Commutative Banach Algebras and Modular Representation Theory

The sun sets sail, by Rob Gonsalves

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Abstract. In a recent paper of Benson and Symonds, a new invariant was introduced for modular representations of a finite group. An interpretation was given as a spectral radius with respect to a Banach algebra completion of the representation ring. Our purpose here is to take these notions further, and investigate the structure of the resulting Banach algebras. Some of the material in that paper is repeated here in greater generality, and for clarity of exposition.

We give an axiomatic definition of an abstract representation ring, and representation ideal. The completion is then a commutative Banach algebra, and the techniques of Gelfand from the 1940s are applied in order to study the space of algebra homomorphisms to $\mathbb{C}$. One surprising consequence of this investigation is that the Jacobson radical and the nil radical of a (complexified) representation ring always coincide.

These notes are intended for representation theorists. So background material on commutative Banach algebras is given in detail, whereas representation theoretic background is more condensed.

Contents

Preface

The purpose of these notes is to study the asymptotics of the direct sum decomposition of tensor products and tensor powers of finite dimensional representations of a finite group in prime characteristic. This takes us into the world of commutative Banach algebras and their spectral theory.

This study grew out of the author's joint work with Symonds [**13**]. In the introduction to that paper, the example was given of the two dimensional indecomposable module M for the cyclic group of order five in characteristic five. In this example, although the dimension of $M^{\otimes n}$ is 2^n, the dimension of the non-projective part of $M^{\otimes n}$ is roughly τ^n, where τ is the golden ratio (see Section 5.1). This led us to define $\gamma(M)$ to be τ in this example, and to interpret it as the reciprocal of the radius of convergence of the corresponding generating function. This, in turn, is interpreted as the spectral radius of $[M]$ as an element of the representation ring, with respect to a suitable norm.

Let $a(G)$ be the representation ring, or Green ring of a finite group G over a field k of characteristic p. This has a free $\mathbb{Z}$-basis consisting of the isomorphism classes of finitely generated indecomposable kG-modules, with addition and multiplication coming from direct sum and tensor product.

The complexification $a_{\mathbb{C}}(G) = \mathbb{C}\otimes_{\mathbb{Z}}a(G)$ is a commutative normed algebra, with the norm coming from dimension. Its completion $\hat{a}(G)$ is a commutative Banach algebra which forms our basic object of study. We look at various quotients of the form $\hat{a}_{\mathfrak{X}}(G) = \hat{a}(G)/\hat{a}(G,\mathfrak{X})$ where $\mathfrak{X}$ is an ideal of indecomposable kG-modules, and examine invariants of modules coming from spectral radius in these quotients. The purpose of this is to obtain information about the asymptotic behaviour of the tensor powers of a finitely generated kG-module.

With the aim of being applicable in a wider set of circumstances, we have formulated the definitions and main theorems in terms of abstract representation rings. The axioms are set up in Definition 1.1.1. In particular, a representation ring $\mathfrak{a}$ comes equipped with a free $\mathbb{Z}$-basis $\{x_i \mid i \in \mathfrak{I}\}$ as an additive group, which is supposed to be thought of as the basis of indecomposable modules in the case of $a(G)$. One of the unexpected consequences of studying the completion of $\mathfrak{a}$ in this generality is the proof in Theorem 3.6.2 that the Jacobson radical and the nilradical of the complexification $\mathfrak{a}_{\mathbb{C}}$ coincide.

Theorem. *If $\mathfrak{a}$ is a representation ring then the Jacobson radical and the nil radical of $\mathfrak{a}_{\mathbb{C}}$ are equal.*

If x is a positive element of a representation ring $\mathfrak{a}$, and $\mathfrak{X} \subset \mathfrak{I}$ is a representation ideal, let $\mathsf{c}^{\mathfrak{X}}_n(x)$ be the dimension of the $\mathfrak{X}$-core of x^n (these concepts are defined in Definitions 1.2.1

and 1.3.2). Then in Section 1.4 we define

$$\gamma_{\mathfrak{X}}(x) = \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}.$$

The following theorem, which to some extent parallels Theorem 1.2 of [**13**], summarises some of the properties of this invariant.

THEOREM. *The invariant $\gamma_{\mathfrak{X}}(x)$ has the following properties:*

(i) *We have* $\gamma_{\mathfrak{X}}(x) = \lim_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)} = \inf_{n\geqslant 1} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}$.
(ii) *We have* $0 \leqslant \gamma_{\mathfrak{X}}(x) \leqslant \dim x$.
(iii) *We have* $\gamma_{\mathfrak{X}}(x) = \dim x$ *if and only if* $\mathsf{core}_{\mathfrak{X}}(x^n) = x^n$ *for all* $n \geqslant 0$.
(iv) *We have* $\gamma_{\mathfrak{X}}(x) = 0$ *if and only if* $x \in \langle\mathfrak{X}\rangle$, *otherwise* $\gamma_{\mathfrak{X}}(x) \geqslant 1$.
(v) *If* $1 \leqslant \gamma_{\mathfrak{X}}(x) < \sqrt{2}$ *then* x *is* $\mathfrak{X}$*-endotrivial.*
(vi) *If* x *is not* $\mathfrak{X}$*-endotrivial and* $\gamma_{\mathfrak{X}}(x) = \sqrt{2}$ *then* $xx^*x \equiv 2x \pmod{\langle\mathfrak{X}\rangle}$.
(vii) *We have* $\gamma_{\mathfrak{X}}(x^*) = \gamma_{\mathfrak{X}}(x)$.
(viii) *We have* $\max\{\gamma_{\mathfrak{X}}(x), \gamma_{\mathfrak{X}}(y)\} \leqslant \gamma_{\mathfrak{X}}(x+y) \leqslant \gamma_{\mathfrak{X}}(x) + \gamma_{\mathfrak{X}}(y)$.
(ix) *If* $a, b \geqslant 0$ *we have* $\gamma_{\mathfrak{X}}(a + bx) = a + b\gamma_{\mathfrak{X}}(x)$.
(x) *We have* $\gamma_{\mathfrak{X}}(xy) \leqslant \gamma_{\mathfrak{X}}(x)\gamma_{\mathfrak{X}}(y)$.
(xi) *We have* $\gamma_{\mathfrak{X}}(x^m) = \gamma_{\mathfrak{X}}(x)^m$.
(xii) *If* $\mathfrak{Y} \subseteq \mathfrak{X}$ *we have* $\gamma_{\mathfrak{X}}(x) \leqslant \gamma_{\mathfrak{Y}}(x)$.

The proofs of the various parts of this theorem may be found in the following places: part (i) in Theorem 1.6.4, (ii) in Lemma 1.4.3, (iii) in Lemma 1.4.9, (iv) in Lemma 1.4.10, (v) and (vi) in Theorem 1.8.1, (vii) in Lemma 1.4.5, (viii) in Theorem 1.6.6, (ix) in Theorem 1.6.8, (x) in Lemma 1.4.7, (xi) in Lemma 1.4.8, and (xii) in Lemma 1.4.12.

In the case where $\mathfrak{a} = a(G)$ and $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, which was the case studied in [**13**], we can say more.

THEOREM. *Let G be a finite group and k a field of characteristic p. Let $\mathfrak{a} = a(G)$ be the representation ring of kG-modules, and $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$ be the representation ideal of projective modules. Then the following properties hold in addition to those discussed above, where we write γ_G for the invariant $\gamma_{\mathfrak{X}_{\mathsf{proj}}}$ applied to $\mathfrak{a} = a(G)$.*

(xiii) $\gamma_G(M) = \dim(M)$ *if and only if there is an element of G of order p acting trivially on M.*
(xiv) *We have* $\gamma_G(M) = \max_{E\leqslant G} \gamma_E(M)$, *where the maximum is taken over elementary abelian subgroups* $E \leqslant G$.
(xv) *If M is endotrivial then $\gamma_G(M) = 1$. In particular, combining this with* (v), *there are no modules M with $1 < \gamma_G(M) < \sqrt{2}$.*
(xvi) *If M is a two dimensional faithful indecomposable module for an elementary abelian group $E \cong (\mathbb{Z}/p)^r$ then $\gamma_E(M) = 2\cos(\pi/p^r)$. In particular, if $p = 2$ and $r = 2$ we have $\gamma_E(M) = \sqrt{2}$, and if $p = 5$ and $r = 1$ we have $\gamma_E(M) = \tau$, the golden ratio.*

The main theme of these notes is that the invariant $\gamma_{\mathfrak{X}}(x)$ described above may be interpreted as a spectral radius of the element x in a quotient of a completion of the complexified representation ring $\mathfrak{a}_{\mathbb{C}}$. This completion is a commutative Banach algebra, whose structure we shall investigate.

We put a norm on $\mathfrak{a}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a}$ by setting

$$\left\| \sum_{i \in \mathfrak{I}} a_i x_i \right\| = \sum_{i \in \mathfrak{I}} |a_i| \dim x_i.$$

This makes $\mathfrak{a}_{\mathbb{C}}$ a commutative normed algebra with identity element $\mathbb{1}$. Its completion $\hat{\mathfrak{a}}$ is therefore a commutative Banach algebra.

If $\mathfrak{X}$ is a representation ideal in $\mathfrak{a}$ then the linear span $\langle \mathfrak{X} \rangle_{\mathbb{C}}$ is an ideal in $\mathfrak{a}_{\mathbb{C}}$, and its closure $\widehat{\langle \mathfrak{X} \rangle}_{\mathbb{C}}$ is an ideal in $\hat{\mathfrak{a}}$. The quotient norm on $\hat{\mathfrak{a}}_{\mathfrak{X}} = \hat{\mathfrak{a}}/\widehat{\langle \mathfrak{X} \rangle}_{\mathbb{C}} \cong \widehat{\mathfrak{a}/\langle \mathfrak{X} \rangle}_{\mathbb{C}}$ is given by

$$\left\| \sum_{i \in \mathfrak{I}} a_i x_i \right\|_{\mathfrak{X}} = \sum_{i \in \mathfrak{I}} |a_i| \dim \mathsf{core}_{\mathfrak{X}}(x_i) = \sum_{i \in \mathfrak{I} \setminus \mathfrak{X}} |a_i| \dim x_i.$$

After introducing the background material on commutative Banach algebras in Chapter 2, we investigate these quotients $\hat{\mathfrak{a}}_{\mathfrak{X}}$ of $\hat{\mathfrak{a}}$ in Chapter 3.

In terms of these quotients, if $x \in \mathfrak{a}_{\geqslant 0}$ then the invariant $\gamma_{\mathfrak{X}}(x)$ is interpreted as the spectral radius of the image of x in $\hat{\mathfrak{a}}_{\mathfrak{X}}$ (Theorem 3.2.5). This enables us to relate $\gamma_{\mathfrak{X}}(x)$ to the species $s \colon \mathfrak{a} \to \mathbb{C}$ of representation rings introduced by Benson and Parker [**12**] and studied in [**7, 38, 48, 49, 57, 58, 74, 93, 94, 95**].

A *species* of $\mathfrak{a}$ is a ring homomorphism $s \colon \mathfrak{a} \to \mathbb{C}$. Such a ring homomorphism extends uniquely to a $\mathbb{C}$-algebra homomorphism $s \colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$, which we also call a species. The species which are continuous with respect to the norm on $\mathfrak{a}_{\mathbb{C}}$ are the *dimension bounded* species, namely the ones that satisfy $|s(x)| \leqslant \dim x$ for every $x \in \mathfrak{a}_{\geqslant 0}$. Such a species extends uniquely to a species $s \colon \hat{\mathfrak{a}} \to \mathbb{C}$. All species of $\hat{\mathfrak{a}}$ are of this form, and are automatically continuous.

If $\mathfrak{X}$ is a representation ideal in $\mathfrak{a}$ then the species which vanish on $\mathfrak{X}$ and extend to species of $\hat{\mathfrak{a}}_{\mathfrak{X}}$ are the $\mathfrak{X}$-*core bounded* ones, namely those that satisfy $|s(x)| \leqslant \dim \mathsf{core}_{\mathfrak{X}}(x)$ for every $x \in \mathfrak{a}_{\geqslant 0}$.

We may now apply a theorem of Gelfand relating the spectral radius to the species.

THEOREM. *Let $x \in \mathfrak{a}_{\geqslant 0}$ and $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. Then $\gamma_{\mathfrak{X}}(x)$ is equal to the supremum of $|s(x)|$, where s runs over the $\mathfrak{X}$-core bounded species $s \colon \mathfrak{a} \to \mathbb{C}$.*

The set of species of $\hat{\mathfrak{a}}_{\mathfrak{X}}$ is topologised with the weak* topology, described in Section 2.5, to form a compact Hausdorff topological space called the *structure space* $\Delta_{\mathfrak{X}}(\mathfrak{a})$. If $\mathfrak{Y} \subseteq \mathfrak{X}$ are represetation ideals then every $\mathfrak{X}$-core bounded species of $\mathfrak{a}$ is $\mathfrak{Y}$-core bounded, and this induces a homeomorphism identifying $\Delta_{\mathfrak{X}}(\mathfrak{a})$ with a closed subset of $\Delta_{\mathfrak{Y}}(\mathfrak{a})$.

Now in some ways the Banach algebra $\hat{\mathfrak{a}}_{\mathfrak{X}}$ looks like the group algebra $\ell^1(\Gamma)$ of a discrete abelian group Γ. This analogy is strongest when $\mathfrak{X} = \mathfrak{X}_{\max}$, the largest representation ideal in $\mathfrak{a}$, consisting of those i for which $[x_i x_{i^*} : \mathbb{1}] > 0$. This is because in this quotient, x_i^* acts as a sort of partial inverse for x_i. In particular, we construct a Hilbert space $H(\mathfrak{a})$ on which $\hat{\mathfrak{a}}_{\max} = \hat{\mathfrak{a}}_{\mathfrak{X}_{\max}}$ acts, in such a way that the action of x^* is the adjoint of the action of x. This gives us an injective map of Banach $*$-algebras $\hat{\mathfrak{a}}_{\max} \to \mathscr{L}(H(\mathfrak{a}))$, the bounded operators on

$H(\mathfrak{a})$. The crucial inequality giving boundedness is Theorem 3.7.5, which turns out to be quite tricky to prove. This says that for $x \in \mathfrak{a}_{\mathbb{C}}$ and $y \in H(\mathfrak{a})$ we have

$$|xy| \leqslant \|x\|_{\max}|y|.$$

We let $C^*_{\max}(\mathfrak{a})$ denote the closure of the image of $\hat{\mathfrak{a}}_{\max} \to \mathscr{L}(H(\mathfrak{a}))$. This is a C^*-algebra, and is the completion of $\mathfrak{a}_{\max}$ with respect to the sup norm

$$\|x\|_{\sup} = \sup_{|y|=1} |xy|.$$

Some consequences of this construction include the statement that there are no non-zero quasinilpotent elements in $\hat{\mathfrak{a}}_{\max}$, and a better understanding of idempotents.

In Chapter 4, we specialise to the situation where G is a finite group, k is a field of characteristic p, and $a(G)$ is the representation ring of kG-modules. In this case, we have further structure coming from restriction and induction, elementary abelian subgroups, and Adams psi operations. In particular, in the case $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, we show that the invariant $\gamma(M)$ is detected on elementary abelian subgroups. One consequence of this is that there cannot exist a module M with $1 < \gamma(M) < \sqrt{2}$.

In Chapter 5, we illustrate this situation with some groups and modules where we can make explicit computations. In particular, in case G is a cyclic group of order p and M is the indecomposable two-dimensional module, we have $\gamma(M) = 2\cos(\pi/p)$. In Section 5.8 we prove the following.

Theorem. *If $G = SL(2, q)$ with q a power of p, amd M is the two-dimensional natural module, then $\gamma(M) = 2\cos(\pi/q)$.*

The proof uses tilting theory. We conjecture that for representations of finite groups, if $\gamma(M) < 2$ then $\gamma(M) = 2\cos(\pi/q)$ for $q \geqslant 2$ an integer.

We illustrate the structure of the space $\Delta(G)$ using the only cases where we can make complete computations, namely $G = V_4$, the Klein four group, and $G = A_4$, the alternating group of degree four, in characteristic two.

Notation. We shall often want to use i as an index, so we write i for the complex number $\sqrt{-1}$.

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CHAPTER 1

Abstract representation rings

1.1. Axioms for representation rings

In this section we formulate the properties of representation rings that we shall need. Our fundamental model is the modular representation ring, or Green ring $a(G)$ of a finite group G over a field k of characteristic p, but there are many other examples. Motivation for the properties (i)–(v) comes from this example, see Proposition 4.2.1. In case the field k is algebraically closed, the stronger property (ii′) below holds, and $a(G)$ is a closed representation ring.

Definition 1.1.1. A *representation ring* consists of a commutative ring $\mathfrak{a}$ whose additive group is a free abelian group with a specified basis consisting of symbols x_i with i in an indexing set $\mathfrak{I}$. The identity element $x_0 = \mathbb{1}$ of $\mathfrak{a}$ is one of the basis elements, corresponding to $0 \in \mathfrak{I}$. Multiplication is given by $x_i x_j = \sum_{k\in\mathfrak{I}} c_{i,j,k} x_k$ where the *structure constants* $c_{i,j,k}$ are non-negative integers, and given $i, j \in \mathfrak{I}$, there are only finitely many k with $c_{i,j,k} \neq 0$. If $x = \sum_{i\in\mathfrak{I}} a_i x_i$ is an element of $\mathfrak{a}$ then we write $[x : x_i]$ for a_i, the *multiplicity* of x_i in x. Thus we have $x = \sum_{i\in\mathfrak{I}} [x : x_i] x_i$ and $x_i x_j = \sum_{k\in\mathfrak{I}} [x_i x_j : x_k] x_k$.

(i) There is an involutive permutation $i \mapsto i^*$ of the indexing set $\mathfrak{I}$, which induces an involutive automorphism of $\mathfrak{a}$ sending $x = \sum_{i\in\mathfrak{I}} a_i x_i$ to $x^* = \sum_{i\in\mathfrak{I}} a_i x_{i^*}$.

(ii) If $[x_i x_j : \mathbb{1}] > 0$ then $j = i^*$.

(iii) If $i \in \mathfrak{I}$ satisfies $c_{i,i^*,0} = 0$ then

$$\sum_{j\in\mathfrak{I}} c_{i,i^*,j} c_{j,i,i} \geqslant 2.$$

In other words, if $[x_i x_{i^*} : \mathbb{1}] = 0$ then $[x_i x_{i^*} x_i : x_i] \geqslant 2$.

(iv) There is a dimension function

$$\dim \colon \mathfrak{a} \to \mathbb{Z},$$

which is a ring homomorphism with the property that for each $i \in \mathfrak{I}$,

$$\dim(x_i) = \dim(x_{i^*}) > 0.$$

Thus $\dim(x) = \dim(x^*)$ for all $x \in \mathfrak{a}$.

(v) There is a non-zero element $\rho \in \mathfrak{a}$ which is a non-negative linear combination of the basis elements, with the property that for all $x \in \mathfrak{a}$, $x\rho = (\dim x)\rho$.
[See also Remark 1.1.3.]

A *closed representation ring* is a representation ring satisfying a stronger version of (ii):

(ii′) If $[x_i x_j : \mathbb{1}] > 0$ then $j = i^*$ and $[x_i x_j : \mathbb{1}] = 1$.

Examples 1.1.2. The modular representation ring of a finite group is the motivating example, and this will be discussed at length in Chapters 4 and 5.

The smallest a representation ring can be is $\mathbb{Z}$ with just one basis element, $x_0 = 1$. The element ρ can be any positive integer, and the dimension function $\mathbb{Z} \to \mathbb{Z}$ is the identity map. If $\rho = 1$, this is the representation ring of the trivial group.

Other natural examples include the representation rings of finite group schemes and finite supergroup schemes.

The following examples of representation rings are somewhat artificial, but will be used later to illustrate some ideas.

(i) Choose an integer $d \geqslant 2$, and let

$$\mathfrak{a} = \mathbb{Z}[u, u^{-1}] \oplus \mathbb{Z}$$

as a sum of a subring and an ideal, where the ideal summand $\mathbb{Z}$ is spanned by ρ. Multiplication is given by $u\rho = u^{-1}\rho = \rho$, $\rho^2 = d\rho$. The index set $\mathfrak{I}$ is $\mathbb{Z} \cup \{\infty\}$ where $x_n = u^n$ for $n \in \mathbb{Z}$ and $x_\infty = \rho$. The involutive permutation on $\mathfrak{I}$ sends n to $-n$ and fixes ∞. The dimension function is given by $\dim u^n = 1$, $\dim \rho = d$.

(ii) Choose an integer $d \geqslant 2$, and let

$$\mathfrak{a} = \mathbb{Z}[u, v]/((uv-1)(u-d), (uv-1)(v-d)).$$

The index set $\mathfrak{I}$ is again $\mathbb{Z} \cup \{\infty\}$. The basis is given by $x_0 = 1$, $x_n = u^n$ and $x_{-n} = v^n$ for $n > 0$, and $x_\infty = \rho = uv - 1$. Again the involutive permutation on $\mathfrak{I}$ sends n to $-n$ and fixes ∞. The dimension function is given by

$$\dim u^n = \dim v^n = d^n, \qquad \dim \rho = d^2 - 1.$$

This example will be used in Section 1.7 as an illustration of the difference between the big Picard group and the small Picard group.

(iii) Let $\mathfrak{a} = \mathbb{Z}[v]/(v^3 - 2v^2)$ with basis $x_0 = 1$, $x_1 = v$, $x_2 = v^2 - v$, $\rho = x_1 + x_2$, and $x_1^* = x_2$, $x_2^* = x_1$. Then $x_1^2 = x_1x_2 = x_2^2 = \rho$, and $\dim x_1 = \dim x_2 = 2$. Note that in this ring we have $(x_2 - x_1)^2 = 0$, so the nil radical of $\mathfrak{a}$ is non-zero.

Remark 1.1.3. Examples which are not covered by our axioms include the representation ring of a compact Lie group, as studied for example by Segal [**85**]. This example satisfies all but axiom (v) of Definition 1.1.1. Anything we do in this work that does not mention projectives does not depend on this axiom, and there are arguments for deleting it, but we have chosen to retain it. The element ρ in that axiom plays the role of the regular representation.

There is an elementary way of enhancing such a representation ring in such a way that axiom (v) holds, but this method is not terribly satisfactory. Namely, given $\mathfrak{a}$ satisfying all but this axiom, and an integer $n \geqslant 2$, then we endow $\mathfrak{a} \oplus \mathbb{Z}$ with the structure of a representation ring with one more basis element ρ satisfying $\rho^* = \rho$, $x\rho = (\dim x)\rho$ for $x \in \mathfrak{a}$, $\dim \rho = n$, and $\rho^2 = n\rho$. Note that $[\rho\rho^* : \mathbb{1}] = 0$ and $[\rho\rho^*\rho : \rho] = n^2 \geqslant 2$, so that axioms (ii) and (iii) are satisfied.

Our argument for retaining axiom (v), however, is that our focus will be on studying representation ideals in representation rings, and examples such as the representation ring of a compact Lie group have no non-zero representation ideals. So most of we do here is vacuous in that case. Axiom (v) does not ensure that there are non-zero representation ideals, but if there are none, then the representation ring is finite dimensional and semisimple, and looks very much like the ordinary character ring of a finite group. We shall call this case an *ordinary representation ring*, and the contrary case a *modular representation ring*, see Definition 1.2.6. In a modular representation ring, there is a unique maximal representation ideal and a unique minimal representation ideal, see Proposition 1.3.9.

The following lemmas give some of the more elementary consequences of the definitions. For this purpose, we introduce one more definition.

Definition 1.1.4. If R is a commutative ring, we write $\mathfrak{a}_R$ for the ring $R \otimes_{\mathbb{Z}} \mathfrak{a}$ obtained by extending scalars to R. Elements of $\mathfrak{a}_{\mathbb{C}}$ are finite sums $\sum_{i \in \mathfrak{I}} a_i x_i$, with $a_i \in R$. We shall mostly be interested in the case $R = \mathbb{C}$, the complex numbers, but other extensions of scalars will occasionally be considered. In case $R = \mathbb{C}$, if $x = \sum_{i \in \mathfrak{I}} a_i x_i$ we define $x^* = \sum_{i \in \mathfrak{I}} \bar{a}_i x_i^*$.

Lemma 1.1.5. *We have $\rho = \rho^*$.*

Proof. By property (i), ρ^* also satisfies property (v). So by properties (iv) and (v) we have $\dim(\rho^*)\rho = \rho^*\rho = \dim(\rho)\rho^*$, and also $\dim(\rho^*) = \dim(\rho) > 0$. $\square$

Lemma 1.1.6. *For all $i \in \mathfrak{I}$ we have $[x_i x_{i^*} x_i : x_i] > 0$.*

Proof. If $[x_i x_{i^*} : \mathbb{1}] = 0$ then this follows from property (iii). If $[x_i x_{i^*} : \mathbb{1}] > 0$ then $[x_i x_{i^*} x_i : x_i] \geqslant [x_i x_{i^*} : \mathbb{1}] > 0$. $\square$

Lemma 1.1.7. *The following are equivalent for a basis element x_i:*

(i) $\dim x_i = 1$.
(ii) $x_i x_{i^*} = \mathbb{1}$.
(iii) *x_i is invertible in $\mathfrak{a}$.*

Proof. (i) $\Rightarrow$ (ii): If $\dim x_i = 1$ then $\dim x_i x_{i^*} = 1$ and $\dim x_i x_{i^*} x_i = 1$. In particular, $[x_i x_{i^*} : \mathbb{1}]$ is either zero or one. If it were zero then by Definition 1.1.1 (iii) we would have $[x_i x_{i^*} x_i : x_i] \geqslant 2$, contradicting $\dim x_i x_{i^*} x_i = 1$, and so we have $[x_i x_{i^*} : \mathbb{1}] = 1$. We have $\dim(x_i x_{i^*} - \mathbb{1}) = 0$, and hence $x_i x_{i^*} = \mathbb{1}$.
(ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (i): If x_i is invertible in $\mathfrak{a}$ then $\dim x_i$ is invertible in $\mathbb{Z}$, hence equal to one. $\square$

The following is the analogue of Proposition 2.2 of [**11**].

Lemma 1.1.8. *Let $i, j, k \in \mathfrak{I}$ with $[x_i x_j : x_k] > 0$. If $[x_i x_{i^*} : \mathbb{1}] = 0$ then $[x_k x_k^* : \mathbb{1}] = 0$.*

Proof. It follows from property (ii) that if $[x_i x_{i^*} : \mathbb{1}] = 0$ then for all $x \in \mathfrak{a}$ we have $[x_i x : \mathbb{1}] = 0$. In particular, $[x_i x_j x_{k^*} : \mathbb{1}] = 0$, and since $[x_i x_j : x_k] > 0$ it follows that $[x_k x_{k^*} : \mathbb{1}] = 0$. $\square$

Definition 1.1.9. If $\mathfrak{a}$ is a representation ring, a *species* of $\mathfrak{a}$ is a ring homomorphism $s \colon \mathfrak{a} \to \mathbb{C}$. A species extends uniquely to a $\mathbb{C}$-algebra homomorphism $s \colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$, which we also call a species.

