REPRESENTATION RINGS OF FINITE GROUPS

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Preface

At the Ottawa 1984 fourth international conference on the representations of algebras ("ICRA IV"), I gave a series of three expository lectures entitled "Modules for finite groups : representation rings, quivers and varieties". The main theme of those lectures was to demonstrate the connections depicted in the following diagram.

$$\begin{tikzcd} & \boxed{\text{Representation Rings}} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \boxed{\begin{array}{l}\text{Group Cohomology \&}\\ \text{Varieties for Modules}\end{array}} \arrow[rr, leftrightarrow] & & \boxed{\text{Almost Split Sequences}} \end{tikzcd}$$

These lectures were written up, and will appear in the proceedings, published by Springer-Verlag in their lecture note series. At that conference, the organizers of the present conference asked me to give a similar series of two lectures here at Durham. For the sake of avoiding exact repetition, what I decided to do was to expand on two of the topics discussed there. In the first lecture, I discuss the existence of nilpotent elements in representation rings, treating this topic as an illustration of the above triangle. This may be regarded as a predigested version of Benson & Carlson [5]. In the second lecture, I go over the basic definitions and properties of the cohomological varieties associated to modular representations, and to illustrate the concepts I describe in some detail how to find the varieties for the indecomposable modules for the dihedral two-groups in characteristic two.

FIRST LECTURE

Nilpotent elements in representation rings

Throughout this lecture, G will be a finite group, and k an algebraically closed field of characteristic p. All modules will be finitely generated. We shall omit to mention that some of the results described here work with suitable modifications for more general rings of coefficients.

1. TENSOR PRODUCTS

Recall that the (modular) representation ring or Green Ring (after J.A. Green, who was the first to make any serious study of its structure) is the complex vector space A(G) having the isomorphism classes [V] of indecomposable kG-modules V as basis. Multiplication is given by $[V] \cdot [W] = [V \otimes W]$, where $V \otimes W$ denotes the tensor product over k, with the usual diagonal G-action. It is known that A(G) is finite dimensional if and only if the Sylow p-subgroups of G are cyclic, and otherwise it is not even Noetherian. For most groups the indecomposable representations are in some sense unclassifiable (for groups with non-cyclic Sylow p-subgroups, the representation type is wild except when $p = 2$ and the Sylow 2-subgroups are dihederal, semidihedral, quaternion or generalized quaternion [8]).

Despite the fact that the tensor product operation has very widespread use in representation theory, in general very little is known about how a tensor product of indecomposable modules breaks up as a direct sum of indecomposables. This information is reflected in basic ring-theoretical properties of A(G). As a first step, we determine when the trivial kG-module k appears as a direct summand of a tensor product. This appears to be one of the keys to understanding nilpotent elements in A(G), as we shall see later in this lecture.

Theorem 1 ([5], Theorem 2.1) If V and W are indecomposable kG-modules then $V \otimes W$ has the trivial module k as a direct summand if and only if

(i) $V \cong W^*$

and (ii) $p \nmid \dim V$.

Moreover, under these conditions k is a summand with multiplicity one.

Sketch of proof The trivial kG-module k is a direct summand of $V \otimes W$ if and only if we can find homomorphisms $k \to V \otimes W$ and $V \otimes W \to k$ with non-zero composite. This happens if and only if the composite map

$$\mathrm{Hom}_{kG}(W^*,V) \xrightarrow{i} V \otimes W \xrightarrow{p} (\mathrm{Hom}_{kG}(V,W^*))^*$$

is non-zero. Associated to this we have a map

$$\mathrm{Hom}_{kG}(W^*,V) \otimes \mathrm{Hom}_{kG}(V,W^*) \xrightarrow{\eta} k$$

with $\eta \neq 0$ if and only if $p \circ i \neq 0$. It turns out that η is just composition followed by trace.

$$\mathrm{Hom}_{kG}(W^*,V) \otimes \mathrm{Hom}_{kG}(V,W^*) \xrightarrow{\circ} \mathrm{End}_{kG}(W^*) \xrightarrow{tr} k$$

Since W^* is indecomposable and k is algebraically closed, every endomorphism is of the form $\lambda I + n$ with n nilpotent. Since $\mathrm{tr}(I) = \dim W^* = \dim W$, for η to be non-zero, we must have $p \nmid \dim W$, and we must have elements $\alpha \in \mathrm{Hom}_{kG}(W^*,V)$ and $\beta \in \mathrm{Hom}_{kG}(V,W^*)$ with $\mathrm{tr}(\beta\circ\alpha) \neq 0$, namely such that $\beta\circ\alpha$ is an isomorphism. Since V is indecomposable this implies that $V \cong W^*$. The statement about multiplicities is not difficult.

2. ALMOST SPLIT SEQUENCES

Theorem 1 above may be translated into a statement about almost split sequences as follows. Recall from [6] or my Ottawa talks that with respect to the bilinear form $(\ ,\ )$ on $A(G)$ given by extending $([U],[V]) = \dim_k \mathrm{Hom}_{kG}(U,V)$ bilinearly, we have the following non-singularity statement. For each indecomposable module V we may find an element $\tau_0(V) \in A(G)$ such that for U indecomposable

$$([U], \tau_0(V)) = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{otherwise.} \end{cases}$$

Namely

$$\tau_0(V) = \begin{cases} [V] - [\mathrm{Rad}\, V] & \text{if } V \text{ is projective} \\ [V] + [\Omega^2 V] - [X_V] & \text{otherwise} \end{cases}$$

where $0 \to \Omega^2 V \to X_V \to V \to 0$ is the almost split sequence terminating in V. In particular, it follows that if x is a non-zero element of $A(G)$, then there exists $y \in A(G)$ with $(x,y) \neq 0$.

