Let $G$ be a finite group and $k$ a field of characteristic $p$ (possibly $p = 0$). Let $a(G) = a_k(G)$ be the Green Ring, or representation ring, formed from the finite-dimensional right $kG$-modules, and let $A(G) = A_k(G) = a_k(G) \otimes \mathbb{C}$. In [1] various concepts associated with $A(G)$ were introduced, including the notions of species, vertex, and origin. It is our purpose here to construct operations $\psi^n$ and $\lambda^n$ on $A(G)$, analogous to the Adams operators and exterior power operators in ordinary character theory (see, e.g., Curtis and Reiner [2, p. 313]). While the $\lambda$-operators are not in fact the same as the exterior power operators when $\text{char}(k) \neq 0$, they do make $A(G)$ into a special $\lambda$-ring (see [3, 4] for definitions and basic results about $\lambda$-rings). The $\psi$-operations will be used to construct the powers of a species (which are again species), thus generalizing the notion of the power maps in ordinary character theory. Finally, we examine the vertices and origins of the powers of a species (Theorem 2).

We begin by constructing the operators $\psi^n$ in the case where $n$ is coprime to $p$. Let $n$ be a natural number coprime to $p$, and $T = \langle \alpha : \alpha^n = 1 \rangle$ be a cyclic group of order $n$. Let $\varepsilon$ be a primitive $n$th root of unity in $k$, and $\eta$ a primitive $n$th root of unity in $\mathbb{C}$. If $X$ is a module for $T \times G$, then we denote by $X_{\varepsilon^i}$ the eigenspace of $\alpha$ on $X$ with eigenvalue $\varepsilon^i$. Then $X_{\varepsilon^i}$ is a $T \times G$-invariant direct summand of $X$, and we have

$$X = \bigoplus_{i=1}^n X_{\varepsilon^i},$$

$$A(T \times G) = A(T) \otimes A(G).$$


\[ \psi^n(V) = \sum_{i=1}^{n} \eta^i \left| \otimes^n(V) \right|_{\overline{\phi}}. \]  

(3)

as an element of \( A(G) \).

**Proposition 1.** If \( V_1 \) and \( V_2 \) are \( kC \)-modules then the following hold:

(i) \( \psi^n(V_1 \oplus V_2) = \psi^n(V_1) + \psi^n(V_2) \).

(ii) \( \psi^n(V_1 \otimes V_2) = \psi^n(V_1) \psi^n(V_2) \).

**Proof.** (i) As a module for \( G \), we have

\[ \otimes^n(V_1 \oplus V_2) = \bigoplus_{i_1=1,2, \ldots, i_n=1,2} (V_{i_1} \otimes \cdots \otimes V_{i_n}). \]

Under the action of \( T \), there are two fixed summands, \( \otimes^n(V_1) \) and \( \otimes^n(V_2) \). Apart from these, each orbit forms a module for \( T \times G \) of the form \( Y \otimes Z \), where \( Y \) is a permutation module for \( T \) on a proper subgroup. Thus as an element of \( A(G) \), \( \sum_{i=1}^{n} \eta^i \left| Y \otimes Z \right|_{\overline{\phi}} = 0 \). Hence the result.

(ii) \( \otimes^n(V_1 \otimes V_2) = \otimes^n(V_1) \otimes \otimes^n(V_2) \). Hence \( \left| \otimes^n(V_1 \otimes V_2) \right|_{\overline{\phi}} = \sum_{j=1}^{n} \left| \otimes^n(V_1) \right|_{\overline{\phi}} \left| \otimes^n(V_2) \right|_{\overline{\phi}} \). Thus we have

\[ \psi^n(V_1 \otimes V_2) = \sum_{i=1}^{n} \eta^i \left| \otimes^n(V_1 \otimes V_2) \right|_{\overline{\phi}} \]

\[ = \sum_{i,j=1}^{n} \eta^i \left| \otimes^n(V_1) \right|_{\overline{\phi}} \eta^j \left| \otimes^n(V_2) \right|_{\overline{\phi}} \]

\[ = \psi^n(V_1) \psi^n(V_2). \]

By Proposition 1, we may extend \( \psi^n \) linearly to give a ring endomorphism of \( A(G) \). In fact, the image under \( \psi^n \) of an element of \( a(G) \) is in \( a(G) \), as Proposition 2 shows.