LEMMA 1.1.10. *Any set of species of $\mathfrak{a}$ is linearly independent.*

PROOF. Let $s_1, \dots, s_n \colon \mathfrak{a} \to \mathbb{C}$ satisfy a non-trivial linear relation

$$\lambda_1 s_1 + \cdots + \lambda_n s_n = 0$$

with n as small as possible. Choose $x \in \mathfrak{a}$ such that $s_1(x) \neq s_2(x)$. Then for all $y \in \mathfrak{a}$ we have

$$\lambda_1 s_1(xy) + \lambda_2 s_2(xy) + \cdots + \lambda_n s_n(xy) = 0,$$

and hence

$$\lambda_1 s_1(x) s_1(y) + \lambda_2 s_2(x) s_2(y) + \cdots + \lambda_n s_n(x) s_n(y) = 0.$$

But also

$$\lambda_1 s_1(x) s_1(y) + \lambda_2 s_1(x) s_2(y) \cdots + \lambda_n s_1(x) s_n(y) = 0.$$

Subtracting gives a shorter non-trivial linear relation, contradicting the minimality of n. $\square$

1.2. Ordinary representation theory

DEFINITION 1.2.1. An element $x \in \mathfrak{a}$ is *non-negative* if each $[x : x_i] \geqslant 0$, and *positive* if, in addition, $x \neq 0$. We write $x \succcurlyeq y$ or $y \preccurlyeq x$ if $x - y$ is non-negative and $x \succ y$ or $y \prec x$ if $x - y$ is positive. We write $\mathfrak{a}_{\succcurlyeq 0}$ for the set of non-negative elements, and $\mathfrak{a}_{\succ 0}$ for the set of positive elements.

DEFINITION 1.2.2. A basis element x_i of a representation ring $\mathfrak{a}$ is said to be *projective indecomposable* if $[\rho : x_i] > 0$. The number of projective indecomposables is finite.

An element $x = \sum_{i \in \mathfrak{I}} a_i x_i$ is said to be *virtually projective* if $a_i \neq 0$ implies that x_i is projective. If in addition $x \succcurlyeq 0$ then we say that x is *projective*.

LEMMA 1.2.3.

(i) *If $x \in \mathfrak{a}_{\succcurlyeq 0}$ and y is projective then xy is projective.*
(ii) *If $x \in \mathfrak{a}$ and y is virtually projective then xy is virtually projective.*

PROOF. (i) It suffices to consider the case $x = x_i$, $y = x_j$ with x_j projective. We have $x_i \rho = (\dim x_i)\rho$, so $x_i \rho$ is a non-negative linear combination of projective basis elements, and hence so is $x_i x_j$.

(ii) follows from (i). $\square$

The following theorem will have a generalisation involving commutative Banach algebras in Theorem 3.9.1. The final statement of the theorem should also be contrasted with Example 1.1.2 (iii).

THEOREM 1.2.4 (Ordinary representation theory). *Suppose that $\mathbb{1}$ is projective in a representation ring $\mathfrak{a}$. Then every element of $\mathfrak{a}$ is virtually projective, and the additive group of $\mathfrak{a}$ has finite rank. If $x \in \mathfrak{a}_{\mathbb{C}}$ such that $xx^* = 0$ then $x = 0$. There are no non-zero nilpotent elements in $\mathfrak{a}_{\mathbb{C}}$.*

PROOF. If $\mathbb{1}$ is projective then for every basis element x_i we have

$$(\dim x_i)[\rho : x_i] = [x_i\rho : x_i] \geqslant [\rho : \mathbb{1}] > 0$$

and so every x_i is a projective indecomposable. Thus every element of $\mathfrak{a}$ is virtually projective, and $\mathfrak{a}$ has finite rank. Next, using Definition 1.1.1 (ii) and the fact that $\rho = \sum_j [\rho : x_j]x_j$, we have

$$[\rho : x_{i^*}][x_i x_{i^*} : \mathbb{1}] = \sum_j [\rho : x_j][x_i x_j : \mathbb{1}] = [x_i\rho : \mathbb{1}] = (\dim x_i)[\rho : \mathbb{1}] > 0$$

and so $[x_i x_{i^*} : \mathbb{1}] > 0$. Set $n_i = [x_i x_{i^*} : \mathbb{1}]$. Then

$$xx^* = \sum_i |a_i|^2 x_i x_{i^*} + \sum_{i \neq j} a_i \bar{a}_j x_i x_{j^*}$$

and the coefficient of $\mathbb{1}$ in this is $\sum n_i |a_i|^2$. This is zero if and only if $x = 0$, so $xx^* = 0$ implies $x = 0$. If $x^2 = 0$ then $(xx^*)(xx^*)^* = x^2 x^{*\,2} = 0$, so $xx^* = 0$ and hence $x = 0$. □

COROLLARY 1.2.5. *Suppose that $\mathbb{1}$ is projective in $\mathfrak{a}$. Then as a $\mathbb{C}$-algebra, $\mathfrak{a}_{\mathbb{C}}$ is a direct sum of a finite number of copies of $\mathbb{C}$, and elements are separated by species $s\colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$.*

PROOF. It follows from Theorem 1.2.4 that $\mathfrak{a}_{\mathbb{C}}$ is finite dimensional and semisimple. Every finite dimensional semisimple commutative algebra over $\mathbb{C}$ has this property. □

DEFINITION 1.2.6. Because of Theorem 1.2.4 and Corollary 1.2.5, we shall say that $\mathfrak{a}$ is an *ordinary representation ring* if $\mathbb{1}$ is projective in $\mathfrak{a}$, and that $\mathfrak{a}$ is a *modular representation ring* if $\mathbb{1}$ is not a projective indecomposable in $\mathfrak{a}$.

The *character table* of an ordinary representation ring $\mathfrak{a}$ is the square table of complex numbers whose rows are indexed by the index set $\mathfrak{I}$ and whose columns are indexed by the species $s\colon \mathfrak{a} \to \mathbb{C}$. The *character* of an element $x = \sum_{i \in \mathfrak{I}} a_i x_i \in \mathfrak{a}$ is the corresponding linear combination of the rows of the character table. Elements of $\mathfrak{a}$ are determined by their characters, and ring operations on elements correspond to pointwise operations on the corresponding characters.

The following property of ordinary representation rings is familiar from representation theory of finite groups. The usual proof in that case is that the inverse of an eigenvalue of a matrix of finite order is its complex conjugate.

THEOREM 1.2.7. *If s is a species of an ordinary representation ring $\mathfrak{a}$ then for all $x \in \mathfrak{a}_{\mathbb{C}}$ we have $s(x^*) = \overline{s(x)}$.*

PROOF. Define a species $\bar{s}^*\colon \mathfrak{a} \to \mathbb{C}$ by $\bar{s}^*(x) = \overline{s(x^*)}$. By Corollary 1.2.5, we may choose a primitive idempotent element $e \in \mathfrak{a}_{\mathbb{C}}$ such that $s(e) = 1$, and $s'(e) = 0$ for all other species s'. If $\bar{s}^* \neq s$ then $\bar{s}^*(e) = 0$ and so $s(e^*) = 0$. So for all species s' of $\mathfrak{a}_{\mathbb{C}}$ we have $s'(ee^*) = 0$, and hence $ee^* = 0$. By Theorem 1.2.4, this implies that $e = 0$, which is a contradiction. Hence $\bar{s}^* = s$, which then implies that for all $x \in \mathfrak{a}_{\mathbb{C}}$ we have $s(x^*) = \overline{s(x)}$. □

1.3. Representation ideals and cores

LEMMA 1.3.1. *If* $x \in \mathfrak{a}_{\succcurlyeq 0}$ *then* $\dim x \geqslant 0$*; if* $x \in \mathfrak{a}_{\succ 0}$ *then* $\dim x > 0$*.*

PROOF. It follows from Definition 1.1.1 (iii) that if $x = \sum_{i \in \mathfrak{I}} a_i x_i$ then

$$\dim x = \sum_{i \in \mathfrak{I}} a_i \dim x_i,$$

and that this is $\geqslant 0$ if all $a_i \geqslant 0$, and > 0 if in addition some $a_i > 0$. □

DEFINITION 1.3.2. A *representation ideal* $\mathfrak{X}$ of a representation ring $\mathfrak{a}$ is a proper subset $\mathfrak{X} \subset \mathfrak{I}$ with the following properties:

(i) If $i \in \mathfrak{X}$, $j \in \mathfrak{I}$ and $k \in \mathfrak{I} \setminus \mathfrak{X}$ then $c_{i,j,k} = 0$. Equivalently, if $i \in \mathfrak{X}$ and there exists $x \in \mathfrak{a}$ such that $[x_i x : x_k] \neq 0$ then $k \in \mathfrak{X}$.
(ii) If $i \in \mathfrak{X}$ then $i^* \in \mathfrak{X}$.

The linear span $\langle \mathfrak{X} \rangle$ of a representation ideal $\mathfrak{X}$ is an ideal in $\mathfrak{a}$. We write $\langle \mathfrak{X} \rangle_{\succcurlyeq 0}$ for $\langle \mathfrak{X} \rangle \cap \mathfrak{a}_{\succcurlyeq 0}$ and $\langle \mathfrak{X} \rangle_{\succ 0}$ for $\langle \mathfrak{X} \rangle \cap \mathfrak{a}_{\succ 0}$. We write $\mathfrak{a}_{\mathfrak{X}}$ for the quotient $\mathfrak{a}/\langle \mathfrak{X} \rangle$, and $\mathfrak{a}_{\mathbb{C},\mathfrak{X}}$ for $\mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a}_{\mathfrak{X}} \cong \mathfrak{a}_{\mathbb{C}}/\langle \mathfrak{X} \rangle_{\mathbb{C}}$.

If $\mathfrak{X}$ is a representation ideal in $\mathfrak{a}$ and $x \in \mathfrak{a}_{\succcurlyeq 0}$ then we can write $x = x' + x''$ where $x' = \sum_{i \in \mathfrak{I} \setminus \mathfrak{X}} a_i x_i$ and $x'' = \sum_{i \in \mathfrak{X}} a_i x_i$. We define the $\mathfrak{X}$-*core* of x to be x', and we denote it by $\mathsf{core}_{\mathfrak{X}}(x)$. In the case $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, we omit the subscript and just write $\mathsf{core}(x)$.

LEMMA 1.3.3. *Let* $\mathfrak{Y} \subseteq \mathfrak{X}$ *be representation ideals in a representation ring* $\mathfrak{a}$ *and let* $x, y, z \in \mathfrak{a}_{\succcurlyeq 0}$*.*

(i) *The product* $xy \succcurlyeq 0$*.*
(ii) *We have* $\mathsf{core}_{\mathfrak{X}}(x) \preccurlyeq \mathsf{core}_{\mathfrak{Y}}(x) \preccurlyeq x$*, and in particular we have*

$$\dim \mathsf{core}_{\mathfrak{X}}(x) \leqslant \dim \mathsf{core}_{\mathfrak{Y}}(x) \leqslant \dim x.$$

(iii) *If* $x \preccurlyeq y$ *then* $\dim x \leqslant \dim y$ *and* $\mathsf{core}_{\mathfrak{X}}(x) \preccurlyeq \mathsf{core}_{\mathfrak{X}}(y)$*. In particular, we have*

$$\dim \mathsf{core}_{\mathfrak{X}}(x) \leqslant \dim \mathsf{core}_{\mathfrak{X}}(y).$$

(iv) *We have* $\mathsf{core}_{\mathfrak{X}}(xy) = \mathsf{core}_{\mathfrak{X}}(\mathsf{core}_{\mathfrak{X}}(x)\mathsf{core}_{\mathfrak{X}}(y)) \preccurlyeq \mathsf{core}_{\mathfrak{X}}(x)\mathsf{core}_{\mathfrak{X}}(y)$*.*
(v) *If* $y \preccurlyeq x$ *and* $x \in \langle \mathfrak{X} \rangle$ *then* $y \in \langle \mathfrak{X} \rangle$*.*
(vi) *If* $y \preccurlyeq x$ *then* $zy \preccurlyeq zx$*.*
(vii) $\mathsf{core}_{\mathfrak{X}}(x^*) = \mathsf{core}_{\mathfrak{X}}(x)^*$*.*

PROOF. Parts (i), (ii) and (iii) follow from the fact that the structure constants $c_{i,j,k}$ of $\mathfrak{a}$ are non-negative.

(iv) The equality follows from the definitions of core and representation ideal, and the inequality follows from (ii).

(v) For each $i \in \mathfrak{I}$, $[x : x_i] \geqslant [y : x_i]$. If $[y : x_i] > 0$ then $[x : x_i] > 0$ and so $i \in \mathfrak{X}$.

(vi) $xz - yz = (x - y)z$ is a product of elements of $\mathfrak{a}_{\succcurlyeq 0}$, and is hence in $\mathfrak{a}_{\succcurlyeq 0}$ by (i).

(vii) This follows from part (ii) of Definition 1.3.2. □

LEMMA 1.3.4. *If* $x \in \mathfrak{a}_{\succcurlyeq 0}$ *then* $xx^*x \succcurlyeq x$*. If also* $[xx^* : \mathbb{1}] = 0$ *then* $xx^*x \succcurlyeq 2x$*.*

Proof. It follows from Lemma 1.1.6 that for each $i \in \mathfrak{I}$ we have $[xx^*x : x_i] \geqslant [x : x_i]$. If $[xx^* : \mathbb{1}] = 0$ then by Definition 1.1.1 (iii) we have $[xx^*x : x_i] \geqslant 2[x : x_i]$. □

Proposition 1.3.5. *Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$, and let $x \in \mathfrak{a}_{\succcurlyeq 0}$.*

(i) *If $xx^* \in \langle \mathfrak{X} \rangle$ then $x \in \langle \mathfrak{X} \rangle$.*
(ii) *If $x^n \in \langle \mathfrak{X} \rangle$ for some $n > 0$ then $x \in \langle \mathfrak{X} \rangle$.*

Proof. (i) If $xx^* \in \langle \mathfrak{X} \rangle$ then $xx^*x \in \langle \mathfrak{X} \rangle$. By Lemma 1.3.4 we have $xx^*x \succcurlyeq x$, and so by Lemma 1.3.3 (v) we have $x \in \langle \mathfrak{X} \rangle$.

(ii) We may suppose that $n \geqslant 2$. If $x^n \in \langle \mathfrak{X} \rangle$ then $x^n x^* = x^{n-2}(xx^*x) \in \langle \mathfrak{X} \rangle$. By Lemma 1.3.4 we have $xx^*x \succcurlyeq x$ and so by Lemma 1.3.3 (vi) we have $x^n x^* \succcurlyeq x^{n-1}$. Applying Lemma 1.3.3 (v), we have $x^{n-1} \in \langle \mathfrak{X} \rangle$. Now apply induction on n. □

Definition 1.3.6. Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$. We say that $x \in \mathfrak{a}_{\succcurlyeq 0}$ is *indecomposable modulo* $\mathfrak{X}$ if $x - x_i \in \langle \mathfrak{X} \rangle$ for some $i \in \mathfrak{I} \setminus \mathfrak{X}$.

Lemma 1.3.7. *If $x, y \in \mathfrak{a}_{\succcurlyeq 0}$ and $xy = 0$ then either $x = 0$ or $y = 0$.*

Proof. If $xy = 0$ then $\dim(x)\dim(y) = \dim(xy) = 0$ and so either $\dim(x) = 0$ or $\dim(y) = 0$. Hence $x = 0$ or $y = 0$. □

An ordinary representation ring has no non-empty representation ideals, since any non-empty representation ideal would have to contain ρ, and then it would have to contain all projective indecomposables including $\mathbb{1}$, whereas representation ideals have to be proper subsets of the basis. The following definition gives some examples of representation ideals in modular representation rings.

Definition 1.3.8. Let $\mathfrak{a}$ be a modular representation ring.

(i) We write $\mathfrak{X}_{\max}$ for the subset $\{i \in \mathfrak{I} \mid [x_i x_{i^*} : \mathbb{1}] = 0\}$ of $\mathfrak{I}$.
(ii) We write $\mathfrak{X}_{\mathsf{proj}}$ for the subset consisting of those $i \in \mathfrak{I}$ for which x_i is projective, see Definition 1.2.2.

We write $\mathfrak{a}_{\max}$ and $\mathfrak{a}_{\mathsf{proj}}$ for the quotients $\mathfrak{a}/\langle \mathfrak{X}_{\max} \rangle$ and $\mathfrak{a}/\langle \mathfrak{X}_{\mathsf{proj}} \rangle$.

Part (i) of the following proposition is the analogue of Lemma 2.5 of [**11**].

Proposition 1.3.9. *Let $\mathfrak{a}$ be a modular representation ring.*

(i) *The subset $\mathfrak{X}_{\max} \subseteq \mathfrak{I}$ is the unique maximal proper subset of $\mathfrak{a}$ that is a representation ideal.*
(ii) *The subset $\mathfrak{X}_{\mathsf{proj}} \subseteq \mathfrak{I}$ is the unique minimal non-empty subset of $\mathfrak{a}$ that is a representation ideal.*

Proof. (i) It follows from Lemma 1.1.8 that $\mathfrak{X}_{\max}$ is a representation ideal. It is a proper subset of $\mathfrak{I}$ since $[\mathbb{1}.\mathbb{1} : \mathbb{1}] = 1$. To see that $\mathfrak{X}_{\max}$ is the unique maximal proper representation ideal, let $\mathfrak{X}$ be a representation ideal containing an element i such that $c_{i,i^*,0} > 0$. Then $x_i x_{i^*} \in \langle \mathfrak{X} \rangle$ and so $0 \in \mathfrak{X}$ and $x_0 = \mathbb{1} \in \langle \mathfrak{X} \rangle$. Thus $\mathfrak{X} = \mathfrak{I}$.

(ii) It follows from Lemma 1.2.3 that $\mathfrak{X}_{\mathsf{proj}}$ is a non-empty representation ideal. Conversely, if $\mathfrak{X}$ is a non-empty representation ideal and $i \in \mathfrak{X}$ then $x_i\rho = (\dim x_i)\rho$. If x_j is projective then $[\rho : x_j] > 0$ and so $[x_i\rho : x_j] > 0$. Thus $\mathfrak{X}_{\mathsf{proj}} \subseteq \mathfrak{X}$. □

Corollary 1.3.10. *If $x, y \in \mathfrak{a}_{\succcurlyeq 0}$ and $xy \in \langle \mathfrak{X}_{\max} \rangle$ then either $x \in \langle \mathfrak{X}_{\max} \rangle$ or $y \in \langle \mathfrak{X}_{\max} \rangle$.*

Proof. If $xy \in \langle \mathfrak{X}_{\max} \rangle$ then by Proposition 1.3.9 (i), we have $xyy^* \in \langle \mathfrak{X}_{\max} \rangle$. If $y \notin \langle \mathfrak{X}_{\max} \rangle$ then by definition of $\mathfrak{X}_{\max}$ we have $[yy^* : \mathbb{1}] > 0$ and so $xyy^* \succcurlyeq x$, and hence $x \in \langle \mathfrak{X}_{\max} \rangle$. □

The statement of Corollary 1.3.10 is not necessarily true of other representation ideals, but at least we have the following.

Proposition 1.3.11. *Let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$, and let $x, y \in \mathfrak{a}_{\succcurlyeq 0}$. If xy is not in $\langle \mathfrak{X} \rangle$ then nor is any product of terms of the form x, x^*, y and y^*.*

Proof. If for example xy^* or xyy^* is in $\langle \mathfrak{X} \rangle$ then so is xyy^*y. But by Lemma 1.3.4 we have $xyy^*y \succcurlyeq xy$, and hence by Lemma 1.3.3 (v) we have $xy \in \langle \mathfrak{X} \rangle$. So an easy inductive argument on the number of terms in the product proves the proposition. □

1.4. The gamma invariant

Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$, and $x \in \mathfrak{a}_{\succcurlyeq 0}$. We define

$$\mathsf{c}_n^{\mathfrak{X}}(x) = \dim \mathsf{core}_{\mathfrak{X}}(x^n)$$

and we form the generating function

$$f_{\mathfrak{X},x}(t) = \sum_{n=0}^{\infty} \mathsf{c}_n^{\mathfrak{X}}(x) t^n.$$

Since $\mathsf{c}_n^{\mathfrak{X}}(x) \leqslant \dim(x^n) = \dim(x)^n$, this power series converges in a disc of radius at least $1/\dim(x)$ around the origin in the complex plane.

Lemma 1.4.1 (Cauchy, Hadamard). *Let $\phi \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{C}$. Then the radius of convergence r of the power series*

$$f(t) = \sum_{n=0}^{\infty} \phi(n) t^n$$

is given by

$$1/r = \limsup_{n \to \infty} \sqrt[n]{|\phi(n)|}.$$

Strictly inside the radius, the convergence is uniform and absolute.

Proof. See for example Conway [**29**], Theorem III.1.3. □

Definition 1.4.2. Let $x \in \mathfrak{a}_{\succcurlyeq 0}$ and let $\mathfrak{X}$ be a representation ideal of $\mathfrak{a}$. We define the gamma invariant of x with respect to $\mathfrak{X}$ to be

$$\gamma_{\mathfrak{X}}(x) = \limsup_{n \to \infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}.$$

By Lemma 1.4.1, this is equal to $1/r$, where r is the radius of convergence of the power series $f_{\mathfrak{X},x}(t)$.

If $\mathfrak{X} = \varnothing$ then $\gamma_{\mathfrak{X}}(x) = \dim x$. In the minimal non-zero case $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, we just write $\gamma(x)$ for $\gamma_{\mathfrak{X}_{\mathsf{proj}}}(x)$. In the maximal case $\mathfrak{X} = \mathfrak{X}_{\max}$, we write $\gamma_{\max}(x)$ for $\gamma_{\mathfrak{X}_{\max}}(x)$.

Lemma 1.4.3. *We have* $0 \leqslant \gamma_{\mathfrak{X}}(x) \leqslant \dim \mathsf{core}_{\mathfrak{X}}(x) \leqslant \dim x$.

Proof. This follows from the inequalities

$$0 \leqslant \mathsf{c}_n^{\mathfrak{X}}(x) \leqslant (\dim \mathsf{core}_{\mathfrak{X}}(x))^n \leqslant (\dim x)^n,$$

see Lemma 1.3.3 (ii) and (iv). □

Lemma 1.4.4. *If* $x \in \mathfrak{a}_{\geqslant 0}$ *and* $m \in \mathbb{N}$ *then* $\gamma_{\mathfrak{X}}(mx) = m\gamma_{\mathfrak{X}}(x)$.

Proof. We have $\mathsf{core}_{\mathfrak{X}}((mx)^n) = m^n \mathsf{core}_{\mathfrak{X}}(x^n)$. So $\mathsf{c}_n^{\mathfrak{X}}(mx) = m^n \mathsf{c}_n(x)$, and hence

$$\sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(mx)} = m \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}.$$ □

Lemma 1.4.5. *We have* $\gamma_{\mathfrak{X}}(x^*) = \gamma_{\mathfrak{X}}(x)$.

Proof. It follows from Lemma 1.3.3 (vii) that

$$\mathsf{core}_{\mathfrak{X}}((x^*)^n) = \mathsf{core}_{\mathfrak{X}}((x^n)^*) = \mathsf{core}_{\mathfrak{X}}(x^n)^*$$

and so $\dim \mathsf{core}_{\mathfrak{X}}((x^*)^n) = \dim \mathsf{core}_{\mathfrak{X}}(x^n)$. Now take nth roots and apply $\limsup\limits_{n\to\infty}$. □

Lemma 1.4.6. *If* $x, y \in \mathfrak{a}_{\geqslant 0}$ *and* $y \in \langle \mathfrak{X} \rangle$ *then* $\gamma_{\mathfrak{X}}(x+y) = \gamma_{\mathfrak{X}}(x)$.

Proof. We have $(x+y)^n = x^n + z$ with $z \in \langle \mathfrak{X} \rangle$, so $\mathsf{core}_{\mathfrak{X}}((x+y)^n) = \mathsf{core}_{\mathfrak{X}}(x^n)$ and $\mathsf{c}_n^{\mathfrak{X}}(x+y) = \mathsf{c}_n^{\mathfrak{X}}(x)$. □

Lemma 1.4.7. *We have* $\gamma_{\mathfrak{X}}(xy) \leqslant \gamma_{\mathfrak{X}}(x)\gamma_{\mathfrak{X}}(y)$.

Proof. It follows from Lemma 1.3.3 (iv) that $\mathsf{c}_n^{\mathfrak{X}}(xy) \leqslant \mathsf{c}_n^{\mathfrak{X}}(x)\mathsf{c}_n^{\mathfrak{X}}(y)$, and so

$$\sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(xy)} \leqslant \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}\,\sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(y)}.$$

Now apply $\limsup\limits_{n\to\infty}$ to both sides. □

Although equality does not generally hold in Lemma 1.4.7, we have the following.

Lemma 1.4.8. *We have* $\gamma_{\mathfrak{X}}(x^m) = \gamma_{\mathfrak{X}}(x)^m$.

Proof. By Lemma 1.4.7 we have $\gamma_{\mathfrak{X}}(x^m) \leqslant \gamma_{\mathfrak{X}}(x)^m$. Conversely, if $n = ms + i$ with $0 \leqslant i < m$ then $x^n = x^i(x^m)^s$ and so

$$\mathsf{c}_n^{\mathfrak{X}}(x) \leqslant (\dim x)^m \mathsf{c}_s^{\mathfrak{X}}(x^m).$$

Thus

$$\sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)} \leqslant \sqrt[ms]{\mathsf{c}_n^{\mathfrak{X}}(x)} \leqslant \sqrt[s]{\dim x}\,\sqrt[m]{\sqrt[s]{\mathsf{c}_s^{\mathfrak{X}}(x^m)}}.$$

Applying $\limsup\limits_{n\to\infty}$, the factor $\sqrt[s]{\dim x}$ tends to 1. It follows that

$$\gamma_{\mathfrak{X}}(x) \leqslant \sqrt[m]{\gamma_{\mathfrak{X}}(x^m)}.$$ □

Lemma 1.4.9. *We have* $\gamma_{\mathfrak{X}}(x) = \dim x$ *if and only if* $\mathsf{core}_{\mathfrak{X}}(x^n) = x^n$ *for all* $n \geqslant 0$.

