad . Then $\mathrm{Stab}_M(c_{10}) = \mathrm{Stab}_D(c_{10}) \in \mathcal{M}(M, H)$ is a subgroup of shape $2^1.2^6 PGL_2(5)$ having as image in M_{24} the centralizer of the involution , and generated by the elements shown on p. 143. So the M-orbit of c_{10} consists of $2^{10}.31878$ cosets called the REGULAR-INVOLUTION cosets, forming the M-orbit F.

Let $c_{11} \in f_1($  $)$, a sextet-line coset.

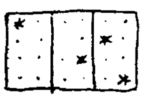
Then $\mathrm{Stab}_H(c_{11}) = \mathrm{Stab}_D(c_{11}) \in \mathcal{M}(H, M)$ is a subgroup of shape $1.1.2^4(2 \times S_3)$ generated by the elements shown on p. 158. The image in $\mathrm{Aut}(M_{22})$ is the stabilizer of the hexagon  in the hexad $\mathcal{G}$. So the H-orbit of c_{11} consists of $2^{13}.3.462$ cosets called the HEXAGON cosets, forming the H-orbit Λ.

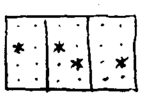
Let $c_{12} = c_5.w.$  $\in \mathfrak{f}_1($  $)$, a sextet-line coset. Then $\mathrm{Stab}_D(c_{12})$ is generated by the elements shown on p. 156. It is not immediately clear whether this group is in $\mathcal{M}(H, M)$, so we postpone further discussion of the H-orbit of c_{12} until p. 117.

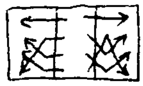
Let $c_{13} = c_{12}.$  $.\alpha$. Then $\mathrm{Stab}_D(c_{13})$ contains as a subgroup the group $\mathfrak{p}$ generated by all but the last of the elements shown on p. 150. Now we use the last relation $f^{-1}x_{23}fx_{23} = 1$ (see p. 89) to obtain a further element of $\mathrm{Stab}_M(c_{13})$.

First we rewrite this relator as

$$m_5^{\,h_1m_2h_2m_3h_3} = m_6$$

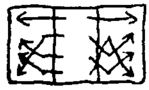
where $m_5 =$  . (6)(15 17 20 5 ∞ 13 4 12 14 7 0 21 11 18 22 3 9 19 5 10 1 2 16)

and $m_6 =$  . (ε)(∞ 17 11 18 4 14 15 13 10 3 9 12 0 2 20 5 22 1 7 19 16 21 8)

Thus $v_\infty h_3^{-1}m_3^{-1}h_2^{-1}m_2^{-1}h_1^{-1}$ is stabilized by m_5. Writing m_2^{-1} as $x_7^{5}.$  .  we see that

$$v_\infty h_3^{-1}m_3^{-1}h_2^{-1}m_2^{-1}h_1^{-1} = \Theta(\varphi).m_3^{-1}h_2^{-1}m_2^{-1}h_1^{-1}$$

$$= c_3h_2^{-1}m_2^{-1}h_1^{-1}$$

$= c_{12}.$  $.h_1^{-1}$

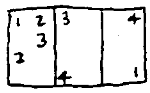
$$= c_{13}.$$

Thus c_{13} is stabilized by $\langle \not{p} , m_5 \rangle$ which has shape $L_2(23)$, lies in the nought-type complement defined by $f_1^{\{6,13,20\}}$ and stabilizes the projective line structure given by the numbering :

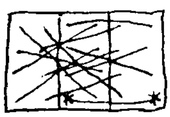
4	10	14	20	12	21
15	0	7	11	5	9
∞	3	13	2	6	19
16	17	18	8	22	1

.

(which of course differs from the standard numbering by an element of M_{24})

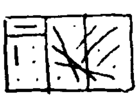
Now let $c_{14} = c_{13}.\alpha.x_7 \in f_1($  $)$, a sextet-line coset. Then $\mathrm{Stab}_D(c_{14})$ is a subgroup of H of shape $2^{0+1}.1.2^4D_8$. A set of generators for this group is given in Appendix F on p. 157 , taking all but the last element there.

But $c_{13}x_7$ is stabilized by the element

$\gamma =$  $\in M$

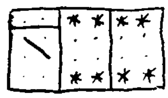
(the image under x_7 of the element $z \to \frac{8z+1}{4z-8}$ of $L_2(23)$ for the above numbering) which is

$$\gamma = \boxed{\begin{matrix} \bar\omega & \bar\omega & 0 \\ 0 & 0 & 0 \end{matrix}}_{\,0} \cdot \begin{pmatrix} 1 & \bar\omega & 0 & 0 & 0 & \omega \\ \bar\omega & 1 & 0 & 0 & 0 & \omega \\ 0 & 0 & \omega & \omega & \omega & 0 \\ 0 & 0 & \omega & 1 & \bar\omega & 0 \\ 0 & 0 & \omega & \bar\omega & 1 & 0 \\ \omega & \omega & 0 & 0 & 0 & \omega \end{pmatrix}_{2A} \in H$$

(acting as  on H's MOG)

and hence the image γ^α of this under α stabilizes c_{14} . This is the last element shown on p. 157 , and

$\mathrm{Stab}_H(c_{14}) = \langle \mathrm{Stab}_D(c_{14}) , \gamma^{\alpha} \rangle \in \mathcal{M}(H , M)$ is a group of shape $2^{0+1}.1.2^4S_4$. Now a 2^81^8 type involution swapping 0 and ∞ on H's MOG fixes an octad pointwise, and is determined by that octad. $\mathrm{Stab}_H(c_{14})$ acts on H's MOG as the stabilizer of an octad and one of the 7 pairs of points defined in this way :



Thus the H-orbit of c_{14} consists of $2^{12}.3.2310$ cosets called the OCTAD-PAIR cosets, forming the H-orbit X .

Now we see that in the group G , we have eight orbits $\Theta, \Phi, \Psi, \Delta, \Xi, \Sigma, X$ and Λ of H on the cosets of M , and a coset c_{12} not in any of them. Since from the character-theoretic calculation H has only 9 orbits on cosets of M , and $|G : M|$ would be too small if $\mathrm{Stab}_H(c_{12}) > \mathrm{Stab}_D(c_{12})$, we must have $\mathrm{Stab}_H(c_{12}) = \mathrm{Stab}_D(c_{12})$ in G , and hence in $\hat{G}$. The image of this group in $\mathrm{Aut}(M_{22})$ is the stabilizer of the hexad ϑ and the duad $\{3,15\}$ in ϑ . Thus the H-orbit of c_{12} consists of $2^{13}.1155$ cosets called the DUAD-HEXAD cosets, forming the H-orbit Γ .

Thus $\mathfrak{p} = \mathrm{Stab}_D(c_{13})$, and $\mathrm{Stab}_M(c_{13}) = \langle \mathfrak{p} , m_5 \rangle \in \mathcal{M}(M , H)$ is a group of shape $L_2(23)$. Thus the M-orbit of c_{13} consists of $2^{11}.40320$ cosets called the PROJECTIVE-LINE cosets, forming the M-orbit L .

Notation for projective-line cosets :

Such a coset is determined by a nought-function f and a numbering of the MOG (conjugate to the standard numbering by an element of M_{24}) on which the stabilizer acts as $L_2(23)$. We write this coset as f(numbering) so that for example $c_{13} = f_1^{\{6,13,20\}}($

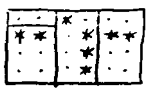
4	10	14	20	12	21
15	0	7	11	5	9
∞	3	13	2	6	19
16	17	18	8	22	1

) ,

and two numberings determine the same coset iff they differ by a projective special linear transformation.

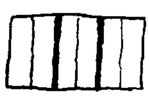
Now let $c_{15} \in \Psi($

0	x	x
x_2+z	x_1+y	x_2+y

) , a Little-Octad coset for the octad  . Then $Stab_D(c_{15})$ is a subgroup of M of shape $2^0.2^6.3(2 \times S_4)$. A set of generators for this group if given in Appendix F on p. 149 . Again, in the group G , we have six orbits I , T , S , Z , F and L of M on cosets of M , and a coset c_{15} not in any of them. Since the character-theoretic calculation shows that M has only 7 orbits on these cosets, and $|G : M|$ would be too small if $Stab_M(c_{15}) > Stab_D(c_{15})$, we must have $Stab_M(c_{15}) = Stab_H(c_{15})$ in G , and hence in $\hat{G}$. The image of this group in M_{24} is the stabilizer of an incident trio and sextet. Thus the M-orbit of c_{15} consists of $2^{11}.26565$ cosets called the TRIO-SEXTET cosets, forming the M-orbit N .

Notation for trio-sextet cosets :

Let the nought-function f_0 be defined by $f_0 = f_1^{\{\infty,22,11\}}$ and let K_0 be the nought-type complement for V in M defined by f_0 . Let Y_0 be the stabilizer in K_0 of the brick trio and the vertical sextet, and let Z_0 be the subgroup of index two in Y_0 obtained by only permitting even permutations of the six columns of M's MOG . Then $\mathrm{Stab}_M(c_{15}) = Z_0 \cup (Y_0 \setminus Z_0)z$, and $N_M(\mathrm{Stab}_M(c_{15})) = \langle Y_0 , z \rangle$ contains $\mathrm{Stab}_M(c_{15})$ to index two. So $\mathrm{Stab}_M(c_{15})$ stabilizes two trio-sextet cosets c_{15} and $c_{15}z$. Thus a pair of trio-sextet cosets is determined by an incident trio t and sextet s , and a nought-function f . We write this pair of cosets as $f(s,t)$. Two nought-functions determine the same pair of cosets iff they are conjugate by the element of V corresponding to the sextet. Thus for example

$f_0($  $) = \{ c_{15} , c_{15}z \}$.

From these definitions we have the following identifications :

$\underline{\Sigma \cap S}$:

The $2^9.3.15$ Little-Syntheme cosets for synthemes in θ are the same as the $2^4.1440$ sextet cosets for sextets cutting the vertical sextet $(2.1^2.0^3)^4(1^4.0^2)^2$.

$\Delta \cap F$:

The $2^6.3.96$ Duad cosets for duads straddling ϑ and its complement are the same as the $2^{10}.18$ Regular-Involution cosets for involutions preserving each column of the vertical sextet.

$\Lambda \cap Z$:

The $2^{13}.3.60$ Hexagon cosets for hexagons in ϑ are the same as the $2^7.11520$ Sextet-line cosets for trios cutting the vertical sextet $(3.1^5)^2(2^4.0^2)$ and sextet lines cutting the vertical sextet $((2.1^2.0^3)^4(1^4.0^2)^2)^3$.

$X \cap Z$:

The $2^{12}.3.90$ Octad-pair cosets for octads disjoint from ϑ and pairs in ϑ are the same as the $2^7.8640$ Sextet-line cosets for trios hitting the vertical sextet $(3.1^5)^2(2^4.0^2)$ and sextet lines cutting the vertical sextet $((2.1^2.0^3)^4(1^4.0^2)^2)^2((3.1.0^4)^2(1^4.0^2)^4)$.

$\Gamma \cap Z$:

The $2^{13}.3.15$ Duad-hexad cosets for the hexad ϑ are the same as the $2^7.2880$ sextet-line cosets for trios hitting the vertical sextet $(2^4.0^2)^3$ and sextet lines hitting the vertical sextet $((2.1^2.0^3)^4(1^4.0^2)^2)^3$.

$\underline{\Gamma \cap L}$:

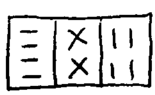
The $2^{13}.3.480$ Duad-hexad cosets for hexads meeting ϑ in two points, one of which is in the duad, are the same as the $2^{11}.5760$ Projective-line cosets for numberings where the tetrads of the vertical sextet all have cross-ratio 2 .

$\underline{\Psi \cap N}$:

The $2^{7}.240$ Little-Octad cosets for octads meeting ϑ in two points are the same as the $2^{11}.15$ Trio-Sextet cosets for the vertical sextet.

Further x_5-x_7 squares and α-x_7 squares show the remaining identifications :

$\underline{\Xi \cap F}$:

The $2^{8}.3.720$ Big-Syntheme cosets for hexads meeting ϑ in a pair which is not in the syntheme are the same as the $2^{10}.540$ Regular-involution cosets for involutions whose centralizer fixes a sextet cutting the vertical one $(2^2.0^4)^6$ and having four orbits in columns of the vertical sextet. (e.g. )

$\underline{\Xi \cap N}$:

The $2^{8}.3.240$ Big-Syntheme cosets for hexads disjoint from ϑ are the same as the $2^{11}.90$ Trio-Sextet cosets for trios incident with the vertical sextet and sextets other than the vertical one.

$\Sigma \cap N$:

The $2^9.3.270$ Little-Syntheme cosets for hexads meeting θ in a pair which does not comprise a duad in the syntheme are the same as the $2^{11}.540$ Trio-Sextet cosets for trios hitting the vertical sextet $(4^2.0^4)(2^4.0^2)^2$ and sextets hitting the vertical sextet $(2^2.0^4)^6$.