With this in mind, Theorem 1 may now be interpreted as saying that for V and W indecomposable,

$$([V \otimes W^*], \tau_0(k)) = \begin{cases} 1 & \text{if } V \cong W \text{ and } p \nmid \dim V \\ 0 & \text{otherwise.} \end{cases}$$

Since $([V \otimes W^*], \tau_0(k)) = ([V], [W].\tau_0(k))$ this means that the following hold.

(i) If $p \mid \dim W$ then $([V], [W].\tau_0(k)) = 0$ for all indecomposable modules V. In particular taking $V = W$ this implies that the connecting homomorphism $\mathrm{Hom}_{kG}(W,W) \to \mathrm{Ext}^1_{kG}(W,\Omega^2k \otimes W)$ is zero, and hence the sequence $0 \to \Omega^2k \otimes W \to X_k \otimes W \to W \to 0$ splits.

(ii) If $p \nmid \dim W$ then for V indecomposable

$$([V], [W].\tau_0(k)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise,} \end{cases}$$

and so the sequence $0 \to \Omega^2k \otimes W \to X_k \otimes W \to W \to 0$ satisfies the defining conditions for an almost split sequence, apart from possibly the indecomposability of $\Omega^2k \otimes W$. This means that if we strip off an injective (= projective) direct summand from the first two terms of this sequence, we are left with an almost split sequence.

We have thus outlined a proof of the following theorem. A different proof appears in Auslander & Carlson [2], which also gives the corresponding result for RG-lattices, where R is a complete discrete valuation ring.

Theorem 2 ([2] Theorem 3.6; [5] Proposition 2.15) Let $0 \to \Omega^2k \to X_k \to k \to 0$ be the almost split sequence terminating in the trivial module k. Let W be an indecomposable kG-module. Then the tensor product

$$0 \to \Omega^2k \otimes W \to X_k \otimes W \to W \to 0$$

has the following properties.

(i) It is either split, or almost split modulo an injective summand.

(ii) It fails to split if and only if $p \nmid \dim W$.

3. NILPOTENT ELEMENTS IN A(G)

We now translate Theorem 1 into information about the structure of $A(G)$. Let us denote by $A(G;p)$ the linear span in $A(G)$ of the indecomposable modules whose dimension is divisible by p.

Theorem 3 ([5], lemma 2.5 and theorem 2.7)

(i) $A(G;p)$ is an ideal in $A(G)$.

(ii) $A(G)/A(G;p)$ has no non-zero nilpotent elements.

Sketch of proof (i) This statement is the same as the statement that if V is indecomposable and $p \mid \dim V$ then for any module W and any direct summand U of $V \otimes W$, we have $p \mid \dim U$. But this follows by applying theorem 1 to each side of the equality $(V \otimes W) \otimes U^* \cong V \otimes (W \otimes U^*)$.

(ii) First suppose $x = \Sigma a_i [V_i] \in A(G)$ with $xx^* \in A(G;p)$, where $x^* = \Sigma \bar{a}_i [V_i^*]$. Then $xx^* = \Sigma |a_i|^2 [V_i \otimes V_i^*] + \sum_{i \neq j} a_i \bar{a}_j [V_i \otimes V_j^*]$ does not involve the trivial module k, and so we may deduce from theorem 1 that $x \in A(G;p)$.

Now suppose $x \in A(G)$ with $x^2 \in A(G;p)$. Let $y = xx^*$. Then $yy^* \in A(G;p)$, and hence $y \in A(G;p)$, and hence $x \in A(G;p)$.

If H is a subgroup of G, we denote by $r_{G,H}$ the restriction map $A(G) \to A(H)$. It can be shown using an induction theorem of Conlon that if the Sylow p-subgroups of G are cyclic then $\bigcap_{H \leq G} r_{G,H}^{-1}(A(H;p)) = 0$. Thus we may deduce in this case that $A(G)$ is semisimple. This was first proved by Green and O'Reilly in the sixties using some long-winded calculations. The above proof is due to Benson & Carlson [5], where more details may be found. To summarize, we have seen that whenever $A(G)$ is finite dimensional it is semisimple.

4. COHOMOLOGICAL CONSTRUCTION OF NILPOTENT ELEMENTS

When $A(G)$ is infinite dimensional the story is quite different. Zemanek [17,18] was the first to show that there can be nilpotent elements in $A(G)$, by explicit construction of modules $V \not\cong W$ with $V \otimes V \oplus W \otimes W \cong V \otimes W \oplus V \otimes W$, so that $([V] - [W])^2 = 0$ in $A(G)$. I shall outline a general construction due to Benson and Carlson [5], using cohomological techniques.

The construction depends on some modules L_ζ introduced by

Carlson [10] for the purpose of studying the cohomological varieties associated to modules. They now represent a standard method of passing from cohomology to representation theory. The definition is as follows. If $0 \neq \zeta \in H^{2n}(G,k) \cong \mathrm{Ext}^{2n}_{kG}(k,k) \cong \mathrm{Hom}_{kG}(\Omega^{2n}k,k)$ (where ΩV represents the kernel of the projective cover of V) then ζ is represented by a surjective homomorphism $\hat{\zeta} : \Omega^{2n}k \to k$, whose kernel we denote by L_ζ. If $\zeta = 0$, we make the convention $L_\zeta = \Omega^{2n}k \oplus \Omega k$. The basic lemma is as follows.