Proof. If $\mathsf{core}_{\mathfrak{X}}(x^n) = x^n$ then $\mathsf{c}_n^{\mathfrak{X}}(x) = (\dim x)^n$ and $\gamma_{\mathfrak{X}}(x) = \dim x$. On the other hand, if $\mathsf{core}_{\mathfrak{X}}(x^n) \neq x^n$ for some $n \geqslant 0$ then we have $\mathsf{core}_{\mathfrak{X}}(x^n) \prec x^n$, and hence $\dim \mathsf{core}_{\mathfrak{X}}(x^n) < (\dim x)^n$. So using Lemma 1.4.8 we have

$$\gamma_{\mathfrak{X}}(x)^n = \gamma_{\mathfrak{X}}(x^n) \leqslant \dim \mathsf{core}_{\mathfrak{X}}(x^n) < (\dim x)^n.$$

Taking nth roots, we have $\gamma_{\mathfrak{X}}(x) < \dim x$. □

Lemma 1.4.10. *We have $\gamma_{\mathfrak{X}}(x) = 0$ if and only if $x \in \langle \mathfrak{X} \rangle$. Otherwise $\gamma_{\mathfrak{X}}(x) \geqslant 1$.*

Proof. If $x \in \langle \mathfrak{X} \rangle$ then $f_{\mathfrak{X},x}(t)$ is the zero power series and so $\gamma_{\mathfrak{X}}(x) = 0$. On the other hand, if $x \notin \langle \mathfrak{X} \rangle$ then by Proposition 1.3.5, no positive power of x is in $\langle \mathfrak{X} \rangle$. So each coefficient of $f_{\mathfrak{X},x}(t)$ is at least 1, and hence $\gamma_{\mathfrak{X}}(x) \geqslant 1$. □

Theorem 1.4.11. (i) *Let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. If $y_1, \dots, y_m \in \mathfrak{a}_{\succ 0}$ with product $y_1 \dots y_n \notin \langle \mathfrak{X} \rangle$ then $\gamma_{\mathfrak{X}}(y_1 + \dots + y_m) \geqslant m$.*
(ii) *If $y_1, \dots, y_m \in \mathfrak{a}_{\succ 0}$ are not in $\langle \mathfrak{X}_{\max} \rangle$ then $\gamma_{\mathfrak{X}}(y_1 + \dots + y_m) \geqslant m$.*

Proof. (i) If $y_1 \dots y_n \notin \langle \mathfrak{X} \rangle$ then by Proposition 1.3.5, neither is any element of the form $y_1^{n_1} \dots y_m^{n_m}$. The element $(y_1 + \dots + y_m)^n$ is therefore a sum of at least m^n elements of $\mathfrak{a}_{\succ 0}$ not in $\langle \mathfrak{X} \rangle$, and so

$$\mathsf{c}_n^{\mathfrak{X}}(y_1 + \dots + y_m) \geqslant m^n$$

and

$$\sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(y_1 + \dots + y_m)} \geqslant m.$$

Now take $\limsup\limits_{n \to \infty}$.

(ii) If $y_1, \dots, y_n$ are not in $\langle \mathfrak{X}_{\max} \rangle$ then by Corollary 1.3.10, nor is $y_1 \dots y_n$. So we are in a position to apply (i). □

Lemma 1.4.12. *If $\mathfrak{X} \subseteq \mathfrak{Y}$ are representation ideals in a representation ring $\mathfrak{a}$ and $x \in \mathfrak{a}_{\succcurlyeq 0}$ then*

$$\gamma_{\mathfrak{Y}}(x) \leqslant \gamma_{\mathfrak{X}}(x).$$

Proof. By Lemma 1.3.3 (ii) we have $\mathsf{c}_n^{\mathfrak{Y}}(x) \leqslant \mathsf{c}_n^{\mathfrak{X}}(x)$ for all $n \geqslant 0$ and so

$$\limsup_{n \to \infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{Y}}(x)} \leqslant \limsup_{n \to \infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}.$$ □

1.5. Pringsheim's Theorem

This section is not logically necessary for the development of the subject, but is closely related to Theorem 3.3.2.

Definition 1.5.1. Given a function $f(t)$ of a complex variable t, we say that it is *analytic* at a point $t = a$ if there is a power series in $t - a$ with a positive radius of convergence, and converging to the value of $f(t)$ in an open neighbourhood of a. We say that a is a *singular point* of $f(t)$ if it not analytic at a.

The following theorem is not so well known. See also Statement (7.21) in Chapter VII of Titchmarsh [**91**].

THEOREM 1.5.2 (Pringsheim). *Suppose that* $\phi\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$*, and that the power series*

$$f(t) = \sum_{n=0}^{\infty} \phi(n) t^n$$

has radius of convergence r*. Then* $t = r$ *is a singular point of* $f(t)$*.*

PROOF. The geometric fact used in the proof of this theorem is that the union in $\mathbb{C}$ of a disc of radius r centred at zero and a disc of positive radius centred at r contains a disc of radius strictly greater than $r/2$ centred at $r/2$.

Expand $f(t)$ as a Taylor series about $t = \frac{r}{2}$:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{r}{2})}{n!} (t - \tfrac{r}{2})^n$$

where $f^{(n)}(t)/n! = \sum_{m \geqslant n} \phi(m) \binom{m}{n} t^{m-n}$. Thus

$$f(t) = \sum_{n=0}^{\infty} (t - \tfrac{r}{2})^n \sum_{m=n}^{\infty} \phi(m) \tbinom{m}{n} (\tfrac{r}{2})^{m-n}.$$

If $t = r$ is not a singular point of $f(t)$ then for ε small enough this converges at $t = r + \varepsilon$. The terms are all non-negative reals, so the sum is absolutely convergent, and we may rearrange the terms to get

$$\begin{aligned} f(r+\varepsilon) &= \sum_{n=0}^{\infty} (\tfrac{r}{2} + \varepsilon)^n \sum_{m=n}^{\infty} \phi(m) \tbinom{m}{n} (\tfrac{r}{2})^{m-n} \\ &= \sum_{m=0}^{\infty} \phi(m) \sum_{n=0}^{m} \tbinom{m}{n} (\tfrac{r}{2} + \varepsilon)^n (\tfrac{r}{2})^{m-n} \\ &= \sum_{m=0}^{\infty} \phi(m) (r+\varepsilon)^m. \end{aligned}$$

The convergence of this sum implies that the radius of convergence of f is larger than r, contradicting the hypotheses of the theorem. □

COROLLARY 1.5.3. *Let* x *be a positive element of a representation ring* $\mathfrak{a}$*, and let* $\mathfrak{X}$ *be a representation ideal of* $\mathfrak{a}$*. If* $x \notin \langle \mathfrak{X} \rangle$*, then the positive real number* $1/\gamma_{\mathfrak{X}}(x)$ *is a singular point of* $f_{\mathfrak{X},x}(t)$*.* □

1.6. Submultiplicative sequences

DEFINITION 1.6.1. We say that a sequence $c_0, c_1, c_2, \dots$ of non-negative real numbers is *submultiplicative* if $c_0 = 1$, and for all $m, n \geqslant 0$ we have $c_{m+n} \leqslant c_m . c_n$.

LEMMA 1.6.2. *If* $\mathfrak{X}$ *is a representation ideal in a representation ring* $\mathfrak{a}$ *and* x *is a positive element of* $\mathfrak{a}$ *that is not in* $\langle \mathfrak{X} \rangle$ *then* $\mathsf{c}_n^{\mathfrak{X}}(x)$ *is a submultiplicative sequence.*

PROOF. This follows from Lemma 1.3.3 (iv). □

LEMMA 1.6.3 (Fekete [**41**]). *If c_n is a submultiplicative sequence then*

$$\limsup_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \sqrt[n]{c_n} = \inf_{n\geqslant 1} \sqrt[n]{c_n}.$$

PROOF. It suffices to show that $\limsup_{n\to\infty} \sqrt[n]{c_n} \leqslant \inf_{n\geqslant 1} \sqrt[n]{c_n}$. If some c_n is equal to zero, then so are all subsequent ones. So we assume that all $c_n > 0$. Suppose that L is a number such that

$$\inf_{n\to\infty} \sqrt[n]{c_n} < L.$$

Then there is an $m \geqslant 1$ with $\sqrt[m]{c_m} < L$. For $n > m$ we use division with remainder to write $n = mq_m + r_m$ with $0 \leqslant r_m < m$. By the definition of submultiplicativity, we have

$$c_n = c_{mq_m+r_m} \leqslant c_{mq_m} c_{r_m} \leqslant (c_m)^{q_m} c_{r_m}.$$

Now $q_m \leqslant n/m$, so $q_m/n \leqslant 1/m$. So we have

$$\sqrt[n]{c_n} \leqslant \sqrt[m]{c_m} \sqrt[n]{c_{r_m}} < L. \sqrt[n]{c_{r_m}}.$$

As n tends to infinity, the numbers $\sqrt[n]{c_0}, \dots, \sqrt[n]{c_{m-1}}$ all tend to one, and so

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leqslant L. \qquad \square$$

THEOREM 1.6.4. *If x is a positive element of of a representation ring $\mathfrak{a}$ and $\mathfrak{X}$ is a representation ideal of $\mathfrak{a}$ then*

$$\gamma_{\mathfrak{X}}(x) = \lim_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)} = \inf_{n\geqslant 1} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}.$$

PROOF. This follows from Lemmas 1.6.2 and 1.6.3. $\square$

LEMMA 1.6.5. *Let a_n, b_n and c_n be sequences of non-negative real numbers, satisfying*

$$c_n \leqslant \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leqslant \limsup_{n\to\infty} \sqrt[n]{a_n} + \limsup_{n\to\infty} \sqrt[n]{b_n}.$$

PROOF. The statement that $\limsup\limits_{n\to\infty} \sqrt[n]{a_n} = \alpha$ implies that for all $\varepsilon > 0$, there exists m such that for all $n \geqslant m$ we have $a_n \leqslant (\alpha+\varepsilon)^n$. Introducing a positive constant A, we can assume that $a_n \leqslant A(\alpha+\varepsilon)^n$ for all $n \geqslant 0$. Similarly, if $\limsup\limits_{n\to\infty} \sqrt[n]{b_n} = \beta$ then for all $\varepsilon > 0$ there exists a positive constant B such that for all $n \geqslant 0$ we have $b_n \leqslant B(\beta+\varepsilon)^n$. Thus for all $\varepsilon > 0$ there is a positive constant $C = AB$ such that for all $n \geqslant 0$ we have

$$c_n \leqslant \sum_{i=0}^{n} \binom{n}{i} A(\alpha+\varepsilon)^i B(\beta+\varepsilon)^{n-i} = C(\alpha+\beta+2\varepsilon)^n,$$

and so $\limsup\limits_{n\to\infty} \sqrt[n]{c_n} \leqslant \alpha + \beta$. $\square$

THEOREM 1.6.6. *Let $x, y \in \mathfrak{a}_{\geqslant 0}$ and let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. Then*

$$\max\{\gamma_{\mathfrak{X}}(x), \gamma_{\mathfrak{X}}(y)\} \leqslant \gamma_{\mathfrak{X}}(x+y) \leqslant \gamma_{\mathfrak{X}}(x) + \gamma_{\mathfrak{X}}(y).$$

Proof. It follows from Lemma 1.3.3 (iv) that

$$\max\{\mathsf{c}_n^{\mathfrak{X}}(x), \mathsf{c}_n^{\mathfrak{X}}(y)\} \leqslant \mathsf{c}_n^{\mathfrak{X}}(x+y) \leqslant \sum_{i=0}^{n} \binom{n}{i} \mathsf{c}_i^{\mathfrak{X}}(x)\mathsf{c}_{n-i}^{\mathfrak{X}}(y).$$

Applying Lemma 1.6.5, we deduce that

$$\max\{\limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)}, \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(y)}\}$$
$$\leqslant \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x+y)} \leqslant \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(x)} + \limsup_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{\mathfrak{X}}(y)},$$

which are the inequalities in the statement of the theorem. □

Proposition 1.6.7. *Suppose that a_n and b_n are submultiplicative sequences. Define a sequence c_n by*

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

Then c_n is also a submultiplicative sequence, and we have

$$\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \sqrt[n]{a_n} + \lim_{n\to\infty} \sqrt[n]{b_n}.$$

Proof. Using the fact that

$$\binom{m+n}{\ell} = \sum_{i+j=\ell} \binom{m}{i}\binom{n}{j}$$

and the submultiplicativity of the sequences a_n and b_n, we have

$$\sum_{\ell=0}^{m+n} \binom{m+n}{\ell} a_\ell b_{m+n-\ell} \leqslant \left(\sum_{i=0}^{m} \binom{m}{i} a_i b_{m-i}\right) \cdot \left(\sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}\right)$$

and so the sequence c_n is submultiplicative.

By Lemma 1.6.5 we have

$$\lim_{n\to\infty} \sqrt[n]{c_n} \leqslant \lim_{n\to\infty} \sqrt[n]{a_n} + \lim_{n\to\infty} \sqrt[n]{b_n}.$$

The reverse inequality is proved similarly. If $\lim_{n\to\infty} \sqrt[n]{a_n} = \alpha$ and $\lim_{n\to\infty} \sqrt[n]{b_n} = \beta$ then given $\varepsilon > 0$ there exist positive constants A and B such that for all $n \geqslant 0$ we have $a_n \geqslant A(\alpha-\varepsilon)^n$ and $b_n \geqslant B(\beta-\varepsilon)^n$. So for all $\varepsilon > 0$ there is a positive constant $C = AB$ such that for all $n \geqslant 0$ we have

$$c_n \geqslant \sum_{i=0}^{n} \binom{n}{i} A(\alpha-\varepsilon)^i B(\beta-\varepsilon)^{n-i} = C(\alpha+\beta-2\varepsilon)^n,$$

and so

$$\lim_{n\to\infty} \sqrt[n]{c_n} \geqslant \alpha+\beta-2\varepsilon = \lim_{n\to\infty} \sqrt[n]{a_n} + \lim_{n\to\infty} \sqrt[n]{b_n} - 2\varepsilon.$$ □

Theorem 1.6.8. *Let $x \in \mathfrak{a}_{\geqslant 0}$ and let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. Then for non-negative integers a and b we have*

$$\gamma_{\mathfrak{X}}(a+bx) = a + b\gamma_{\mathfrak{X}}(x).$$

Proof. We begin with the case $a = 0$. In this case we have $\mathsf{core}_{\mathfrak{X}}(bx)^n = b^n \mathsf{core}_{\mathfrak{X}}(x^n)$ and so $\mathsf{c}_n^{\mathfrak{X}}(bx) = b^n \mathsf{c}_n^{\mathfrak{X}}(x)$. Thus

$$f_{\mathfrak{X},bx}(t) = \sum_{n=0}^{\infty} b^n \mathsf{c}_n^{\mathfrak{X}}(x) t^n = f_{\mathfrak{X},x}(bt)$$

and $\gamma_{\mathfrak{X}}(bx) = b\gamma_{\mathfrak{X}}(x)$.

It now suffices to show that $\gamma_{\mathfrak{X}}(1 + x) = 1 + \gamma_{\mathfrak{X}}(x)$. We have

$$\mathsf{c}_n^{\mathfrak{X}}(1 + x) = \sum_{i=0}^{n} \binom{n}{i} \mathsf{c}_i^{\mathfrak{X}}(x).$$

So we can apply Proposition 1.6.7 with $a_n = \mathsf{c}_n^{\mathfrak{X}}(x)$, $b_n = 1$ and $c_n = \mathsf{c}_n^{\mathfrak{X}}(1 + x)$. $\square$

1.7. The Picard group

Definition 1.7.1. Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$. We say that an element x of $\mathfrak{a}_{\succcurlyeq 0}$ is $\mathfrak{X}$-*endotrivial* if $xx^* - \mathbb{1} \in \langle \mathfrak{X} \rangle$. Note that this implies that $[xx^* : \mathbb{1}] = 1$, and hence $xx^* \succcurlyeq \mathbb{1}$. In case $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, we just say that x is *endotrivial*.

We begin by collecting some properties of $\mathfrak{X}$-endotrivial elements.

Lemma 1.7.2. (i) *If $x \in \mathfrak{a}_{\succcurlyeq 0}$ is $\mathfrak{X}$-endotrivial then x is indecomposable modulo $\mathfrak{X}$ and $x \notin \langle \mathfrak{X}_{\max} \rangle$.*

(ii) *If $x \in \mathfrak{a}_{\succcurlyeq 0}$ and xx^* is $\mathfrak{X}$-endotrivial then so is x.*

Proof. (i) If $x = y + z$ with $y, z \in \mathfrak{a}_{\succcurlyeq 0}$ and $y, z \notin \langle \mathfrak{X} \rangle$ then $xx^* = yy^* + yz^* + zy^* + zz^*$, and by Proposition 1.3.5 (i), neither yy^* nor zz^* is in $\langle \mathfrak{X} \rangle$. But then $xx^* - \mathbb{1}$ cannot be in $\langle \mathfrak{X} \rangle$, contradicting the definition of $\mathfrak{X}$-endotrivial. So x is indecomposable modulo $\mathfrak{X}$.

The fact that x is not in $\langle \mathfrak{X}_{\max} \rangle$ follows from $[xx^* : \mathbb{1}] > 0$.

(ii) If xx^* is $\mathfrak{X}$-endotrivial then xx^* is indecomposable modulo $\mathfrak{X}$, and xx^* is not in $\langle \mathfrak{X}_{\max} \rangle$, by (i). By Proposition 1.3.9, $\mathfrak{X}_{\max}$ is a representation ideal, and so x is not in $\langle \mathfrak{X}_{\max} \rangle$. Thus by definition of $\mathfrak{X}_{\max}$, we have $[xx^* : \mathbb{1}] > 0$. Since xx^* is indecomposable modulo $\mathfrak{X}$, this implies that $xx^* - \mathbb{1} \in \langle \mathfrak{X} \rangle$, and hence x is $\mathfrak{X}$-endotrivial. $\square$

Theorem 1.7.3. *Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$, and let x be a non-negative element of $\mathfrak{a}$. If $\gamma_{\mathfrak{X}}(x) = 1$ then x is $\mathfrak{X}$-endotrivial.*

Proof. Suppose that $\gamma_{\mathfrak{X}}(x) = 1$. Then $\gamma_{\mathfrak{X}}(x^*) = 1$, and by Lemma 1.4.7 we have $\gamma_{\mathfrak{X}}(xx^*) \leqslant 1$ and $\gamma_{\mathfrak{X}}(xx^*x) \leqslant 1$. If $\gamma_{\mathfrak{X}}(xx^*) = 0$ then $[xx^* : \mathbb{1}] = 0$ and so by Lemma 1.3.4 we have $xx^*x \succcurlyeq 2x$. But then using Theorem 1.6.8 we have $1 \geqslant \gamma_{\mathfrak{X}}(xx^*x) \geqslant \gamma_{\mathfrak{X}}(2x) = 2$, a contradiction. It follows that $\gamma_{\mathfrak{X}}(xx^*) = 1$ and $[xx^* : \mathbb{1}] = 1$. Thus $xx^* - \mathbb{1} \succcurlyeq 0$, and again using Theorem 1.6.8, we have $\gamma_{\mathfrak{X}}(xx^* - 1) = 0$. By Lemma 1.4.10, we have $xx^* - \mathbb{1} \in \langle \mathfrak{X} \rangle$. $\square$

Definition 1.7.4. The *big Picard group* modulo $\mathfrak{X}$ of a representation ring $\mathfrak{a}$, denoted $\mathbb{P}\mathsf{ic}_{\mathfrak{X}}(\mathfrak{a})$, is the set of $\mathfrak{X}$-endotrivial elements of $\mathfrak{a}_{\succcurlyeq 0}$. The *small Picard group* modulo $\mathfrak{X}$, denoted $\mathsf{Pic}_{\mathfrak{X}}(\mathfrak{a})$, is the set of elements of $\mathfrak{a}_{\succcurlyeq 0}$ satisfying $\gamma_{\mathfrak{X}}(x) = 1$. These are both abelian groups under multiplication modulo $\mathfrak{X}$, with the inverse of x being given by x^*. The two versions of the Picard group do not have to be equal, but in examples coming from representation theory, they often are. In Example 1.1.2 (ii), the elements u and v are endotrivial, but $\gamma(u) = \gamma(v) = d \geqslant 2$, so in this case the big and small Picard groups are not equal. We write $\mathbb{P}\mathsf{ic}_{\max}(\mathfrak{a})$, $\mathsf{Pic}_{\max}(\mathfrak{a})$ and $\mathbb{P}\mathsf{ic}(\mathfrak{a})$, $\mathsf{Pic}(\mathfrak{a})$ for the cases $\mathfrak{X} = \mathfrak{X}_{\max}$ and $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$.

The following is a generalisation of Theorem 3.5 of Carlson [**23**], and shows that the only positive idempotent modulo a representation ideal $\mathfrak{X}$ is the identity element.

THEOREM 1.7.5. *Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$. If $x \in \mathfrak{a}_{\succcurlyeq 0}$, $x \notin \langle\mathfrak{X}\rangle$, but $x^2 - x \in \langle\mathfrak{X}\rangle_{\succcurlyeq 0}$, then $x - \mathbb{1} \in \langle\mathfrak{X}\rangle_{\succcurlyeq 0}$.*

PROOF. By Lemma 1.4.6, the hypotheses imply that $\gamma_{\mathfrak{X}}(x^2) = \gamma_{\mathfrak{X}}(x)$. Using Lemmas 1.4.8, this becomes $\gamma_{\mathfrak{X}}(x)^2 = \gamma_{\mathfrak{X}}(x)$. By Lemma 1.4.10 we have $\gamma_{\mathfrak{X}}(x) \neq 0$, and hence $\gamma_{\mathfrak{X}}(x) = 1$. Using Theorem 1.7.3, we deduce that x is $\mathfrak{X}$-endotrivial. Since $x^2 - x \in \langle\mathfrak{X}\rangle$ we deduce that $(x^2 - x)x^* \in \langle\mathfrak{X}\rangle$ and so $x - \mathbb{1} \in \langle\mathfrak{X}\rangle$. Thus $[x : \mathbb{1}] \geqslant 1$, and so $x - \mathbb{1} \succcurlyeq 0$. $\square$

1.8. Elements with small gamma invariant

Let $x \in \mathfrak{a}_{\succcurlyeq 0}$ and let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. We saw in Lemma 1.4.10 that if $\gamma_{\mathfrak{X}}(x) = 0$ then $x \in \langle\mathfrak{X}\rangle$, and that otherwise $\gamma_{\mathfrak{X}}(x) \geqslant 1$. Furthermore, we saw in Theorem 1.7.3 that if $\gamma_{\mathfrak{X}}(x) = 1$ then x is $\mathfrak{X}$-endotrivial. We strengthen this in the following theorem, to show that if $1 \leqslant \gamma_{\mathfrak{X}}(x) < \sqrt{2}$ then x is $\mathfrak{X}$-endotrivial.

THEOREM 1.8.1. *If $x \in \mathfrak{a}_{\succ 0}$ is not $\mathfrak{X}$-endotrivial then $\gamma_{\mathfrak{X}}(xx^*) \geqslant 2$, and $\gamma_{\mathfrak{X}}(x) \geqslant \sqrt{2}$. In the case where $\gamma_{\mathfrak{X}}(x) = \sqrt{2}$, we have $\gamma_{\mathfrak{X}}(xx^*) = 2$, and $xx^*x - 2x \in \langle\mathfrak{X}\rangle$.*

If, furthermore, $x \notin \langle\mathfrak{X}_{\max}\rangle$ then $xx^* = 1 + y$ with $\gamma_{\mathfrak{X}}(y) = 1$. In particular, y is $\mathfrak{X}$-endotrivial.

PROOF. We divide into two cases, according as $[xx^* : \mathbb{1}] = 0$ or $[xx^* : \mathbb{1}] > 0$.

If $[xx^* : \mathbb{1}] = 0$ then $x \in \langle\mathfrak{X}_{\max}\rangle$, and by Lemma 1.3.4 we have $xx^*x \succcurlyeq 2x$ and so $(xx^*)^2 \succcurlyeq 2xx^*$. It follows using Lemma 1.3.3 (iii) that $\gamma_{\mathfrak{X}}((xx^*)^2) \geqslant 2\gamma_{\mathfrak{X}}(xx^*)$. Since $\gamma_{\mathfrak{X}}(xx^*) > 0$ it follows that $\gamma_{\mathfrak{X}}(xx^*) \geqslant 2$. Using Lemma 1.4.7 we have

$$2 \leqslant \gamma_{\mathfrak{X}}(xx^*) \leqslant \gamma_{\mathfrak{X}}(x)\gamma_{\mathfrak{X}}(x^*) = \gamma_{\mathfrak{X}}(x)^2$$

and so $\gamma_{\mathfrak{X}}(x) \geqslant \sqrt{2}$. If $\gamma_{\mathfrak{X}}(x) = \sqrt{2}$ this shows that $\gamma_{\mathfrak{X}}(xx^*) = 2$. Set $v = xx^*x - 2x$. If $v \notin \langle\mathfrak{X}\rangle$, then by Proposition 1.3.5 (i), $vv^* \notin \langle\mathfrak{X}\rangle$. This implies that $vx^*xx^* = vv^* + 2vx^* \notin \langle\mathfrak{X}\rangle$ and so $vx \notin \langle\mathfrak{X}\rangle$. By Theorem 1.4.11 (i), we then have $2\sqrt{2} \geqslant \gamma_{\mathfrak{X}}(xx^*x) = \gamma_{\mathfrak{X}}(v + 2x) \geqslant 3$, a contradiction. Thus $v \in \langle\mathfrak{X}\rangle$.

On the other hand, if $[xx^* : \mathbb{1}] > 0$ then $x \notin \langle\mathfrak{X}_{\max}\rangle$ and $xx^* = \mathbb{1} + y$ with $y \succcurlyeq 0$. Since x is not $\mathfrak{X}$-endotrivial we have $y \notin \langle\mathfrak{X}\rangle$ and hence $\gamma_{\mathfrak{X}}(y) \geqslant 1$. Thus by Theorem 1.4.11, $\gamma_{\mathfrak{X}}(xx^*) \geqslant 2$. Again this shows that $\gamma_{\mathfrak{X}}(x) \geqslant \sqrt{2}$, and if $\gamma_{\mathfrak{X}}(x) = \sqrt{2}$ then $\gamma_{\mathfrak{X}}(xx^*) = 2$ and $xx^* = \mathbb{1} + y$ with $\gamma_{\mathfrak{X}}(y) = 1$. So y is $\mathfrak{X}$-endotrivial, and in particular indecomposable modulo $\mathfrak{X}$. Furthermore, $y = y^*$, so $y^2 - \mathbb{1} \in \langle\mathfrak{X}\rangle$. Thus $(xy)x^* = y + y^2$, which is $\mathbb{1} + y$ plus an element of $\langle\mathfrak{X}\rangle$. Hence $(xy - x)x^* \in \langle\mathfrak{X}\rangle$, and so $(xy - x)(xy - x)^* \in \langle\mathfrak{X}\rangle$, and by Proposition 1.3.5 (i), $xy - x \in \langle\mathfrak{X}\rangle$. Finally, we have $xx^*x = (\mathbb{1} + y)x$ and so $xx^*x - 2x \in \langle\mathfrak{X}\rangle$. $\square$

EXAMPLE 1.8.2. Let $d \geqslant 2$ be an integer, and let $\mathfrak{a}$ be the representation ring with basis $x_0 = 1$, $x_1 = x$, $x_2 = y$ and $x_3 = \rho$ with multiplication table

$$\begin{array}{|llll}\hline 1 & x & y & \rho \\ x & 1+y+\rho & x & d\rho \\ y & x & 1 & \rho \\ \rho & d\rho & \rho & (d^2-2)\rho.\end{array}$$

with $x^* = x$, $y^* = y$, $\rho^* = \rho$, $\dim x = d$, $\dim y = 1$ and $\dim \rho = d^2 - 2$. This is an example of Theorem 1.8.1, with $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}} = \mathfrak{X}_{\max}$ and $x \notin \mathfrak{X}_{\max}$. We have $\gamma_{\mathfrak{X}}(x) = \sqrt{2}$, $\gamma_{\mathfrak{X}}(y) = 1$ and $\gamma_{\mathfrak{X}}(\rho) = 0$.