$\Sigma \cap Z$:

The $2^9.3.180$ Little-Syntheme cosets for hexads meeting θ in two points comprising a duad of the syntheme are the same as the $2^7.2160$ Sextet-line cosets for trios hitting the vertical sextet $(2^4.0^2)^3$ and sextet lines cutting the vertical sextet $((2.1^2.0^3)^4(1^4.0^2)^2)^2((2^2.0^4)^6)$.

$\Sigma \cap F$:

The $2^9.3.240$ Little-Syntheme cosets for hexads disjoint from θ are the same as the $2^{10}.360$ Regular-Involution cosets for involutions whose centralizer fixes a sextet cutting the vertical one $(2^2.0^4)^6$ and having no orbits in the columns of the vertical sextet.

$\Gamma \cap F$:

The $2^{13}.3.360$ Duad-Hexad cosets for hexads meeting θ in two points neither of which is in the duad are the same as the $2^{10}.8640$ Regular-Involution cosets for involutions whose centralizer fixes a sextet cutting the vertical one $(2.1^2.0^3)^4(1^4.0^2)^2$ and having nc orbits in the columns of the vertical sextet.

$\Gamma \cap N(1)$:

The $2^{13}.3.60$ Duad-Hexad cosets for hexads meeting θ in two points comprising the duad are the same as the $2^{11}.720$ Trio-Sextet cosets for trios hitting the vertical sextet $(2^4.0^2)^3$ and sextets hitting the vertical one $(2^2.0^4)^6$.

$\Gamma \cap N(2)$:

The $2^{13}.3.240$ Duad-Hexad cosets for hexads disjoint from θ are the same as the $2^{11}.2880$ Trio-Sextet cosets for trios hitting the vertical sextet $(2^4.0^2)(3.1^5)^2$ and sextets hitting the vertical one $(3.1.0^4)^2(1^4.0^2)^4$.

$X \cap F(1)$:

The $2^{12}.3.60$ Octad-Pair cosets for octads meeting θ in four points and the pair also in θ are the same as the $2^{10}.720$ Regular-Involution cosets for involutions whose centralizer fixes a sextet cutting the vertical one $(2^2.0^4)^6$ and having 6 orbits in columns of the vertical sextet.

$X \cap F(2)$:

The $2^{12}.3.360$ Octad-Pair cosets for octads meeting θ in four points and the pair disjoint from θ are the same as the $2^{10}.4320$ Regular-Involution cosets for involutions whose centralizer fixes a sextet cutting the vertical one $(3.1.0^4)^2(1^4.0^2)^4$.

$X \cap N(1)$:

The $2^{12}.3.120$ Octad-Pair cosets for octads disjoint from θ and pair also disjoint from θ are the same as the $2^{11}.720$ Trio-Sextet cosets for trios hitting the vertical sextet $(4^2.0^4)(2^4.0^2)^2$ and sextets hitting the vertical one $(3.1.0^4)^2(1^4.0^2)^4$.

$X \cap N(2)$:

The $2^{12}.3.720$ Octad-Pair cosets for octads meeting θ in two points and pairs disjoint from θ are the same as the $2^{11}.4320$ Trio-Sextet cosets for trios hitting the vertical sextet $(2^4.0^2)^3$ and sextets hitting the vertical one $(2.1^2.0^3)^4(1^4.0^2)^2$.

$X \cap L$:

The $2^{12}.3.960$ Octad-Pair cosets for octads hitting θ in two points and pairs meeting θ in one point are the same as the $2^{11}.5760$ Projective-line cosets for numberings where the tetrads of the vertical sextet all have cross-ratio 5 .

$\Lambda \cap F$:

The $2^{13}.3.720$ Hexagon cosets for hexagons intersecting θ in two opposite points of the hexagon are the same as the $2^{10}.17280$ Regular-Involution cosets for involutions whose centralizer fixes a sextet cutting the vertical one $(2.1^2.0^3)^4(1^4.0^2)^2$ and having two orbits in columns of the vertical sextet.

$\Lambda \cap N$:

The $2^{13}.3.1440$ Hexagon cosets for hexagons intersecting ϑ in two points at distance two on the hexagon are the same as the $2^{11}.17280$ Trio-Sextet cosets for trios meeting the vertical sextet $(2^4.0^2)(3.1^5)^2$ and sextets meeting the vertical one $(2.1^2.0^3)^4(1^4.0^2)^2$.

$\Lambda \cap L(1)$:

The $2^{13}.3.960$ Hexagon cosets for hexagons disjoint from ϑ are the same as the $2^{11}.11520$ Projective-line cosets for the numberings where the tetrads of the vertical sextet all have cross-ratio 3 .

$\Lambda \cap L(2)$:

The $2^{13}.3.1440$ Hexagon cosets for hexagons hitting ϑ in two adjacent points of the hexagon are the same as the $2^{11}.17280$ Projective-line cosets for numberings where four of the tetrads of the vertical sextet have cross-ratio 4 and the other two have cross-ratio 5 .

This completes the proof that $\hat{G} \cong G$.

Index of Notation

Appendix A

The character table of J_4 in 'Atlas' notation.

(see [5])

J4

ind	1A	2A	2B	3A	4A	4B	4C	5A	6A	6B	6C	7A	B**	8A	8B	8C	10A	10B	11A	11B	12A
	867755710	217993	18166	2661	5406	98	43	6	2661	2	2			1					31		
	46077562880	95040	57920	120	720	304	008	720	120	304	304	840	840	280	768	512	960	80	944	242	192
p power		A	A	A	A	A	B	A	AA	AA	AB	A	A	A	B	B	AA	AB	A	A	BA
p' part		A	A	A	A	A	A	A	AA	AA	AB	A	A	A	A	A	AA	AB	A	A	AA
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1333	53	-11	10	-11	5	-3	3	-10	2	-2	b7	**	1	-3	1	3	-1	2	2	-2
0	1333	53	-11	10	-11	5	-3	3	-10	2	-2	**	b7	1	-3	1	3	-1	2	2	-2
0	299367	-153	231	45	-89	7	7	-3	45	-3	-3	4b7	**	-1	-1	-1	-3	1	13	2	1
0	299367	-153	231	45	-89	7	7	-3	45	-3	-3	**	4b7	-1	-1	-1	-3	1	13	2	1
0	887778	738	-606	45	34	-14	10	3	45	-3	-3	b7	**	6	2	-2	3	-1	1	1	1
0	887778	738	-606	45	34	-14	10	3	45	-3	-3	**	b7	6	2	-2	3	-1	1	1	1
+	889111	2071	727	55	87	39	-1	6	55	7	7	-1	-1	-5	7	3	6	2	3	3	3
0	1187145	585	-375	-45	-55	-7	17	0	-45	3	3	5b7	**	5	1	-3	0	0	14	3	-1
0	1187145	585	-375	-45	-55	-7	17	0	-45	3	3	**	5b7	5	1	-3	0	0	14	3	-1
+	1776838	2808	120	99	120	24	8	8	99	3	3	1	1	0	8	0	8	0	3	3	3
0	3403149	-627	-627	-66	77	13	-3	14	66	6	-6	5b7	**	5	-3	-3	-2	-2	24	2	2
0	3403149	-627	-627	-66	77	13	-3	14	66	6	-6	**	5b7	5	-3	-3	-2	-2	24	2	2
+	4290927	1647	175	141	175	31	7	-8	-99	-3	13	4	4	-5	-1	3	-8	0	25	3	1
0	32307363	-6237	99	0	99	51	-21	3	0	0	0	6b7	**	-9	3	-1	3	-1	0	0	0
0	32307363	-6237	99	0	99	51	-21	3	0	0	0	**	6b7	-9	3	-1	3	-1	0	0	0
0	32897107	-5293	-749	10	211	35	-21	-3	-10	2	-2	4b7	**	-1	3	-1	-3	1	-10	1	-2
0	32897107	-5293	-749	10	211	35	-21	-3	-10	2	-2	**	4b7	-1	3	-1	-3	1	-10	1	-2
+	35411145	10185	3465	105	265	25	49	0	105	9	9	0	0	5	1	5	0	0	11	0	1
+	35411145	10185	3465	105	265	25	49	0	105	9	9	0	0	5	1	5	0	0	11	0	1
+	95288172	25452	364	231	44	108	28	7	-189	15	-5	0	0	4	-4	-4	7	-1	-54	1	-1
+	230279749	11333	6853	-308	197	37	21	14	308	-4	4	0	0	5	-3	-3	-2	-2	51	-4	-4
+	259775040	6720	6720	210	320	64	0	0	-210	-6	6	0	0	0	0	0	0	0	19	-3	2
+	259775040	6720	6720	210	320	64	0	0	-210	-6	6	0	0	0	0	0	0	0	19	-3	2
+	300364800	34650	-7910	420	-550	42	-14	0	0	12	-8	0	0	-10	-6	6	0	0	56	1	-4
+	366159104	-2816	-1792	440	768	0	0	-6	-440	-8	8	-4	-4	0	0	0	-6	-2	36	3	0
+	393877506	-10494	7106	561	66	-110	-22	1	-99	9	5	1	1	6	2	-2	1	1	0	0	-3
+	394765284	-9756	-2716	309	804	4	-28	-1	-351	-3	-7	4	4	4	4	-4	-1	-1	1	1	-3
+	460559498	24458	6986	329	-54	90	-14	-7	329	-7	-7	0	0	6	2	6	-7	1	-19	3	-3
+	493456605	16605	7645	-120	285	29	13	0	540	0	4	5	5	5	-3	-3	0	0	-29	4	0
+	690839247	23247	-10801	21	79	15	-49	7	441	-3	17	0	0	-1	-1	-1	7	-1	32	-1	1
+	786127419	16443	3003	252	187	-37	-21	14	-252	12	-12	0	0	-5	3	3	-2	-2	-22	0	4
+	786127419	16443	3003	-126	187	-37	-21	14	126	-6	6	0	0	-5	3	3	-2	-2	-22	0	-2
+	786127419	16443	3003	-126	187	-37	-21	14	126	-6	6	0	0	-5	3	3	-2	-2	-22	0	-2
+	789530568	49608	-440	-111	-440	40	-8	8	-351	-15	1	1	1	0	-8	0	8	0	2	2	1
+	385257856	24192	-8064	0	384	0	0	21	0	0	0	0	0	0	0	0	-3	1	-45	-1	0
+	385257856	24192	-8064	0	384	0	0	21	0	0	0	0	0	0	0	0	-3	1	-45	-1	0
+	1016407168	27776	9856	-56	384	0	0	-7	56	8	-8	0	0	0	0	0	1	1	29	-4	0
+	1016407168	27776	9856	-56	384	0	0	-7	56	8	-8	0	0	0	0	0	1	1	29	-4	0
+	1085604531	-17229	-9533	330	307	19	35	6	330	-6	-6	4	4	-5	3	-5	6	2	33	0	-2
+	1089007630	14400	-3520	-330	320	-64	-64	0	-90	6	-10	5	5	0	0	0	0	0	57	2	2
+	1182518964	-32076	10164	99	-396	-12	-28	-1	99	3	3	4	4	4	-4	4	-1	-1	0	0	3
+	1183406741	-31339	341	-154	341	101	-35	1	-154	-10	-10	6	6	1	-3	1	1	1	0	0	2
+	1183406741	39061	6677	440	-363	-75	-27	6	-440	-8	8	-1	-1	5	-3	-3	6	2	0	0	0
+	1184295852	-29268	10284	-99	-276	12	-20	7	-99	-3	-3	5	5	4	4	4	7	-1	3	3	-3
+	1445942610	-13230	6930	0	210	-30	42	0	0	0	0	0	0	-10	-6	-2	0	0	-9	2	0
+	1445942610	-13230	6930	0	210	-30	42	0	0	0	0	0	0	-10	-6	-2	0	0	-9	2	0
+	1445942610	-13230	6930	0	210	-30	42	0	0	0	0	0	0	-10	-6	-2	0	0	-9	2	0
+	1509863773	-5027	-7139	-176	-99	45	-27	-7	-176	16	16	5	5	1	-3	-7	-7	1	0	0	0
+	1579061136	31632	4752	-384	528	-48	-43	-14	384	0	0	-5	-5	0	0	0	2	2	4	4	0
+	1842237992	1064	3752	-385	-472	8	56	42	-385	-1	-1	0	0	0	8	0	-6	2	45	1	-1
+	1903741279	26719	-737	385	-737	-17	7	-6	385	1	1	6	6	-5	7	-5	-6	-2	0	0	1
+	1981808640	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-24	-2	0
+	1981808640	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-24	-2	0
+	1981808640	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-24	-2	0
+	2001151845	-10395	-10395	0	165	-27	21	0	0	0	0	0	0	5	-3	5	0	0	0	0	0
+	2001151845	-10395	-10395	0	165	-27	21	0	0	0	0	0	0	5	-3	5	0	0	0	0	0
+	2001151845	-10395	-10395	0	165	-27	21	0	0	0	0	0	0	5	-3	5	0	0	0	0	0
+	2267824128	-49152	0	384	0	0	0	8	384	0	0	-4	-4	0	0	0	8	0	20	-2	0
+	2692972480	-34880	3520	-230	-320	64	-64	0	-230	10	10	-5	-5	0	0	0	0	0	-11	0	-2
+	2727495848	25256	-4312	-385	-88	8	56	-42	-385	-1	-1	0	0	0	8	0	6	-2	0	0	-1
+	3054840657	1617	-495	231	-495	33	-7	-8	231	-9	-9	-6	-6	5	1	-3	-8	0	0	0	3

