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Lambda and Psi Operations on Green Rings

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Let G be a finite group and k a field of characteristic p (possibly $p=0$). Let $a(G)=a_k(G)$ be the Green Ring, or representation ring, formed from the finite-dimensional right kG-modules, and let $A(G)=A_k(G)=a_k(G)\otimes_{\mathbb{Z}}\mathbb{C}$. In [1] various concepts associated with $A(G)$ were introduced, including the notions of species, vertex, and origin. It is our purpose here to construct operations ψ^n and λ^n on $A(G)$, analogous to the Adams operators and exterior power operators in ordinary character theory (see, e.g., Curtis and Reiner [2, p. 313]). While the λ-operators are not in fact the same as the exterior power operators when $\operatorname{char}(k)\neq 0$, they do make $A(G)$ into a special λ-ring (see [3, 4] for definitions and basic results about λ-rings). The ψ-operations will be used to construct the powers of a species (which are again species), thus generalizing the notion of the power maps in ordinary character theory. Finally, we examine the vertices and origins of the powers of a species (Theorem 2).

We begin by constructing the operators ψ^n in the case where n is coprime to p. Let n be a natural number coprime to p, and $T=\langle\alpha : \alpha^n=1\rangle$ be a cyclic group of order n. Let ε be a primitive nth root of unity in k, and η a primitive nth root of unity in $\mathbb{C}$. If X is a module for $T\times G$, then we denote by X_{ε^i} the eigenspace of α on X with eigenvalue ε^i. Then X_{ε^i} is a $T\times G$-invariant direct summand of X, and we have

$$X=\bigoplus_{i=1}^{n} X_{\varepsilon^i}, \tag{1}$$

$$A(T\times G)=A(T)\bigotimes_{\mathbb{C}} A(G). \tag{2}$$

Given a module V for G, we let T act on $\otimes^n(V)$ by permuting the tensor multiplicands. Then $\otimes^n(V)$ is a module for $T\times G$, and $[\otimes^n(V)]_{\varepsilon^i}\cong[\otimes^n(V)]_{\varepsilon^j}$ as G-modules whenever $\langle\alpha^i\rangle=\langle\alpha^j\rangle$. We define

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$$\psi^n(V) = \sum_{i=1}^{n} \eta^i |\otimes^n(V)|_{\varepsilon^i}, \tag{3}$$

as an element of $A(G)$.

PROPOSITION 1. *If V_1 and V_2 are kC-modules then the following hold*:

(i) $\psi^n(V_1 \oplus V_2) = \psi^n(V_1) + \psi^n(V_2)$.

(ii) $\psi^n(V_1 \otimes V_2) = \psi^n(V_1)\, \psi^n(V_2)$.

Proof. (i) As a module for G, we have

$$\otimes^n(V_1 \oplus V_2) = \bigoplus_{\substack{i_1 = 1,2 \\ \vdots \\ i_n = 1,2}} (V_{i_1} \otimes \cdots \otimes V_{i_n}).$$

Under the action of T, there are two fixed summands, $\otimes^n(V_1)$ and $\otimes^n(V_2)$. Apart from these, each orbit forms a module for $T \times G$ of the form $Y \otimes Z$, where Y is a permutation module for T on a proper subgroup. Thus as an element of $A(G)$, $\sum_{i=1}^{n} \eta^i [Y \otimes Z]_{\varepsilon^i} = 0$. Hence the result.

(ii) $\otimes^n(V_1 \otimes V_2) = \otimes^n(V_1) \otimes \otimes^n(V_2)$. Hence $|\otimes^n(V_1 \otimes V_2)|_{\varepsilon^i} = \sum_{j=1}^{n} |\otimes^n(V_1)|_{\varepsilon^j} |\otimes^n(V_2)|_{\varepsilon^{i-j}}$. Thus we have

$$\begin{aligned} \psi^n(V_1 \otimes V_2) &= \sum_{i=1}^{n} \eta^i |\otimes^n(V_1 \otimes V_2)|_{\varepsilon^i} \\ &= \sum_{i,j=1}^{n} \eta^j |\otimes^n(V_1)|_{\varepsilon^j}\, \eta^{i-j} |\otimes^n(V_2)|_{\varepsilon^{i-j}} \\ &= \psi^n(V_1)\, \psi^n(V_2). \quad \blacksquare \end{aligned}$$

By Proposition 1, we may extend ψ^n linearly to give a ring endomorphism of $A(G)$. In fact, the image under ψ^n of an element of $a(G)$ is in $a(G)$, as Proposition 2 shows.