The next proposition involves the algebraic integer $\alpha \approx 2.839286755\ldots$, which is the real root of the polynomial $X^3 - 4X^2 + 4X - 2$. The other two roots are complex conjugate.

PROPOSITION 1.8.3. *If $x \in \langle \mathfrak{X}_{\max} \rangle_{\succ 0}$ and $xx^*x - 2x \notin \langle \mathfrak{X} \rangle$ then $\gamma_{\mathfrak{X}}(xx^*) \geqslant \alpha$.*

PROOF. Suppose that $\gamma_{\mathfrak{X}}(xx^*) < \alpha$. Since $x \in \langle \mathfrak{X}_{\max} \rangle$ we have $xx^*x - 2x \succcurlyeq 0$. If $xx^*x - 2x \notin \langle \mathfrak{X} \rangle$ then we may write $xx^*x = 2x + v$ with $v \in \langle \mathfrak{X}_{\max} \rangle_{\succ 0}$ and $v \notin \langle \mathfrak{X} \rangle$. By Proposition 1.3.5 (i) we have $vv^* \notin \langle \mathfrak{X} \rangle$, so $vx^*xx^* = vv^* + 2vx^* \notin \mathfrak{X}$ and hence $vx \notin \mathfrak{X}$. By Proposition 1.3.11, the elements xx^*, vx^*, vv^* and vv^*xx^* are not in $\langle \mathfrak{X} \rangle$. Since $v \in \langle \mathfrak{X}_{\max} \rangle$, we have $vv^*v = 2v + w$ with $w \succcurlyeq 0$. Thus we have

$$\begin{aligned}(xx^*)(xx^*) &= 2xx^* + vx^* \\ (xx^*)(vx^*) &= 2vx^* + vv^* \\ (xx^*)(vv^*) &= xx^*vv^* \\ (xx^*)(xx^*vv^*) &= 2vx^* + 2xx^*vv^* + wx^*.\end{aligned}$$

Ignoring the term wx^*, multiplication by xx^* on the linear span of the four elements xx^*, vx^*, vv^* and xx^*vv^* is given by the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The element $(xx^*)^n$ is therefore a sum of at least

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} A^{n-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

positive terms not in $\langle \mathfrak{X} \rangle$. Note that this argument does not depend on these four elements being linearly independent. It follows from Perron–Frobenius theory that $\gamma_{\mathfrak{X}}(xx^*)$ is at least as large as the largest positive real eigenvalue of A. The characteristic equation of A is $t^3 - 4t^2 + 4t - 2 = 0$, and so the largest positive real eigenvalue is the algebraic integer α. $\square$

THEOREM 1.8.4. *If $x \in \mathfrak{a}_{\succcurlyeq 0}$ with $2 \leqslant \gamma_{\mathfrak{X}}(xx^*) < 1 + \sqrt{2}$ then $xx^* = 1 + y$ and y is $\mathfrak{X}$-endotrivial.*

Proof. If $x \notin \mathfrak{X}_{\max}$ then $xx^* = 1 + y$ with $1 \leqslant \gamma_{\mathfrak{X}}(y) < \sqrt{2}$. By Theorem 1.8.1, y is $\mathfrak{X}$-endotrivial.

If $x \in \mathfrak{X}_{\max}$ then $xx^*x \succcurlyeq 2x$. If $xx^*x - 2x \in \langle\mathfrak{X}\rangle$ then $(xx^*)^2 - 2(xx^*) \in \langle\mathfrak{X}\rangle$ and then by Lemma 1.4.6, we have $\gamma_{\mathfrak{X}}(xx^*) = 2$. So we may suppose that $xx^*x - 2x \notin \langle\mathfrak{X}\rangle$. We are now in the situation of Proposition 1.8.3, and since $\alpha > 1 + \sqrt{2}$, we are done. $\square$

1.9. Algebraic elements

Algebraic modules for finite groups were first studied by Alperin [**4**]; see also §II.5 of Feit [**40**], as well as Berger [**15, 16**], Craven [**31, 32**], Feit [**39**], Gill [**47**]. The following definition generalises this.

Definition 1.9.1. Let $\mathfrak{X}$ be a representation ideal in a representation ring $\mathfrak{a}$. An element $x \in \mathfrak{a}_{\geqslant 0}$ is said to be *algebraic* modulo $\mathfrak{X}$ if x satisfies some monic equation with integer coefficients:

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0\mathbb{1} = 0$$

with $n \geqslant 1$, in the quotient ring $\mathfrak{a}_{\mathfrak{X}} = \mathfrak{a}/\langle\mathfrak{X}\rangle$. If $\mathfrak{X} = \varnothing$, we just say that x is *algebraic.* The *minimal equation* of x is the monic equation of least degree satisfied by x.

Lemma 1.9.2. *If x and y are algebraic modulo $\mathfrak{X}$ then so are $x + y$ and xy. For a non-negative element $x \in \mathfrak{a}$, the following are equivalent:*

(i) *x is algebraic modulo $\mathfrak{X}$.*
(ii) *The additive group of the subring of $\mathfrak{a}_{\mathfrak{X}}$ generated by x is free abelian of finite rank.*
(iii) *The additive group of the subring of $\mathfrak{a}_{\mathfrak{X}}$ generated by x and x^* is free abelian of finite rank.*
(iv) *There are only finitely many basis elements x_i, $i \in \mathfrak{I}\backslash\mathfrak{X}$ such that for some $m, n \geqslant 0$ we have $[x^m x^{*n} : x_i] > 0$.*
(v) *x is contained in a representation subring $\mathfrak{a}'$ of $\mathfrak{a}$, containing $\langle\mathfrak{X}\rangle$, such that the additive group of $\mathfrak{a}'_{\mathfrak{X}} = \mathfrak{a}'/\langle\mathfrak{X}\rangle$ is free abelian of finite rank, containing the x_i, $i \in \mathfrak{I} \setminus \mathfrak{X}$ such that for some $m, n \geqslant 0$ we have $[x^m x^{*n} : x_i] > 0$.*

Proof. (i) $\Leftrightarrow$ (ii): If x is algebraic modulo $\mathfrak{X}$ then the additive group of the subring of $\mathfrak{a}_{\mathfrak{X}}$ generated by x is spanned by $\mathbb{1}, x, x^2, \ldots, x^{n-1}$. Conversely, suppose that the additive group of the subring generated by x has finite rank. Look at the ascending chain of additive subgroups whose ith term is generated by $\mathbb{1}, x, \ldots, x^i$. This ascending chain has to terminate, so for some value of n, the element x^n is in the additive subgroup generated by $\mathbb{1}, x, \ldots, x^{n-1}$. This gives us a degree n monic equation with integer coefficients satisfied by x.

If x and y are algebraic modulo $\mathfrak{X}$ then the y is algebraic over the subring of $\mathfrak{a}_{\mathfrak{X}}$ generated by x, and so as a module over that ring, the subring generated by x and y is finitely generated. It follows that it is finitely generated as an abelian group, i.e., has finite rank, so every element of the subring generated by x and y is algebraic. In particular, $x + y$ and xy are algebraic.

(ii) $\Leftrightarrow$ (iii): If x is algebraic then so is x^*, with the same minimal equation. Therefore the subring generated by x and x^* has finite rank. Conversely, if the subring generated by x and x^* has finite rank, so does the subring generated by x.

The equivalence of (iii) and (iv) is obvious. It is also obvious that (v) implies (ii). Finally, to see that (iv) implies (v), we look at the subring $\mathfrak{a}'$ of $\mathfrak{a}$ spanned by the x_i such that $i \in \mathfrak{X}$, or x_i is projective, or for some $m, n \geqslant 0$ we have $[x^m x^{*n} : x_i] > 0$. This is a representation subring $\mathfrak{a}$ with the required properties. $\square$

LEMMA 1.9.3. *An element $x \in \mathfrak{a}_{\geqslant 0}$ is algebraic if and only if it is algebraic modulo $\mathfrak{X}_{\mathsf{proj}}$.*

PROOF. This follows from Lemma 1.9.2, since there are only finitely many projective indecomposables. $\square$

1.10. The maximal quotient

Let $\mathfrak{a}$ be a representation ring. In this section we examine some properties of the quotient $\mathfrak{a}_{\max} = \mathfrak{a}/\langle \mathfrak{X}_{\max}\rangle$, which has a basis consisting of the x_i with $i \in \mathfrak{I} \setminus \mathfrak{X}_{\max}$.

DEFINITION 1.10.1. We define $n_i = [x_i x_{i^*} : \mathbb{1}]$. Then $n_i > 0$ if and only if $i \in \mathfrak{I} \setminus \mathfrak{X}_{\max}$. If $\mathfrak{a}$ is a closed representation ring, then $n_i = 1$ for $i \in \mathfrak{I} \setminus \mathfrak{X}_{\max}$.

LEMMA 1.10.2. *For $i, j, k \in \mathfrak{I}$ we have*

$$[x_i x_j x_k : \mathbb{1}] = n_k [x_i x_j : x_{k^*}].$$

In particular, $[x_i x_j x_k : \mathbb{1}] = 0$ unless $i, j, k \notin \mathfrak{X}_{\max}$.

PROOF. This follows from

$$[x_i x_j x_k : \mathbb{1}] = \sum_{\ell} [x_i x_j : x_\ell][x_\ell x_k : \mathbb{1}]$$

and property (ii) in Definition 1.1.1. $\square$

The following is the analogue of Problem (4.12), at the end of Chapter 4 of Isaacs [**59**].

LEMMA 1.10.3. *For $i, j, k \in \mathfrak{I} \setminus \mathfrak{X}_{\max}$ we have*

$$[x_i x_j : x_k] \leqslant (\max\{n_i, n_j\}/n_k) \dim x_k.$$

In particular, if $\mathfrak{a}$ is a closed representation ring then

$$[x_i x_j : x_k] \leqslant \dim x_k.$$

PROOF. Swapping the roles of i and j if necessary, we may assume that

$$\frac{\dim x_i}{n_i} \leqslant \frac{\dim x_j}{n_j}.$$

Then using Lemma 1.10.2 we have

$$[x_i x_j : x_k] = [x_i x_j x_{k^*} : \mathbb{1}]/n_k = n_j [x_i x_{k^*} : x_j]/n_k.$$

Now

$$\dim x_i \dim x_k \geqslant [x_i x_{k^*} : x_j] \dim x_j$$

and so

$$[x_i x_j : x_k] \leqslant \frac{n_j}{n_k} \frac{\dim x_i \dim x_k}{\dim x_j} = \frac{n_i(\dim x_i/n_i)(\dim x_k/n_k)}{(\dim x_j/n_j)} \leqslant n_i \dim x_k/n_k. \qquad \square$$

The following result will be the model for the proof of the sharper Theorem 3.7.5.

LEMMA 1.10.4. *For $i, j \in \mathfrak{I} \setminus \mathfrak{X}_{\max}$ we have*

$$\sum_{k \in \mathfrak{I} \setminus \mathfrak{X}_{\max}} \frac{n_k}{n_j} [x_i x_j : x_k]^2 = [x_{i^*} x_i x_j : x_j] \leqslant (\dim x_i)^2.$$

PROOF. Using Lemma 1.10.2, we have

$$[x_{i^*} x_i x_j : x_j] = \sum_k [x_{i^*} x_j : x_k][x_i x_k : x_j] = \sum_k \frac{n_k}{n_j} [x_i x_j : x_k]^2.$$

We also have $[x_{i^*} x_i x_j : x_j] \dim x_j \leqslant (\dim x_i)^2 \dim x_j$. Now divide by $\dim x_j$. $\square$

CHAPTER 2

Commutative Banach algebras

Most of the usual examples of Banach spaces and Banach algebras are not relevant to our purpose. We therefore eschew their description in favour of brevity. The only examples of interest to us in this work are completions of representation rings, and their quotients. We examine the group ring of $\mathbb{Z}$ via the theory of Fourier series, because of the similarity to our context.

2.1. Banach spaces

Throughout, we work over the complex numbers. Most of the background we need concerning commutative Banach algebras was developed in the paper of Gelfand [**45**]. As references, we use Berberian [**14**], Folland [**42**], Gelfand, Raikov and Shilov [**46**], Kaniuth [**61**], Lax [**66**], and Rickart [**81**]. Beware that terminology varies among these sources. In particular, in [**46**] a normed vector space is assumed to be complete; we shall follow the other references for our definition of normed vector space.

Definition 2.1.1. A *normed vector space* is a complex vector space V together with a *norm* $V \to \mathbb{R}$, $v \mapsto \|v\|$, satisfying

(i) positivity: $\|x\| \geqslant 0$, with $\|x\| = 0$ if and only if $x = 0$,
(ii) subadditivity: $\|x + y\| \leqslant \|x\| + \|y\|$, and
(iii) homogeneity: $\|cx\| = |c|\|x\|$.

A norm gives rise to a metric on V via $d(x, y) = \|x - y\|$, and we say that V is *complete* with respect to the norm if this metric space is complete. A complete normed vector space is called a *Banach space*.

Definition 2.1.2. If V and W are normed spaces, and $f\colon V \to W$ is a linear map, we define the *sup norm* of f, $\|f\|_{\mathrm{sup}}$ to be the supremum of $\|f(v)\|$ as v runs over elements of V with $\|v\| \leqslant 1$. If $\|f\|_{\mathrm{sup}} < \infty$, we say that f is *bounded*.

Lemma 2.1.3. *For a linear map of normed spaces $V \to W$ the following are equivalent:*

(i) *f is bounded,*
(ii) *f is continuous,*
(iii) *f is continuous at some point $v \in V$.*

Proof. We have $\|f(x - y)\| \leqslant \|f\|_{\mathrm{sup}}\|x - y\|$, and so (i) implies (ii). Clearly (ii) implies (iii). Finally, to prove (iii) implies (i), if f is continuous at v then given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|f(v + x) - f(v)\| < \varepsilon$. By linearity, $\|f(x)\| < \varepsilon$, and so $\|f\|_{\mathrm{sup}} < \varepsilon/\delta$. Thus f is bounded. □

Example 2.1.4. We write $\ell^1(\mathbb{Z})$ for the space of functions $x\colon \mathbb{Z} \to \mathbb{C}$ with the property that $\|x\| = \sum_{n\in\mathbb{Z}} |x(n)| < \infty$. This norm makes $\ell^1(\mathbb{Z})$ into a Banach space. We can identify this with the space of absolutely convergent Fourier series $f\colon S^1 \to \mathbb{C}$, via $f(e^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} x(n)e^{\mathrm{i}n\theta}$. Here, S^1 is the unit circle in $\mathbb{C}$, parametrized by angle θ. The coefficients $x(n)$ can be recovered from f by the Fourier inversion formula:

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}\theta})e^{-\mathrm{i}n\theta}d\theta.$$

We shall continue with this as a running example throughout this chapter, and we shall make use of it in the context of representation rings in a later chapter.

Lemma 2.1.5. *The metric completion $\hat{V}$ of a normed vector space V is a Banach space in which V is a dense subspace.*

Proof. The termwise sum of two Cauchy sequences in V, and a scalar multiple of a Cauchy sequence, are again Cauchy sequences. These operations are compatible with the equivalence relation on Cauchy sequences. □

Definition 2.1.6. If W is a closed subspace of a normed vector space V then the *quotient norm* on the space V/W is defined via

$$\|x + W\| = \inf_{w\in W} \|x + w\|.$$

Lemma 2.1.7. *Let W be a closed subspace of a normed vector space V.*

(i) *Definition 2.1.6 defines a norm on the quotient space V/W.*
(ii) *If V is complete then so are W and V/W.*
(iii) *The natural map from the completion $\widehat{V/W}$ of V/W to the quotient of the completions $\hat{V}/\hat{W}$ is an isometric isomorphism.*

Proof. (i) To prove positivity of the quotient norm, let u be an element of V/W. Clearly $\|u\| \geqslant 0$. If $\|u\| = 0$ then there is a sequence of elements $v_1, v_2, \dots$ of V lying in the coset $u \in V/W$ such that $\lim\limits_{n\to\infty} \|v_n\| = 0$. Let $y_n = v_1 - v_n \in W$. Then $\lim\limits_{n\to\infty} \|v_1 - y_n\| = 0$ and so $\lim\limits_{n\to\infty} y_n = v_1$. It follows that v_1 is in the closure of W, and hence in W. Since it was chosen to be in the coset $u \in V/W$, we have $u = 0$.

To prove subadditivity, let $u_1, u_2 \in V/W$. Then by definition of quotient norm, given $\varepsilon > 0$ we may choose representatives $v_1, v_2 \in V$ of u_1, u_2 such that $\|v_1\| < \|u_1\| + \varepsilon$ and $\|v_2\| < \|u_2\| + \varepsilon$. Then

$$\|u_1 + u_2\| \leqslant \|v_1 + v_2\| \leqslant \|v_1\| + \|v_2\| < \|u_1\| + \|u_2\| + 2\varepsilon.$$

Since this is true for all $\varepsilon > 0$, we have $\|u_1 + u_2\| \leqslant \|u_1\| + \|u_2\|$.

Finally, homogeneity of the quotient norm is trivial to verify.

(ii) Completeness of W is clear since a closed subspace of a complete metric space is complete. For the quotient V/W, we argue as follows. Given a Cauchy sequence $x_n + W$ in V/W, we may replace with a subsequence satisfying $d(x_n + W, x_{n+1} + W) < 2^{-(n+1)}$, or equivalently $\|(x_n - x_{n+1}) + W\| < 2^{-(n+1)}$. By definition of the norm on the quotient, we may inductively choose a sequence of elements $w_n \in W$ so that $w_1 = 0$ and for $n \geqslant 1$ we have

$$\|(x_n + w_n) - (x_{n+1} + w_{n+1})\| = \|(x_n - x_{n+1}) + (w_n - w_{n+1})\| < 2^{-n}.$$

Then the sequence $x_n + w_n$ is a Cauchy sequence in V. If x is its limit, then $x + W$ is the limit of the $x_n + W$ in V/W.

(iii) The canonical map $V \to \hat{V} \to \hat{V}/\hat{W}$ factors through V/W, and the induced map $V/W \to \hat{V}/\hat{W}$ is an isometric embedding. Since V is dense in $\hat{V}$, V/W is dense in $\hat{V}/\hat{W}$. So $\widehat{V/W}$ is a complete, dense subspace of $\hat{V}/\hat{W}$, and hence the embedding is an isomorphism. □

2.2. Banach algebras

We shall only be interested in Banach algebras with a multiplicative identity, so we make this part of the definition. If we wish to talk of *Banach algebras without a unit*, then that is what we shall call them.

Definition 2.2.1. A (unital) *normed algebra* is an associative algebra A over $\mathbb{C}$ with identity $\mathbb{1}$, together with a norm $A \to \mathbb{R}$, $x \mapsto \|x\|$, satisfying the conditions (i)–(iii) for a normed vector space given in Definition 2.1.1, together with

(iv) submultiplicativity: for all $x, y \in A$, we have $\|xy\| \leqslant \|x\|\|y\|$.
(v) normalisation: $\|\mathbb{1}\| = 1$.

A *Banach algebra* is a normed algebra that is complete with respect to the norm. A *commutative Banach algebra* is a Banach algebra which is commutative as an abstract ring.

Lemma 2.2.2. *The metric completion $\hat{A}$ of a normed algebra A is a Banach algebra in which A is a dense subalgebra.*

Proof. By Lemma 2.1.5, $\hat{A}$ is a Banach space. The termwise product of two Cauchy sequences in A is again a Cauchy sequence in A, and this is compatible with the equivalence relation on Cauchy sequences. □

Example 2.2.3. The Banach space $\ell^1(\mathbb{Z})$ discussed in Example 2.1.4 can be made into a Banach algebra by putting on it the *convolution product* $(x * y)(n) = \sum_{i+j=n} x(i)y(j)$. This corresponds to pointwise multiplication of Fourier series, $fg(e^{\mathrm{i}\theta}) = f(e^{\mathrm{i}\theta})g(e^{\mathrm{i}\theta})$ where $f(e^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} x(n)e^{\mathrm{i}n\theta}$ and $g(e^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} y(n)e^{\mathrm{i}n\theta}$.

The identity element of this Banach algebra is the function $\mathbb{1}$ given by $\mathbb{1}(0) = 1$, $\mathbb{1}(n) = 0$ if $n \neq 0$. Let $u\colon \mathbb{Z} \to \mathbb{C}$ be the function $u(1) = 1$, $u(n) = 0$ if $n \neq 1$. It is easy to check that for $j \in \mathbb{Z}$, $u^j : \mathbb{Z} \to \mathbb{C}$ is the function $u^j(j) = 1$, $u^j(n) = 0$ if $n \neq j$. The algebra $\ell^1(\mathbb{Z})$ is the completion of $\mathbb{C}[u, u^{-1}]$ with respect to the norm, and has $\mathbb{C}[u, u^{-1}]$ as a dense subalgebra. The algebra $\ell^1(\mathbb{Z})$ is sometimes called the *Wiener algebra*.

Lemma 2.2.4. (i) *If I is a closed ideal in a Banach algebra A, then A/I is a Banach algebra with the quotient norm*

$$\|x + I\| = \inf_{y\in I} \|x + y\|.$$

The canonical map $\pi\colon A \to A/I$ is continuous, with $\|\pi\| = 1$.

(ii) *If $J \leqslant I \leqslant A$ are closed ideals then the natural map $(A/J)/(I/J) \to A/I$ is an isometric isomorphism.*

(iii) *If A is a normed algebra and I is a closed ideal then the natural map of completions $\widehat{A/I} \to \hat{A}/\hat{I}$ is an isometric isomorphism.*

PROOF. (i) By Lemma 2.1.7 (ii), A/I is a Banach space.

To prove submultiplicativity of the norm, let x_1, x_2 be elements of A/I. Given $\varepsilon > 0$ we may choose representatives $y_1, y_2 \in A$ of x_1, x_2 such that $\|y_1\| < \|x_1\|+\varepsilon$ and $\|y_2\| < \|x_2\|+\varepsilon$. Then

$$\|x_1x_2\| \leqslant \|y_1y_2\| \leqslant \|y_1\|\|y_2\| < \|x_1\|\|x_2\| + \varepsilon(\|x_1\| + \|x_2\| + \varepsilon).$$

Since this is true for all $\varepsilon > 0$, we have $\|x_1x_2\| \leqslant \|x_1\|\|x_2\|$.

The identity element of A/I is the coset of $\mathbb{1} \in A$. To prove normalisation, the unit element clearly has norm at most one, and by submultiplicativity it has norm at least one or norm zero. Since I is a proper subspace of A, the norm is not zero, therefore it is one.

Part (ii) is an elementary verification using the definition of the norm, and part (iii) follows from Lemma 2.1.7 (iii). □

2.3. Spectrum

If an element x of a Banach algebra has both a left inverse u and a right inverse v, then $u = uxv = v$. We then say that x is *invertible* with *inverse* $u = v$.

PROPOSITION 2.3.1. *Invertible elements of a Banach algebra A form an open subset. More precisely, if x is invertible and $\|x - y\| < 1/\|x^{-1}\|$ then y is invertible, and the inverse is given by an absolutely convergent power series.*

PROOF. Set $z = x^{-1}(x - y)$, so that $y = x(1 - z)$ and $\|z\| \leqslant \|x^{-1}\|\|x - y\| < 1$. Then the series $1 + z + z^2 + \cdots$ converges to an inverse of $1 - z$, and hence $x^{-1}(1 + z + z^{-1} + \cdots)$ converges to an inverse of y. □

EXAMPLE 2.3.2. Taking $x = \mathbb{1}$ in Proposition 2.3.1, we see that the open ball of radius one centred at $\mathbb{1}$ consists of invertible elements.

COROLLARY 2.3.3. *Every maximal ideal of a Banach algebra A is closed.*

PROOF. By Proposition 2.3.1, the closure of an ideal does not contain $\mathbb{1}$. A maximal ideal is therefore equal to its closure. □

DEFINITION 2.3.4. If x is an element of a Banach algebra A, we write $\mathsf{Spec}(x)$ for the *spectrum* of x, namely the set of $\lambda \in \mathbb{C}$ such that $x - \lambda\mathbb{1}$ is not invertible in A. If we wish to emphasise the ambient Banach algebra A, we write $\mathsf{Spec}_A(x)$.

REMARK 2.3.5. From its definition, the spectrum of an element x of a Banach algebra A only depends on the algebraic structure of A and not on the metric or topological structure.

EXAMPLE 2.3.6. Continuing Example 2.2.3, let us determine the spectrum of $u \in \ell^1(\mathbb{Z})$. We claim that $u - \lambda\mathbb{1}$ is invertible if and only if $|\lambda| \neq 1$. To see this, we note that for $x \in \ell^1(\mathbb{Z})$,

$$((u - \lambda\mathbb{1}) * x)(n) = x(n - 1) - \lambda x(n).$$

So for x to be inverse to $u - \lambda\mathbb{1}$, we need

$$x(n-1) - \lambda x(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Solving these equations inductively, we get

$$x(n) = \begin{cases} \lambda^{-n} x(0) & n \geqslant 0 \\ \lambda^{n-1} x(-1) & n < 0, \end{cases}$$

and $x(-1) - \lambda x(0) = 1$. For this to satisfy $\sum_{n\in\mathbb{Z}} |x(n)| < \infty$, if $|\lambda| > 1$ there is a unique solution, with $x(-1) = 0$ and $x(0) = -\lambda^{-1}$. If $|\lambda| < 1$ there is also a unique solution, with $x(-1) = 1$ and $x(0) = 0$. Finally, if $|\lambda| = 1$ there is no solution.

The conclusion is that $\mathsf{Spec}(u) = S^1$, the unit circle in $\mathbb{C}$.