<u>Lemma</u> ([11]; [5], Theorem 3.3) Suppose V is a kG-module, and suppose $\zeta \in H^{2n}(G,k)$ annihilates $\mathrm{Ext}^*_{kG}(V,V)$ (cup-product action; note that it is enough to check that ζ annihilates the identity element in $\mathrm{Ext}^0_{kG}(V,V) = \mathrm{End}_{kG}(V)$). Then

$$L_\zeta \otimes V \cong \Omega^{2n}V \oplus \Omega V \oplus (\text{projective}).$$

Now suppose L_ζ happens to be periodic with period two (i.e. $\Omega^2 L_\zeta \cong L_\zeta$ but $\Omega L_\zeta \not\cong L_\zeta$), and suppose ζ annihilates $\mathrm{Ext}^*_{kG}(L_\zeta,L_\zeta)$. It happens that this forces G to have p-rank one or two, but that under these conditions there are many examples of such behaviour. We shall list some below. According to the lemma, we have

$$\begin{aligned} L_\zeta \otimes L_\zeta &\cong \Omega^{2n}L_\zeta \oplus \Omega L_\zeta \oplus (\text{projective}) \\ &\cong L_\zeta \oplus \Omega L_\zeta \oplus (\text{projective}). \end{aligned}$$

Applying Ω twice to this, we obtain

$$\Omega L_\zeta \otimes L_\zeta \cong \Omega L_\zeta \oplus L_\zeta \oplus (\text{projective})$$

and $$\Omega L_\zeta \otimes \Omega L_\zeta \cong L_\zeta \oplus \Omega L_\zeta \oplus (\text{projective}).$$

Hence $([L_\zeta] - [\Omega L_\zeta])^2$ is a linear combination of projective modules. If L_ζ and ΩL_ζ have the same Brauer character (for a p-group this simply says that they have the same dimension) then $([L_\zeta] - [\Omega L_\zeta])^2 = 0$; otherwise we would have to adjust by some linear combination of projective modules to force the square to be zero (note that the linear span in $A(G)$ of the projective modules is a finite dimensional direct summand, so that we may project onto its complement).

We thus have the following theorem, which is a restricted version of [5], Theorem 3.4.

Theorem 4 Suppose $\zeta \in H^{2n}(G,k)$ has the following properties.

(i) L_ζ is periodic with period two.

(ii) L_ζ and ΩL_ζ have the same Brauer character.

(iii) ζ annihilates $\mathrm{Ext}^*_{kG}(L_\zeta, L_\zeta)$.

Then $[L_\zeta] - [\Omega L_\zeta]$ is a non-zero nilpotent element of $A(G)$.

Examples (a) For G an elementary abelian group of order p^2, p odd, and $k = \bar{\mathbb{F}}_p$, we have $H^*(G,k) = \Lambda(x_1,x_2) \otimes k[y_1,y_2]$ with $\deg x_1 = \deg x_2 = 1$ and $\deg y_1 = \deg y_2 = 2$. Then for each $(\alpha : \beta) \in \mathbb{P}^1(k)$ and each $n \geq 1$, the element $(\alpha y_1 + \beta y_2)^n \in H^{2n}(G,k)$ satisfies the conditions of theorem 4. The ideal in $A(G)$ generated by the corresponding nilpotent elements is an infinite dimensional nilpotent ideal. The same also works for $C_2 \times C_4$.

(b) For G dihedral of order 2^n, $n \geq 3$, and $k = \bar{\mathbb{F}}_2$, we have $H^*(G,k) = k[x,y,z]/(xy)$ with $\deg x = \deg y = 1$ and $\deg z = 2$. If $\alpha \neq 0 \neq \beta$ and $n \geq 1$ then the element $(\alpha x^2 + \beta y^2)^n \in H^{2n}(G,k)$ satisfies the conditions of theorem 4, and so again we get an infinite dimensional nilpotent ideal in $A(G)$.

(c) If G is a generalized quaternion group, the above construction produces nilpotent elements, but not in such vast quantities. The problem is that all kG-modules are periodic.

(d) If G is an elementary abelian 2-group, the above method produces no nilpotent elements at all, since every periodic module has period one. For G elementary abelian of order four, it turns out that there are no nilpotent elements (Conlon [12]). For G elementary abelian of order 2^n, $n \geq 3$, it is still an open problem as to whether $A(G)$ has nilpotent elements. Since there are standard methods for passing from subquotients to the whole group, this is essentially the only open case.

Remark The above nilpotent elements all square to zero. I do not know whether there are ever nilpotent elements of order greater than two, although this seems likely.

SECOND LECTURE

VARIETIES FOR MODULES : A SAMPLE CALCULATION

In this lecture I shall describe the theory of cohomological varieties associated to modular representations, and give the details of a sample calculation to illustrate the concepts. These varieties grew out of work of Quillen on the structure of equivariant cohomology rings [14,15], and have been developed by Carlson, Alperin, Evens, Avrunin, Scott [1,3,9,10,11] etc.

1. Definitions and basic properties

Let G be a finite group and k an algebraically closed field of characteristic p. Denote by $H^{\cdot}(G,k) \cong \mathrm{Ext}^{\cdot}_{kG}(k,k)$ the cohomology ring $H^{*}(G,k)$ if $p = 2$, and $H^{ev}(G,k)$, the even part of the cohomology ring if $p \neq 2$. Since the cohomology ring is graded commutative (i.e. $xy = (-1)^{\deg(x)\deg(y)}yx$), $H^{\cdot}(G,k)$ is a commutative ring, and so we may form the maximal ideal spectrum $X_G = \mathrm{Max}\ H^{\cdot}(G,k)$. Since the cohomology ring is finitely generated, we may view X_G as a concrete affine variety in the usual way. Namely if $H^{\cdot}(G,k) = k[x_1,\ldots, x_n]/I$ for some homogeneous generators x_i and homogeneous ideal I, then X_G is the variety in $\mathbb{A}^n(k)$ given by the simultaneous zeros of the polynomials in I. In particular, X_G is a homogeneous variety (a union of lines through the origin, where the origin corresponds to the ideal of elements of positive degree) and we may form a projective variety $\bar{X}_G = \mathrm{Proj}\ H^{\cdot}(G,k)$ of one smaller dimension.