**Proposition 2.** For \( d \) dividing \( n \), let \( \varepsilon_d \) be a primitive \( d \)th root of unity in \( k \). Then

\[ \psi^n(V) = \sum_{d \mid n} \mu(d) \left| \otimes^n(V) \right|_{\varepsilon_d}. \]

(Here, \( \mu \) is the Möbius function of multiplicative number theory.)
Proof. Whenever $\langle x^i \rangle = \langle x^i \rangle$, $|\otimes^n(V)|_{e^t} \cong |\otimes^n(V)|_{e^t}$. Thus

$$\psi^n(V) = \sum_{i=1}^n \eta^i |\otimes^n(V)|_{e^i} = \sum_{d|n} \left( \sum_{i,\delta = 1 \ldots d \wedge i < \delta} \epsilon_{\delta} \right) |\otimes^n(V)|_{e^d} = \sum_{d|n} \mu(d) |\otimes^n(V)|_{e^d}.$$

Considering $\psi^n$ as an endomorphism of $a(G)$ in this way, we have

**Proposition 3.** For $x \in a(G)$, $q$ a prime not equal to $p$, we have

$$\psi^q(x) \equiv x^{q^2} \pmod{q}.$$

Proof. By Proposition 2, we have

$$\psi^q(x) = [x^{q^2}]_1 - [x^{q^2}]_{e_q}.$$

Since $x^{q^2} = [x^{q^2}]_1 + (q-1)[x^{q^2}]_{e_q} + q(q-1)[x^{q^2}]_{e_{q^2}} + \cdots + q^{q-1}(q-1)[x^{q^2}]_{e_{q^r}}$, the result follows.

**Example.** If $p \neq 2$, we have

$$\psi^2(V) = S^2(V) - A^2(V).$$

Thus in particular, if $V$ is irreducible then the Frobenius–Schur indicator is defined by

$$\text{Ind}(V) = (1, \psi^2(V)) = +1 \text{ if } V \text{ is orthogonal,}$$

$$= -1 \text{ if } V \text{ is symplectic,}$$

$$= 0 \text{ otherwise.}$$

(Recall that $(\ , \ )$ is the inner product on $A(G)$ given by linearly extending $(M, N) = \dim_k \text{Hom}_{kG}(M, N)$.)

**Definition.** We define the $n$th power of a species $s$ of $A(G)$, for $n \in \mathbb{N} \setminus p\mathbb{N}$, via

$$(s^n, x) = (s, \psi^n(x)).$$

Proposition 1 shows that $s^n$ is again a species of $A(G)$.

**Proposition 4.** If $b$ is a Brauer species of $A(G)$ (see [1, 6.12])

**corresponding to a $p'$-element $g$, then $h^n$ is the Brauer species corresponding to $g^n$.**
Proof: Let $V$ be a $kG$-module, and let $b'$ be the Brauer species corresponding to $g^n$. We may choose a basis $v_1, \ldots, v_r$ of $V$ consisting of eigenvectors of $g$. Let $v_i g = \lambda_i v_i$. Then as $k\langle g \rangle$-modules, $V = \oplus \langle v_i \rangle$, and so 

\[(b^n, V) = (b, \psi^n(V)) = \left( b, \sum_{i=1}^{r} \psi^n(\langle v_i \rangle) \right) = \sum_{i=1}^{r} (b, \psi^n(\langle v_i \rangle)) = \sum_{i=1}^{r} \lambda_i^n = (b', V). \]

We now wish to prove that $\psi^m \psi^n = \psi^{mn}$. We start off with a lemma.

**Lemma 1.** Let $S_n$ denote the symmetric group on $n$ letters. Then there is a subgroup $T_n$ of $S_n$ having the properties:

- (i) $T_n$ contains a cyclic group of order $n$ which is transitive on the $n$ letters.
- (ii) If $n = n_1 n_2$ then $T_n$ contains the direct product of the cyclic groups of orders $n_1$ and $n_2$, in its direct product action on the $n$ letters.
- (iii) If a prime $q$ divides $\lvert T_n \rvert$ then $q$ also divides $n$.