Theorem 2.3.7 (Spectral Theorem, Gelfand [**45**]). *If $x \in A$ then $\mathsf{Spec}(x)$ is a non-empty closed bounded subset of $\mathbb{C}$. Outside of $\mathsf{Spec}(x)$, the inverse $(x-\lambda\mathbb{1})^{-1}$ is an analytic function of λ.*

Proof. By Proposition 2.3.1, $\mathbb{C} \setminus \mathsf{Spec}(x)$ is open, and so $\mathsf{Spec}(x)$ is closed.

We have

$$(x - \lambda\mathbb{1})^{-1} = -\lambda^{-1}(\mathbb{1} + \lambda^{-1}x + \lambda^{-2}x^2 + \dots), \tag{2.3.8}$$

and the right hand side converges if $\|\lambda^{-1}x\| < 1$, namely $|\lambda| > \|x\|$, so the spectrum is contained in a closed circle of radius $\|x\|$.

In a neighbourhood of any particular $\lambda_0 \in \mathbb{C}\setminus\mathsf{Spec}(x)$, the inverse is given by an absolutely convergent power series, which is hence an analytic function of λ.

Suppose that $\mathsf{Spec}(x)$ is empty. Then $(x - \lambda\mathbb{1})^{-1}$ is defined and analytic for all $\lambda \in \mathbb{C}$. Then by Cauchy's theorem, for any closed contour we have $\frac{1}{2\pi i}\oint (x-\lambda\mathbb{1})^{-1} d\lambda = 0$. However, taking the integral around a circular contour of radius bigger than $\|x\|$, integrating the power series (2.3.8) term by term, and using Cauchy's integral formula, we get $-\mathbb{1}$. This contradiction shows that $\mathsf{Spec}(x)$ is non-empty. □

Corollary 2.3.9 (Gelfand–Mazur Theorem). *If A is a Banach algebra in which every non-zero element is invertible then A is isomorphic to $\mathbb{C}$ as a $\mathbb{C}$-algebra.*

Proof. If $x \in A$, then by Theorem 2.3.7 there exists $\lambda \in \mathbb{C}$ such that $x - \lambda\mathbb{1}$ is not invertible. Therefore $x - \lambda\mathbb{1} = 0$ and so $x = \lambda\mathbb{1}$. So every element is a multiple of $\mathbb{1}$. It is now easy to check that the map sending x to λ is a Banach algebra isomorphism $A \to \mathbb{C}$. □

Corollary 2.3.10. *Every maximal ideal of a commutative Banach algebra A is the kernel of an algebra homomorphism $A \to \mathbb{C}$.*

Proof. By Corollary 2.3.3, every maximal ideal I is closed. By Lemma 2.2.4, the quotient A/I is a Banach algebra in which every non-zero element is invertible. So by Corollary 2.3.9 the quotient is isomorphic to $\mathbb{C}$. □

Corollary 2.3.10 is closely related to the following.

Proposition 2.3.11 (Automatic continuity). *If A is a commutative Banach algebra and $\phi\colon A \to \mathbb{C}$ is an algebra homomorphism then for all $x \in A$ we have $|\phi(x)| \leqslant \|x\|$. In particular, ϕ is continuous with respect to the norm.*

Proof. Suppose that $|\phi(x)| > \|x\|$. Setting $y = x/\phi(x)$, we have $\|y\| < 1$ and $\phi(y) = 1$. So by Proposition 2.3.1, $\mathbb{1} - y$ is invertible with inverse the sum of the convergent power series $\mathbb{1} + y + y^2 + \cdots$. On the other hand, $\phi(\mathbb{1} - y) = \phi(\mathbb{1}) - \phi(y) = 1 - 1 = 0$. This contradiction proves that $|\phi(x)| \leqslant \|x\|$. $\square$

2.4. Spectral radius

Definition 2.4.1. Let A be a commutative Banach algebra. The *spectral radius* of $x \in A$ is defined to be $\rho(x) = \sup_{\lambda \in \mathsf{Spec}(x)} |\lambda|$.

Proposition 2.4.2 (Spectral radius formula, Gelfand [**45**]). *If A is a Banach algebra and $x \in A$ then the spectral radius $\rho(x)$ is related to the norm by the formula*

$$\rho(x) = \limsup_{n\to\infty} \sqrt[n]{\|x^n\|} = \lim_{n\to\infty} \sqrt[n]{\|x^n\|} = \inf_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

Proof. By Theorem 2.3.7, $(x - \lambda\mathbb{1})^{-1}$ is an analytic function of λ outside of $\mathsf{Spec}(x)$. So to find the spectral radius of x, we must find the exact radius of convergence of the power series (2.3.8). By the Cauchy–Hadamard formula (Lemma 1.4.1) the radius of convergence in the variable λ^{-1} is given by

$$1/r = \limsup_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

Now by axiom (iv) in Definition 2.2.1, the sequence $\|x^n\|$ is a submultiplicative sequence, and so we can apply Fekete's lemma 1.6.3 to deduce that

$$\limsup_{n\to\infty} \sqrt[n]{\|x^n\|} = \lim_{n\to\infty} \sqrt[n]{\|x^n\|} = \inf_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

The series therefore converges for $|\lambda^{-1}| < r$ and diverges for $|\lambda^{-1}| > r$. Inverting, it converges for $|\lambda| > 1/r$ and diverges for $|\lambda| < 1/r$. It follows that the spectral radius is equal to $1/r$. $\square$

Remark 2.4.3. If B is a closed subalgebra of a Banach algebra A and $x \in B$ then $\mathsf{Spec}_A(x) \subseteq \mathsf{Spec}_B(x)$, since if $x - \lambda\mathbb{1}$ is not invertible in A then it is not invertible in B. Although it can happen that $\mathsf{Spec}_A(x) \neq \mathsf{Spec}_B(x)$, Proposition 2.4.2 shows that the spectral radius is the same in A as in B.

Lemma 2.4.4. *If x and y are elements of A then*

(i) $\rho(xy) \leqslant \rho(x)\rho(y)$,
(ii) $\rho(x + y) \leqslant \rho(x) + \rho(y)$.

Proof. Part (i) is straightforward, so we prove part (ii). Using Lemma 1.6.5 and Proposition 2.4.2, we have

$$\begin{aligned}\rho(x+y) &= \limsup_{n\to\infty} \sqrt[n]{\|(x+y)^n\|}\\ &= \limsup_{n\to\infty} \sqrt[n]{\|\textstyle\sum_{i=0}^n \binom{n}{i} x^i y^{n-i}\|}\end{aligned}$$

$$\leqslant \limsup_{n\to\infty} \sqrt[n]{\textstyle\sum_{i=0}^{n} \binom{n}{i} \|x^i\| \|y^{n-i}\|}$$
$$\leqslant \limsup_{n\to\infty} \sqrt[n]{\|x^n\|} + \limsup_{n\to\infty} \sqrt[n]{\|y^n\|}. \qquad \square$$

THEOREM 2.4.5. *An element x of a commutative Banach algebra A is invertible if and only if $\phi(x) \neq 0$ for all algebra homomorphisms $\phi\colon A \to \mathbb{C}$.*

PROOF. If x is invertible then for every algebra homomorphism $\phi\colon A \to \mathbb{C}$ we have $\phi(x)\phi(x^{-1}) = \phi(xx^{-1}) = \phi(\mathbb{1}) = 1$ and so $\phi(x) \neq 0$. On the other hand, if x is not invertible then x generates an ideal in A. By Zorn's lemma, this ideal is contained in some maximal ideal I of A. By Corollary 2.3.3, I is closed in A, and therefore using Lemma 2.2.4, the quotient A/I is a Banach algebra in which every non-zero element is invertible. By Corollary 2.3.9, we have $A/I \cong \mathbb{C}$. There is therefore an algebra homomorphism $\phi\colon A \to \mathbb{C}$ with kernel I, and then we have $\phi(x) = 0$. $\square$

COROLLARY 2.4.6. *If x is an element of a commutative Banach algebra A then $\mathsf{Spec}(x)$ is the set of values of $\phi(x)$ as ϕ runs over the algebra homomorphisms $A \to \mathbb{C}$. The spectral radius $\rho(x)$ is equal to $\sup\limits_{\phi\colon A\to\mathbb{C}} |\phi(x)|$.*

PROOF. It follows from Theorem 2.4.5 that $x - \lambda\mathbb{1}$ is not invertible if and only if there exists an algebra homomorphism $\phi\colon A \to \mathbb{C}$ such that $\phi(x) = \lambda$. $\square$

COROLLARY 2.4.7. *If x is an invertible element of a commutative Banach algebra A then $\mathsf{Spec}(x^{-1}) = \{\lambda^{-1} \mid \lambda \in \mathsf{Spec}(x)\}$.*

PROOF. This follows from Corollary 2.4.6, since if $\phi\colon A \to \mathbb{C}$ is an algebra homomorphism then $\phi(x^{-1}) = \phi(x)^{-1}$.

Alternatively, we have $x^{-1} - \lambda^{-1}\mathbb{1} = -x^{-1}\lambda^{-1}(x - \lambda\mathbb{1})$, so $x - \lambda\mathbb{1}$ is invertible if and only if $x^{-1} - \lambda^{-1}\mathbb{1}$ is invertible. $\square$

EXAMPLE 2.4.8. Continuing Example 2.3.6, recall that $\ell^1(\mathbb{Z})$ is the completion of the algebra $\mathbb{C}[u, u^{-1}]$ with respect to the ℓ^1 norm. A $\mathbb{C}$-algebra homomorphism $\phi\colon \mathbb{C}[u, u^{-1}] \to \mathbb{C}$ is determined by $\phi(u)$, and this may be any complex number except zero. The ones that extend to $\ell^1(\mathbb{Z})$ are those with $|\phi(u)| = 1$.

PROPOSITION 2.4.9. *Let x be an element of A with spectral radius r. Then r is an element of $\mathsf{Spec}(x)$ if and only if $\mathbb{1} + x$ has spectral radius $1 + r$.*

PROOF. The set $\mathsf{Spec}(\mathbb{1} + x)$ is the set of $1 + \lambda$ with $\lambda \in \mathsf{Spec}(x)$. This is contained in a disc of radius r centred at $1 \in \mathbb{C}$. The only point in this disc at distance $1 + r$ from the origin is the real number $1 + r$. Using the fact that $\mathsf{Spec}(x)$ is closed (Theorem 2.3.7), we see that the spectral radius of $\mathbb{1} + x$ is $1 + r$ if and only if $r \in \mathsf{Spec}(x)$. $\square$

We end this section by noting that there are various interesting subsets of the spectrum that are important in the subject.

DEFINITION 2.4.10. The *point spectrum* of an element x of a Banach algebra A is the set of $\lambda \in \mathbb{C}$ such that multiplication $x - \lambda\mathbb{1}$ is not injective; in other words, $x - \lambda\mathbb{1}$ is a divisor of zero.

The *peripheral spectrum* of x is the set of $\lambda \in \mathsf{Spec}(x)$ such that $|\lambda|$ is equal to the spectral radius of x. This is a non-empty closed subset of the circle whose radius is the spectral radius.

2.5. The structure space

DEFINITION 2.5.1. If A is a commutative Banach algebra, the *structure space*[1] $\Delta(A)$ is the set of algebra homomorphisms $\phi\colon A \to \mathbb{C}$, endowed with the *weak* topology*. This is the topology defined by the following basic open neighbourhoods of an element $\phi \in \Delta(A)$. For each finite list of elements $x_1, \dots, x_n \in A$ and for each $\varepsilon > 0$, we have a basic open neighbourhood $[\phi; x_1, \dots, x_n; \varepsilon]$ of ϕ in $\Delta(A)$ consisting of those $\phi'\colon A \to \mathbb{C}$ such that $|\phi(x_i) - \phi'(x_i)| < \varepsilon$ for $i = 1, \dots, n$.

LEMMA 2.5.2. *If we put the product topology on $Q = \prod_{x\in A}\mathbb{C}$ then the natural map $\Delta(A) \to Q$ sending ϕ to $\prod_{x\in A}\phi(x)$ is injective. The weak* topology on $\Delta(A)$ is the subspace topology for this embedding. It is the coarsest topology with the property that for all $x \in A$ the evaluation map*

$$\hat{x}\colon \Delta(A) \to \mathbb{C}$$

sending ϕ to $\phi(x)$ is continuous.

PROOF. The map $\Delta(A) \to Q$ is injective because a homomorphisms ϕ is determined by its values at elements of A. The basic open neighbourhoods $[\phi; x_1, \dots, x_n; \varepsilon]$ defining the weak* topology are the inverse images of the basic open neighbourhoods of the image of ϕ in Q in the product topology. □

PROPOSITION 2.5.3. *With the weak* topology, $\Delta(A)$ is a compact Hausdorff topological space.*

PROOF. Let Q be as in Lemma 2.5.2, and let Q' be the subset of Q given by the product over $x \in A$ of the closed disc of radius $\|x\|$ centred at the origin in $\mathbb{C}$. Then by Proposition 2.3.11, the image of $\Delta(A)$ in Q lies in Q'. By Tychonoff's theorem Q' is compact, so it remains to show that the image of $\Delta(A)$ is closed in Q'.

If $\phi = \prod_{x\in A}\phi(x)$ is in the closure of the image of $\Delta(A)$ in Q', we must show that the map $x \mapsto \phi(x)$ is an algebra homomorphism. Given $\varepsilon > 0$ and $x, y \in A$ there exists $\phi' \in \Delta(A)$ such that

$$\begin{aligned}
|\phi(\mathbb{1}) - \phi'(\mathbb{1})| &= |\phi(\mathbb{1}) - \mathbb{1}| < \varepsilon,\\
|\phi(x) - \phi'(x)| &< \varepsilon,\\
|\phi(y) - \phi'(y)| &< \varepsilon,\\
|\phi(xy) - \phi'(xy)| &= |\phi(xy) - \phi'(x)\phi'(y)| < \varepsilon.
\end{aligned}$$

Then

$$|\phi(xy) - \phi(x)\phi(y)| \leqslant |\phi(xy) - \phi'(x)\phi'(y)|$$

[1]Other names for this in the literature are the *carrier space*, the *spectrum*, the *Gelfand space*, and the *maximal ideal space* of A.

$$
\begin{aligned}
&+ |\phi'(x)\phi'(y) - \phi'(x)\phi(y)| + |\phi'(x)\phi(y) - \phi(x)\phi(y)| \\
&< \varepsilon + |\phi'(x)|\varepsilon + \varepsilon|\phi(y)| \\
&\leqslant \varepsilon(1 + \|x\| + |\phi(y)|).
\end{aligned}
$$

This is true for all $\varepsilon > 0$, and so $\phi(xy) = \phi(x)\phi(y)$. Similar arguments show that $\phi(\mathbb{1}) = 1$, $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(\lambda x) = \lambda\phi(x)$. □

DEFINITION 2.5.4. Let $C(\Delta(A), \mathbb{C})$ be the algebra of continuous maps from the compact Hausdorff space $\Delta(A)$ to $\mathbb{C}$. The *Gelfand representation*, also called the *canonical representation*, of a commutative Banach algebra A is the map $\Gamma_A \colon A \to C(\Delta(A), \mathbb{C})$ sending x to the continuous function $\hat{x} \colon \Delta(A) \to \mathbb{C}$ given by

$$\hat{x} \colon \phi \mapsto \phi(x).$$

The map $\hat{x}$ is called the *Gelfand transform* of x.

REMARK 2.5.5. By Corollary 2.4.6, for $x \in A$ the image of $\hat{x}$ is

$$\hat{x}(\Delta(A)) = \mathsf{Spec}(x) \subseteq \mathbb{C}.$$

EXAMPLE 2.5.6. Continuing Example 2.4.8, we have $\Delta(\ell^1(\mathbb{Z})) = S^1$. The map

$$\Gamma_{\ell^1(\mathbb{Z})} \colon \ell^1(\mathbb{Z}) \to C(S^1, \mathbb{C})$$

is given by $\hat{x}(e^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} x(n)e^{\mathrm{i}n\theta}$.

As an application of our running example we can prove the following theorem about Fourier series.

THEOREM 2.5.7 (Wiener). *If $f(e^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} x(n)e^{\mathrm{i}n\theta}$ with $\sum_{n\in\mathbb{Z}} |x(n)| < \infty$, and if $f(e^{\mathrm{i}\theta}) \neq 0$ for all $e^{\mathrm{i}\theta} \in S^1$, then $1/f(e^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} y(n)e^{\mathrm{i}n\theta}$ with $\sum_{n\in\mathbb{Z}} |y(n)| < \infty$.*

PROOF. By Example 2.5.6, we have $f = \hat{x}$ with $x \in \ell^1(\mathbb{Z})$. If $f(e^{\mathrm{i}\theta}) \neq 0$ for all $e^{\mathrm{i}\theta} \in S^1$ then by Theorem 2.4.5, x is invertible in $\ell^1(\mathbb{Z})$. Let y be its inverse. Then $1/f = \hat{y}$. □

2.6. Closed ideals and the structure space

Let A be a commutative Banach algebra and let I be a closed ideal in A. Write $i \colon I \to A$ for the inclusion and $q \colon A \to A/I$ for the quotient homomorphism.

We define $\Delta(I)$ to be the set of *non-zero* homomorphisms from I to $\mathbb{C}$, where I is regarded as a Banach algebra without a unit.

LEMMA 2.6.1. *If $\phi \colon I \to \mathbb{C}$ is a non-zero homomorphism then ϕ extends uniquely to a homomorphism $A \to \mathbb{C}$.*

PROOF. Since ϕ is non-zero and $\mathbb{C}$ is a field, there exists an element $y \in I$ such that $\phi(y) = 1$. If $\psi \colon A \to \mathbb{C}$ is an extension of ϕ to A then for $x \in A$ we have $xy \in I$ and

$$\psi(x) = \psi(x)\phi(y) = \psi(x)\psi(y) = \psi(xy) = \phi(xy).$$

So ψ is determined by ϕ. It remains to check that the ψ defined this way is an algebra homomorphism. We have

$$\psi(\lambda x) = \phi(\lambda xy) = \lambda\phi(xy) = \lambda\psi(x)$$

$$
\begin{aligned}
\psi(x_1+x_2) &= \phi((x_1+x_2)y) = \phi(x_1y+x_2y)\\
&= \phi(x_1y)+\phi(x_2y) = \psi(x_1)+\psi(x_2)\\
\psi(x_1x_2) &= \phi(x_1x_2y) = \phi(x_1x_2y)\phi(y)\\
&= \phi(x_1yx_2y) = \phi(x_1y)\phi(x_2y) = \psi(x_1)\psi(x_2).
\end{aligned}
$$

Thus ψ is indeed an algebra homomorphism. □

We can also look at the subalgebra $I_+ = \mathbb{C}\oplus I$ of A generated by $\mathbb{1}$ and I. Every non-zero homomorphism $I \to \mathbb{C}$ extends uniquely to an algebra homomorphism $I_+ \to \mathbb{C}$, since $\mathbb{1}$ has to go to $1 \in \mathbb{C}$. But there is one more algebra homomorphism denoted $\star\colon I_+ \to \mathbb{C}$, which is identically zero on I. Thus $\Delta(I_+) = \Delta(I) \cup \{\star\}$.

There are obvious maps $q^*\colon \Delta(A/I) \to \Delta(A)$ and $i^*\colon \Delta(A) \to \Delta(I_+)$. The following lemma shows us that these maps allow us to identify $\Delta(I_+)$ with the quotient space $\Delta(A)/\Delta(A/I)$ with basepoint $\star$.

LEMMA 2.6.2. (i) *The map q^* is injective, open and continuous with image*

$$q^*(\Delta(A/I)) = \{\phi \in \Delta(A) \mid i^*(\phi) = \star\}.$$

(ii) *$\Delta(I_+)$ is the one point compactification of $\Delta(I)$ with basepoint $\star$. The map i^* is surjective, and identifies $\Delta(I_+)$ with the quotient of $\Delta(A)$ by the image of q^*.*

PROOF. (i) Clearly q^* is injective with image $(i^*)^{-1}(\star) \subseteq \Delta(A)$. To check that it is a homeomorphic onto its image, we note that

$$
\begin{aligned}
q^*([\phi; x_1+I,\ldots,x_n+I;\varepsilon]) &= \{q^*(\phi') \mid |\phi'(x_i+I)-\phi(x_i+I)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\}\\
&= \{\psi \mid |\psi(x_i)-\phi(q(x_i))| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\}\\
&= [q^*(\phi); x_1,\ldots,x_n;\varepsilon].
\end{aligned}
$$

(ii) Denote the basic open neighbourhoods of ϕ in $\Delta(I_+)$ by $[\phi; x_1,\ldots,x_n;\varepsilon]_+$. Then for $\phi \in \Delta(I)$ we have

$$
[\phi; x_1,\ldots,x_n;\varepsilon]_+ = \begin{cases} [\phi; x_1,\ldots,x_n;\varepsilon] & |\phi(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\\ [\phi; x_1,\ldots,x_n;\varepsilon] \cup \{\star\} & \text{otherwise.} \end{cases}
$$

For $\star$ we have

$$
\begin{aligned}
[\star; x_1,\ldots,x_n;\varepsilon]_+ &= \{\star\} \cup \{\phi \in \Delta(I) \mid |\phi(x_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\}\\
&= \bigcap_{i=1}^{n} [\star; x_i;\varepsilon]_+.
\end{aligned}
$$

For each x_i, the complement of $[\star; x_i;\varepsilon]_+$ is closed in $\Delta(I_+)$ and hence compact by Proposition 2.5.3. So the complement of $[\star; x_1,\ldots,x_n;\varepsilon]$ in $\Delta(I_+)$ is a finite union of compact sets, and hence compact. It follows that $\Delta(I_+)$ is the one point compactification of $\Delta(I)$.

Lemma 2.6.1 shows that every homomorphism $\phi\colon I_+ \to \mathbb{C}$ apart from $\star$ extends uniquely to an algebra homomorphism $\hat{\phi}\colon A \to \mathbb{C}$ which is not in the image of q^*. On the other hand, $\star\colon I_+ \to \mathbb{C}$ is the image in $\Delta(I_+)$ of every element of $q^*(\Delta(A/I))$.

Putting these statements together with the description above of the topology on $\Delta(I_+)$, we see that $\Delta(I_+)$ is homeomorphic with the quotient of $\Delta(A)$ by the image of q^*. $\square$

2.7. Systems of generators

Definition 2.7.1. We say that a set K of elements of A is a *system of generators* if the smallest closed subalgebra of A containing K (and the identity element) is the entire algebra A. In other words, the subalgebra generated by K is dense in A.

Lemma 2.7.2. *If K is a system of generators of A then the weak* topology on $\Delta(A)$ is generated by the open sets $[\phi; x_1, \dots, x_n; \varepsilon]$ with $x_1, \dots, x_n \in K$.*

Proof. We must show that every open neighbourhood $[\phi; y_1, \dots, y_m; \varepsilon]$ of ϕ contains one of these open sets. Since polynomials in elements of K are dense in A, there is a finite list $x_1, \dots, x_n$ of elements of K and polynomials $f_1, \dots, f_m$ in n variables such that for $1 \leqslant i \leqslant m$ we have $\|y_i - f_i(x_1, \dots, x_n)\| < \varepsilon/3$. Then by Proposition 2.3.11, for any ϕ' we have

$$|\phi'(y_i) - f_i(\phi'(x_1), \dots, \phi'(x_n))| < \varepsilon/3.$$

Choose $\delta > 0$ such that

$$[\phi; x_1, \dots, x_n; \delta] \subseteq [\phi; f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n); \varepsilon/3].$$

Then for all $\phi' \in [\phi; x_1, \dots, x_n; \delta]$ and $1 \leqslant i \leqslant m$ we have

$$|f_i(\phi'(x_1), \dots, \phi'(x_n)) - f_i(\phi(x_1), \dots, \phi(x_n)| < \varepsilon/3$$

and hence

$$\begin{aligned} |\phi'(y_i) - \phi(y_i)| \leqslant{} & |\phi'(y_i) - f_i(\phi'(x_1), \dots, \phi'(x_n))| \\ & + |f_i(\phi'(x_1), \dots, \phi'(x_n)) - f_i(\phi(x_1), \dots, \phi(x_n))| \\ & + |f_i(\phi(x_1), \dots, \phi(x_n)) - \phi(y_i)| \\ <{} & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

It follows that $[\phi; x_1, \dots, x_n; \delta] \subseteq [\phi; y_1, \dots, y_m; \varepsilon]$. $\square$

Definition 2.7.3. If K is a system of generators for A and $K = \{y_1, \dots, y_n\}$ is a finite set, we say that A is *finitely generated*.

Remark 2.7.4. If A is finitely generated by $K = \{y_1, \dots, y_n\}$, then we have a map $\Delta(A) \to \mathbb{C}^n$ given by $\hat{y}_1, \dots, \hat{y}_n$. This is a homeomorphism from $\Delta(A)$ to a compact subset of $\mathbb{C}^n$. It turns out that the image can be characterised by the property of being "polynomially convex," a notion weaker than convexity, see Stout [**89**]. The complement of a polynomially convex subset of $\mathbb{C}^n$ is always $(n-1)$-connected, by a theorem of Forsternič [**43**].

2.8. The Jacobson radical

DEFINITION 2.8.1. An element x of a Banach algebra A is said to be *quasi-nilpotent* if $\lim_{n\to\infty} \sqrt[n]{\|x^n\|} = 0$. In the literature, this is sometimes also called *topologically nilpotent.* It follows from Lemma 1.6.5 that the sum of two quasi-nilpotent elements is quasi-nilpotent. Since a linear multiple of a quasi-nilpotent element is also nilpotent, it follows that the quasi-nilpotent elements form a linear subspace. In fact, we shall see that they form a closed ideal.

DEFINITION 2.8.2. The *Jacobson radical* of a ring A, denoted $J(A)$, is the intersection of its maximal right ideals, or equivalently the intersection of its maximal left ideals. The ring A is said to be *semisimple* if $J(A) = 0$.

PROPOSITION 2.8.3. *In the case where A is a commutative Banach algebra, the maximal ideals are closed, and are the kernels of (automatically continuous) algebra homomorphisms $A \to \mathbb{C}$. The Jacobson radical $J(A)$ is the intersection of the kernels of these algebra homomorphisms. In particular, A is semisimple if and only if its elements are separated by algebra homomorphisms $A \to \mathbb{C}$.*

PROOF. This follows from Corollaries 2.3.3 and 2.3.10 and Proposition 2.3.11. $\square$

THEOREM 2.8.4. *For an element x in a commutative Banach algebra A, the following are equivalent:*

(i) *x is quasi-nilpotent.*
(ii) *The spectral radius of x is zero.*
(iii) *For every algebra homomorphism $\phi\colon A \to \mathbb{C}$ we have $\phi(x) = 0$.*
(iv) *The image of $\hat{x}$ is $\{0\}$.*
(v) *x is in the Jacobson radical $J(A)$.*

PROOF. The equivalence of (i) and (ii) follows from the spectral radius formula, Proposition 2.4.2. The equivalence of (ii), (iii) and (iv) follows from Corollary 2.4.6. The equivalence of (iii) and (v) follows from Proposition 2.8.3. $\square$

2.9. Banach $*$-algebras

DEFINITION 2.9.1. A *$*$-algebra* A is an algebra with a star operation $x \mapsto x^*$ satisfying

(i) involutory: for all $x \in A$ we have $x^{**} = x$.
(ii) antilinear: for all $\lambda \in \mathbb{C}$ and $x, y \in A$ we have $(x+y)^* = x^* + y^*$ and $(\lambda x)^* = \bar{\lambda} x^*$.
(iii) multiplicative antiautomorphism: for all $x, y \in A$ we have $(xy)^* = y^* x^*$.