192 BB AB 12B	48 CC AC 12C	84 AA AA 14A	84 BA BA B**	56 AB AB 14C	56 BB BB D**	30 AA AA 15A	32 C A 16A	160 AA AA 20A	160 AA AA B*	42 BA AA 21A	42 AA BA B**	264 AA AA 22A	22 BB BB 22B	23 A A 23A	48 BB AB 24A	48 BB AB B*
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	-b7	**	b7	**	0	-1	-1	-1	b7	**	-2	0	-1	0	0
2	0	**	-b7	**	b7	0	-1	-1	-1	**	b7	-2	0	-1	0	0
1	1	-2b7	**	0	0	0	-1	1	1	b7	**	1	0	-1	-1	-1
1	1	**	-2b7	0	0	0	-1	1	1	**	b7	1	0	-1	-1	-1
1	1	b7	**	b7	**	0	0	-1	-1	b7	**	1	-1	1	-1	-1
1	1	**	b7	**	b7	0	0	-1	-1	**	b7	1	-1	1	-1	-1
3	-1	-1	-1	-1	-1	0	1	2	2	-1	-1	3	1	0	1	1
-1	-1	-b7	**	b7	**	0	-1	0	0	-b7	**	2	-1	0	1	1
-1	-1	**	-b7	**	b7	0	-1	0	0	**	-b7	2	-1	0	1	1
3	-1	1	1	1	1	-1	0	0	0	1	1	3	-1	0	-1	-1
-2	0	b7	**	b7	**	-1	1	2	2	-b7	**	0	0	0	0	0
-2	0	**	b7	**	b7	-1	1	2	2	**	-b7	0	0	0	0	0
1	1	2	2	0	0	1	1	0	0	1	1	-3	-1	1	-1	-1
0	0	0	0	-2b7	**	0	1	-1	-1	0	0	0	0	-1	0	0
0	0	0	0	**	-2b7	0	1	-1	-1	0	0	0	0	-1	0	0
2	0	2b7	**	0	0	0	1	1	1	b7	**	-2	-1	0	0	0
2	0	**	2b7	0	0	0	1	1	1	**	b7	-2	-1	0	0	0
1	1	0	0	0	0	0	-1	0	0	0	0	-1	0	0	1	1
1	1	0	0	0	0	0	-1	0	0	0	0	-1	0	0	1	1
3	1	0	0	0	0	1	0	-1	-1	0	0	-2	1	0	-1	-1
4	0	0	0	0	0	2	1	2	2	0	0	3	0	0	0	0
-2	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	2r3	-2r3
-2	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-2r3	2r3
0	-2	0	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
0	0	-2	-2	0	0	0	0	-2	-2	-1	-1	0	1	1	0	0
1	-1	-1	-1	1	1	1	0	1	1	1	1	0	0	-1	-1	-1
1	-1	2	2	0	0	-1	0	-1	-1	1	1	1	1	0	1	1
-3	1	0	0	0	0	-1	0	1	1	0	0	5	1	0	-1	-1
-4	-2	1	1	1	1	0	1	0	0	-1	-1	-5	0	0	0	0
-3	-1	0	0	0	0	1	-1	-1	-1	0	0	4	1	0	-1	-1
-4	0	0	0	0	0	2	-1	2	2	0	0	-2	0	0	0	0
2	0	0	0	0	0	-1	-1	2	2	0	0	-2	0	0	0	0
2	0	0	0	0	0	-1	-1	2	2	0	0	-2	0	0	0	0
1	1	-1	-1	1	1	-1	0	0	0	1	1	-2	0	0	1	1
0	0	0	0	0	0	0	0	2r5-1	*	0	0	3	-1	0	0	0
0	0	0	0	0	0	0	0	*	2r5-1	0	0	3	-1	0	0	0
0	0	0	0	0	0	-1	0	2r5-1	*	0	0	1	0	0	0	0
0	0	0	0	0	0	-1	0	*	2r5-1	0	0	1	0	0	0	0
-2	2	-2	-2	0	0	0	-1	2	2	1	1	-3	0	0	0	0
2	2	1	1	1	1	0	0	0	0	-1	-1	1	0	0	0	0
3	-1	-2	-2	0	0	-1	0	-1	-1	1	1	0	0	0	-1	-1
2	-2	0	0	-2	-2	1	-1	1	1	0	0	0	0	0	0	0
0	0	1	1	-1	-1	0	1	2	2	-1	-1	0	0	0	0	0
-3	1	-1	-1	1	1	1	0	-1	-1	-1	-1	3	-1	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0
0	0	-1	-1	1	1	-1	-1	1	1	-1	-1	0	0	0	0	0
0	0	-1	-1	-1	-1	1	0	-2	-2	1	1	-4	0	0	0	0
-1	-1	0	0	0	0	0	0	-2	-2	0	0	-3	1	0	-1	-1
1	1	0	0	-2	-2	0	1	-2	-2	0	0	0	0	-1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	-1	0	0
0	0	2	2	0	0	-1	0	0	0	-1	-1	-4	0	1	0	0
-2	2	1	1	-1	-1	0	0	0	0	1	1	1	0	0	0	0
-1	-1	0	0	0	0	0	0	2	2	0	0	0	0	0	-1	-1
3	-1	0	0	2	2	1	-1	0	0	0	0	0	0	0	1	1

28 CC AC 28A	28 DC BC B**	29 A A 29A	30 AAA AAA 30A	31 A A 31A	31 A A B*5	31 A A C*25	66 AA AA 33A	66 AA AA B*	35 BA AA 35A	35 AA BA B**	37 A A 37A	37 A A B*2	37 A A C*4	40 AA AA 40A	40 BA AA B*	42 ABA AAA 42A	42 BAA BBA B**	43 A A 43A	43 A A B*6	43 A A C*36	44 AA AA 44A	66 AAA AAA 66A	66 BAA BAA B*
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
-b7	**	-1	0	0	0	0	-1	-1	b7	**	1	1	1	1	1	-b7	**	0	0	0	0	1	1
**	-b7	-1	0	0	0	0	-1	-1	**	b7	1	1	1	1	1	**	-b7	0	0	0	0	1	1
0	0	0	0	0	0	0	1	1	-b7	**	0	0	0	-1	-1	b7	**	1	1	1	-1	1	1
0	0	0	0	0	0	0	1	1	**	-b7	0	0	0	-1	-1	**	b7	1	1	1	-1	1	1
b7	**	1	0	0	0	0	1	1	b7	**	0	0	0	1	1	b7	**	0	0	0	1	1	1
**	b7	1	0	0	0	0	1	1	**	b7	0	0	0	1	1	**	b7	0	0	0	1	1	1
-1	-1	0	0	0	0	0	0	0	-1	-1	1	1	1	0	0	-1	-1	0	0	0	-1	0	0
b7	**	1	0	0	0	0	-1	-1	0	0	0	0	0	0	0	-b7	**	1	1	1	0	-1	-1
**	b7	1	0	0	0	0	-1	-1	0	0	0	0	0	0	0	**	-b7	1	1	1	0	-1	-1
1	1	0	-1	-1	-1	-1	0	0	1	1	0	0	0	0	0	1	1	-1	-1	-1	-1	0	0
-b7	**	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	b7	**	0	0	0	0	0	0
**	-b7	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	**	b7	0	0	0	0	0	0
0	0	0	1	0	0	0	-2	-2	-1	-1	0	0	0	0	0	-1	-1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	b7	**	-1	-1	-1	1	1	0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	**	b7	-1	-1	-1	1	1	0	0	1	1	1	0	0	0
0	0	0	0	0	0	0	-1	-1	-b7	**	0	0	0	-1	-1	-b7	**	0	0	0	2	1	1
0	0	0	0	0	0	0	-1	-1	**	-b7	0	0	0	-1	-1	**	-b7	0	0	0	2	1	1
0	0	-1	0	0	0	0	-b33	*	0	0	-1	-1	-1	0	0	0	0	0	0	0	1	-b33	*
0	0	-1	0	0	0	0	*	-b33	0	0	-1	-1	-1	0	0	0	0	0	0	0	1	*	-b33
0	0	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	-2	-2
0	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0
0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	1	-1	-1
0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	1	-1	-1
0	0	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	1	0	0	0	-2	0	0
-1	-1	-1	1	0	0	0	0	0	1	1	0	0	0	1	1	-1	-1	0	0	0	0	0	0
0	0	0	-1	0	0	0	1	1	-1	-1	0	0	0	-1	-1	-1	-1	0	0	0	1	1	1
0	0	0	-1	0	0	0	-1	-1	0	0	0	0	0	1	1	0	0	0	0	0	1	-1	-1
-1	-1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	0	0	0	-1	1	1
0	0	0	1	0	0	0	-1	-1	0	0	0	0	0	-1	-1	0	0	0	0	0	2	1	1
0	0	1	-2	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	1	1	0	0	0	-b33	*	0	0	0	0	0	0	0	0	0	0	0	0	0	b33	*
0	0	1	1	0	0	0	*	-b33	0	0	0	0	0	0	0	0	0	0	0	0	0	*	b33
-1	-1	0	-1	0	0	0	-1	-1	1	1	0	0	0	0	0	-1	-1	0	0	0	0	1	1
0	0	-1	0	1	1	1	0	0	0	0	0	0	0	r5	-r5	0	0	0	0	0	-1	0	0
0	0	-1	0	1	1	1	0	0	0	0	0	0	0	-r5	r5	0	0	0	0	0	-1	0	0
0	0	1	1	0	0	0	-1	-1	0	0	0	0	0	-r5	r5	0	0	0	0	0	-1	1	1
0	0	1	1	0	0	0	-1	-1	0	0	0	0	0	r5	-r5	0	0	0	0	0	-1	1	1
0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	1	1	0	0	0	-1	0	0
-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	-2	-2
0	0	0	-1	1	1	1	0	0	-1	-1	0	0	0	-1	-1	1	1	1	1	1	0	0	0
0	0	0	1	0	0	0	0	0	1	1	-1	-1	-1	1	1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	0	0	1	1	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c43	*6	*36	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*36	c43	*6	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	*6	*36	c43	1	0	0
1	1	1	-1	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	0	0	0	0	0	0
1	1	0	-1	0	0	0	1	1	0	0	0	0	0	0	0	-1	-1	0	0	0	0	-1	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	-1	-1	-1	0	0	0	0	-c37	*2	*4	0	0	0	0	1	1	1	0	0	0
0	0	-1	0	-1	-1	-1	0	0	0	0	*4	-c37	*2	0	0	0	0	1	1	1	0	0	0
0	0	-1	0	-1	-1	-1	0	0	0	0	*2	*4	-c37	0	0	0	0	1	1	1	0	0	0
0	0	1	0	c31	*5	*25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	*25	c31	*5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	*5	*25	c31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	1	1	1	-1	-1	1	1	0	0	0	0	0	-1	-1	0	0	0	0	-1	-1
-1	-1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	1	-1	-1	-1	-1	1	1
0	0	-1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	-1	-1	-1	0	0	0

APPENDIX B

TWO GENERATORS FOR J_4

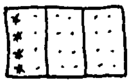
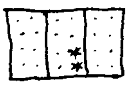
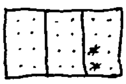
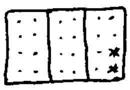
Appendix C

Rank of 1+x for x a representative of each conjugacy class of element of J_4 in the 112-dimensional representation over GF(2) .

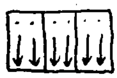
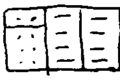
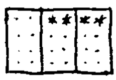
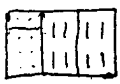
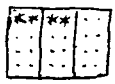
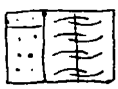
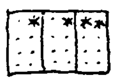
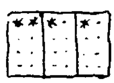
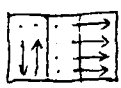
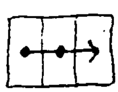
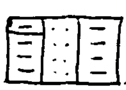
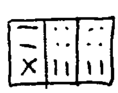
J_4-class of elt. x	Rank(1+x)	J_4-class of elt. x	Rank(1+x)	J_4-class of elt. x	Rank(1+x)
1A	0	12B	100	31A	110
2A	50	12C	102	31B	110
2B	56	14A	104	31C	110
3A	72	14B	104	33A	112
4A	80	14C	104	33B	112
4B	80	14D	104	35A	112
4C	84	15A	104	35B	112
5A	88	16A	104	37A	108
6A	92	20A	104	37B	108
6B	90	20B	104	37C	108
6C	92	21A	108	40A	108
7A	96	21B	108	40B	108
7B	96	22A	110	42A	110
8A	96	22B	106	42B	110
8B	96	23A	110	43A	112
8C	96	24A	106	43B	112
10A	98	24B	106	43C	112
10B	100	28A	108	44A	110
11A	110	28B	108	66A	112
11B	100	29A	112	66B	112
12A	100	30A	108		

Appendix D

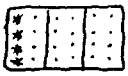
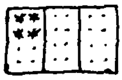
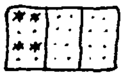
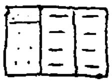
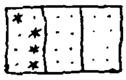
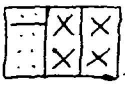
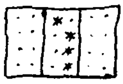
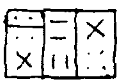
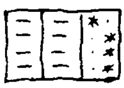
Some elements of J_4 written in the notations for those of H , M , P and L in which they lie. Elements of H are written as the product of a vector representing an element of E and a 6 X 6 matrix representing an element of F (c.f. p. 18), and also the action on H's MOG is given.