Now if V is a KG-module, we may think of $\mathrm{Ext}^{*}_{kG}(V,V)$ as equivalence classes of long exact sequences beginning and ending in V, where the equivalence relation is generated by morphisms of long exact sequences which are isomorphisms on the end terms (see Maclane[13]). The Yoneda splice of long exact sequences gives $\mathrm{Ext}^{*}_{kG}(V,V)$ a ring structure, which may be non-commutative. In fact this ring may have complete matrix rings as quotients. However, there is a natural map $H^{\cdot}(G,k) \cong \mathrm{Ext}^{\cdot}_{kG}(k,k) \to \mathrm{Ext}^{*}_{kG}(V,V)$ given by tensoring long exact sequences with V. The image of this map lies in the centre of $\mathrm{Ext}^{*}_{kG}(V,V)$, and $\mathrm{Ext}^{*}_{kG}(V,V)$ is finitely generated as a module over the image. Thus if we let $\tilde{X}_G(V) = \mathrm{Max}\ Z\ \mathrm{Ext}^{*}_{kG}(V,V)$, the spectrum of maximal ideals of the centre of this ring, then we have a map of varieties $\tilde{X}_G(V) \to X_G$. We denote by $X_G(V)$ the image of this map, so that the map $\tilde{X}_G(V) \to X_G(V)$ is finite (the preimage of a point is a finite set). Also, since $X_G(V)$ is a (closed) homogeneous subvariety of X_G, we may form the corresponding projective subvariety $\bar{X}_G(V)$ of $\bar{X}_G$.

Example If G is elementary abelian of order p^n, then

$$H^*(G,k) = \begin{cases} k[x_1,\ldots,x_n] & \text{if } p = 2 \\ \Lambda(x_1,\ldots,x_n) \otimes k[y_1,\ldots,y_n] & \text{if } p \neq 2 \end{cases}$$

where $\deg(x_i) = 1$, $\deg(y_i) = 2$, and $\Lambda(x_1,\ldots,x_n)$ denotes the exterior algebra. Thus if we denote by J the radical of nilpotent elements in $H^\cdot(G,k)$ then $H^\cdot(G,k)/J$ is a polynomial ring in n variables (x_i if $p = 2$ and y_i if $p \neq 2$) and so $X_G \cong \mathbb{A}^n(k)$, $\bar{X}_G \cong \mathbb{P}^{n-1}(k)$.

For more general G, Quillen [14,15] has shown that

$$X_G \cong \lim_{\substack{\rightarrow \\ E\leq G}} X_E.$$

The direct limit is taken over the set of all elementary abelian subgroups E of G, and the maps are those induced by conjugations and inclusions. The "isomorphism " is a homeomorphism in the Zariski topology, or an "inseparable isogeny" at the coordinate ring level. Avrunin and Scott [3] have shown that the appropriate generalization to modules is also true:

$$X_G(V) = \lim_{\substack{\rightarrow \\ E\leq G}} X_E(V\downarrow_E).$$

From the definitions and results given so far, it would seem that the varieties $X_G(V)$ are very difficult to calculate. It turns out that there is an alternative formulation which makes calculation much easier. By the above result of Avrunin and Scott, it suffices to treat the case where G is elementary abelian. In this case, we have the following rank variety introduced by Carlson [10].

Let Y_G be the affine space $J(kG)/J^2(kG) \cong \mathbb{A}^n$ (where $|G| = p^n$), and let $Y_G(V)$ denote the image modulo J^2 of $\{0\} \cup \{\alpha \in J(kG) : V\downarrow_{\langle 1+\alpha\rangle}$ is not a free $k\langle 1+\alpha\rangle$ - module$\}$ (it turns out that this is a union of cosets of J^2). There is a natural isomorphism

$X_G \cong Y_G$ with the property that for all kG-modules V, the image of $X_G(V)$ is equal to $Y_G(V)$. This was conjectured by Carlson and proved by Avrunin and Scott, as the difficult step in proving their theorem mentioned above. The proof of statement (ii) of Theorem 1 below also needed this fact. The variety $Y_G(V)$ is easy to calculate as the set of zeros of minors of certain matrices.

The following is a list of properties of the $X_G(V)$ which we shall be using. Some of these properties are quite difficult to prove.

Theorem 1 Let V and W be kG-modules.

(i) $X_G(V \oplus W) = X_G(V) \cup X_G(W)$

(ii) (Avrunin-Scott [3]) $X_G(V \otimes W) = X_G(V) \cap X_G(W)$

(iii) The dimension of the variety $X_G(V)$ is equal to the complexity of V. Namely if $\ldots \to P_2 \to P_1 \to P_0 \to V \to 0$ is a minimal resolution of V, the complexity is defined to be the order of the pole of the rational function $\Sigma t^i \dim P_i$ at $t = 1$. This measures the rate of growth of the resolution. Thus the complexity is zero if and only if V is projective, and one if and only if V is periodic, and so on.

(iv) (Carlson [11]) If V is indecomposable then $\bar{X}_G(V)$ is connected in the Zariski topology.