A *normed $*$-algebra* is a normed algebra which is simultaneously a $*$-algebra, in such a way that the star operation is continuous with respect to the norm. A *Banach $*$-algebra* A is a normed $*$-algebra that is complete with respect to the norm.

LEMMA 2.9.2. *The metric completion $\hat{A}$ of a normed $*$-algebra A is a Banach $*$-algebra in which A is a dense subalgebra.*

PROOF. By Lemma 2.2.2, $\hat{A}$ is a Banach algebra. Applying the star operation to a Cauchy sequence yields another Cauchy sequence. This preserves the equivalence relation, and hence defines a star operation on $\hat{A}$. The properties in Definition 2.9.1 for this star operation on $\hat{A}$ follow from those properties on A. $\square$

LEMMA 2.9.3. *If x is an element of a Banach $*$-algebra A then* $\mathsf{Spec}(x^*) = \overline{\mathsf{Spec}(x)}$.

PROOF. This follows from the fact that $x - \lambda\mathbb{1}$ is invertible if and only if

$$(x - \lambda\mathbb{1})^* = x^* - \bar{\lambda}\mathbb{1}$$

is invertible. $\square$

EXAMPLE 2.9.4. The Banach algebra $\ell^1(\mathbb{Z})$ of Example 2.2.3 is a Banach $*$-algebra with star operation

$$x^*(n) = \overline{x(-n)}.$$

This star operation swaps u and u^{-1}. In terms of Fourier series, this star operation is given by $f^*(e^{\mathrm{i}\theta}) = \overline{f(e^{-\mathrm{i}\theta})}$. We shall see later in this section that this is an example of a *symmetric* Banach $*$-algebra.

DEFINITION 2.9.5. A $*$-*ideal* in a Banach $*$-algebra is an ideal I such that for all $x \in I$ we have $x^* \in I$.

LEMMA 2.9.6.
(i) *If I is a closed $*$-ideal in a Banach $*$-algebra A, then A/I is a Banach $*$-algebra with the quotient norm, and with the star operation $(x + I)^* = x^* + I$.*
(ii) *If $J \leqslant I \leqslant A$ are closed $*$-ideals then then natural map $(A/J)/(I/J) \to A/I$ is an isometric isomorphism of Banach $*$-algebras.*
(iii) *If A is a normed $*$-algebra and I is a closed ideal then the natural map of completions $\widehat{A/I} \to \hat{A}/\hat{I}$ is an isometric isomorphism of Banach $*$-algebras.*

PROOF. The proof is the same as the proof of Lemma 2.2.4, but keeping track of the star operation. $\square$

DEFINITION 2.9.7. An element x is *self-conjugate* if $x = x^*$.

REMARK 2.9.8. Given any element x, we can write x uniquely as $y + \mathrm{i}z$ where y and z are self-conjugate elements. Namely, we have $y = (x + x^*)/2$ and $z = (x - x^*)/2\mathrm{i}$. Then we have $x^* = y - \mathrm{i}z$.

DEFINITION 2.9.9. If A is a commutative Banach $*$-algebra and $\phi\colon A \to \mathbb{C}$ is an algebra homomorphism, we define the *conjugate* of ϕ to be the algebra homomorphism $\phi^*\colon A \to \mathbb{C}$ defined by

$$\phi^*(x) = \overline{\phi(x^*)}.$$

This defines a continuous involutary automorphism on the structure space.

PROPOSITION 2.9.10. *The following conditions on a commutative Banach $*$-algebra A are equivalent:*

(i) *Every algebra homomorphism $\phi\colon A \to \mathbb{C}$ is self conjugate. In other words, for every $x \in A$ we have*

$$\phi(x^*) = \overline{\phi(x)}.$$

(ii) *For every algebra homomorphism $\phi\colon A \to \mathbb{C}$, and every self-conjugate $x \in A$, $\phi(x)$ is real.*
(iii) *Every element of the form $x^*x + \mathbb{1}$ is invertible in A.*
(iv) *If $x \in A$ is self-conjugate then $x - \mathsf{i}\mathbb{1}$ is invertible.*
(v) *For every $x \in A$, the spectral radius satisfies $\rho(x^*x) = \rho(x)^2$.*

Proof. Remark 2.9.8 shows that (i) and (ii) are equivalent.

If (i) holds, then for $x \in A$ and $\phi\colon A \to \mathbb{C}$ we have

$$\phi(x^*x + \mathbb{1}) = \phi(x^*)\phi(x) + 1 = \overline{\phi(x)}\phi(x) + 1 = |\phi(x)|^2 + 1 > 0.$$

By Theorem 2.4.5 it follows that for every $x \in A$ the element $x^*x + \mathbb{1}$ is invertible, and so (iii) holds.

Next, we show that (iii) implies (iv). If x is self-conjugate then

$$(x - \mathsf{i}\mathbb{1})^*(x - \mathsf{i}\mathbb{1}) = (x + \mathsf{i}\mathbb{1})(x - \mathsf{i}\mathbb{1}) = x^2 + \mathbb{1} = x^*x + \mathbb{1}$$

is invertible, and hence so is $x - \mathsf{i}\mathbb{1}$.

Nest, we prove that (iv) implies (ii). If x is self-conjugate, $\phi\colon A \to \mathbb{C}$, and $a + \mathsf{i}b \in \mathbb{C}$ with $b \neq 0$, we must show that $\phi(x) \neq a + \mathsf{i}b$. So it is enough to show that $x - (a + \mathsf{i}b)\mathbb{1}$ is invertible. This follows from (iv) by writing $x - (a + \mathsf{i}b)\mathbb{1} = b((x - a\mathbb{1})/b - \mathsf{i}\mathbb{1})$.

To prove that (i) implies (v), if $\phi\colon A \to \mathbb{C}$ is an algebra homomorphism and (i) holds then

$$\phi(x^*x) = \phi(x^*)\phi(x) = \overline{\phi(x)}\phi(x) = |\phi(x)|^2.$$

So by Corollary 2.4.6,

$$\rho(x^*x) = \sup_{\phi\colon A \to \mathbb{C}} |\phi(x^*x)| = \sup_{\phi\colon A \to \mathbb{C}} |\phi(x)|^2 = \rho(x)^2.$$

Finally, to prove that (v) implies (ii), we suppose that (v) holds but (ii) does not hold, and deduce a contradiction. So there exists a self-conjugate element x such that $\phi(x) = \alpha + \beta\mathsf{i}$ with α and β real and β non-zero. Replacing x by $\beta^{-1}(x - \alpha\mathbb{1})$, we may assume that $\phi(x) = \mathsf{i}$. If a is a positive real number then $(a\mathbb{1} - \mathsf{i}x)^* = a\mathbb{1} + \mathsf{i}x$. We have $\phi(a\mathbb{1} - \mathsf{i}x) = a + 1$, and so by Corollary 2.4.6, it follows that $a + 1 \leqslant \rho(a\mathbb{1} - \mathsf{i}x)$. If (v) holds then using Lemma 2.4.4 we have

$$(a + 1)^2 \leqslant \rho(a\mathbb{1} - \mathsf{i}x)^2 = \rho((a\mathbb{1} + \mathsf{i}x)(a\mathbb{1} - \mathsf{i}x)) = \rho(a^2\mathbb{1} + x^2) \leqslant a^2 + \rho(x)^2,$$

and hence $2a + 1 \leqslant \rho(x)^2$. This holds for all $a > 0$, which is the desired contradiction. □

Remark 2.9.11. Condition (ii) of Proposition 2.9.10 may be interpreted as saying that the spectrum of a self-conjugate element of a Banach $*$-algebra satisfying these equivalent conditions is real (cf. Corollary 2.4.6), whereas for a more general Banach $*$-algebra the spectrum of a self-conjugate element is merely symmetric about the real axis (Lemma 2.9.3).

Definition 2.9.12. A Banach $*$-algebra is said to be *symmetric* if condition (iii) of Proposition 2.9.10 is satisfied, and *Hermitian* if every self-conjugate element has a real spectrum. These conditions are equivalent for commutative Banach $*$-algebras by the Proposition, but are not equivalent without the commutativity assumption. Nonetheless, a theorem of Shirali [**86, 87**] shows that even for a non-commutative Banach $*$-algebra, Hermitian implies symmetric.

EXAMPLE 2.9.13. For an example of a symmetric Banach $*$-algebra, see the algebra $\ell^1(\mathbb{Z})$ of Example 2.9.4. It is an easy exercise to check that this satisfies condition (i) of Proposition 2.9.10.

For an example of a non-symmetric Banach $*$-algebra, consider the algebra of continuous functions from the closed disc $D = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ to $\mathbb{C}$ which are holomorphic on the interior of D, with pointwise addition and multiplication, and with $f^*(z) = \overline{f(\bar{z})}$. The spectrum of f is its image, so for example the element $f(z) = z$ is self-conjugate; its spectrum is symmetric about the real axis, but not real.

COROLLARY 2.9.14. *If A is a commutative symmetric Banach $*$-algebra an I is a closed $*$-ideal then A/I is a commutative symmetric Banach $*$-algebra.*

PROOF. Condition (iv) of Proposition 2.9.10 is inherited by A/I. □

THEOREM 2.9.15. *If e is an idempotent in a commutative symmetric Banach $*$-algebra then $e = e^*$.*

PROOF. Let

$$z = 1 + (e^* - e)^*(e^* - e) = 1 - e - e^* + 2ee^*.$$

Then we have $z = z^*$ and

$$ez = e - e - ee^* + 2ee^* = ee^*.$$

Similarly, $e^*z = ee^*$, so $ez = e^*z$. But z is invertible by Proposition 2.9.10, and so $e = e^*$. □

2.10. C^*-algebras

DEFINITION 2.10.1. A C^*-*algebra* is a Banach $*$-algebra A in which for all $x \in A$ we have $\|x^*x\| = \|x\|^2$.

THEOREM 2.10.2. *If A is a commutative C^*-algebra then A is a symmetric Banach $*$-algebra.*

PROOF. We shall verify that condition (iv) of Proposition 2.9.10 holds. If x is self-adjoint and $x - \mathrm{i}\mathbb{1}$ is not invertible then i is in the spectrum of x, and hence for every real $a > 0$, $a+1 = a - \mathrm{i}^2$ is in the spectrum of $a\mathbb{1} - \mathrm{i}x$. So by Corollary 2.4.6 we have $a+1 \leqslant \|a\mathbb{1} - \mathrm{i}x\|$. Hence

$$(a+1)^2 \leqslant \|a\mathbb{1} - \mathrm{i}x\|^2 = \|(a\mathbb{1} + \mathrm{i}x)(a\mathbb{1} - \mathrm{i}x)\| = \|a^2\mathbb{1} + x^2\| \leqslant a^2 + \|x\|^2.$$

So we have $\|x\|^2 \geqslant 2a + 1$ for every $a > 0$, which is absurd. Hence $x - \mathrm{i}\mathbb{1}$ is invertible. □

EXAMPLE 2.10.3. The symmetric Banach $*$-algebra $\ell^1(\mathbb{Z})$ (see Example 2.9.4) is the completion of $\mathbb{C}[u, u^{-1}]$ with respect to the ℓ^1 norm, with star operation sending u to u^{-1}. To see that this is not a C^*-algebra, we let $x = u^2 + u - 1$. Then $\|x^2\| = 7$ while $\|xx^*\| = 5$.

On the other hand, if Z is any compact Hausdorff topological space then the algebra $C(Z)$ of continuous functions on Z is a commutative C^*-algebra. For the norm we take $\|x\| = \sup_{z\in Z} |x(z)|$, and the $*$ operation is defined by $x^*(z) = \overline{x(z)}$. We have

$$\|x^*x\| = \sup_{z\in Z}(\overline{x(z)}x(z)) = \sup_{z\in Z} |x(z)|^2 = \big(\sup_{z\in Z} |x(z)|\big)^2 = \|x\|^2.$$

Remark 2.10.4. The reader will notice that the proof of Theorem 2.10.2 is similar to the proof of (v) implies (ii) in Proposition 2.9.10. Indeed, in the light of the following theorem, we could have just invoked that proposition.

Theorem 2.10.5. *If x is an element of a commutative C^*-algebra then its spectral radius is equal to $\|x\|$.*

Proof. We use the spectral radius formula, Proposition 2.4.2. Let $y = x^*x$, so that $y^* = y$. Then the spectral radius of y is

$$\rho(y) = \lim_{n\to\infty} \sqrt[n]{\|(x^*x)^n\|} = \lim_{n\to\infty} \sqrt[n]{\|(x^n)^*(x^n)\|} = \lim_{n\to\infty} \sqrt[n]{\|x^n\|^2} = \Big(\lim_{n\to\infty} \sqrt[n]{\|x^n\|}\Big)^2 = \rho(x)^2.$$

So it suffices to prove that $\rho(y) = \|y\| = \|x\|^2$.

We have $\|y^2\| = \|y^*y\| = \|y\|^2$, and by induction, for all $k \geqslant 0$ we have $\|y^{2^k}\| = \|y\|^{2^k}$. Thus

$$\rho(y) = \lim_{k\to\infty} \sqrt[2^k]{\|y^{2^k}\|} = \lim_{k\to\infty} \sqrt[2^k]{\|y\|^{2^k}} = \|y\|.$$ □

Corollary 2.10.6. *The Jacobson radical of a commutative C^*-algebra is zero. Thus the only quasi-nilpotent element is zero.*

Proof. By Theorems 2.8.4, if x is in the Jacobson radical then the spectral radius is zero. By Theorem 2.10.5 the spectral radius of x is $\|x\|$ so $\|x\| = 0$. This implies that $x = 0$. By Theorem 2.8.4, quasi-nilpotent elements lie in the Jacobson radical, and therefore the only quasi-nilpotent element is zero. □

2.11. Hilbert space

Definition 2.11.1. A *Hilbert space* is a complex vector space H with an inner product $\langle -, - \rangle \colon H \times H \to \mathbb{C}$, satisfying

(i) Linearity in the first variable: $\langle ax, y\rangle = a\langle x, y\rangle$ and $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$,
(ii) Conjugate symmetry: $\langle y, x\rangle = \overline{\langle x, y\rangle}$,
(iii) Positive definiteness: if $x \neq 0$ then $\langle x, x\rangle > 0$,
(iv) Completeness: With respect to the norm coming from the inner product $|x| = \sqrt{\langle x, x\rangle}$, H is complete.

Thus a Hilbert space is a particularly rigid kind of Banach space.

Lemma 2.11.2. *If $x \in H$ then $|x| = \sup_{|w|=1} |\langle x, w\rangle| = \sup_{|w|=1} |\langle w, x\rangle|$.*

PROOF. We prove the first equality, as the second follows from the fact that by conjugate symmetry we have $|\langle w,x\rangle| = |\langle x,w\rangle|$.

If $w = x/|x|$ then $|w| = 1$ and $\langle x,w\rangle = |x|$. Conversely, for any $w \in H$ with $|w| = 1$ we set $w' = w - \frac{\langle x,w\rangle}{\langle x,x\rangle}x$. Then

$$\langle x,w'\rangle = \langle x,w\rangle - \frac{\langle x,w\rangle}{\langle x,x\rangle}\langle x,x\rangle = 0.$$

So

$$1 = |w|^2 = |w'|^2 + \left|\frac{\langle x,w\rangle}{\langle x,x\rangle}x\right|^2,$$

and hence $\left|\frac{\langle x,w\rangle}{\langle x,x\rangle}x\right| \leqslant 1$. This implies that $|\langle x,w\rangle| \leqslant \frac{|\langle x,x\rangle|}{|x|} = |x|$. □

LEMMA 2.11.3 (Parallelogram identity). *If u and v are elements of a Hilbert space H then*

$$|u+v|^2 + |u-v|^2 = 2|u|^2 + 2|v|^2.$$

PROOF. Expand $\langle u+v,u+v\rangle$ and $\langle u-v,u-v\rangle$ using bilinearity. □

LEMMA 2.11.4. *If V is a closed subspace of a Hilbert space H then every coset of V in H contains a unique element orthogonal to V.*

PROOF. Let w be an element of the coset. We let d be the infimum of the values of $|w|$ as w runs over elements the coset. Then there is a sequence of elements $w_1, w_2, \dots$ of such elements with $\lim_{i\to\infty}|w_i| = d$. We claim that $w_1, w_2, \dots$ is a Cauchy sequence in H. To see this, we apply the parallelogram identity to w_i and w_j:

$$|w_i + w_j|^2 + |w_i - w_j|^2 = 2|w_i|^2 + 2|w_j|^2,$$

which gives

$$|w_i - w_j|^2 = 2|w_i|^2 + 2|w_j|^2 - 4|\tfrac{1}{2}(w_i + w_j)|^2.$$

Since $\frac{1}{2}(w_i + w_j)$ is an element of the same coset, we have $|\frac{1}{2}(w_i + w_j)| \geqslant d$. If $|w_i| < d + \varepsilon$ and $|w_j| < d + \varepsilon$, we obtain

$$|w_i - w_j|^2 < 2(d+\varepsilon)^2 + 2(d+\varepsilon)^2 - 4d^2 = 4\varepsilon(2d+\varepsilon).$$

So as $\varepsilon \to 0$ we have $|w_i - w_j| \to 0$, and $w_1, w_2, \dots$ is a Cauchy sequence. Now H is complete and V is closed, so V is complete and hence the coset is complete. So this Cauchy sequence has a limit w, and we have $|w| = d$.

If $0 \neq v \in V$ let $w' = w - \frac{\langle v,w\rangle}{\langle v,v\rangle}v$. Then $\langle v,w'\rangle = 0$ and so

$$|w|^2 = |w'|^2 + \left|\frac{\langle v,w\rangle}{\langle v,v\rangle}v\right|^2.$$

Since $|w'|^2 \geqslant |w|^2$ it follows that $\langle v,w\rangle = 0$. Thus w is orthogonal to V. If w_0 is another element of the coset orthogonal to V then $w - w_0$ is in V and orthogonal to V, and hence equal to zero. □

THEOREM 2.11.5 (Fréchet–Riesz). *Given a continuous $\mathbb{C}$-algebra homomorphism $\phi\colon H \to \mathbb{C}$ there exists a unique element $y \in H$ such that for all $x \in H$ $\phi(x) = \langle x,y\rangle$.*

PROOF. Let V be the kernel of ϕ. Since ϕ is continuous, V is a closed subspace of H of codimension one. The set of elements $x \in H$ such that $\phi(x) = 1$ is a coset of V. Using Lemma 2.11.4 there is an element y in this coset which is orthogonal to all $v \in V$. Every element of H can be written as $v + \lambda y$ with $v \in V$ and $\lambda \in \mathbb{C}$, and we have

$$\phi(v + \lambda y) = \phi(v) + \lambda\phi(y) = \lambda = \langle v + \lambda y, y\rangle.$$

Finally, for uniqueness, if y' is another then $y - y'$ is both in V and orthogonal to V, and hence equal to zero. $\square$

DEFINITION 2.11.6. If $f\colon H \to H$ is a linear transformation, we set

$$\|f\|_{\sup} = \sup_{|x|=1} |f(x)|.$$

If $\|f\|_{\sup} < \infty$, we say that f is a *bounded operator* on H. By Lemma 2.1.3, bounded operators are continuous. We write $\mathscr{L}(H)$ for the set of bounded operators on H. We shall see in Theorem 2.11.8 that $\mathscr{L}(H)$ has the structure of a C^*-algebra.

THEOREM 2.11.7 (Adjoints). *Let $f\colon H \to H$ be a bounded operator. Then there is a unique bounded operator f^*, called the* adjoint *of f, such that for all x and y in H we have*

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle.$$

We have $f^{**} = f$, $(\lambda f + \mu f')^* = \bar\lambda f^* + \bar\mu f'^*$, $(f \circ f')^* = f'^* \circ f^*$, $\|f^* \circ f\|_{\sup} = \|f\|^2_{\sup}$, and $\|f^*\|_{\sup} = \|f\|_{\sup}$.

PROOF. For $x \in H$, the map $\phi\colon H \to \mathbb{C}$ given by $\phi(y) = \langle f(x), y\rangle$ is a continuous $\mathbb{C}$-algebra homomorphism. By Theorem 2.11.5, there exists a unique element $z \in H$ such that for all $x \in H$ we have $\langle f(x), y\rangle = \langle x, z\rangle$. We define $f^*(x) = z$. To see that f^* is linear, we have

$$\begin{aligned}\langle x, f^*(\lambda y + \mu z)\rangle &= \langle f(x), \lambda y + \mu z\rangle \\ &= \bar\lambda\langle f(x), y\rangle + \bar\mu\langle f(x), z\rangle \\ &= \langle x, \lambda f^*(y) + \mu f^*(z)\rangle,\end{aligned}$$

so that by uniqueness, $f^*(\lambda y + \mu z) = \lambda f^*(y) + \mu f^*(z)$. It is now easy to check that $f^{**} = f$ and $(\lambda f + \mu f')^* = \bar\lambda f^* + \bar\mu f'^*$.

For all x and y in H we have

$$\langle (f \circ f')(x), y\rangle = \langle f(f'(x), y\rangle = \langle f'(x), f^*(y)\rangle = \langle x, f'^*(f^*(y))\rangle = \langle x, (f'^* \circ f^*)(y)\rangle,$$

and so $(f \circ f')^* = f'^* \circ f^*$.

Using Lemma 2.11.2, we have

$$\begin{aligned}\|f^* \circ f\|_{\sup} &= \sup_{|y|=1} |f^* \circ f(y)| \\ &= \sup_{|x|=1,|y|=1} \langle x, f^*(f(y))\rangle \\ &= \sup_{|x|=1,|y|=1} \langle f(x), f(y)\rangle\end{aligned}$$

$$= \sup_{|x|=1} |f(x)|^2 = \|f\|_{\sup}^2.$$

Thus $\|f^* \circ f\|_{\sup} = \|f\|_{\sup}^2$. Similarly, using Lemma 2.11.2 we have

$$\begin{aligned}
\|f^*\|_{\sup} &= \sup_{|y|=1} |f^*(y)| \\
&= \sup_{|x|=1,|y|=1} |\langle x, f^*(y)\rangle| \\
&= \sup_{|x|=1,|y|=1} |\langle f(x), y\rangle| \\
&= \sup_{|x|=1} |f(x)| = \|f\|_{\sup}.
\end{aligned}$$

Thus $\|f^*\|_{\sup} = \|f\|_{\sup}$. It follows that f^* is bounded. □

THEOREM 2.11.8. *With the operation of composition, the sup norm, and the star operation of taking adjoints, $\mathscr{L}(H)$ is a C^*-algebra.*

PROOF. The sum of two bounded operators is a bounded operator, with

$$\|f + f'\|_{\sup} = \sup_{|x|=1} |f(x) + f'(x)| \leqslant \sup_{|x|=1} |f(x)| + \sup_{|x|=1} |f'(x)| = \|f\|_{\sup} + \|f'\|_{\sup}.$$

Positivity and homogeneity is easily checked, so $\mathscr{L}(H)$ is a normed space. The limit of a Cauchy sequence of bounded operators is a bounded operator, so $\mathscr{L}(H)$ is a Banach space.

The composite of two bounded operators is a bounded operator, with

$$\|f \circ f'\|_{\sup} = \sup_{|x|=1} |f(f'(x))| \leqslant \sup_{|y|=\|f'\|_{\sup}} |f(y)| = \|f\|_{\sup}\|f'\|_{\sup}.$$

The identity linear transformation is bounded with sup norm one. So $\mathscr{L}(H)$ is a Banach algebra. Using Theorem 2.11.7, the star operation is involutory, antilinear, and an anti-automorphism, so it is a Banach $*$-algebra. Finally, by the same theorem it also satisfies $\|f^* \circ f\|_{\sup} = \|f\|_{\sup}^2$, so it is a C^*-algebra. □

DEFINITION 2.11.9. An *action* of a Banach $*$-algebra A on a Hilbert space H is a continuous $*$-algebra homomorphism $A \to \mathscr{L}(H)$. The action is *faithful* if this map is injective.

THEOREM 2.11.10. *If A is a Banach $*$-algebra and $A \to \mathscr{L}(H)$ is an action on a Hilbert space H then the closure of the image of A in $\mathscr{L}(H)$ is a C^*-algebra. If A is commutative then every quasi-nilpotent element of A is in the kernel of the action on H.*

CHAPTER 3

Completions of representation rings

3.1. The norm on a representation ring

DEFINITION 3.1.1. Let $\mathfrak{a}$ be a representation ring (see Definition 1.1.1), and let $\mathfrak{a}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a}$ (see Definition 1.1.4). If $x = \sum_{i \in \mathfrak{I}} a_i x_i \in \mathfrak{a}_{\mathbb{C}}$ then we define the *weighted ℓ^1 norm* of x to be

$$\|x\| = \sum_{i \in \mathfrak{I}} |a_i| \dim x_i.$$

LEMMA 3.1.2. *The map $\mathfrak{a}_{\mathbb{C}} \to \mathbb{R}$ given by $x \mapsto \|x\|$ satisfies Definition 2.2.1* (i)–(v), *and hence makes $\mathfrak{a}_{\mathbb{C}}$ a normed algebra. Furthermore, the star operation taking $x = \sum_{i \in \mathfrak{I}} a_i x_i$ to $x^* = \sum_{i \in \mathfrak{I}} \bar{a}_i x_{i^*}$ makes $\mathfrak{a}_{\mathbb{C}}$ into a commutative normed $*$-algebra, see Definition 2.9.1.*

PROOF. To verify that submultiplicativity holds, let $x = \sum_{i \in \mathfrak{I}} a_i x_i$ and $y = \sum_{j \in \mathfrak{I}} b_j x_j$. We have $x_i x_j = \sum_{k \in \mathfrak{I}} c_{i,j,k} x_k$ with $c_{i,j,k}$ non-negative integers. So

$$xy = \sum_{k \in \mathfrak{I}} \Big(\sum_{i,j \in \mathfrak{I}} c_{i,j,k} a_i b_j \Big) x_k$$

and hence

$$\begin{aligned}
\|xy\| &= \sum_{k \in \mathfrak{I}} \Big| \Big(\sum_{i,j \in \mathfrak{I}} c_{i,j,k} a_i b_j \Big) \Big| \dim x_k \\
&\leqslant \sum_{k \in \mathfrak{I}} \Big(\sum_{i,j \in \mathfrak{I}} c_{i,j,k} |a_i b_j| \Big) \dim x_k \\
&= \sum_{i,j \in \mathfrak{I}} |a_i b_j| \dim x_i x_j \\
&= \sum_{i \in \mathfrak{I}} |a_i| \dim x_i \sum_{j \in \mathfrak{I}} |b_j| \dim x_j \\
&= \|x\| \|y\|.
\end{aligned}$$

The remaining axioms for the norm and star operation are easy to verify. $\square$

DEFINITION 3.1.3. We define $\hat{\mathfrak{a}}$ to be the completion of $\mathfrak{a}_{\mathbb{C}}$ with respect to the norm defined above. By Lemmas 2.9.2 and 3.1.2, $\hat{\mathfrak{a}}$ is a commutative Banach $*$-algebra associated to the representation ring $\mathfrak{a}$, and $\mathfrak{a}_{\mathbb{C}}$ is a dense subalgebra of $\hat{\mathfrak{a}}$.