H		M	P	L
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_1$	←		L_0	$a \wedge b$
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0$	←		$L_2 \cdot 3_{1\,\omega}$	$(a \wedge b + a \wedge c) \cdot \begin{pmatrix} 10000 \\ 01000 \\ 00100 \\ 10010 \\ 10001 \end{pmatrix}$
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_0$	←		$L_1 \cdot 3_{\bar{\omega}\,0} \cdot 5_{1\,1} \cdot (\infty 5)(04)(12)(36)$	$(a \wedge b + a \wedge d) \cdot \begin{pmatrix} 10000 \\ 01000 \\ 10100 \\ 00010 \\ 10001 \end{pmatrix}$
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_0$	←		$L_2 \cdot 3_{1\,\bar{\omega}} \cdot 5_{1\,1} \cdot (\infty 5)(04)(12)(36)$	$(a \wedge b + a \wedge c + a \wedge d) \cdot \begin{pmatrix} 10000 \\ 01000 \\ 10100 \\ 10010 \\ 10001 \end{pmatrix}$
$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_0$	←		$L_2 \cdot 3_{\omega\,0}$	$(a \wedge b + a \wedge c + a \wedge e) \cdot \begin{pmatrix} 10000 \\ 01000 \\ 00100 \\ 00010 \\ 10001 \end{pmatrix}$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_0$	←		$L_1 \cdot 3_{\bar{\omega}\,0}$	$(a \wedge b + a \wedge d + a \wedge e) \cdot \begin{pmatrix} 10000 \\ 01000 \\ 00100 \\ 00010 \\ 10001 \end{pmatrix}$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_0$	←		$L_2 \cdot 3_{\bar{\omega}\,0}$	$(a \wedge b + a \wedge c + a \wedge d + a \wedge e) \cdot \begin{pmatrix} 10000 \\ 01000 \\ 00100 \\ 00010 \\ 10001 \end{pmatrix}$

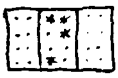
<u>The Subgroup $E = O_2(H)$</u>

H		M	P	L
diag(ω)	←		$(\infty)(0)$ $(142)(356)$	$\begin{pmatrix}01000\\11000\\00100\\00010\\00001\end{pmatrix}$
$\begin{pmatrix}1&1&1&0&1&1\\1&1&1&1&0&1\\1&1&1&1&1&0\\0&1&1&1&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\end{pmatrix}_{2A}$	 2A		$0_1\ 0$	$c\wedge d$
$\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}_{2A}$	 2A		$0_{\varpi}\ 0$	$(a\wedge b + c\wedge d + c\wedge e)\cdot\begin{pmatrix}10000\\01000\\00100\\00010\\00101\end{pmatrix}$
$\begin{pmatrix}1&\omega&\varpi&0&\omega&\varpi\\\varpi&1&\omega&\varpi&0&\omega\\\omega&\varpi&1&\omega&\varpi&0\\0&\omega&\varpi&1&\omega&\varpi\\\varpi&0&\omega&\varpi&1&\omega\\\omega&\varpi&0&\omega&\varpi&1\end{pmatrix}_{2A}$	 2A		$L_0\cdot(0\omega)(1\varpi)$	-
$\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}_{2A}$	 2A		$L_0\cdot(01)(\omega\varpi)$	$(c\wedge d + d\wedge e)\cdot\begin{pmatrix}10000\\01000\\00100\\00010\\00011\end{pmatrix}$
$\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0\end{pmatrix}_{3A}$	 3A		-	$\begin{pmatrix}10000\\01000\\00010\\00110\\00001\end{pmatrix}$
$\begin{pmatrix}0&0&1&1&1&0\\1&\varpi&\omega&\omega&\omega&\omega\\1&\omega&\varpi&\varpi&\varpi&\varpi\\0&0&0&1&1&1\\0&0&0&\varpi&1&0\\0&0&0&\omega&1&0\end{pmatrix}_{3A}$	 3A		$(1\varpi\omega)\in\Sigma$	-
$(\omega\leftrightarrow\varpi)_{2A}$	 2A		$0_1\ \ 1\cdot\begin{matrix}(\infty 0)(16)\\(23)(45)\end{matrix}$	$\begin{pmatrix}10000\\11000\\00100\\00010\\00001\end{pmatrix}$

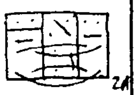
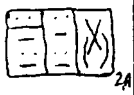
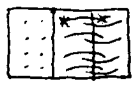
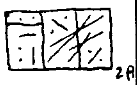
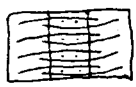
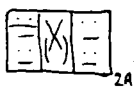
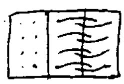
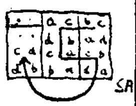
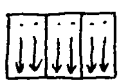
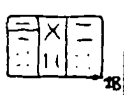
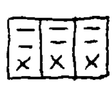
<u>**The Subgroup M ∩ F**</u>

H		M	P	L
$\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}_1$	(z)		L_0	$a_\wedge b$
$\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}_0$	←		L_4	$a_\wedge c$
$\begin{bmatrix}0&\omega&\omega\\0&\omega&\omega\end{bmatrix}_0$	←		L_2	$b_\wedge c$
$\begin{pmatrix}1&1&1&0&1&1\\1&1&1&1&0&1\\1&1&1&1&1&0\\0&1&1&1&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\end{pmatrix}_{2A}$			$O_1\ 0$	$c_\wedge d$
$\begin{pmatrix}1&1&1&0&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\\0&1&1&1&1&1\\1&1&1&1&0&1\\1&1&1&1&1&0\end{pmatrix}_{2A}$			$O_\omega\ 0$	$(a_\wedge b + c_\wedge e).$ $\begin{pmatrix}10000\\01000\\00100\\00010\\00101\end{pmatrix}$
$\begin{pmatrix}1&0&0&0&0&0\\1&0&0&0&0&1\\1&0&0&0&1&0\\0&1&1&1&1&1\\1&0&1&0&0&0\\1&1&0&0&0&0\end{pmatrix}_{2A}$			$O_0\ 1$	$c_\wedge d + c_\wedge e$
$\begin{pmatrix}1&1&1&0&1&1\\0&0&0&1&1&0\\0&0&0&1&0&1\\0&0&0&1&0&0\\0&1&0&1&0&0\\0&0&1&1&0&0\end{pmatrix}_{2A}$			$O_0\ \omega$	$(a_\wedge b + c_\wedge d).$ $\begin{pmatrix}10000\\01000\\00100\\00110\\00101\end{pmatrix}$
$\begin{bmatrix}1&0&1\\1&0&1\end{bmatrix}_0$	←		$3_1\ 0$	$a_\wedge d$
$\begin{bmatrix}\bar\omega&\bar\omega&1\\\bar\omega&\bar\omega&1\end{bmatrix}_0$	←		$3_\omega\ 0$	$(a_\wedge b + a_\wedge c + b_\wedge c + a_\wedge e).$ $\begin{pmatrix}10000\\01000\\00100\\00010\\10001\end{pmatrix}$

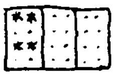
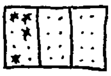
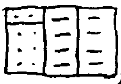
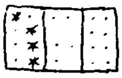
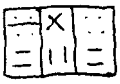
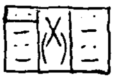
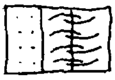
The Subgroup $O_2(P)$

H		M	P	L
$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}_0$	←		$3_{0\ 1}$	$a_\wedge d + a_\wedge e$
$\begin{bmatrix}0 & \omega & \bar\omega\\ 1 & \omega & \bar\omega\end{bmatrix}_0$	←		$3_{0\ \omega}$	$(a_\wedge b + a_\wedge c + b_\wedge c + a_\wedge d).\begin{pmatrix}10000\\01000\\00100\\10010\\10001\end{pmatrix}$
$\begin{bmatrix}\omega & 0 & \omega\\ \omega & 0 & \omega\end{bmatrix}_0$	←		$5_{1\ 0}$	$b_\wedge d$
$\begin{bmatrix}\omega & \bar\omega & \bar\omega\\ \omega & \bar\omega & \bar\omega\end{bmatrix}_0$	←		$5_{\omega\ 0}$	$(a_\wedge b + a_\wedge c + b_\wedge e).\begin{pmatrix}10000\\01000\\00100\\00010\\01001\end{pmatrix}$
$\begin{bmatrix}\omega & 0 & \omega\\ \bar\omega & 1 & \bar\omega\end{bmatrix}_0$	←		$5_{0\ 1}$	$b_\wedge d + b_\wedge e$
$\begin{bmatrix}0 & \bar\omega & 1\\ \omega & \bar\omega & 1\end{bmatrix}_0$	←		$5_{0\ \omega}$	$(a_\wedge b + a_\wedge c + b_\wedge d).\begin{pmatrix}10000\\01000\\00100\\01010\\01001\end{pmatrix}$

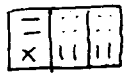
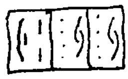
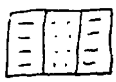
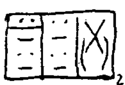
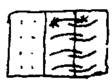
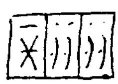
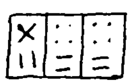
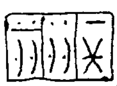
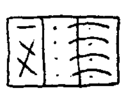
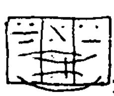
The Subgroup $O_2(P)$ contd.

H		M	P	L
$\begin{pmatrix}1&0&0&0&1&1\\0&1&0&1&0&0\\0&0&1&1&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}_{2A}$		–	Σ (∞0)	$\begin{pmatrix}10000\\01000\\00100\\00011\\00001\end{pmatrix}$
$\begin{pmatrix}0&1&1&0&0&0\\0&1&0&0&0&0\\1&1&0&0&0&0\\0&1&0&0&0&1\\1&0&1&1&1&1\\0&1&0&1&0&0\end{pmatrix}_{2A}$			(01)	$\begin{pmatrix}10000\\01000\\00100\\00010\\00011\end{pmatrix}$
$\begin{pmatrix}1&0&0&0&1&1\\1&\bar\omega&\omega&1&1&1\\1&\omega&\bar\omega&1&1&1\\0&0&0&1&1&1\\0&0&0&0&\omega&\bar\omega\\0&0&0&0&\bar\omega&\omega\end{pmatrix}_{2A}$			$(1\,\omega)$	-
$\begin{pmatrix}0&0&1&0&1&0\\1&1&1&1&0&1\\1&0&0&0&1&0\\0&0&0&0&1&1\\0&0&0&0&1&0\\0&0&0&1&1&0\end{pmatrix}_{2A}$			$(\omega\,\bar\omega)$	$d_\wedge e$
$\begin{pmatrix}0&1&1&1&1&1\\1&0&\omega&0&\bar\omega&0\\1&0&\bar\omega&0&\omega&0\\1&0&1&0&1&0\\1&\bar\omega&1&1&1&\omega\\1&\omega&1&1&1&\bar\omega\end{pmatrix}_{5A}$	 $= x_5$ (see p.42)	–	$(1\,\bar\omega\,\omega\,0\,\infty)$	-
-	-	 $= x_7$	Λ (∞)(0123456)	$\begin{pmatrix}11100\\11000\\01100\\00010\\00001\end{pmatrix}$
$\mathrm{diag}(\omega)_{3A}$	←		(∞)(0) (142)(356)	$\begin{pmatrix}01000\\11000\\00100\\00010\\00001\end{pmatrix}$
$\begin{pmatrix}1&1&1&0&1&1\\0&0&0&1&1&0\\0&1&0&1&0&0\\0&0&0&1&0&0\\0&0&0&1&0&1\\0&0&0&1&1&0\end{pmatrix}_{2B}$			(∞0)(16) (23)(45)	$(c_\wedge e).\begin{pmatrix}10000\\11000\\00100\\00010\\00001\end{pmatrix}$

The Subgroups Σ and $\Lambda \leqslant P$

H		M	P	L
$\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}_1$	(z)		L_0	$a \wedge b$
$\begin{bmatrix}0&1&1\\0&1&1\end{bmatrix}_0$	←		L_4	$a \wedge c$
$\begin{bmatrix}0&\omega&\omega\\0&\omega&\omega\end{bmatrix}_0$	←		L_2	$b \wedge c$
$\begin{bmatrix}1&0&1\\1&0&1\end{bmatrix}_0$	←		$3_{1\;0}$	$a \wedge d$
$\begin{bmatrix}\omega&0&\omega\\\omega&0&\omega\end{bmatrix}_0$	←		$5_{1\;0}$	$b \wedge d$
$\begin{pmatrix}1&1&1&0&1&1\\1&1&1&1&0&1\\1&1&1&1&1&0\\0&1&1&1&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\end{pmatrix}_{2A}$	 $_{2A}$		$0_{1\;0}$	$c \wedge d$
$\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix}_0$	←		$3_{1\;1}$	$a \wedge e$
$\begin{bmatrix}0&0&0\\\omega&\omega&\omega\end{bmatrix}_0$	←		$5_{1\;1}$	$b \wedge e$
$\begin{pmatrix}1&1&1&0&1&1\\0&0&1&1&0&0\\0&1&0&1&0&0\\0&0&0&1&0&0\\0&0&0&1&0&1\\0&0&0&1&1&0\end{pmatrix}_{2A}$	 $_{2A}$		$0_{1\;1}$	$c \wedge e$
$\begin{pmatrix}0&0&1&0&1&0\\1&1&1&1&0&1\\1&0&0&0&1&0\\0&0&0&0&1&1\\0&0&0&0&1&0\\0&0&0&1&1&0\end{pmatrix}_{2A}$	 $_{2A}$		$(\omega\ \bar{\omega}) \in \Sigma$	$d \wedge e$