PROPOSITION 2. *For d dividing n, let ε_d be a primitive* dth *root of unity in k. Then*

$$\psi^n(V) = \sum_{d \mid n} \mu(d) |\otimes^n(V)|_{\varepsilon_d}.$$

(*Here, μ is the Möbius function of multiplicative number theory.*)

Proof. Whenever $\langle\alpha^i\rangle = \langle\alpha^j\rangle$, $[\otimes^n(V)]_{\varepsilon^i} \cong [\otimes^n(V)]_{\varepsilon^j}$. Thus

$$\begin{aligned}\psi^n(V) &= \sum_{i=1}^{n} \eta^i [\otimes^n(V)]_{\varepsilon^i}\\ &= \sum_{d\mid n} \left(\sum_{\substack{(i,d)=1\\ 1\leqslant i\leqslant d}} (\varepsilon_d)^i \right) [\otimes^n(V)]_{\varepsilon_d}\\ &= \sum_{d\mid n} \mu(d)[\otimes^n(V)]_{\varepsilon_d}. \quad \blacksquare\end{aligned}$$

Considering ψ^n as an endomorphism of $a(G)$ in this way, we have

PROPOSITION 3. *For $x \in a(G)$, q a prime not equal to p, we have*

$$\psi^{q^t}(x) \equiv x^{q^t} \qquad (\text{mod } q).$$

Proof. By Proposition 2, we have

$$\psi^{q^t}(x) = [x^{q^t}]_1 - [x^{q^t}]_{\varepsilon_q}.$$

Since $x^{q^t} = [x^{q^t}]_1 + (q-1)[x^{q^t}]_{\varepsilon_q} + q(q-1)[x^{q^t}]_{\varepsilon_{q^2}} + \cdots + q^{t-1}(q-1)[x^{q^t}]_{\varepsilon_{q^t}}$, the result follows. $\blacksquare$

EXAMPLE. If $p \neq 2$, we have

$$\psi^2(V) = S^2(V) - \Lambda^2(V).$$

Thus in particular, if V is irreducible then the Frobenius–Schur indicator is defined by

$$\begin{aligned}\operatorname{Ind}(V) = (1, \psi^2(V)) &= +1 \quad \text{if } V \text{ is orthogonal,}\\ &= -1 \quad \text{if } V \text{ is symplectic,}\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

(Recall that $(\ ,\ )$ is the inner product on $A(G)$ given by linearly extending $(M, N) = \dim_k \operatorname{Hom}_{kG}(M, N)$.)

DEFINITION. We define the nth power of a species s of $A(G)$, for $n \in \mathbb{N}\setminus p\mathbb{N}$, via

$$(s^n, x) = (s, \psi^n(x)).$$

Proposition 1 shows that s^n is again a species of $A(G)$.

PROPOSITION 4. *If b is a Brauer species of $A(G)$ (see [1, 6.12]) corresponding to a p'-element g, then b^n is the Brauer species corresponding to g^n.*

Proof. Let V be a kG-module, and let b' be the Brauer species corresponding to g^n. We may choose a basis $v_1,\dots,v_r$ of V consisting of eigenvectors of g. Let $v_i g = \lambda_i v_i$. Then as $k\langle g\rangle$-modules, $V = \oplus\langle v_i\rangle$, and so

$$(b^n, V) = (b, \psi^n(V)) = \left(b, \sum_{i=1}^{r} \psi^n(\langle v_i\rangle)\right) = \sum_{i=1}^{r} (b, \psi^n(\langle v_i\rangle))$$

$$= \sum_{i=1}^{r} \lambda_i^n = (b', V). \quad \blacksquare$$

We now wish to prove that $\psi^m\psi^n = \psi^{mn}$. We start off with a lemma.