We can think of elements of $\hat{\mathfrak{a}}$ concretely as possibly infinite linear combinations $\sum_{i \in \mathfrak{I}} a_i x_i$ where $\sum_{i \in \mathfrak{I}} |a_i| \dim x_i < \infty$. Note that any such sum automatically has countable support, by the following lemma.

Lemma 3.1.4. *Let $\{r_\alpha\}_{\alpha\in J}$ be a collection of positive real numbers indexed by a set J. If the sums $\sum_{\alpha\in I} r_\alpha$ over finite subsets $I\subseteq J$ are bounded above then J is countable.*

Proof. For $n>0$, the subset $J_n\subseteq J$ consisting of those $\alpha\in J$ for which $r_\alpha>1/n$ is finite, and $J=\bigcup_{n>0} J_n$ is a countable union of finite sets. $\square$

However, $\hat{\mathfrak{a}}$ is usually not separable:

Definition 3.1.5. A Banach algebra A is *separable* if there is a countable subset K of A such that the closure of the subalgebra generated by K is the whole of A.

Proposition 3.1.6. *The Banach algebra $\hat{\mathfrak{a}}$ is separable if and only if the index set $\mathfrak{I}$ is countable.* $\square$

Remark 3.1.7. Another basis for $\mathfrak{a}_{\mathbb{C}}$ consists of the elements x_j for $j=j^*\in\mathfrak{I}$, together with the elements $(x_j+x_j^*)/2$ and $(x_j-x_j^*)/2\mathsf{i}$ for $j\neq j^*\in\mathfrak{I}$. The elements in this basis are self-conjugate. Their linear span $\mathfrak{a}_{\mathbb{R}}$ is a real normed algebra, whose completion $\hat{\mathfrak{a}}_{\mathbb{R}}$ is a real Banach algebra with the property that $\mathbb{C}\otimes_{\mathbb{R}}\hat{\mathfrak{a}}_{\mathbb{R}}\cong\hat{\mathfrak{a}}$, with star operation coming from complex conjugation on the first tensor factor.

3.2. Norms and cores

Definition 3.2.1. Let $\mathfrak{X}\subset\mathfrak{I}$ be a representation ideal in a representation ring $\mathfrak{a}$. Then $\langle\mathfrak{X}\rangle_{\mathbb{C}}=\mathbb{C}\otimes_{\mathbb{Z}}\langle\mathfrak{X}\rangle$ is an ideal in $\mathfrak{a}_{\mathbb{C}}=\mathbb{C}\otimes_{\mathbb{Z}}\mathfrak{a}$, and its closure $\widehat{\langle\mathfrak{X}\rangle_{\mathbb{C}}}$ in $\hat{\mathfrak{a}}$ is a closed ideal. We write $\hat{\mathfrak{a}}_{\mathfrak{X}}$ for the quotient $\hat{\mathfrak{a}}/\widehat{\langle\mathfrak{X}\rangle_{\mathbb{C}}}$. We write $\hat{\mathfrak{a}}_{\max}$ for $\hat{\mathfrak{a}}_{\mathfrak{X}_{\max}}$ and $\hat{\mathfrak{a}}_{\mathsf{proj}}$ for $\hat{\mathfrak{a}}_{\mathfrak{X}_{\mathsf{proj}}}$ (see Definition 1.3.8).

By Lemma 2.2.4 (iii), $\hat{\mathfrak{a}}_{\mathfrak{X}}$ is isometrically isomorphic to the completion $\widehat{\mathfrak{a}_{\mathbb{C},\mathfrak{X}}}$ of $\mathfrak{a}_{\mathbb{C},\mathfrak{X}}$ with respect to the quotient norm. It is easy to check that the quotient norm on $\hat{\mathfrak{a}}_{\mathfrak{X}}$ is given by

$$\left\|\sum_{i\in\mathfrak{I}} a_i x_i\right\|_{\mathfrak{X}}=\sum_{i\in\mathfrak{I}}|a_i|\dim\mathsf{core}_{\mathfrak{X}}(x_i)=\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}}|a_i|\dim x_i. \tag{3.2.2}$$

We can think of elements of $\hat{\mathfrak{a}}_{\mathfrak{X}}$ concretely as possibly infinite (but necessarily countably supported, see Lemma 3.1.4) linear combinations $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}} a_i x_i$ where

$$\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}}|a_i|\dim x_i<\infty.$$

Lemma 3.2.3. *If $x,y\in\mathfrak{a}_{\geqslant 0}$ then $\|x+y\|_{\mathfrak{X}}=\|x\|_{\mathfrak{X}}+\|y\|_{\mathfrak{X}}$.*

Proof. This is clear from the definition. $\square$

Lemma 3.2.4. *If $x\in\mathfrak{a}_{\geqslant 0}$ then the quotient norm on the image of x in $\hat{\mathfrak{a}}_{\mathfrak{X}}$ is equal to $\dim\mathsf{core}_{\mathfrak{X}}(x)$. Thus $\gamma_{\mathfrak{X}}(x)=\lim\limits_{n\to\infty}\sqrt[n]{\|x^n\|_{\mathfrak{X}}}$.*

Proof. This follows from (3.2.2). $\square$

Theorem 3.2.5. *If $x\in\mathfrak{a}_{\geqslant 0}$ then $\gamma_{\mathfrak{X}}(x)$ is equal to the spectral radius of the image of x in $\hat{\mathfrak{a}}_{\mathfrak{X}}$.*

Proof. This follows from Lemma 3.2.4 and Proposition 2.4.2. $\square$

Lemma 3.2.6. *If $\mathfrak{Y} \leqslant \mathfrak{X}$ are representation ideals in $\mathfrak{a}$ then the closure I of $\langle\mathfrak{X}\rangle_{\mathbb{C}}/\langle\mathfrak{Y}\rangle_{\mathbb{C}}$ in $\hat{\mathfrak{a}}_{\mathfrak{Y}}$ is a closed ideal, and the quotient $(\hat{\mathfrak{a}}_{\mathfrak{Y}})/I$, with the quotient norm, is a commutative Banach algebra isometrically isomorphic to $\hat{\mathfrak{a}}_{\mathfrak{X}}$.*

Proof. This follows from Lemma 2.2.4. □

3.3. Spectrum, species, structure space

Definition 3.3.1. Let $\mathfrak{X} \subset \mathfrak{I}$ be a representation ideal in a representation ring $\mathfrak{a}$. If $x \in \hat{\mathfrak{a}}_{\mathfrak{X}}$, we write $\mathsf{Spec}_{\mathfrak{X}}(x)$ for it spectrum as in Definition 2.3.4. This is a closed, bounded subset of $\mathbb{C}$ by Theorem 2.3.7. For $x \in \mathfrak{a}$, we write $\mathsf{Spec}_{\mathfrak{X}}(x)$ for the spectrum of the image of x in $\hat{\mathfrak{a}}_{\mathfrak{X}}$.

Theorem 3.3.2. *If $x \in \mathfrak{a}_{\succcurlyeq 0}$ then the spectral radius $\gamma_{\mathfrak{X}}(x)$ is an element of $\mathsf{Spec}_{\mathfrak{X}}(x)$.*

Proof. By Theorem 1.6.8, the spectral radius $\gamma_{\mathfrak{X}}(\mathbb{1} + x)$ is equal to $1 + \gamma_{\mathfrak{X}}(x)$. The theorem now follows from Proposition 2.4.9. □

Remark 3.3.3. This theorem is a special case of a theorem about the spectrum of a positive operator on a Banach lattice. For a more direct proof in that context, see for example Theorem 7.9 of Abramovich and Aliprantis [**1**]; but in fact our approach through Proposition 2.4.9 also works in this generality.

Recall the definition of species from Definition 1.1.9.

Definition 3.3.4. Let $\mathfrak{a}$ be a representation ring and $\mathfrak{a}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a}$. A *species* of $\mathfrak{a}$ is a ring homomorphism $s\colon \mathfrak{a} \to \mathbb{C}$. A species of $\mathfrak{a}$ extends uniquely to give a $\mathbb{C}$-algebra homomorphism $s\colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$, which we call a species of $\mathfrak{a}_{\mathbb{C}}$.

Let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. We say that a species s of $\mathfrak{a}$ is $\mathfrak{X}$-*core bounded* if for all $x \in \mathfrak{a}_{\succcurlyeq 0}$ we have

$$|s(x)| \leqslant \dim \mathsf{core}_{\mathfrak{X}}(x).$$

Of course, this only needs checking on the basis elements x_i, $i \in \mathfrak{I}$.

If $\mathfrak{X} = \varnothing$ is the empty representation ideal, we have $\mathsf{core}_{\varnothing}(x) = x$, and we call a $\varnothing$-core bounded species a *dimension bounded species.* Thus a species s is dimension bounded if and only if for all $i \in \mathfrak{I}$ we have

$$|s(x_i)| \leqslant \dim(x_i).$$

So s is $\mathfrak{X}$-core bounded if and only if it is dimension bounded and vanishes on x_i for $i \in \mathfrak{X}$.

If $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, we call an $\mathfrak{X}_{\mathsf{proj}}$-core bounded species a *core bounded species.*

Theorem 3.3.5. *For a species $s\colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$, the following are equivalent:*

(i) *s is $\mathfrak{X}$-core bounded.*
(ii) *For all $x \in \mathfrak{a}_{\mathbb{C}}$ we have $|s(x)| \leqslant \|x\|_{\mathfrak{X}}$.*
(iii) *s vanishes on every x_i with $i \in \mathfrak{X}$ and is continuous with respect to the norm on $\mathfrak{a}_{\mathbb{C}}/\langle\mathfrak{X}\rangle_{\mathbb{C}}$.*
(iv) *s vanishes on $\langle\mathfrak{X}\rangle_{\mathbb{C}}$ and extends to an algebra homomorphism $\hat{\mathfrak{a}}_{\mathfrak{X}} \to \mathbb{C}$.*

PROOF. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear, and the implication (iv) $\Rightarrow$ (i) follows from Proposition 2.3.11. So it remains to prove that (i) $\Rightarrow$ (ii). Suppose that s is core-bounded, and write $x = \sum_{i\in\mathfrak{I}} a_i x_i$. Then $s(x) = \sum_{i\in\mathfrak{I}} a_i s(x_i)$ and so

$$|s(x)| \leqslant \sum_{i\in\mathfrak{I}} |a_i||s(x_i)| \leqslant \sum_i |a_i| \dim \mathsf{core}_{\mathfrak{X}}(x_i) = \|x\|_{\mathfrak{X}}. \qquad \square$$

COROLLARY 3.3.6. *A species $s\colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ is dimension bounded if and only if it is continuous with respect to the norm on $\mathfrak{a}_{\mathbb{C}}$.* $\square$

THEOREM 3.3.7. *For $x \in \mathfrak{a}$, the spectrum $\mathsf{Spec}_{\mathfrak{X}}(x)$ is the set of values of $s(x)$ as x runs over the $\mathfrak{X}$-core bounded species of $\mathfrak{a}$. The spectral radius is*

$$\gamma_{\mathfrak{X}}(x) = \max_{\substack{s\colon \mathfrak{a}\to\mathbb{C} \\ \mathfrak{X}\text{-core bounded}}} |s(x)|.$$

There is an $\mathfrak{X}$-core bounded species s with $\gamma_{\mathfrak{X}}(x) = s(x)$.

PROOF. This follows from Corollary 2.4.6 and Theorems 3.2.5, 3.3.2 and 3.3.5. $\square$

DEFINITION 3.3.8. A species s of $\mathfrak{a}$ is said to be a *Brauer species* if $s(x_i) \neq 0$ for some projective basis element x_i (see Definition 1.2.2).

EXAMPLE 3.3.9. The dimension function is always a Brauer species. This is because it is a ring homomorphism, and is non-zero on ρ and therefore on some projective basis element.

PROPOSITION 3.3.10. *A Brauer species is determined by its value on the projective basis elements. The Brauer species form a finite set, whose cardinality is at most the number of projective basis elements. Every dimension bounded species is either a Brauer species or a core bounded species, but not both.*

PROOF. By Lemma 2.6.1, a Brauer species is determined by its value on the projective basis elements. It therefore cannot be core bounded, because such species vanish on projective basis elements. Since distinct species are linearly independent, it follows that the number of Brauer species is at most the number of projective basis elements.

On the other hand, if s is not a Brauer species then it vanishes on the ideal $\langle \mathfrak{X}_{\mathsf{proj}} \rangle$ of projectives, and is therefore core bounded by Theorem 3.3.5. $\square$

REMARK 3.3.11. Example 1.1.2 (iii) shows that the number of Brauer species can be strictly less than the number of projective basis elements. In this example, there are two projective basis elements but only one Brauer species, namely the dimension function.

Similarly, we have the following.

PROPOSITION 3.3.12. *Let $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. Then the dimension bounded species of $\mathfrak{a}$ fall into two disjoint subsets, the $\mathfrak{X}$-core bounded species, and the species which take non-zero value on some x_i with $i \in \mathfrak{X}$. The latter are determined by their values on the elements x_i, $i \in \mathfrak{X}$.*

PROOF. The proof is essentially the same as the proof of Proposition 3.3.10. $\square$

Definition 3.3.13. We write $\Delta_{\mathfrak{X}}(\mathfrak{a})$ for the structure space $\Delta(\hat{\mathfrak{a}}_{\mathfrak{X}})$ of the commutative Banach $*$-algebra $\hat{\mathfrak{a}}_{\mathfrak{X}}$. In the case $\mathfrak{X} = \mathfrak{X}_{\max}$, we write $\Delta_{\max}(\mathfrak{a})$, and in the case $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$ we write $\Delta_{\mathsf{proj}}(\mathfrak{a})$.

Theorem 3.3.14. *Let $x \in \mathfrak{a}_{\geqslant 0}$ and $\mathfrak{X}$ be a representation ideal in $\mathfrak{a}$. The structure space $\Delta_{\mathfrak{X}}(\mathfrak{a})$ may be identified with the set of $\mathfrak{X}$-core bounded species of $\mathfrak{a}$, or equivalently of $\mathfrak{a}_{\mathbb{C}}$, with the weak* topology. It is a compact Hausdorff space.*

Proof. Using Theorem 3.3.5, we identify the species of $\hat{\mathfrak{a}}_{\mathfrak{X}}$ with the $\mathfrak{X}$-core bounded species of $\mathfrak{a}$, or equivalently of $\mathfrak{a}_{\mathbb{C}}$. Then the theorem follows from Proposition 2.5.3. □

3.4. Symmetric representation rings

Definition 3.4.1. We say that a representation ring $\mathfrak{a}$ is *symmetric* if the completion $\hat{\mathfrak{a}}$ of $\mathfrak{a}_{\mathbb{C}}$ is a symmetric Banach $*$-algebra, see Definition 2.9.12.

Proposition 3.4.2. *For a representation ring $\mathfrak{a}$, the following are equivalent.*

(i) *$\mathfrak{a}$ is symmetric.*
(ii) *Every dimension bounded species of $\mathfrak{a}$ is self conjugate. In other words, for every dimension bounded species $s\colon \mathfrak{a} \to \mathbb{C}$ and every basis element x_i, we have $s(x_i^*) = \overline{s(x_i)}$.*
(iii) *For every $x \in \hat{\mathfrak{a}}$, the spectral radius ρ satisfies $\rho(x^*x) = \rho(x)^2$.*

If these hold then for every $x \in \mathfrak{a}_{\geqslant 0}$ and every representation ideal $\mathfrak{X}$ of $\mathfrak{a}$ we have

$$\gamma_{\mathfrak{X}}(xx^*) = \gamma_{\mathfrak{X}}(x)^2.$$

Proof. The equivalence of (i), (ii) and (iii) follows from the equivalence of (i), (iii) and (v) in Proposition 2.9.10. By Theorem 3.2.5, $\gamma_{\mathfrak{X}}(x)$ is the spectral radius of the image of x in $\hat{\mathfrak{a}}_{\mathfrak{X}}$, so using Corollary 2.9.14, condition (iii) implies the last statement. □

It would be advantageous to have a condition for symmetry that is easier to check. For example, we do not know whether the modular representation ring of a finite group is symmetric. At least we have the following.

Example 3.4.3. Theorem 1.2.7 implies that an ordinary representation ring is symmetric.

3.5. Algebraic elements revisited

Theorem 3.5.1. *Let $\mathfrak{a}$ be a representation ring, let $x \in \mathfrak{a}_{\geqslant 0}$ be algebraic modulo $\mathfrak{X}_{\max}$, and let $\mathfrak{a}'$ be the representation subring of $\mathfrak{a}$ described in Lemma 1.9.2 (v). Then there is a unique species*

$$s\colon \mathfrak{a}'_{\max} = \mathfrak{a}'/\langle \mathfrak{X}_{\max}\rangle \to \mathbb{C}$$

such that for $x \in \mathfrak{a}'_{\max,\geqslant 0}$, we have $\gamma_{\max}(x) = s(x)$.

Proof. Let $\mathfrak{J} \subseteq \mathfrak{I} \setminus \mathfrak{X}_{\max}$ be the set of basis elements of $\mathfrak{a}'$ not in $\mathfrak{X}_{\max}$, and set $y = \sum_{j\in\mathfrak{J}} x_j$. Let B be the matrix $(b_{i,j})$ where $yx_i \equiv \sum_{j\in\mathfrak{J}} b_{i,j}x_j \pmod{\langle \mathfrak{X}_{\max}\rangle}$ for $i \in \mathfrak{J}$. We claim that the entries of B are strictly positive. To see this, for $i, j \in \mathfrak{J}$ we have

$$[x_{i^*}x_jx_i : x_j] \geqslant [x_{i^*}x_i : \mathbb{1}] > 0.$$

So there exists $k \in \mathfrak{J}$ with $[x_{i^*}x_j : x_k] > 0$ and $[x_k x_i : x_j] > 0$, and hence $b_{i,j} = [yx_i : x_j] > 0$.

It follows from the fact that the entries of B are strictly positive, that we may apply the Perron–Frobenius theorem. This says that there is an eigenvalue λ of B which is positive real, and larger in absolute value than all the other eigenvalues, and that the eigenspace is one dimensional, spanned by a vector $u = \sum_{i\in\mathfrak{J}} a_i x_i$ with all the a_i positive real. We thus have $yu = \lambda u$.

Given $x \in \mathfrak{a}'_{\max}$, we have $yxu = xyu = \lambda xu$, and so xu is another non-zero eigenvector of multiplication by y. Since the eigenspace is one dimensional, we have $xu = s(x)u$ for some $s(x) \in \mathbb{C}$. It is now easy to check that s is a ring homomorphism from $\mathfrak{a}'_{\max}$ to $\mathbb{C}$.

For $x \in \mathfrak{a}'_{\max,\succcurlyeq 0}$, $s(x)$ is the spectral radius of x as an element of $\mathfrak{a}'_{\max}$. By the spectral radius formula, Proposition 2.4.2, the spectral radius of x regarded as an element of $\mathfrak{a}_{\max}$ is the same as its spectral radius regarded as an element of $\mathfrak{a}'_{\max}$. It now follows from Theorem 3.2.5 that $s(x) = \gamma_{\max}(x)$. $\square$

3.6. Quasi-nilpotent elements

We suppose, for the purposes of the next theorem, that we are in the following situation. We are given a commutative, associative $\mathbb{C}$-algebra A with a vector space basis $\{x_i,\ i \in \mathfrak{J}\}$ satisfying $x_i x_j = \sum_k c_{i,j,k} x_k$ where the structure constants $c_{i,j,k}$ are non-negative integers. We are also given an algebra homomorphism $d\colon A \to \mathbb{C}$ such that each $d(x_i)$ is a positive integer. We put a norm on A by setting

$$\|\sum_i a_i x_i\| = \sum_i |a_i| d(x_i).$$

As in Lemma 3.1.2, this does indeed define a norm. Under these circumstances, the following theorem shows that quasi-nilpotent elements are nilpotent. I would like to thank Pavel Etingof for suggesting this method of proof.

THEOREM 3.6.1. *If $a = \sum_i a_i x_i \in A$ satisfies $\sqrt[n]{\|a^n\|} \to 0$ as $n \to \infty$ then a is nilpotent.*

PROOF. We suppose that a is not nilpotent, and obtain a contradiction. Write $a = a' + \mathrm{i}a''$ in such a way that the coefficients of x_i in a', a'' are real. Then the element $\bar{a} = a' - \mathrm{i}a''$ also satisfies the hypothesis. By Lemma 1.6.5, elements satisfying the hypothesis form a linear subspace of A. It follows that a' and a'' also satisfy the hypothesis. Furthermore, if a' and a'' are both nilpotent, then so is a. Therefore we may assume without loss of generality that the coefficients a_i of a are real.

Since $a = \sum_i a_i x_i$ is a finite sum, we let V be the real linear span in A of those x_i with $a_i \neq 0$. Then V is a finite dimensional $\mathbb{R}$-vector subspace of A. Consider the elements of V of the form $b = \sum_i b_i x_i$ with $b_i \in \mathbb{Z}$. These form a lattice Λ in V. It follows that there is a constant C such that given any element v of V there is an element of Λ at distance at most C from v. In particular, given $q \in \mathbb{Z}$ there is an element b of Λ within distance at most C from qa. So looking at the elements of the form $b - qa$ with $q \in \mathbb{Z}$, $b \in \Lambda$, there have to be two such, at an arbitrarily small distance from each other. Taking the difference, we see that given $\varepsilon > 0$ we can choose $q > 0$ and b with $\|b - qa\| < \varepsilon$, and hence $\|\frac{b}{q} - a\| < \frac{\varepsilon}{q}$.

The nilpotent elements of V form a linear subspace, which is therefore a closed subset in the norm topology. Since a is not nilpotent, we may choose b as above such that $\frac{b}{q}$ is also not nilpotent. So for all $n > 0$ we have $\|b^n\| \geqslant 1$ (because b has integer coefficients) and hence $\|(\frac{b}{q})^n\| \geqslant \frac{1}{q^n}$.

Now we have

$$\begin{aligned}\tfrac{1}{q^n} &\leqslant \| \left(\tfrac{b}{q}\right)^n \| = \|(a + (\tfrac{b}{q} - a))^n\| = \left\|\textstyle\sum_{i=0}^n \binom{n}{i} a^i (\tfrac{b}{q} - a)^{n-i}\right\| \\ &\leqslant \textstyle\sum_{i=0}^n \binom{n}{i} \|a^i\| \; \|(\tfrac{b}{q} - a)^{n-i}\|.\end{aligned}$$

Applying Lemma 1.6.5, we deduce that

$$\tfrac{1}{q} = \limsup_{n\to\infty} \sqrt[n]{\tfrac{1}{q^n}} \leqslant \limsup_{n\to\infty} \sqrt[n]{\|a^n\|} + \limsup_{n\to\infty} \sqrt[n]{\|(\tfrac{b}{q} - a)^n\|} < 0 + \tfrac{\varepsilon}{q} = \tfrac{\varepsilon}{q}.$$

So choosing $\varepsilon \leqslant 1$, we obtain a contradiction. Hence a is nilpotent. $\square$

We apply this theorem to characterise the nilpotent elements in a representation ring in terms of species. Recall that the *nil radical* of a commutative ring is the ideal of nilpotent elements.

THEOREM 3.6.2. *Let $\mathfrak{a}$ be a representation ring. An element $x \in \mathfrak{a}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{a}$ is nilpotent if and only if for every dimension bounded species $s\colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ we have $s(x) = 0$. Thus the Jacobson radical of $\mathfrak{a}_{\mathbb{C}}$ is equal to the nil radical.*

PROOF. We embed $\mathfrak{a}_{\mathbb{C}}$ in $\hat{\mathfrak{a}}$. The species of $\hat{\mathfrak{a}}$ are the dimension bounded species, so by Theorem 2.8.4 the intersection of the kernels of the dimension bounded species is the set of quasi-nilpotent elements of $\hat{\mathfrak{a}}$. It follows from Theorem 3.6.1 that a quasi-nilpotent element $x \in \mathfrak{a}_{\mathbb{C}}$ is nilpotent. $\square$

REMARK 3.6.3. Since $\mathfrak{a}_{\mathbb{C}}$ is not necessarily Noetherian, we cannot conclude from the theorem that if $\mathfrak{m}$ is a maximal ideal of $\mathfrak{a}_{\mathbb{C}}$ then $\mathfrak{a}_{\mathbb{C}}/\mathfrak{m} \cong \mathbb{C}$. For example, letting R be the polynomial algebra $\mathbb{C}[v, \{u_\lambda\}_{\lambda\in\mathbb{C}}]$, given any non-zero element $x \in R$, there is an algebra homomorphism $s\colon R \to \mathbb{C}$ such that $s(x) \neq 0$. On the other hand, there is a surjective algebra homomorphism $\phi\colon R \to \mathbb{C}(t)$ sending v to t and u_λ to $(t - \lambda)^{-1}$, and $R/\mathsf{Ker}(\phi) \cong \mathbb{C}(t)$.

The following generalises Theorem 2.7 of [**11**], with essentially the same proof.

THEOREM 3.6.4. *There are no non-zero nilpotent elements in $\hat{\mathfrak{a}}_{\max}$.*

PROOF. Suppose that $x \in \hat{\mathfrak{a}}_{\max}$ is nilpotent, and write $x = \sum_{i\in\mathfrak{X}\setminus\mathfrak{X}_{\max}} a_i x_i$. Let $n_i = [x_i x_{i^*} : \mathbb{1}] > 0$. Then $x^* = \sum_i \bar{a}_i x_i^*$. We have

$$xx^* = \sum_i |a_i|^2 x_i x_i^* + \sum_{i\neq j} a_i \bar{a}_j x_i x_j^*,$$

and the coefficient of $\mathbb{1}$ in this is $\sum_i n_i |a_i|^2$. This is zero if and only if $x = 0$, so $xx^* = 0$ implies $x = 0$. If $x^2 = 0$ then $(xx^*)(xx^*)^* = x^2 x^{*\,2} = 0$, so $xx^* = 0$ and hence $x = 0$. $\square$

REMARK 3.6.5. We shall use the proof of Theorem 3.6.4 as motivation for the introduction of the trace map. This will eventually enable us to prove that there are no quasi-nilpotent elements in $\hat{\mathfrak{a}}_{\max}$, see Theorem 3.9.1.