(v) (Carlson [10]) If $0 \neq \zeta \in H^{2n}(G,k) \cong \mathrm{Ext}^{2n}_{kG}(k,k) \cong \mathrm{Hom}_{kG}(\Omega^{2n}k,k)$, we define L_ζ to be the kernel of the homomorphism $\Omega^{2n}k \to k$ (as in the last lecture). Then $X_G(L_\zeta)$ is the hypersurface $X_G(\zeta)$ given by taking the zeros of ζ regarded as an element of the coordinate ring of X_G.

Proofs of these statements may be found in [1,3,9,10,11]. It should be remarked that it follows from (ii) and (v) of this theorem that every homogeneous (closed) subvariety of X_G is of the form $X_G(V)$ (express the variety as an intersection of hypersurfaces, and take the tensor product of the corresponding L_ζ's).

We now turn to our sample calculation to illustrate the above concepts. We have chosen a class of groups for which we have a good understanding of the set of indecomposable modules, namely the dihedral two-groups. The results of these calculations appear without proof in the appendix of [4], p.185.

2. Cohomology of the dihedral two-groups

For the rest of this lecture, let $G = \langle u,v : u^2 = v^2 = (uv)^{2^{n-1}} = 1\rangle$ be the dihedral group of order 2^n, and let k be an algebraically closed field of characteristic two. Then $H^*(G,k) = k[x,y,z]/(xy)$, where $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. We choose the labelling in such a way that the generator $x \in H^1(G,\mathbb{F}_2) = \mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})$ corresponds to the subgroup $\langle uvu,v\rangle$ of index two, while y corresponds to $\langle u,vuv\rangle$. Thus $\bar{X}_G = \mathrm{Proj}(k[x,y,z]/(xy)) = \mathbb{P}^1_a \cup \mathbb{P}^1_b$, where $\mathbb{P}^1_a$ and $\mathbb{P}^1_b$ are projective lines over k intersecting in the common point at infinity : $\mathbb{P}^1_a \cap \mathbb{P}^1_b = \{\infty_a = \infty_b\}$. We choose the notation so that $\mathbb{P}^1_a = \mathrm{Proj}(k[x,z])$ and $\mathbb{P}^1_b = \mathrm{Proj}(k[y,z])$, and so that $\lambda x^2 + \mu y^2 + z = 0$ is the equation of the pair of points $\{\lambda_a,\mu_b\} \subseteq (\mathbb{P}^1_a \cup \mathbb{P}^1_b)\backslash\{\infty\}$.

Let $H_a = \langle u,w\rangle$ and $H_b = \langle v,w\rangle$ where $w = (uv)^q$, $q = 2^{n-2}$, be representatives of the two conjugacy classes of elementary abelian subgroups of order four in G. Then $H^*(H_a,k) = k[x_a,z_a]$ and $H^*(H_b,k) = k[y_b,z_b]$ with $\deg(x_a) = \deg(z_a) = \deg(y_b) = \deg(z_b) = 1$. We choose the notation so that $x_a = \mathrm{res}_{G,H_a}(x)$, $y_b = \mathrm{res}_{G,H_b}(y)$, z_a corresponds to the subgroup $\langle u\rangle$ of index two in H_a, and z_b corresponds to the subgroup $\langle v\rangle$ in H_b. Then the restriction maps are as follows.

Generator of $H^*(G,k)$	Image under res_{G,H_a}	Image under res_{G,H_b}
x	x_a	0
y	0	y_b
z	$z_a(x_a+z_a)$	$z_b(y_b+z_b)$

Let $\hat{\mathbb{P}}^1_a = \mathrm{Proj}\ H^*(H_a,k)$, labelled in such a way that u corresponds to $\hat{0}_a \in \hat{\mathbb{P}}^1_a$, w corresponds to $\hat{\infty}_a$, and uw corresponds to $\hat{1}_a$. Similarly we label $\hat{\mathbb{P}}^1_b = \mathrm{Proj}\ H^*(H_b,k)$ in such a way that v corresponds to $\hat{0}_b \in \hat{\mathbb{P}}^1_b$, w corresponds to $\hat{\infty}_b$, and vw corresponds to $\hat{1}_b$. The maps $t_{H_a,G} = \mathrm{res}^*_{G,H_a} : \hat{\mathbb{P}}^1_a \to \mathbb{P}^1_a$ and $t_{H_b,G} = \mathrm{res}^*_{G,H_b} : \hat{\mathbb{P}}^1_b \to \mathbb{P}^1_b$ are given by

$$t_{H_a,G}(\hat{\lambda}_a) = \lambda_a(1 + \lambda_a)$$

$$t_{H_b,G}(\hat{\mu}_b) = \mu_b(1 + \mu_b).$$

3. MODULES FOR THE DIHEDRAL TWO-GROUPS

The indecomposable kG-modules $(G = D_{2^n})$ were first classified by Bondarenko [7], but we shall rather use the description given in Ringel [16]. First we describe the finite dimensional indecomposable modules for the infinite dihedral group

$$\tilde{G} = \langle u,v : u^2 = v^2 = 1\rangle$$

and then we indicate which are modules for the quotient group $G = D_{2^n} = \langle u,v : u^2 = v^2 = 1,\ (uv)^q = (vu)^q\rangle$.