<u>The Subgroup $O_2(L)$</u>

H		M	P	L
$(\omega \leftrightarrow \bar{\omega})_{2A}$	 $_{2A}$		$0_1\ \ 1\cdot(\infty 0)(16)(23)(45)$	$\begin{pmatrix}10000\\11000\\00100\\00010\\00001\end{pmatrix}$
$\boxed{\begin{matrix}0&0&0\\ \omega&0&\omega\end{matrix}}_0$	$\leftarrow$		$6_1\ \ 1\cdot(\infty 6)(02)(14)(35)$	$\begin{pmatrix}10000\\01000\\01100\\00010\\00001\end{pmatrix}$
$\begin{pmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}_{2A}$	 $_{2A}$		$0_{\omega\ \bar{\omega}}$	$\begin{pmatrix}10000\\01000\\00100\\00110\\00001\end{pmatrix}$
$\begin{pmatrix}0&1&1&0&0&0\\0&1&0&0&0&0\\1&1&0&0&0&0\\0&1&0&0&0&1\\1&0&1&1&1&1\\0&1&0&1&0&0\end{pmatrix}_{2A}$	 $_{2A}$		$(01)\in\Sigma$	$\begin{pmatrix}10000\\01000\\00100\\00010\\00011\end{pmatrix}$
$\mathrm{diag}(\omega).(\omega\leftrightarrow\bar{\omega}) = (1\leftrightarrow\bar{\omega})_{2A}$	 $_{2A}$		$0_1\ \ 1\cdot(\infty 0)(15)(26)(34)$	$\begin{pmatrix}11000\\01000\\00100\\00000\\00001\end{pmatrix}$
-	-		$2_1\ \ 1\cdot(\infty 2)(03)(14)(56)$	$\begin{pmatrix}10000\\01100\\00100\\00010\\00001\end{pmatrix}$
$\begin{pmatrix}0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{pmatrix}_{2A}$	 $_{2A}$		-	$\begin{pmatrix}10000\\01000\\00110\\00010\\00001\end{pmatrix}$
$\alpha = \begin{pmatrix}1&0&0&0&1&1\\0&1&0&1&0&0\\0&0&1&1&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}_{2A}$	 $_{2A}$	-	$(\infty 0)\in\Sigma$	$\begin{pmatrix}10000\\01000\\00100\\00011\\00001\end{pmatrix}$

<u>The Subgroup L_1</u>

Appendix E

Orbits of M , H and $D = M \cap H$
on right cosets of M in J_4.
(due to S.Norton)

→ orbits of H

↓ Orbits of M

	Ⓗ	Φ	Ψ	Δ	Ξ	Σ	Γ	X	Λ
I	1 (95)								
T	60 (95)	720 (104)		2880 (112)	11520 (113)				
S	16 (97)	1440 (105)	3840 (107)			23040 (119)			
Z		5760 (109)	7680 (110)	23040 (112)	138240 (113)	276480 (122)	368640 (120)	1105920 (120)	1474560 (120)
F				18432 (120)	552960 (121)	368640 (122)	8847360 (122)	737280 4423680 (123)	17694720 (124)
N			30720 (121)		184320 (121)	1105920 (122)	1474560 5898240 (123)	1474560 8847360 (124)	35389440 (125)
L							11796480 (121)	11796480 (124)	23592960 35389440 (125)

Sizes of the 36 orbits of D on the cosets of M
(numbers in brackets refer to page numbers where the identifications are made)

Appendix F

Generating sets for the stabilizers in M and H of representatives of the orbits of these groups on the cosets of M .

In each case a generating set is given in such a way that each composition factor in a suitable composition series is covered in turn.

M-Orbits

Trivial Orbit I :

One coset v_∞ , stabilizer M of shape $2^{11}M_{24}$.

Trio Orbit T :

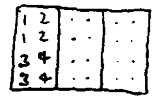
$2^2.3795$ cosets, representative $v_0 = f_1$(brick trio) . Stabilizer shape $2^9.2^6(S_3 \times L_3(2))$ generated by even $\mathcal{C}$*-sets in V hitting each octad of the brick trio evenly, and the trio group in K for the brick trio.

Sextet Orbit S :

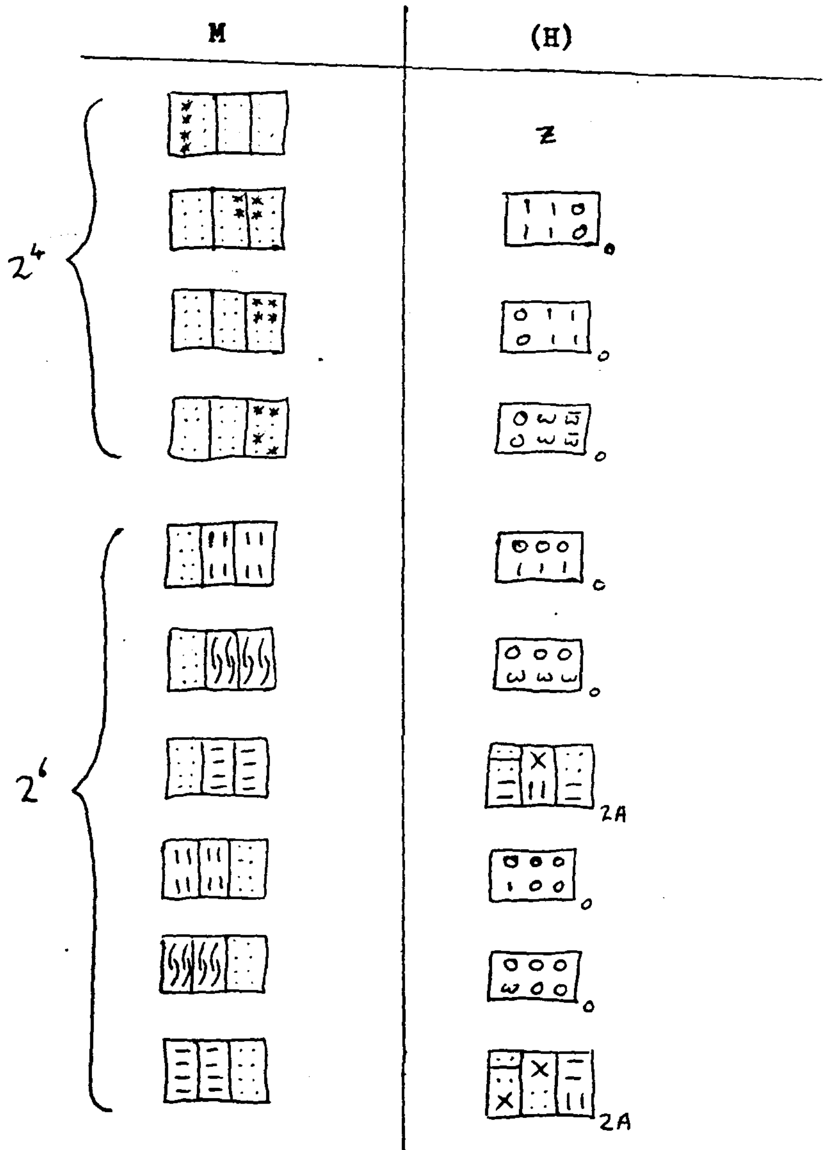
$2^4.1771$ cosets, representative f_1(vertical sextet) . Stabilizer shape $2^7.2^6.3S_6$ generated by even $\mathcal{C}$*-sets in V hitting each tetrad of the vertical sextet with the same parity (i.e. the parity subgroup E_1 (see p. 27)) and the sextet group in K for the vertical sextet.

Sextet-line Orbit Z :

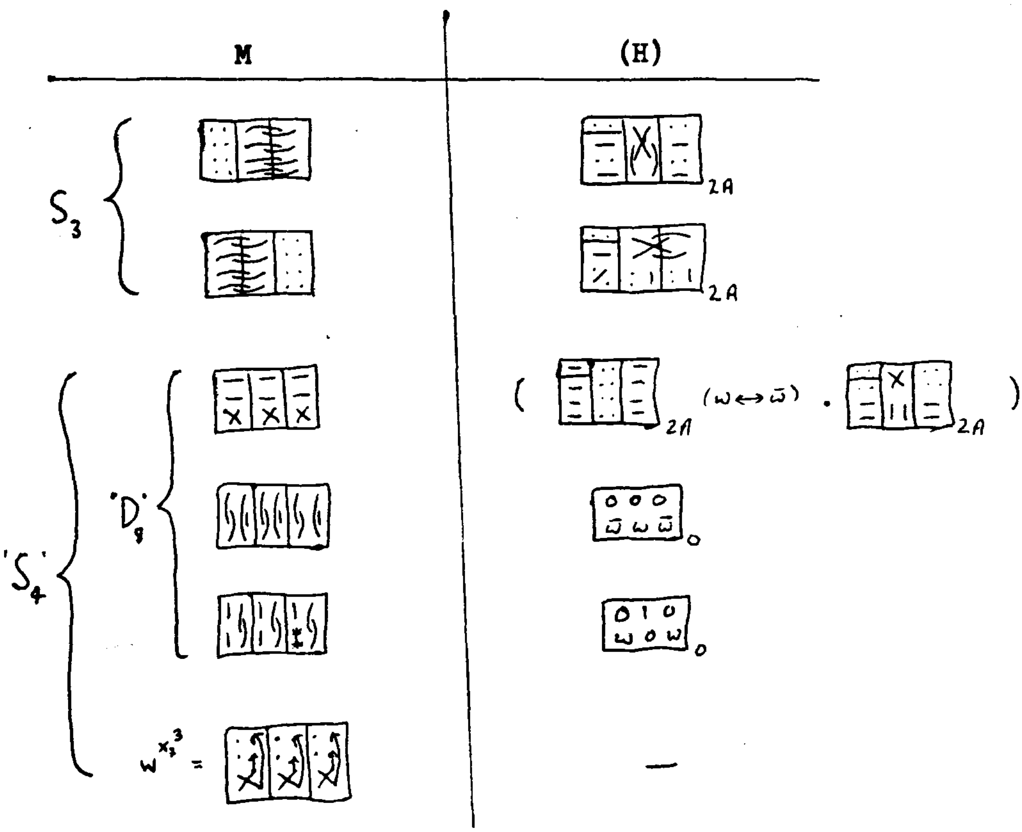
$2^7.26565$ cosets, representative

$c_5 = \Phi\left(\begin{array}{|ccc|}\hline 0 & x & x \\ x_2+z & x_1+y & x_2+y \\ \hline\end{array} : x, y, z \in GF(2)\right) \in f_1($  $)$

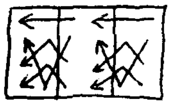
Stabilizer shape $2^4.2^6(S_3 \times S_4)$ generated by