**Proof.** Let $n = \prod p_i^{\beta_i}$. Then we have a subgroup 

\[ \prod S_{p_i^{\beta_i}} \leq S_n, \]

with direct product action on the $n$ points. Let $P_i$ be a Sylow $p_i$-subgroup of $S_{p_i^{\beta_i}}$, and let

\[ T_n = \prod P_i \leq \prod S_{p_i^{\beta_i}}. \]

Then properties (i) and (iii) are clearly satisfied. To check property (ii), let $n = n_1 n_2$ with $n_1 = \prod p_i^{\beta_i}$, $n_2 = \prod p_i^{\gamma_i}$, and $\beta_i + \gamma_i = \alpha_i$. Let $Q_i \times R_i$ denote a Sylow $p$-subgroup of $S_{p_i^{\beta_i}} \times S_{p_i^{\gamma_i}} \leq S_{p_i^{\alpha_i}}$, with $Q_i \times R_i \leq P_i$. Then $\prod Q_i \times \prod R_i \leq S_{n_1} \times S_{n_2}$ contains the appropriate direct product of cyclic groups.

**Theorem 1.** 

$\psi^m \psi^n = \psi^{mn}$.

**Proof.** Without loss of generality, we may assume that $k$ is a splitting field for $T_{mn}$. Thus by property (iii), $p$ does not divide $\lvert T_{mn} \rvert$, and so the central idempotents for $kT_{mn}$ are in natural one-to-one correspondence with those for $CT_{mn}$, and $kT_{mn}$ is semisimple.

By properties (i) and (ii) of $T_{mn}$, and the definition of the $\psi$ operators, $\psi^m \psi^n(V)$ and $\psi^{mn}(V)$ are of the form $\sum \lambda_i (\otimes^{\alpha_i}(V) \cdot e_i)$ and
\[ \sum \lambda'(\otimes^m(V) \cdot e_i), \] where the \( e_i \) are the primitive central idempotents for \( T_{mn} \), and the \( \lambda \) and \( \lambda' \) are independent of \( V \). Moreover, the \( \lambda \) and \( \lambda' \) may both be expressed in terms of induced characters from the subgroups of \( T_{mn} \) given in the definition, and hence if we keep \( m \) and \( n \) constant and vary \( p \) over primes not dividing \( mn \), the \( \lambda \) and \( \lambda' \) do not vary. Thus it is sufficient to prove the result in the case where \( p \) divides neither \( mn \) nor \( |G| \). In this case, every species is a Brauer species, and modules are characterized by the values of Brauer species. By Proposition 4, we have

\[
(b, \psi^m\psi^n(V)) = (b^m, \psi^n(V))
\]

\[
= ((b^m)^n, V)
\]

\[
= (b^{mn}, V)
\]

\[
= (b, \psi^{mn}(V)).
\]

Thus the \( \lambda \) and \( \lambda' \) are equal, and the result is proved.

**Corollary.** The \( \psi^n \) commute with each other.

We now extend the definition of \( \psi^n \) to include all \( n \in \mathbb{N} \). Let \( F \) denote the Frobenius map on \( a(G) \) or \( A(G) \). Thus if \( V \) is a module, \( F(V) \) is the module with the same addition and the same group action, but with scalar multiplication defined by first raising the field element to the \( p \)th power, and then applying the old scalar multiplication. The map \( F \) commutes with \( \psi^n \) for \( n \) coprime to \( p \), and so we may define for any \( n \in \mathbb{N} \) with \( n = p^a \cdot n_0 \) and \( n_0 \) coprime to \( p \),

\[
\psi^n(V) = F^a\psi^{n_0}(V).
\]

It is easy to check that Propositions 1 and 4, and Theorem 1 remain valid with this definition.

Now we are in a position to define the \( \lambda \)-operations. Unfortunately, these are only defined on \( A(G) \), and not on \( a(G) \), as the example following the definition shows.