LEMMA 1. *Let S_n denote the symmetric group on n letters. Then there is a subgroup T_n of S_n having the properties:*

(i) *T_n contains a cyclic group of order n which is transitive on the n letters.*

(ii) *if $n = n_1 n_2$ then T_n contains the direct product of the cyclic groups of orders n_1 and n_2, in its direct product action on the n letters.*

(iii) *If a prime q divides $|T_n|$ then q also divides n.*

Proof. Let $n = \prod p_i^{\alpha_i}$. Then we have a subgroup

$$\prod S_{p_i^{\alpha_i}} \leqslant S_n,$$

with direct product action on the n points. Let P_i be a Sylow p_i-subgroup of $S_{p_i^{\alpha_i}}$, and let

$$T_n = \prod P_i \leqslant \prod S_{p_i^{\alpha_i}}.$$

Then properties (i) and (iii) are clearly satisfied. To check property (ii), let $n = n_1 n_2$ with $n_1 = \prod p_i^{\beta_i}$, $n_2 = \prod p_j^{\gamma_i}$, and $\beta_i + \gamma_i = \alpha_i$. Let $Q_i \times R_i$ denote a Sylow p-subgroup of $S_{p_i^{\beta_i}} \times S_{p_i^{\gamma_i}} \leqslant S_{p_i^{\alpha_i}}$, with $Q_i \times R_i \leqslant P_i$. Then $\prod Q_i \times \prod R_i \leqslant S_{n_1} \times S_{n_2}$ contains the appropriate direct product of cyclic groups. $\blacksquare$

THEOREM 1.

$$\psi^m\psi^n = \psi^{mn}.$$

Proof. Without loss of generality, we may assume that k is a splitting field for T_{mn}. Thus by property (iii), p does not divide $|T_{mn}|$, and so the central idempotents for kT_{mn} are in natural one–one correspondence with those for $\mathbb{C}T_{mn}$, and kT_{mn} is semisimple.

By properties (i) and (ii) of T_{mn}, and the definition of the ψ operators, $\psi^m\psi^n(V)$ and $\psi^{mn}(V)$ are of the form $\sum \lambda_i(\otimes^{mn}(V)\cdot e_i)$ and

$\sum \lambda_i'(\otimes^{mn}(V) \cdot e_i)$, where the e_i are the primitive central idempotents for T_{mn}, and the λ_i and λ_i' are independent of V. Moreover, the λ_i and λ_i' may both be expressed in terms of induced characters from the subgroups of T_{mn} given in the definition, and hence if we keep m and n constant and vary p over primes not dividing mn, the λ_i and λ_i' do not vary. Thus it is sufficient to prove the result in the case where p divides neither mn nor $|G|$. In this case, every species is a Brauer species, and modules are characterized by the values of Brauer species. By Proposition 4, we have

$$\begin{aligned}(b, \psi^m\psi^n(V)) &= (b^m, \psi^n(V)) \\ &= ((b^m)^n, V) \\ &= (b^{mn}, V) \\ &= (b, \psi^{mn}(V)).\end{aligned}$$

Thus the λ_i and λ_i' are equal, and the result is proved. ▮

COROLLARY. *The ψ^n commute with each other.*

We now extend the definition of ψ^n to include all $n \in \mathbb{N}$. Let F denote the Frobenius map on $a(G)$ or $A(G)$. Thus if V is a module, $F(V)$ is the module with the same addition and the same group action, but with scalar multiplication defined by first raising the field element to the pth power, and then applying the old scalar multiplication. The map F commutes with ψ^n for n coprime to p, and so we may define for any $n \in \mathbb{N}$ with $n = p^a \cdot n_0$ and n_0 coprime to p,

$$\psi^n(V) = F^a\psi^{n_0}(V).$$

It is easy to check that Propositions 1 and 4, and Theorem 1 remain valid with this definition.

Now we are in a position to define the λ-operations. Unfortunately, these are only defined on $A(G)$, and not on $a(G)$, as the example following the definition shows.