Let $\mathcal{W}$ be the set of words in the letters a, b, a^{-1} and b^{-1} such that a and a^{-1} are always followed by b or b^{-1} and vice-versa, together with the "zero length words" 1_a and 1_b. If C is a word, we define C^{-1} as follows. $(1_a)^{-1} = 1_b$, $(1_b)^{-1} = 1_a$; and otherwise, we reverse the order of the letters in the word and invert each letter according to the rule $(a^{-1})^{-1} = a$, $(b^{-1})^{-1} = b$. Let $\mathcal{W}_1$ be the set obtained from $\mathcal{W}$ by identifying each word with its inverse.

The n^{th} power of a word of even length is obtained by juxtaposing n copies of the word. Let $\mathcal{W}'$ be the subset of $\mathcal{W}$ consisting of all words of even non-zero length which are not powers of smaller words. Let $\mathcal{W}_2$ be the set obtained from $\mathcal{W}'$ by identifying each word with its inverse and with its images under cyclic permutations of the letters, $\ell_1 \ldots \ell_n \to \ell_n \ell_1 \ldots \ell_{n-1}$.

The following is a list of all the isomorphism types of indecomposable $k\tilde{G}$-modules.

Modules of the first kind These are in one-one correspondence with elements of $\mathcal{W}_1$. Let $C = \ell_1 \ldots \ell_n \in \mathcal{W}$. Let $M(C)$ be a vector space over k with basis $z_0, \ldots, z_n$ on which $\tilde{G}$ acts according to the schema

$$kz_0 \xleftarrow{\ell_1} kz_1 \xleftarrow{\ell_2} kz_2 \ldots kz_{n-1} \xleftarrow{\ell_n} kz_n$$

where x acts as "1 + a" and y acts as "1 + b". For example, if $C = ab^{-1}aba^{-1}$ then the schema is

$$kz_0 \xleftarrow{a} kz_1 \xrightarrow{b} kz_2 \xleftarrow{a} kz_3 \xleftarrow{b} kz_4 \xrightarrow{a} kz_5$$

and the representation is given by

$$x \mapsto \begin{pmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ && 1 &&& \\ && 1 & 1 && \\ &&&& 1 & 1 \\ &&&&& 1 \end{pmatrix} \qquad y \mapsto \begin{pmatrix} 1 &&&&& \\ & 1 & 1 &&& \\ && 1 &&& \\ &&& 1 && \\ &&& 1 & 1 & \\ &&&&& 1 \end{pmatrix}$$

It is clear that $M(C) \cong M(C^{-1})$.

<u>Modules of the second kind</u>. These are in one-one correspondence with elements of $\mathcal{W}_2 \times \mathcal{V}$ where

$$\mathcal{V} = \{(V,\phi) : V \text{ is a vector space over } k \text{ and } \phi \text{ is an indecomposable automorphism of } V\}$$

(since we are only dealing with the case where k is algebraically closed, an indecomposable automorphism of a vector space is simply a Jordan block). If $(C,(V,\phi)) \in \mathcal{W}' \times \mathcal{V}$ with $C = \ell_1 \ldots \ell_n$, let $M(C,V,\phi)$ be the vector space $\bigoplus_{i=0}^{n-1} V_i$ with $V_i \cong V$ on which G acts according to the schema

$$\begin{tikzcd} V_0 \arrow[rrrrr, bend right, "\ell_n = \mathrm{id}"'] & V_1 \arrow[l, "\ell_1 = \phi"'] & V_2 \arrow[l, "\ell_2 = \mathrm{id}"'] & \cdot\ \cdot & V_{n-2} & V_{n-1} \arrow[l, "\ell_{n-1} = \mathrm{id}"'] \end{tikzcd}$$

where again x acts as "1 + a" and y acts as "1 + b" as above. It is clear that if C and C' represent the same element of $\mathcal{W}_2$ then $M(C,V,\phi) \cong M(C',V,\phi)$.

A module represents the quotient group G if and only if either

(i) the module is of the first kind and the corresponding word does not contain $(ab)^q$, $(ba)^q$ or their inverses,

(ii) the module is of the second kind and no power of the corresponding word contains $(ab)^q$, $(ba)^q$ or their inverses, or

(iii) the module is the projective indecomposable module $M((ab)^q(ba)^{-q},k,id)$ (of the second kind).

Ringel [16] also calculated which of the above modules are periodic. It turns out that a module of the first kind $M(C)$ is periodic if and only if $C \sim (ab)^{q-1}a$ or $(ba)^{q-1}b$, while all modules of the second kind are periodic.

4. THE VARIETIES FOR THE INDECOMPOSABLE kD_{2n}-MODULES

The following theorem gives the varieties for the indecomposable kG-modules in terms of the above classification.

Theorem 2 (i)
$$\bar{X}_G(M(C)) = \begin{cases} \mathbb{P}^1_a \cup \mathbb{P}^1_b & \text{if } C \sim a^{\pm 1} \dots b^{\pm 1} \\ \mathbb{P}^1_b & \text{if } C \sim a^{\pm 1} \dots a^{\pm 1} \\ & \quad \text{but } C \not\sim (ab)^{q-1}a \\ \mathbb{P}^1_a & \text{if } C \sim b^{\pm 1} \dots b^{\pm 1} \\ & \quad \text{but } C \not\sim (ba)^{q-1}b \\ \{0_b\} & \text{if } C \sim (ab)^{q-1}a \\ \{0_a\} & \text{if } C \sim (ba)^{q-1}b \end{cases}$$

(ii)
$$\bar{X}_G\left(M\left(C, \begin{pmatrix} \lambda & 1 & & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ 0 & & \cdot & 1 \\ & & & \lambda \end{pmatrix}\right)\right) = \begin{cases} \{\infty\} & \text{unless } C \sim a^{-1}b(ab)^{q-1}, \\ & \quad b^{-1}a(ba)^{q-1} \text{ or } (ab)^q(ba)^{-q} \\ \lambda_a & \text{if } C \sim a^{-1}b(ab)^{q-1} \\ \lambda_b & \text{if } C \sim b^{-1}a(ba)^{q-1} \end{cases}$$

(iii) $\bar{X}_G(M((ab)^q(ba)^{-q},k,id)) = \phi$.