**Definition.**

\[
\lambda^n(x) = \frac{1}{n!} \begin{vmatrix}
\psi^1(x) & 1 \\
\psi^2(x) & \psi^1(x) & 2 \\
\psi^3(x) & \psi^2(x) & \psi^1(x) & 3 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\psi^n(x) & \ldots & \ldots & \ldots & \psi^1(x)
\end{vmatrix}
\]
It follows from [3, pp. 49, 54], Proposition 1, and Theorem 1, that with these \(\lambda\)-operations, \(A(G)\) is a special \(\lambda\)-ring. In particular, if we let

\[
\lambda_n(x) = \sum_{n=0}^{\infty} t^n \lambda_n(x) \in A(G)[[t]],
\]

then

\[
\frac{d}{dt} \log \lambda_n(x) = \sum_{n=1}^{\infty} (-1)^n \psi^{n-1}(x) t^n.
\]

**Example.** For \(n < p\), \(\lambda^n(x)\) is just the exterior \(n\)th power of \(x\). However, for example, when \(n = p = 2\), we have

\[
\lambda^2(x) = \frac{1}{2}(x^2 - F^2(x)).
\]

**Proposition 5.** The subring \(a(G) \otimes \mathbb{Z}(1/p)\) of \(A(G)\) is closed under the \(\lambda\)-operations, and is hence a special \(\lambda\)-ring in its own right.

**Proof:** This follows from Proposition 3, Theorem 1 and [5, Prop. 1.2].

Next, we examine the effect of \(\psi^n\) on origins and vertices of species. The notation is as in [1, Sect. 7].

**Definition.** If \(H\) is a \(p\)-hypoelementary group and \(n = p^m \cdot n_0\) with \(n_0\) coprime to \(p\), we let \(H^{n_0}\) denote the unique subgroup of index \((|H|, n_0)\) in \(H\).

Let \(s_{H,b}\) and \(e_{H,b}\) be as in [1, Sect. 7].

**Lemma 2.** \((s_{H,b})^n = s_{H[n],b^n}\).

**Proof:** Let \(V\) be a trivial source \(kG\)-module and let \(V_H = W_1 \oplus W_2\), where \(W_1\) is a direct sum of modules with vertex \(O_p(H)\) and \(W_2 \in A'(G, H)\). Then by Proposition 1, \(\psi^n(V) \downarrow_H = \psi^n(V_H) = \psi^n(W_1) + \psi^n(W_2)\). \(\psi^n(W_1)\) is a linear combination of trivial source modules with vertex \(O_p(H)\) and \(\psi^n(W_2) \in A'(G, H)\). Thus

\[
((s_{H,b})^n, V) = (s_{H,b}, \psi^n(V)) = (b, \psi^n(W_1)) = (b^n, W_1) = (s_{H[n],b^n}, V).
\]

**Lemma 3.**

\[
\psi^n(e_{H,b}) = \sum e_{H',b'},
\]

where the sum runs over one representative of each \(G\)-conjugacy class of pairs \((H', b')\) with \((H')^{n_0} = H\) and \((b')^n = b\).
Proof:

\[ \psi^n(e_{H,b}) = \sum_{(H',b')} (s_{H',b'}, \psi^n(e_{H',b})) e_{H',b'} \]

(Here, the sum runs over one representative of each \( G \)-conjugacy class of pairs \((H', b')\).) Hence

\[ \psi^n(e_{H,b}) = \sum_{(H',b')} (s_{(H')^n,(b')^n}, e_{H',b'}) e_{H',b'} \]

by Lemma 2.

Thus the coefficient of \( e_{H',b'} \) is one if \(((H')^n,(b')^n)\) is \( G \)-conjugate to \((H, b)\), and zero otherwise.  

**Theorem 2.**

(i) If \( H \) is an origin of \( s \), then \( H^{(n)} \) is an origin of \( s^n \).

(ii) If \( D \) is a vertex of \( s \), then \( D \) is also a vertex of \( s^n \).

**Proof.**

(i) If \( H \) is an origin of \( s \), then for some Brauer species \( b \) of \( H \), \( (s, e_{H,b}) = 1 \). Thus by Lemma 3,

\[ (s^n, e_{H^{(n)},b^n}) = (s, \psi^n(e_{H^{(n)},b^n})) = 1. \]

Hence \( H^{(n)} \) is an origin for \( s^n \).

(ii) By [1, Theorem 7.8], we may take \( D = O_p(H) \), and the result follows from (i).

**Example.** The representation table for the cyclic group of order 3 is

<table>
<thead>
<tr>
<th></th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Applying the operator \( \psi^2 \), we have

\[ s_1^2 = s_1, s_2^2 = s_3^2 = s_2, \]

whereas for any prime \( q \neq 2 \), we have

\[ s_i^q = s_i, i = 1, 2, 3. \]

In general what is the role played by the primes \( q \) dividing \( p - 1 \) in this theory?
REFERENCES