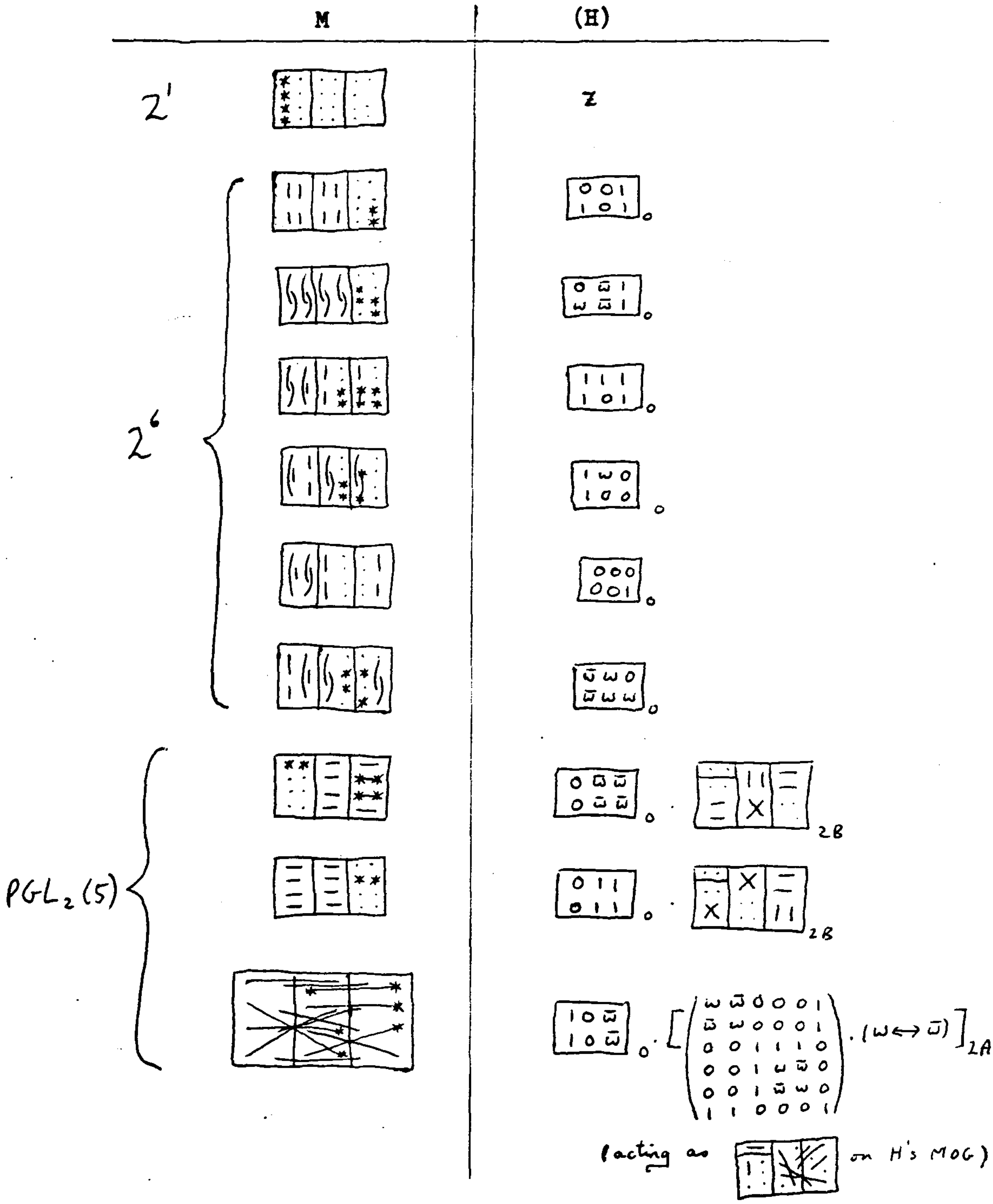
(continued on next page)


M
(H)
S3
2A
2A
'S4'
'D8'
(ω↔ω̄)
2A
2A
ω̄ ω ω̄
0
0 1 0
ω 0 ω
0
w^{x3^3} =
—

Regular-Involution Orbit F :

$2^{10}.31878$ cosets, representative $c_{10} = c_6 \cdot$  $\cdot\alpha$

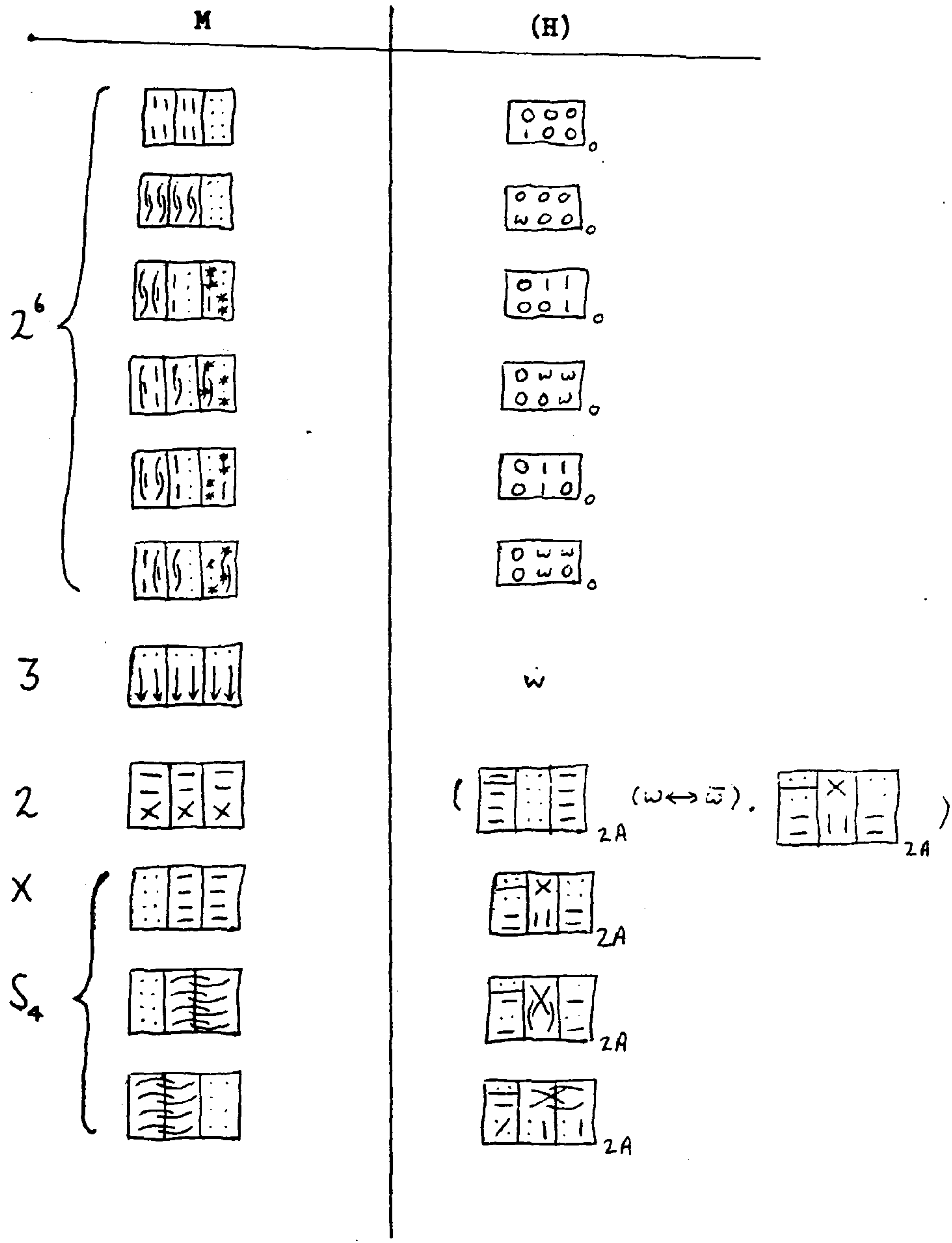
Stabilizer shape $2^1.2^6PGL_2(5)$ generated by



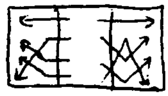
Trio-sextet orbit N :

$2^{11}.26565$ cosets, representative $c_{15} \in \Psi\left(\begin{array}{|ccc|}\hline 0 & x & x \\ x_2+z & x_1+y & x_2+y \\ \hline\end{array}\right)$

Stabilizer shape $2^0.2^6.3(2 \times S_4)$ generated by



Projective-Line Orbit L :

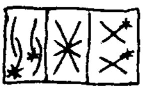
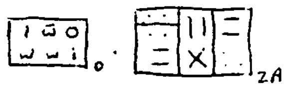
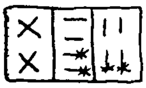
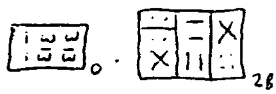
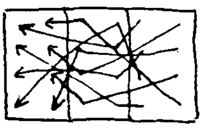
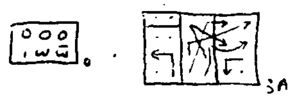
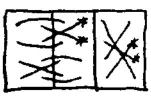
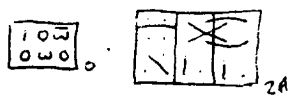
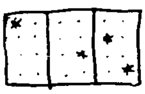
$2^{11}.40320$ cosets, representative $c_{13} = c_{12}.$  $.\alpha$

$= \mathfrak{f}_1^{\{6,13,20\}}($

4	10	14	20	12	21
15	0	7	11	5	9
∞	3	13	2	6	19
16	17	18	8	22	1

$)$

Stabilizer shape $1.L_2(23)$ generated by

	M	(transformation in $L_2(23)$ for above numbering)	(H)
$L_2(23)$, S_4		$z \to \frac{4z+1}{z-4}$	
		$z \to \frac{1-6z}{6+z}$	
		$z \to \frac{5z+4}{z-2}$	()
		$z \to \frac{8-5z}{5+z}$	
	$m_5 =$  . (6)(15 17 20 5 ∞ 13 4 12 14 7 0 21 11 18 22 3 9 19 5 10 1 2 16)	$z \to z+1$	—

H-Orbits

Hexad Orbit Θ :

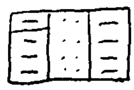
77 cosets, representative $\Theta(\vartheta) = v_\infty$.

Stabilizer shape $2^{1+12}.3.2^4S_6$ generated by E and the stabilizer in F of the hexad ϑ .

Big-Octad Orbit Φ :

$2^3.3.330$ cosets, representative $c_1 = \Phi\left(\begin{array}{|ccc|} X & Y & Z \\ x & y & z \end{array} : x, y, z \in GF(2) \right)$

$= f_1^{a,b}\left(\begin{array}{|c|c|} \mathcal{O} & a \\ \hline & b \end{array} \right)$

Stabilizer shape $2^{1+9}.1.2^4L_3(2)$ generated by the centralizer in F of the element $\omega \leftrightarrow \bar{\omega}$ () ,

of shape $2^1.2^4L_4(2)$ and the subgroup

$E(c_1) = \left\{ \begin{array}{|ccc|} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{array}_{\,i} \;\middle|\; \lambda_1+\lambda_5+\lambda_6,\ \lambda_2+\lambda_4+\lambda_6,\ \lambda_3+\lambda_4+\lambda_5 \text{ are in } GF(2);\ i = 0 \text{ or } 1 \right\}$

of E of shape 2^{1+9} .

Little-Octad Orbit Ψ:

$2^7.330$ cosets, representative $c_3 = f_1($

u	v	w	w	w	w
v	u	x	x	x	x
v	u	y	y	y	y
v	u	z	z	z	z

$)$

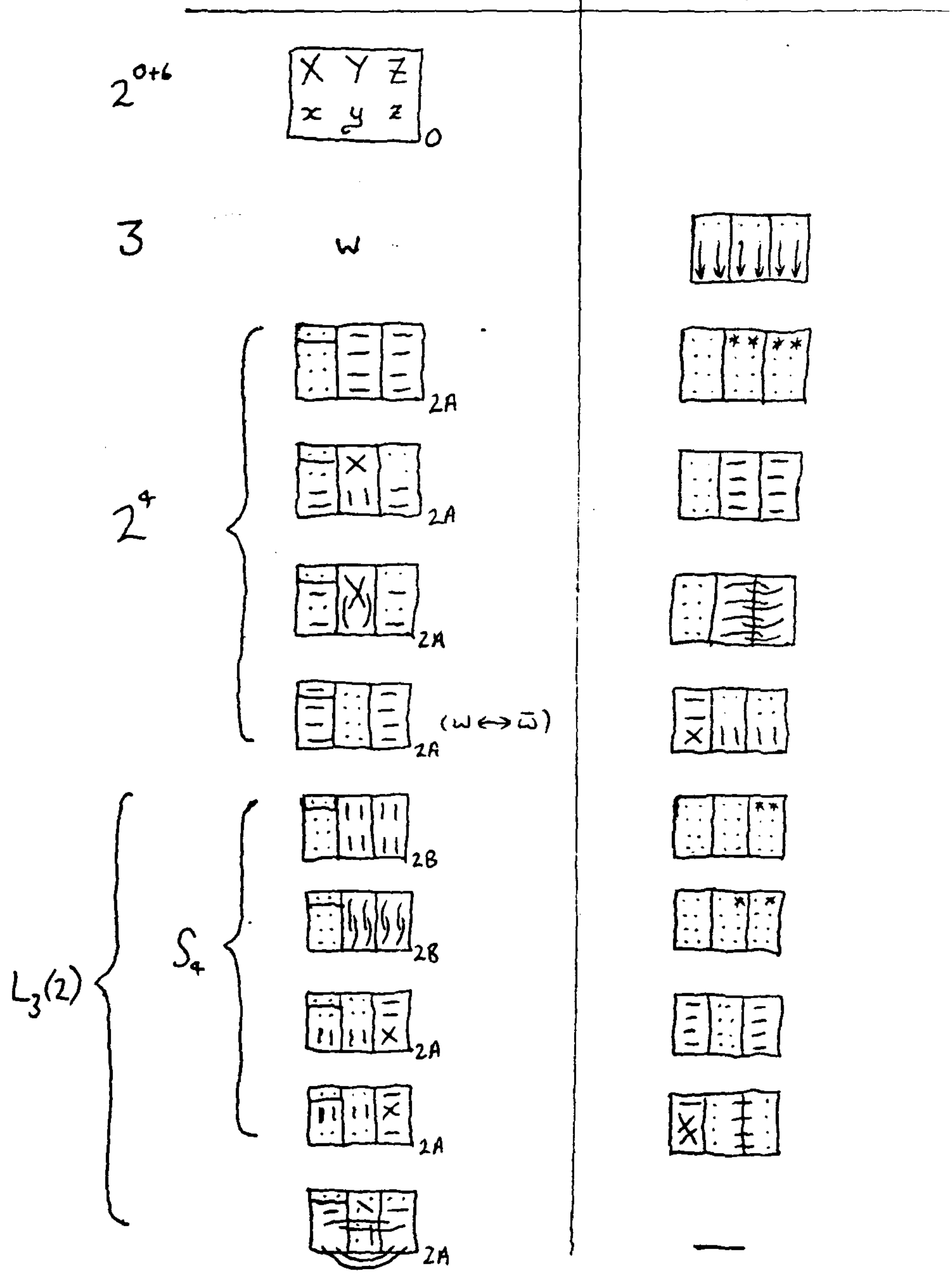
$\in \Psi($

X	Y	Z
x	y	z

$)$

Stabilizer shape $2^{0+6}.3.2^4L_3(2)$ generated by

H	(M)