We shall prove Theorem 2 by dealing with the various cases in separate lemmas. The following lemma deals with the first case of (i).

Lemma 1. If V is an indecomposable kG-module with $\dim(V)$ odd, then $\bar{X}_G(V) = \bar{X}_G$.

<u>Proof</u>. If $\dim(V)$ is odd, then for each shifted subgroup $\langle 1+\alpha\rangle$ of an elementary abelian subgroup E of G, $V{\downarrow}_{\langle 1+\alpha\rangle}$ is not free. Thus $Y_E(V{\downarrow}_E) = Y_E$, hence $X_E(V{\downarrow}_E) = X_E$, and so $X_G(V) = \varinjlim_E X_E(V{\downarrow}_E) = X_G$.

<u>Lemma 2</u>. A kG-module $M(C)$ of the first kind is periodic if and only if $C \sim (ab)^{q-1}a$ or $C \sim (ba)^{q-1}b$.

<u>Proof</u>. As mentioned at the end of section 3, this was proved by Ringel in [16].

<u>Lemma 3</u>. (i) $M(a^{\pm 1} \ldots a^{\pm 1}){\downarrow}_{\langle u\rangle}$ is free while $M(a^{\pm 1} \ldots a^{\pm 1}){\downarrow}_{\langle v\rangle}$ is not.

(ii) $M(b^{\pm 1} \ldots b^{\pm 1}){\downarrow}_{\langle v\rangle}$ is free while $M(b^{\pm 1} \ldots b^{\pm 1}){\downarrow}_{\langle u\rangle}$ is not.

<u>Proof</u>. This follows from the explicit description of the action of u and v on these modules given by the schema. Thus it can be seen, for example, that $M(a^{\pm 1} \ldots a^{\pm 1}){\downarrow}_{\langle v\rangle}$ has exactly two non-projective summands, corresponding to the basis elements occuring at the end of the schema.

<u>Lemma 4</u>. (i) $$\bar{X}_G(M(a^{\pm 1} \ldots a^{\pm 1})) = \begin{cases} \{0_b\} & \text{if } C \sim (ab)^{q-1}a \\ \mathbb{P}^1_b & \text{otherwise} \end{cases}$$

(ii) $$\bar{X}_G(M(b^{\pm 1} \ldots b^{\pm 1})) = \begin{cases} \{0_a\} & \text{if } C \sim (ba)^{q-1}b \\ \mathbb{P}^1_a & \text{otherwise.} \end{cases}$$

<u>Proof</u>. Carlson's connectedness theorem (part (iv) of theorem 1) states that if V is indecomposable then $\bar{X}_G(V)$ is connected in the Zariski topology. As explained in section 1, we may calculate $\bar{X}_G(V)$ by restrictions to cyclic subgroups $\langle 1+\alpha\rangle$ of kG. Thus it follows that $\bar{X}_G(M(a^{\pm 1} \ldots a^{\pm 1}))$ is a connected subvariety of $\bar{X}_G$ containing the point 0_b but not 0_a, by Lemma 3. If $C \sim (ab)^{q-1}a$ then by Lemma 2, $M(C)$ is periodic, and so by part (iii) of Theorem 1, $\bar{X}_G(M(C)) = \{0_b\}$. For all other choices of $C = a^{\pm 1} \ldots a^{\pm 1}$, $M(C)$ is not periodic, and so $\dim(\bar{X}_G(M(C))) = 1$. Thus $\bar{X}_G(M(C)) = \mathbb{P}^1_b$. Statement (ii) is proved in the same way.

We have now completed the proof of part (i) of Theorem 1, and so we turn to the non-projective modules of the second kind. According to section 8 of [16], these are all periodic of period 1 or 2, and so by Carlson's connectedness theorem their variety consists of a single point in each case.

Lemma 5. Suppose V is a non-projective indecomposable kG-module $M(C,\phi)$ of the second kind. Then $\bar{X}_G(V) = \{\infty\}$ unless $C \sim a^{-1}b(ab)^{q-1}$ or $C \sim b^{-1}a(ba)^{q-1}$, in which cases we have $\bar{X}_G(V) \subseteq \mathbb{P}^1_a\backslash\{\infty\}$, resp. $\bar{X}_G(V) \subseteq \mathbb{P}^1_b\backslash\{\infty\}$.

Proof. Suppose $\bar{X}_G(V) \neq \{\infty\}$. By the above remark, $\bar{X}_G(V)$ is either a point in $\mathbb{P}^1_a\backslash\{\infty\}$ or a point in $\mathbb{P}^1_b\backslash\{\infty\}$. Suppose without loss of generality that we are in the former case. Then $\bar{X}_{H_b}(V{\downarrow}_{H_b}) = \phi$, and so $\bar{X}_H(V{\downarrow}_H) = \phi$, where $H = \langle uvu,v\rangle$ of index two in G. Hence by part (iii) of Theorem 1, $V{\downarrow}_H$ is projective.

Now when dealing with modules for a p-group, we can distinguish projective modules from non-projective modules by the action of the norm element (i.e. the sum of all the group elements as an element of the group algebra). For the norm element acts as zero on all non-projective indecomposable modules. Thus the rank of the corresponding matrix in a given representation is at most the dimension divided by the order of the group, with equality if and only if the representation is projective.