DEFINITION.

$$\lambda^n(x) = \frac{1}{n!}\begin{vmatrix} \psi^1(x) & 1 & & & & 0 \\ \psi^2(x) & \psi^1(x) & 2 & & & \\ \psi^3(x) & \psi^2(x) & \psi^1(x) & 3 & & \\ \vdots & & & & \ddots & \\ & & & & & n-1 \\ \psi^n(x) & & \cdots & & & \psi^1(x) \end{vmatrix}$$

It follows from [3, pp. 49, 54], Proposition 1, and Theorem 1, that with these λ-operations, $A(G)$ is a special λ-ring. In particular, if we let

$$\lambda_t(x) = \sum_{n=0}^{\infty} t^n \lambda^n(x) \in A(G)[[t]],$$

then

$$\frac{d}{dt} \log \lambda_t(x) = \sum_{n=1}^{\infty} (-1)^n \psi^{n+1}(x)\, t^n.$$

EXAMPLE. For $n < p$, $\lambda^n(x)$ is just the exterior nth power of x. However, for example, when $n = p = 2$, we have

$$\lambda^2(x) = \tfrac{1}{2}(x^2 - F^2(x)).$$

PROPOSITION 5. *The subring $a(G) \otimes_{\mathbb{Z}} \mathbb{Z}(1/p)$ of $A(G)$ is closed under the λ-operations, and is hence a special λ-ring in its own right.*

Proof. This follows from Proposition 3, Theorem 1 and [5, Prop. 1.2]. ∎

Next, we examine the effect of ψ^n on origins and vertices of species. The notation is as in [1, Sect. 7].

DEFINITION. If H is a p-hypoelementary group and $n = p^a \cdot n_0$ with n_0 coprime to p, we let $H^{[n]}$ denote the unique subgroup of index $(|H|, n_0)$ in H.

Let $s_{H,b}$ and $e_{H,b}$ be as in [1, Sect. 7].

LEMMA 2. $(s_{H,b})^n = s_{H^{[n]},b^n}$.

Proof. Let V be a trivial source kG-module and let $V\downarrow_H = W_1 \oplus W_2$, where W_1 is a direct sum of modules with vertex $O_p(H)$ and $W_2 \in A'(G, H)$. Then by Proposition 1, $\psi^n(V)\downarrow_H = \psi^n(V\downarrow_H) = \psi^n(W_1) + \psi^n(W_2)$, $\psi^n(W_1)$ is a linear combination of trivial source modules with vertex $O_p(H)$, and $\psi^n(W_2) \in A'(G, H)$. Thus

$$\begin{aligned} ((s_{H,b})^n, V) &= (s_{H,b}, \psi^n(V)) \\ &= (b, \psi^n(W_1)) \\ &= (b^n, W_1) \\ &= (s_{H^{[n]},b^n}, V). \quad ∎ \end{aligned}$$

LEMMA 3.

$$\psi^n(e_{H,b}) = \sum e_{H',b'},$$

where the sum runs over one representative of each G-conjugacy class of pairs (H', b') with $(H')^{[n]} = H$ and $(b')^n = b$.

Proof.

$$\psi^n(e_{H,b}) = \sum_{\substack{\text{all} \\ (H',b')}} (s_{H',b'}, \psi^n(e_{H,b}))\, e_{H',b'}.$$

(Here, the sum runs over one representative of each G-conjugacy class of pairs (H', b').) Hence

$$\psi^n(e_{H,b}) = \sum (s_{(H')^{[n]},(b')^n}, e_{H,b})\, e_{H',b'},$$

by Lemma 2.

Thus the coefficient of $e_{H',b'}$ is one if $((H')^{[n]}, (b')^n)$ is G-conjugate to (H, b), and zero otherwise. ∎

THEOREM 2. (i) *If H is an origin of s, then $H^{[n]}$ is an origin of s^n.*

(ii) *If D is a vertex of s, then D is also a vertex of s^n.*

Proof. (i) If H is an origin of s, then for some Brauer species b of H, $(s, e_{H,b}) = 1$. Thus by Lemma 3,

$$(s^n, e_{H^{[n]},b^n}) = (s, \psi^n(e_{H^{[n]},b^n})) = 1.$$

Hence $H^{[n]}$ is an origin for s^n.

(ii) By [1, Theorem 7.8], we may take $D = O_p(H)$, and the result follows from (i). ∎

EXAMPLE. The representation table for the cyclic group of order 3 is

s_1	s_2	s_3
1	1	1
2	−1	1
3	0	0

Applying the operator ψ^2, we have

$$s_1^2 = s_1,\ s_2^2 = s_3^2 = s_2,$$

whereas for any prime $q \neq 2$, we have

$$s_i^q = s_i,\ i = 1, 2, 3.$$

In general what is the role played by the primes q dividing $p - 1$ in this theory?

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