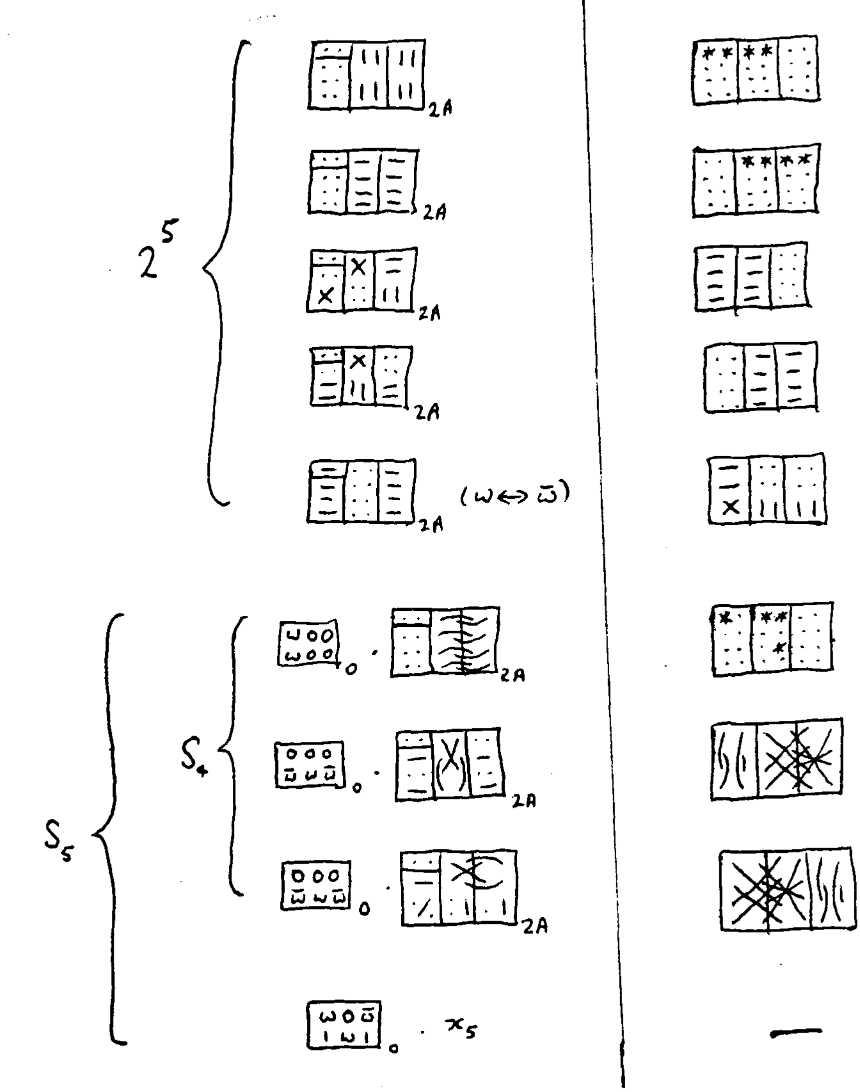


Duad Orbit Δ :

$2^6.3.231$ cosets, representative $c_6 = f_1\left(\begin{array}{cc|cc|cc} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 3 & 1 & 1 & 2 & 2 \\ 3 & 3 & 1 & 1 & 2 & 2 \end{array}\right)$

Stabilizer shape $2^{1+6}.1.2^5S_5$ generated by

H	(M)
2^{1+6} $\left\{ \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \end{bmatrix}_i : \lambda_1, \lambda_4 \in GF(2),\ \lambda_2+\lambda_3+\lambda_5+\lambda_6 = 0,\right.$ and $\lambda_2, \lambda_3, \lambda_5, \lambda_6$ are either all in $GF(2)$ or all in $GF(4)\setminus GF(2)$; $\left. i = 0 \text{ or } 1 \right\}$	



Big-Syntheme Orbit :

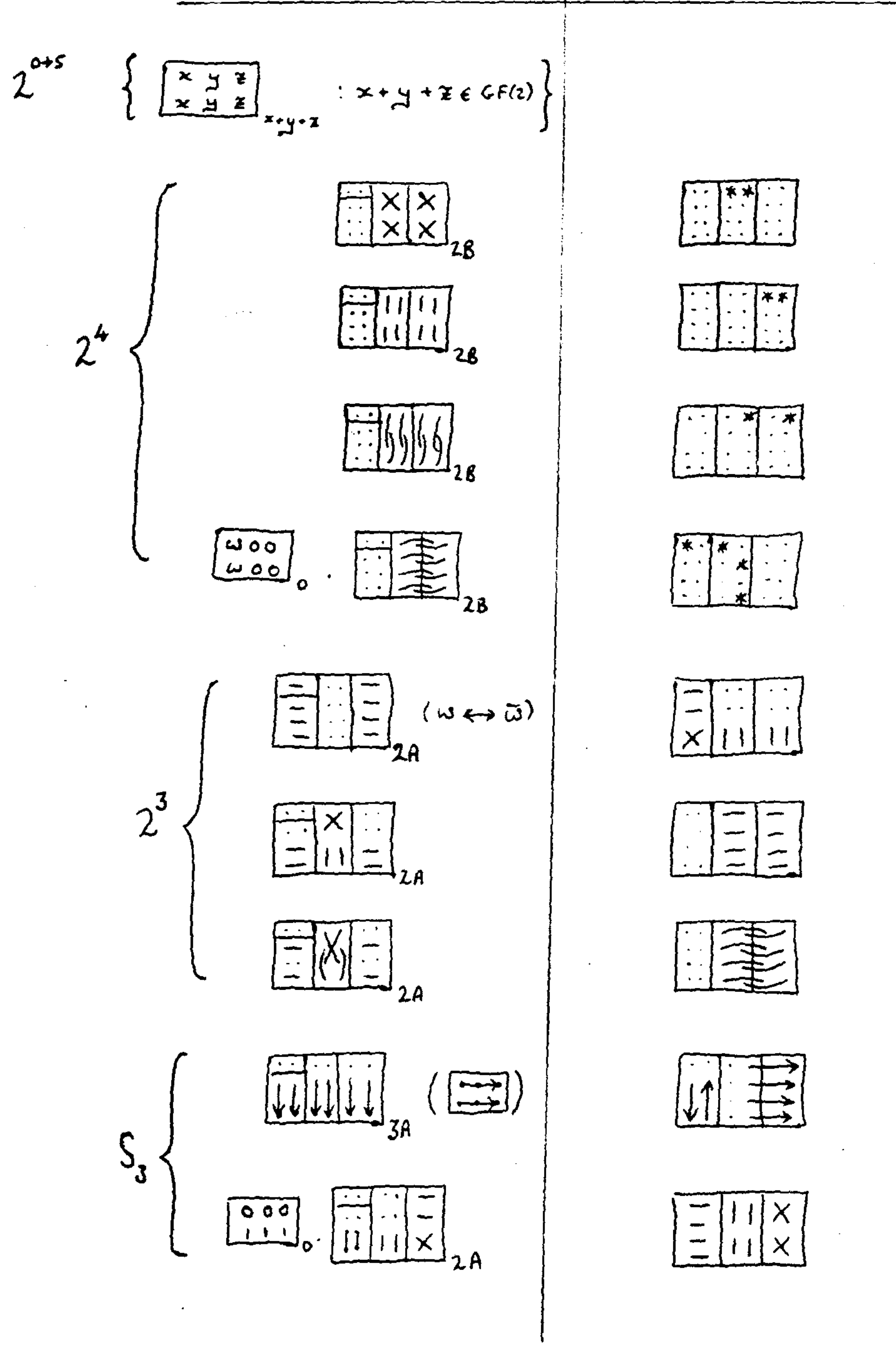
$2^8.3.1155$ cosets, representative $c_8 = f_1($

1	2	3	3	3	3
2	1	3	3	3	3
2	1	2	2	2	2
2	1	1	1	1	1

$)$

Stabilizer shape $2^{0+5}.1.2^4(2^3S_3)$ generated by

H | (M)



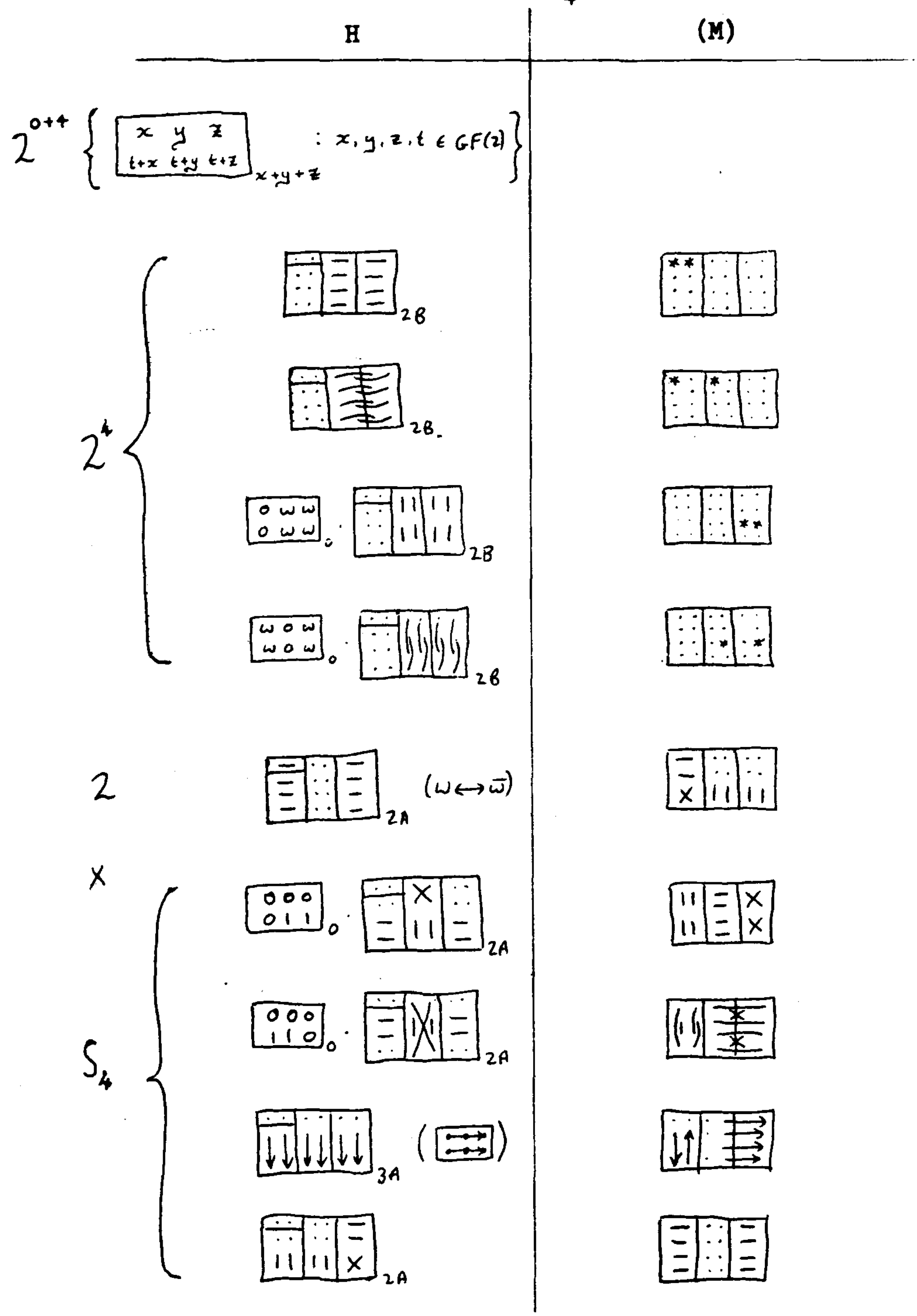
Little-Syntheme Orbit Σ :

$2^9.3.1155$ cosets, representative $c_9 = f_1($

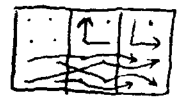
u	u	u	v	w	x
v	v	u	v	w	x
w	x	z	y	y	y
x	w	y	z	z	z