The norm element of H is

$$n_H = [(1+uvu)(1+v)]^{q/2}.$$

Let $X = 1+u$ and $Y = 1+v$, so that $X^2 = Y^2 = 0$. Then

$$\begin{aligned} n_H &= [(Y + XY + YX + XYX)Y]^{q/2} \\ &= (YXY + XYXY)^{q/2} \\ &= (YX)^{q-1}Y + (XY)^q. \end{aligned}$$

Since V is a non-projective indecomposable kG-module, $(XY)^q$, which is the norm element of kG, acts as zero. Since $V{\downarrow}_H$ is a projective kH-module, n_H, which we now know to act in the same way as $(YX)^{q-1}Y$, acts as a matrix whose rank is $\dim(V)/|H|$. So the rank of $(YX)^{q-1}Y$ on V is

$\dim(V)/2q$. Looking at the description of how X and Y act on V according to the schema given in section 3, we see that this is impossible unless we have $C \sim [a^{-1}b(ab)^{q-1}]^r$ for some $r \geq 1$ (recall that $(ba)^q$ must not appear in any power of C). Now since a word in $\mathcal{W}'$ is not allowed to be a power of a word of smaller length, we have $r = 1$.

A similar argument shows that in the case where $\bar{X}_G(V)$ is a point in $\mathbb{P}^1_b\backslash\{\infty\}$, then $C \sim b^{-1}a(ba)^{q-1}$.

To complete the proof of Theorem 2, we must identify some modules of the form L_ζ , and use part (v) of Theorem 1.

Lemma 6 Let $\zeta = (\lambda x^2 + \mu y^2 + z)^r \in H^{2r}(G,k)$. Then

$$L_\zeta \cong M(a^{-1}b(ab)^{q-1}, \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}) \oplus M(b^{-1}a(ba)^{q-1}, \begin{pmatrix} \mu & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}),$$

where the matrices on the right hand side of this equation are $r \times r$ matrices.

Proof. It is easy to see by direct calculation or by looking at section 8 of [16] that $\Omega^{2r}(k) \cong M((b^{-1}a(ba)^{q-1})^{-r}(a^{-1}b(ab)^{q-1})^r)$, a module of dimension $4qr+1$. According to the schema, we have an ordered basis $z_0, \ldots, z_{4qr}$ corresponding to this word.

With respect to this basis, $y^{2(r-s)}z^s \in H^{2r}(G,k)$ corresponds to the homomorphism from $\Omega^{2r}(k)$ to k sending z_{2qs} to 1 and all other z_i to zero, while $x^{2(r-s)}z^s$ corresponds to the homomorphism sending $z_{2q(2r-s)}$ to 1 and all other z_i to zero. We shall show that

$$L_\zeta \cap \langle z_0, \ldots, z_{2qr}\rangle \cong M(b^{-1}a(ba)^{q-1}, \begin{pmatrix} \mu & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{pmatrix}), \text{ while}$$

$$L_\zeta \cap \langle z_{2qr}, \ldots, z_{4qr}\rangle \cong M(a^{-1}b(ab)^{q-1}, \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}).$$

In fact, we shall only show the latter, since the former is symmetrically identical. In terms of the schema for

$$M(a^{-1}b(ab)^{q-1}, \begin{pmatrix} \lambda & 1 & & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot\ 1 \\ 0 & & & \lambda \end{pmatrix}), \text{ we wish to take}$$

$V_i = \langle z_{2qr+i}, z_{2q(r+1)+i}, \ldots, z_{2q(2r-1)+i}\rangle$ for $1 \leqslant i \leqslant 2q-1$, and $V_0 = L_\zeta \cap \langle z_{2qr}, z_{2q(r+1)}, \ldots, z_{4qr}\rangle$, taking as basis for V_0 the images of $z_{2q(r+1)}, \ldots, z_{4qr}$ under the map $L_\zeta \cap \langle z_{2qr}, z_{2q(r+1)}, \ldots, z_{4qr}\rangle \xrightarrow{\cong} \langle z_{2qr}\rangle + (L_\zeta \cap \langle z_{2qr}, \ldots, z_{4qr}\rangle)/\langle z_{2qr}\rangle$. In terms of these bases, the map $\phi : V_1 \to V_0$ of the schema has as its matrix

$$\phi = \begin{pmatrix} r\lambda & \binom{r}{2}\lambda^2 & \ldots & \lambda^r \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \ldots 1 & 0 \end{pmatrix}$$

Some elementary linear algebra shows that this is conjugate to the matrix

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot\ 1 \\ 0 & & & \lambda \end{pmatrix}$$

Thus we have produced two submodules of L_ζ of the appropriate isomorphism types, which intersect in $\{0\}$ and span L_ζ. This completes the proof.

We may now complete the proof of theorem 2. It follows from lemma 6 and part (v) of Theorem 1 that

$$\bar{X}_G(M(a^{-1}b(ab)^{q-1}, \begin{pmatrix} \lambda & 1 & & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot\ 1 \\ 0 & & & \lambda \end{pmatrix}) \oplus M(b^{-1}a(ba)^{q-1}, \begin{pmatrix} \mu & 1 & & 0 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \cdot\ 1 \\ 0 & & & \mu \end{pmatrix}))$$

$= \{\lambda_a, \mu_b\}$. Comparing these statements for different values of λ and μ, and using part (i) of Theorem 1, we see that part (ii) of Theorem 2 holds. Part (iii) follows from part (iii) of Theorem (i).

Open Problem Calculate the ring structure of $A(D_{2^n})$. It would be interesting to understand the nilpotent elements in this ring.

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