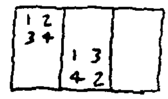
$)$

Stabilizer shape $2^{0+4}.1.2^4(2 \times S_4)$ generated by

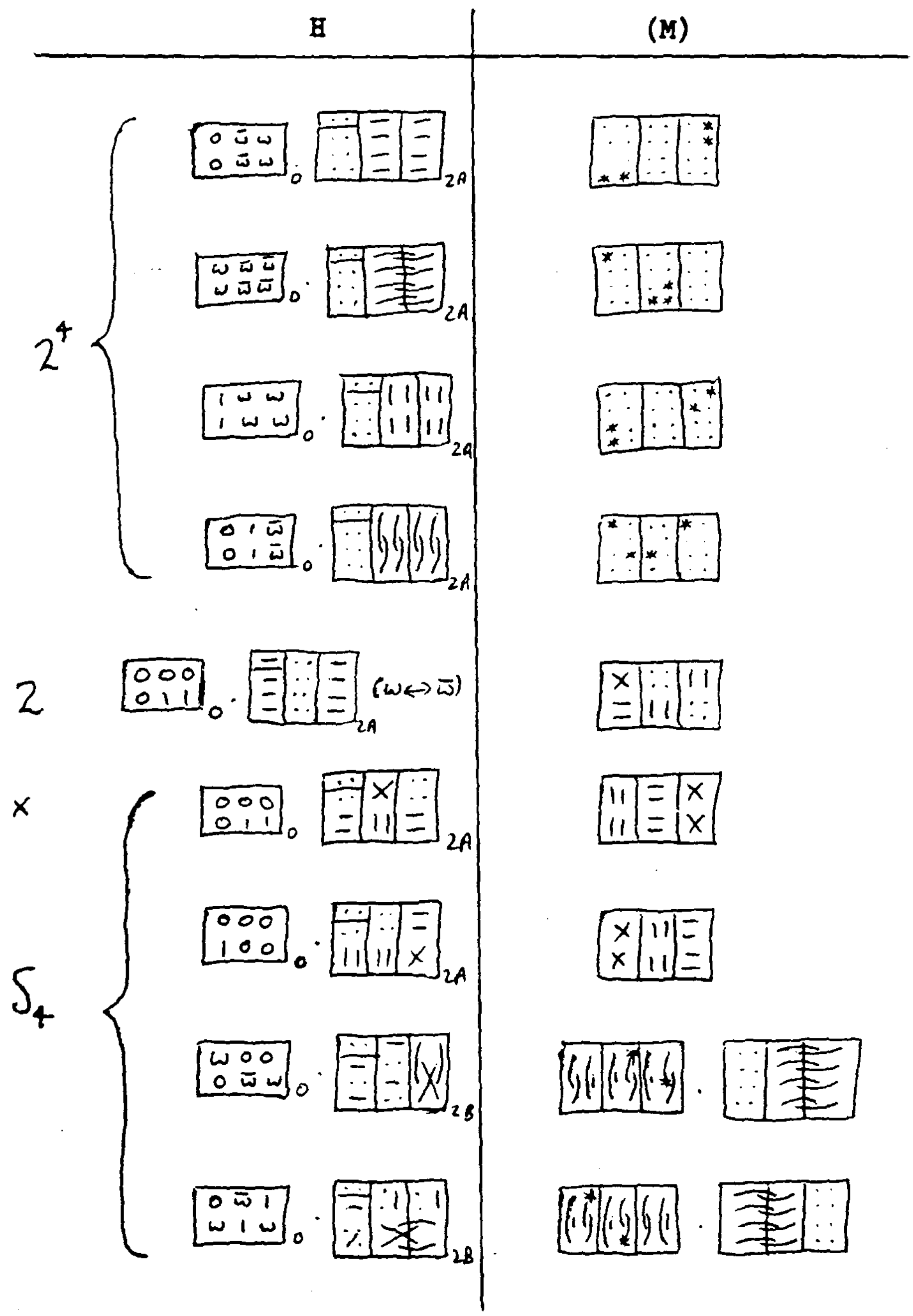


Duad-Hexad Orbit Γ :

$2^{13}.3.1155$ cosets, representative $c_{12} = c_5.w.$ 

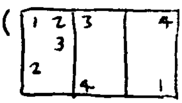
$\in f_1($  $)$

Stabilizer shape $1.1.2^4(2 \times S_4)$ generated by

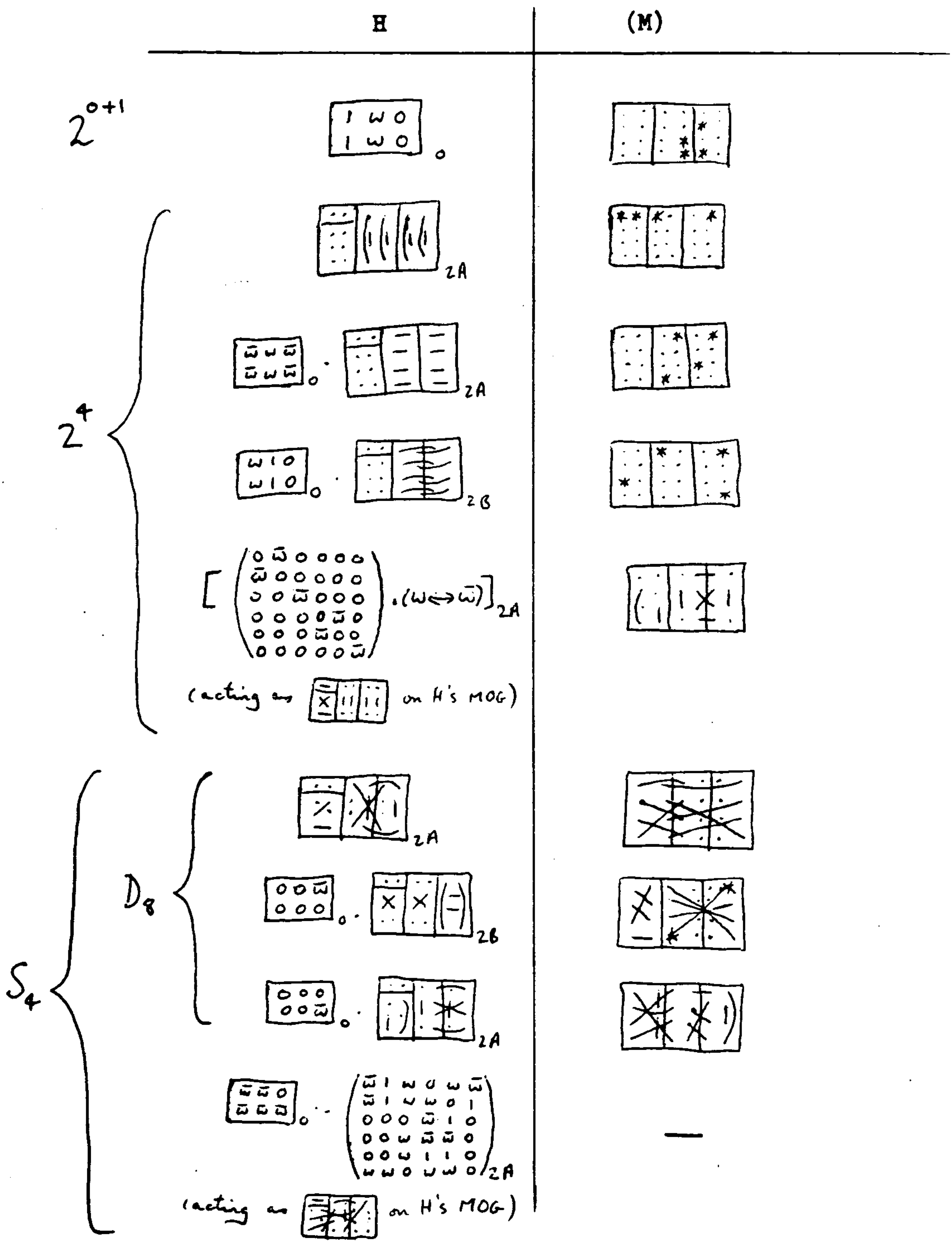


Octad-Pair Orbit X :

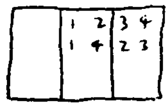
$2^{12}.3.2310$ cosets, representative $c_{14} = c_{13}.\alpha.x_7$

$\in f_1($  $)$

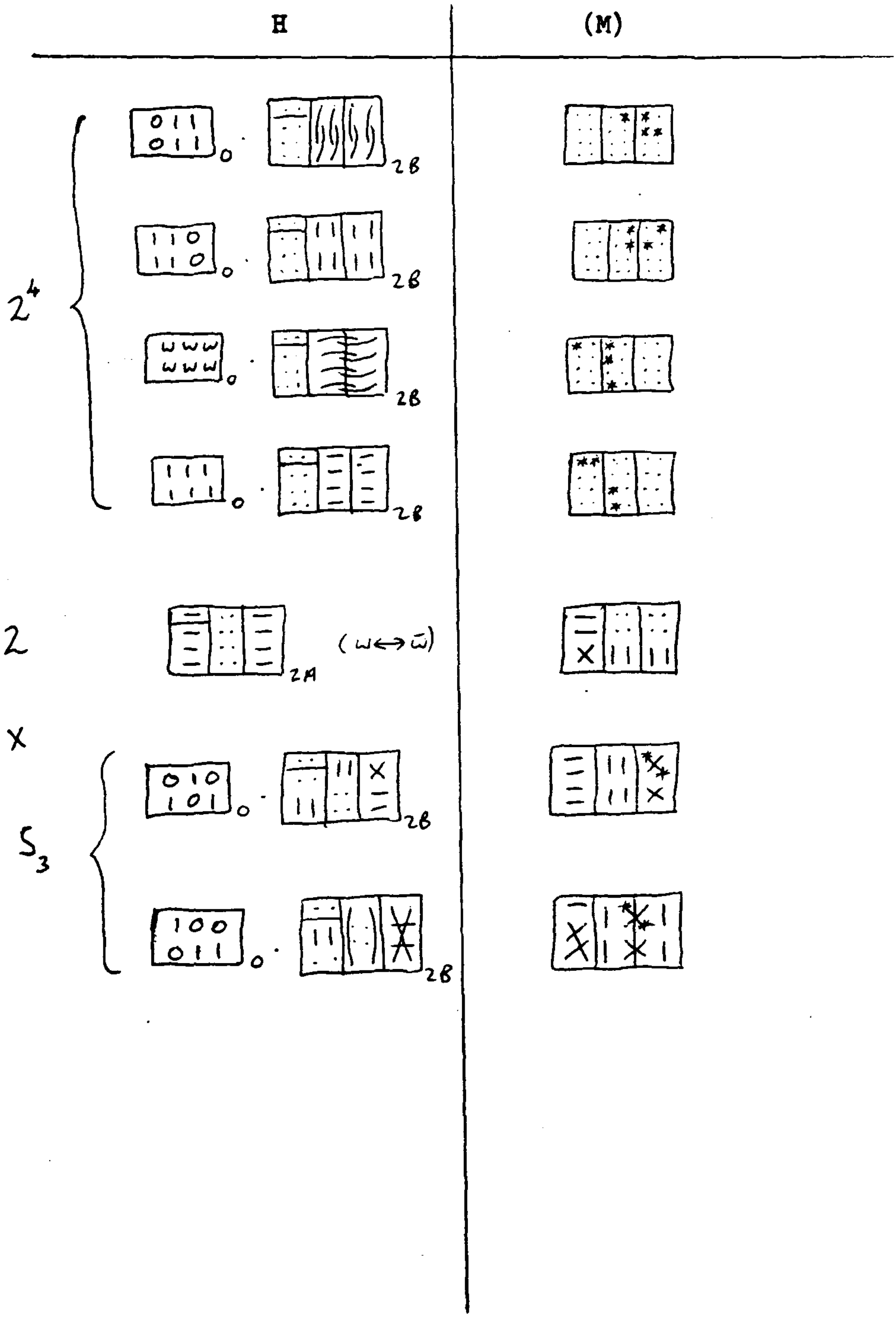
Stabilizer shape $2^{0+1}.1.2^4S_4$ generated by



Hexagon Orbit Λ :

$2^{13}.3.462$ cosets, representative $c_{11} \in f_1($  $)$

Stabilizer shape $1.1.2^4(2 \times S_3)$ generated by



Appendix G

The 2-modular character tables

of M_{22} , $M_{22}.2$, $3M_{22}$ and $3M_{22}.2$

in 'Atlas' notation.

(see [5])

	443520	36	5	7	7	11	11		
p power		A	A	A	A	A	A		
p' part		A	A	A	A	A	A		
ind	1A	3A	5A	7A	B**	11A	B**	fus	ind
+	1	1	1	1	1	1	1	:	+
o	10	1	0	b7	**	−1	−1	:	o
o	10	1	0	**	b7	−1	−1	:	o
−	34	−2	−1	−1	−1	1	1	:	−
o	70	−2	0	0	0	−1+b11	**	|	+
o	70	−2	0	0	0	**	−1+b11	|	
−	98	−1	−2	0	0	−1	−1	:	−
	1	3	5	7	7	11	11		
	:3	:1	:3	:3	:3	:3	:3		
o2	6	0	1	−1	−1	−b11	**	**	+
o2	15	0	0	1	1	−1+b11	**	**	+
o2	45	0	0	b7	**	1	1	|	o2
o2	45	0	0	**	b7	1	1	**	
o2	64	0	−1	0	0	1−b11	**	**	+
o2	364	0	−1	−1	−1	−1	−1	**	+

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