3.7. Action on Hilbert space

Recall that for a representation ring $\mathfrak{a}$, we have the completion $\hat{\mathfrak{a}}_{\max}$ with respect to the norm $\| \ \|_{\max}$. In this section we investigate an action of $\hat{\mathfrak{a}}_{\max}$ on a Hilbert space $H(\mathfrak{a})$. The crucial inequality allowing us to do this is given in Theorem 3.7.5, which turns out to be quite tricky to prove. We begin with the following definitions, which are suggested by the proof of Theorem 3.6.4.

Definition 3.7.1. The *trace map* $\mathsf{Tr}\colon \mathfrak{a} \to \mathbb{Z}$ is defined by

$$\mathsf{Tr}\left(\sum_{i\in\mathfrak{I}} a_i x_i\right) = a_0$$

(recall that x_0 is the basis element $\mathbb{1}$ of $\mathfrak{a}$). This extends to a trace map $\mathsf{Tr}\colon \mathfrak{a}_{\mathbb{C}} \to \mathbb{C}$ given by the same formula.

Recall that n_i is defined to be $[x_i x_{i^*} : \mathbb{1}]$, and that $n_i = 0$ if and only if $i \in \mathfrak{X}_{\max}$.

Lemma 3.7.2. *We have* $\mathsf{Tr}(x_i x_{i^*}) = n_i$*, and* $\mathsf{Tr}(x_i x_{j^*}) = 0$ *for* $i \neq j$*.*

Proof. This follows from Definition 1.1.1 (ii). □

Definition 3.7.3. We define the *weighted* ℓ^2 *norm* on $\mathfrak{a}_{\mathbb{C},\max} = \mathfrak{a}_{\mathbb{C}}/\langle \mathfrak{X}_{\max}\rangle_{\mathbb{C}}$ to be

$$\left|\sum_{i\in\mathfrak{I}} a_i x_i\right| = \sqrt{\sum_{i\in\mathfrak{I}} n_i |a_i|^2} = \sqrt{\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} n_i |a_i|^2}.$$

This is associated to the inner product

$$\left\langle \sum_{i\in\mathfrak{I}} a_i x_i, \sum_{i\in\mathfrak{I}} b_i x_i \right\rangle = \sum_{i\in\mathfrak{I}} n_i a_i \bar{b}_i = \sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} n_i a_i \bar{b}_i.$$

The completion of $\mathfrak{a}_{\mathbb{C},\max}$ with respect to the weighted ℓ^2 norm is a Hilbert space which is denoted $H(\mathfrak{a})$. We can think of elements of $H(\mathfrak{a})$ as countably supported infinite sums $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} a_i x_i$, with $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} n_i|a_i|^2 < \infty$, and with trace and inner product given by the same formulas as above.

Lemma 3.7.4. *For* $x, y \in H(\mathfrak{a})$ *we have*

(i) $\langle x, y\rangle = \mathsf{Tr}(xy^*)$
(ii) $\langle xy, z\rangle = \langle y, x^* z\rangle$

Proof. (i) If $x = \sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} a_i x_i$ and $y = \sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} b_i x_i$ then

$$xy^* = \sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} a_i \bar{b}_i x_i x_{i^*} + \sum_{i\neq j\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} a_i \bar{b}_j x_i x_{j^*}.$$

By Lemma 3.7.2, the trace of this is equal to $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} n_i a_i \bar{b}_i$, which is $\langle x, y\rangle$.

(ii) Using (i) we have

$$\langle xy, z\rangle = \mathsf{Tr}(xyz^*) = \mathsf{Tr}(y(x^*z)^*) = \langle y, x^*z\rangle. \qquad \square$$

Theorem 3.7.5. *For $x \in \mathfrak{a}_{\mathbb{C}}$, $y \in H(\mathfrak{a})$ we have $|xy| \leqslant \|x\|_{\max}|y|$.*

Proof. The model for this proof is the inequality in Lemma 1.10.4, but the details are much more complicated.

First we treat the case $x = x_i$ and $y = \sum_{j\in\mathfrak{I}} b_j x_j$. We have

$$\begin{aligned}
|x_i y|^2 &= \langle x_i y, x_i y\rangle \\
&= \langle x_{i^*} x_i y, y\rangle \\
&= \sum_{j,k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} b_j \bar{b}_k \langle x_{i^*} x_i x_j, x_k\rangle \\
&\leqslant \sum_{j,k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} |b_j||b_k| \langle x_{i^*} x_i x_j, x_k\rangle \\
&= \sum_{j,k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} \frac{|b_j|n_j}{\dim x_j}\,\frac{|b_k|n_k}{\dim x_k}\,\langle x_{i^*} x_i x_j, x_k\rangle\,\frac{\dim x_j}{n_j}\,\frac{\dim x_k}{n_k}
\end{aligned}$$

For $j \neq k$, the (j,k) term and the (k,j) term in this sum are equal. So using the inequality

$$2\,\frac{|b_j|n_j}{\dim x_j}\,\frac{|b_k|n_k}{\dim x_k} \leqslant \left(\frac{|b_j|n_j}{\dim x_j}\right)^2 + \left(\frac{|b_k|n_k}{\dim x_k}\right)^2$$

we have

$$\begin{aligned}
|x_i y|^2 &\leqslant \sum_{j,k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} \left(\frac{|b_j|n_j}{\dim x_j}\right)^2 \langle x_{i^*} x_i x_j, x_k\rangle\,\frac{\dim x_j}{n_j}\,\frac{\dim x_k}{n_k} \\
&= \sum_{j,k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} \frac{|b_j|^2 n_j}{\dim x_j}\,\langle x_{i^*} x_i x_j, x_k\rangle\,\frac{\dim x_k}{n_k}.
\end{aligned}$$

Now we also have

$$\begin{aligned}
\sum_{k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} \langle x_{i^*} x_i x_j, x_k\rangle\,\frac{\dim x_k}{n_k} &= \sum_{k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} [x_{i^*} x_i x_j : x_k] n_k\,\frac{\dim x_k}{n_k} \\
&= \sum_{k\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} [x_{i^*} x_i x_j : x_k] \dim x_k \\
&= \dim \mathsf{core}_{\max}(x_{i^*} x_i x_j) \\
&\leqslant (\dim \mathsf{core}_{\max}(x_i))^2 \dim x_j
\end{aligned}$$

and so we get

$$\begin{aligned}
|x_i y|^2 &\leqslant \sum_{j\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} \frac{|b_j|^2 n_j}{\dim x_j}\,(\dim \mathsf{core}_{\max}(x_i))^2 \dim x_j \\
&= \sum_{j\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} |b_j|^2 n_j (\dim \mathsf{core}_{\max}(x_i))^2 \\
&= (\dim \mathsf{core}_{\max}(x_i))^2 \langle y, y\rangle \\
&= \|x_i\|_{\max}^2 |y|^2.
\end{aligned}$$

Taking square roots of both sides, we obtain $|x_i y| \leqslant \|x_i\|_{\max}|y|$.

Finally, in general if $x = \sum_{i\in\mathfrak{I}} a_i x_i$ then using the case proved above, we have

$$\begin{aligned} |xy| &= \left|\sum_{i\in\mathfrak{I}} a_i x_i y\right| \leqslant \sum_{i\in\mathfrak{I}} |a_i||x_i y| \leqslant \sum_{i\in\mathfrak{I}} |a_i|\|x_i\|_{\max}|y| \\ &= \left\|\sum_{i\in\mathfrak{I}} a_i x_i\right\|_{\max} |y| = \|x\|_{\max}|y|. \end{aligned}$$ □

Remark 3.7.6. Setting $y = \mathbb{1}$ in the theorem, we have $|x| \leqslant \|x\|_{\max}$. In particular, every infinite sum $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} a_i x_i$ in $\hat{\mathfrak{a}}_{\max}$ is in $H(\mathfrak{a})$. So we have a norm decreasing (continuous) injective map of Banach $*$-algebras $\hat{\mathfrak{a}}_{\max} \hookrightarrow H(\mathfrak{a})$, with dense image.

Proposition 3.7.7. *For $x \in \mathfrak{a}_{\mathbb{C}}$ the map $y \mapsto xy$ of $H(\mathfrak{a})$ is bounded. Elements of $\langle \mathfrak{X}_{\max}\rangle$ act as zero, and so we have a map $\mathfrak{a}_{\mathbb{C},\max} \to \mathscr{L}(H(\mathfrak{a}))$.*

Proof. This follows from the inequality in Theorem 3.7.5. □

Definition 3.7.8. We write $\|x\|_{\sup}$ for the sup norm of x under the map

$$\mathfrak{a}_{\mathbb{C},\max} \to \mathscr{L}(H(\mathfrak{a}))$$

sending $x \in \mathfrak{a}_{\mathbb{C}}$ to the map $y \mapsto xy$ of $H(\mathfrak{a})$. This is given by

$$\|x\|_{\sup} = \sup_{|y|=1} |xy|.$$

Theorem 3.7.9. *For $x \in \mathfrak{a}_{\mathbb{C},\max}$ we have $|x| \leqslant \|x\|_{\sup} \leqslant \|x\|_{\max}$. The map $\mathfrak{a}_{\mathbb{C},\max} \to \mathscr{L}(H(\mathfrak{a}))$ is a continuous $*$-homomorphism of normed $*$-algebras, and extends to an injective continuous $*$-homomorphism of Banach $*$-algebras $\hat{\mathfrak{a}}_{\max} \to \mathscr{L}(H(\mathfrak{a}))$.*

Proof. Theorem 3.7.5 shows that $\|x\|_{\sup} \leqslant \|x\|_{\max}$, so by Lemma 2.1.3 this map is continuous, and hence extends to a map $\hat{\mathfrak{a}}_{\max} \to \mathscr{L}(H(\mathfrak{a}))$. By Lemma 3.7.4 (ii), this map preserves the star operation. The action of x on $\mathbb{1} \in H(\mathfrak{a})$ shows that $|x| \leqslant \|x\|_{\sup}$, and that this map is injective, see Remark 3.7.6. □

3.8. The C^*-algebra $C^*_{\max}(\mathfrak{a})$

Definition 3.8.1. We saw in Theorem 3.7.9 that the map $\mathfrak{a}_{\mathbb{C},\max} \to \mathscr{L}(H(\mathfrak{a}))$ sending an element x to left multiplication by x is a continuous map of normed $*$-algebras, and extends to an injective map of Banach $*$-algebras $\hat{\mathfrak{a}}_{\max} \to \mathscr{L}(H(\mathfrak{a}))$. We let $C^*_{\max}(\mathfrak{a})$ denote the closure of the image of this map. This is a commutative C^*-algebra.

Definition 3.8.2. A species $s\colon \mathfrak{a} \to \mathbb{C}$ is *sup bounded* if for all $x \in \mathfrak{a}_{\geqslant 0}$ we have

$$|s(x)| \leqslant \|x\|_{\sup} = \sup_{|y|=1} |xy|.$$

Proposition 3.8.3. *For an $\mathfrak{X}_{\max}$-core bounded species of $\mathfrak{a}$, the following are equivalent:*

(i) *s is sup bounded,*
(ii) *s is continuous with respect to the sup norm,*

(iii) *s extends to a $\mathbb{C}$-algebra homomorphism $C^*_{\max}(\mathfrak{a}) \to \mathbb{C}$.*

This leads us to another invariant of an element $x \in \mathfrak{a}_{\succcurlyeq 0}$, namely the spectral radius in the sup norm. By Theorem 2.10.5, this is equal to $\|x\|_{\sup}$:

$$\gamma_{\sup}(x) = \|x\|_{\sup} = \sup_{|y|=1} |xy|.$$

This is really an invariant of the image of x in $\mathfrak{a}_{\max}$, and we have

$$\gamma_{\sup}(x) \leqslant \gamma_{\max}(x).$$

This invariant has one advantage over the others we have been examining. Namely,

THEOREM 3.8.4. *For $x \in \mathfrak{a}_{\succcurlyeq 0}$ we have*

$$\gamma_{\sup}(xx^*) = \gamma_{\sup}(x)^2.$$

PROOF. By Theorem 2.11.8, $\mathscr{L}(H(\mathfrak{a}))$ is a C^*-algebra with respect to the sup norm. In particular, in accordance with Definition 2.10.1, we have $\|xx^*\|_{\sup} = \|x\|^2_{\sup}$. □

REMARK 3.8.5. The C^*-algebra $C^*_{\max}(\mathfrak{a})$ is analogous to the C^*-algebra $C^*(\Gamma)$ of a discrete abelian group Γ.

Recall that in general, if Γ is a discrete group and $x \in \ell^1(\Gamma)$ then we look at all actions of $\ell^1(\Gamma)$ on a Hilbert space, and take the *C^*-norm* $\|x\|_{C^*}$ to be the supremum of the sup norms in these actions. We have $\|x\|_{C^*} \leqslant \|x\|_{\ell^1}$, and we define the C^*-algebra $C^*(\Gamma)$ of Γ to be the completion of $\ell^1(\Gamma)$ with respect to this norm.

If we restrict our attention to the action of $\ell^1(\Gamma)$ on the Hilbert space $\ell^2(\Gamma)$ by convolution, rather than using all actions on Hilbert spaces, then we obtain a possibly smaller norm called the *reduced C^*-norm*,

$$\|x\|_{C^*_r} = \sup_{|y|=1} |x * y|,$$

and the *reduced C^*-algebra* $C^*_r(\Gamma)$ is the completion of $\ell^1(\Gamma)$ with respect to this norm.

The group Γ is said to be *amenable* if there exists a finitely additive measure on Γ which is invariant under left multiplications. If Γ is amenable then the C^*-norm and the reduced C^*-norm are equal, and we have $C^*(\Gamma) = C^*_r(\Gamma)$. In particular, all abelian groups are amenable, and so if Γ is abelian then $C^*(\Gamma)$ is the closure of the image of the action of $\ell^1(\Gamma)$ on $\ell^2(\Gamma)$.

3.9. Quasi-nilpotent elements revisited

THEOREM 3.9.1. *There are no non-zero quasi-nilpotent elements in $\hat{\mathfrak{a}}_{\max}$.*

PROOF. By Theorem 3.7.9 and Definition 3.8.1, we have a continuous injective map from $\hat{\mathfrak{a}}_{\max}$ to $\mathscr{L}(H(\mathfrak{a}))$, and the closure of its image is the commutative C^*-algebra $C^*_{\max}(\mathfrak{a})$. If x is a quasi-nilpotent element of $\hat{\mathfrak{a}}_{\max}$ then its image in $C^*_{\max}(\mathfrak{a})$ is also quasi-nilpotent and hence, by Corollary 2.10.6, equal to zero. □

COROLLARY 3.9.2. *If x is a non-zero element of $\hat{\mathfrak{a}}_{\max}$ then there is a sup bounded (and hence $\mathfrak{X}_{\max}$-core bounded) species s with $s(x) \neq 0$.*

PROOF. This follows from Theorem 2.8.4, Proposition 3.8.3, and Theorem 3.9.1. □

Corollary 3.9.3. (i) *The Jacobson radical of* $\hat{\mathfrak{a}}_{\max}$ *is zero.*
(ii) *The Jacobson radical of* $\mathfrak{a}_{\mathbb{C}}/\langle \mathfrak{X}_{\max}\rangle$ *is zero.*

Proof. (i) By the previous corollary, every non-zero element x lies outside some maximal ideal I of $\hat{\mathfrak{a}}_{\max}$ with $\hat{\mathfrak{a}}_{\max}/I$ isomorphic to $\mathbb{C}$ via an $\mathfrak{X}_{\max}$-core bounded species s.

(ii) If x is a non-zero element of $\mathfrak{a}_{\mathbb{C}}/\langle \mathfrak{X}_{\max}\rangle \subseteq \hat{\mathfrak{a}}_{\max}$ then the species $s\colon \mathfrak{a}_{\mathbb{C}}/\langle \mathfrak{X}_{\max}\rangle \to \mathbb{C}$ of part (i) is surjective, and x is not in its kernel. So $x \notin J(\mathfrak{a}_{\mathbb{C}}/\langle \mathfrak{X}_{\max}\rangle)$. □

3.10. Idempotents

In this section we examine idempotents in $\hat{\mathfrak{a}}_{\max}$ and in $\mathfrak{a}_{K,\max} = K \otimes_{\mathbb{Z}} \mathfrak{a}_{\max}$ with K a field. We show that if $e \in \hat{\mathfrak{a}}_{\max}$ is idempotent and not equal to zero or one then $0 < \mathsf{Tr}(e) < 1$; and if $e \in \mathfrak{a}_{K,\max}$ then $\mathsf{Tr}(e)$ is in the ground field of K.

The following is the analogue for representation rings of a theorem of Kaplansky on group rings (unpublished, but see the end of §II.3 of Kaplansky [**62**], Lemma 2 of Montgomery [**69**], or §2.1 of Passman [**75**]).

Theorem 3.10.1. *If* $e \in \hat{\mathfrak{a}}_{\max}$ *is idempotent,* $e \neq 0, 1$ *then* $0 < \mathsf{Tr}(e) < 1$.

Proof. We have $e \in \hat{\mathfrak{a}}_{\max} \subseteq C^*_{\max}(\mathfrak{a})$. Now $C^*_{\max}(\mathfrak{a})$ is a C^*-algebra, and hence a symmetric Banach $*$-algebra. By Theorem 2.9.15 we have $e = e^*$ and so $e = e^*e$. Now using Lemma 3.7.4, we have

$$\mathsf{Tr}(e) = \mathsf{Tr}(e^*e) = \langle e, e\rangle > 0.$$

Since $1 - e$ is also an idempotent, we have $0 < \mathsf{Tr}(e) < 1$. □

Corollary 3.10.2. *There are no non-trivial idempotents in* $\mathfrak{a}_{\max}$.

Proof. If $e \in \mathfrak{a}_{\max}$ then $\mathsf{Tr}(e)$ is a rational integer. By Theorem 3.10.1 it follows that if e is idempotent then e is equal to zero or one. □

Corollary 3.10.3. *Let* K *be a field of characteristic zero. If* e *is an idempotent in* $\mathfrak{a}_{K,\max}$ *then* $\mathsf{Tr}(e)$ *is a totally real algebraic element of* K*. Every element of* $\mathbb{C}$ *satisfying its minimal equation is a real number between* 0 *and* 1.

Proof. Let $e = \sum_{i=1}^n a_i x_i$ with $i \in \mathfrak{I} \setminus \mathfrak{X}$, and let $K_0 = \mathbb{Q}(a_1, \dots, a_n)$. As an abstract field, $\mathbb{C}$ is an algebraic closure of an infinite transcendental extension of $\mathbb{Q}$. For every field embedding $K_0 \to \mathbb{C}$, Theorem 3.10.1 shows that the image of $\mathsf{Tr}(e)$ is a real number lying between 0 and 1. If $\mathsf{Tr}(e)$ were transcendental, there would exist an embedding $K_0 \to \mathbb{C}$ taking $\mathsf{Tr}(e)$ to a non-real number, and therefore $\mathsf{Tr}(e)$ is algebraic. Moreover, given any complex number satisfying its minimal equation, again there exists a field embedding $K_0 \to \mathbb{C}$ taking $\mathsf{Tr}(e)$ there. □

In the case of group rings, Zalesskii [**97**] has shown that $\mathsf{Tr}(e)$ has to be rational. The proof does not appear to extend to our situation.

Corollary 3.10.4. *Let* K *be a field of characteristic zero whose only totally real subfield is* $\mathbb{Q}$*, and let* $\mathcal{O}_K$ *be its ring of integers. Then there are no idempotents in* $\mathfrak{a}_{\mathcal{O}_K,\max}$ *other than* 0 *and* 1.

Proof. Let e be an idempotent in $\mathfrak{a}_{\mathcal{O}_K,\max}$. Then by Corollary 3.10.3, $\mathsf{Tr}(e)$ is a totally real element of $\mathcal{O}_K$. By the hypothesis on K, it follows that $\mathsf{Tr}(e)$ is in $\mathbb{Q} \cap \mathcal{O}_K = \mathbb{Z}$. By Theorem 3.10.1 it follows that $e = 0$ or $e = 1$. $\square$

CHAPTER 4

Representation rings of finite groups

4.1. Preliminaries on kG-modules

Let G be a finite group and k a field of characteristic p. Throughout this text, we only consider finitely generated kG-modules. When we write the tensor product $M \otimes N$ of two kG-modules M and N we mean the tensor product over the field k, $M \otimes_k N$, with diagonal G-action. Thus $g(m \otimes n) = gm \otimes gn$ for $g \in G$, $m \in M$ and $n \in N$. We write M^* for the linear dual of M, with G-action given by $(g(f))(m) = f(g^{-1}(m))$. We write k for the trivial kG-module, namely a copy of the field k on which all elements of G act as the identity.

The various parts of the following proposition are due to Benson and Carlson [**11**] and Auslander and Carlson [**6**].

PROPOSITION 4.1.1. *Let M be a kG-module.*

(i) *M is isomorphic to a direct summand of $M \otimes M^* \otimes M$.*
(ii) *If p divides the dimension of M then $M \otimes M^* \otimes M$ has a direct summand isomorphic to $M \oplus M$.*
(iii) *If p does not divide the dimension of M then $M \otimes M^*$ has a direct summand isomorphic to k.*
(iv) *If M and N are indecomposable and $M \otimes N$ has a direct summand isomorphic to k then $N \cong M^*$.*

PROOF. Let $\{m_1, \dots, m_n\}$ be a basis for M and $\{f_1, \dots, f_n\}$ the dual basis of M^*. Thus $\sum_i f_i(m_i) = n = \dim(M)$, and for $m \in M$ we have $m = \sum_i f_i(m)m_i$.

For (i) we have maps $M \to M \otimes M^* \otimes M$ given by $m \mapsto \sum_i m \otimes f_i \otimes m_i$ and $M \otimes M^* \otimes M \to M$ given by $m \otimes f \otimes m' \mapsto f(m)m'$, with composite the identity on M.

For (ii) (cf. Proposition 4.9 in Auslander and Carlson [**6**], where this is proved with the further hypothesis that M is indecomposable, but this hypothesis is not used in the proof), we have maps $M \oplus M \to M \otimes M^* \otimes M$ given by

$$(m, m') \mapsto \sum_i (m \otimes f_i \otimes m_i + m_i \otimes f_i \otimes m')$$

and $M \otimes M^* \otimes M \to M \oplus M$ given by $m \otimes f \otimes m' \mapsto (f(m)m', f(m')m)$. If M has dimension divisible by p then the composite is the identity on $M \oplus M$.

For (iii), we have maps $k \to M \otimes M^*$ given by $1 \mapsto \sum_i m_i \otimes f_i$ and $M \otimes M^* \to k$ given by $m \otimes f \mapsto f(m)$. If p does not divide the dimension of M then the composite of these maps is non-zero.

For (iv), we have maps $k \to M \otimes N \to k$ whose composite is non-zero. Associated to these is the map

$$\mathsf{Hom}_{kG}(M \otimes N, k) \times \mathsf{Hom}_{kG}(k, M \otimes N) \to k$$

given by composition. Adjointly,

$$\mathsf{Hom}_{kG}(N, M^*) \times \mathsf{Hom}_{kG}(M^*, N) \to k$$

is given by composition followed by trace. Thus there are maps $M^* \to N \to M^*$ whose composite has non-zero trace. Since M^* is indecomposable, endomorphisms which are not isomorphisms have zero trace, so the composite is an isomorphism. Finally, since N is indecomposable, both maps must be isomorphisms. □

4.2. The representation ring $a(G)$

Let G be a finite group and k a field of characteristic p. Let $a(G)$ be the representation ring, or *Green ring* of kG. This has as a basis the symbols $[M_i]$, where M_i is a indecomposable kG-module, and i is in a suitable indexing set $\mathfrak{I}$. The symbol $[M_i]$ only depends on the isomorphism class of M_i. If $M = \bigoplus_i n_i M_i$ then we write $[M]$ for $\sum_i n_i[M_i] \in a(G)$. By the Krull–Schmidt theorem, this is well defined, and identifies the non-negative elements of $a(G)$ with the isomorphism classes $[M]$ of kG-modules. The multiplication in $a(G)$ is then given on non-negative elements by the tensor product, $[M][N] = [M \otimes_k N]$, and extended bilinearly to all elements.

We shall also be interested in the complexification of the representation ring, $a_{\mathbb{C}}(G) = \mathbb{C} \otimes_{\mathbb{Z}} a(G)$.

Proposition 4.2.1. *The representation ring $a(G)$ of a finite group over a field k is an example of a representation ring in the sense of Definition 1.1.1. It is an ordinary representation ring in the sense of Definition 1.2.6 if k has characteristic zero, or prime characteristic p not dividing $|G|$, and a modular representation ring if k has characteristic p dividing $|G|$.*

Proof. For property (i), we set $x_i = [M_i]$, and define the star operation by letting M_{i^*} be the dual module M_i^* of M_i. Property (ii) follows from Proposition 4.1.1 (iv). Proposition 4.1.1 (iii) shows that if $M_i \otimes M_{i^*}$ does not have a direct summand isomorphic to k then the $\dim M_i$ is not divisible by p, and then Proposition 4.1.1 (ii) shows that $M_i \otimes M_{i^*} \otimes M_i$ has a direct summand isomorphic to $M_i \oplus M_i$; this proves that property (iii) holds. The dimension fuction for property (iv) is defined so that the dimension of $[M_i]$ is its dimension as a k-vector space. For Property (v), the role of the element ρ is played by the regular representation $[kG]$. For any kG-module M, the tensor product $M \otimes kG$ is isomorphic to a direct sum of $\dim M$ copies of kG, so we have $[M][kG] = (\dim[M]).[kG]$ in $a(G)$.

If k has characteristic zero, or prime characteristic p dividing $|G|$, then the surjective augmentation map $kG \to k$ is split by the map $k \to kG$ sending 1 to $\frac{1}{|G|}\sum_{g \in G} g$. So the trivial module k is projective, and hence $\mathbb{1}$ is a projective basis element of $a(G)$. On the other hand, if k has characteristic dividing $|G|$ then the augmentation map $kG \to k$ does not split, so the trivial module is not projective, and therefore $\mathbb{1}$ is not a projective basis element of $a(G)$. □

Remark 4.2.2. Proposition 4.2.1 generalises in an obvious way to finite group schemes. For finite supergroup schemes, it is usual to take the super dimension function to be the dimension of the even part minus the dimension of the odd part. However, this fails axiom (iv), as the dimension function is not positive. Instead, one needs to take the naïve dimension, namely the dimension of the even part plus the dimension of the odd part. With this definition, the representation ring of a finite supergroup scheme satisfies the axioms of Definition 1.1.1. However, for the analogue of Proposition 4.1.1, the super dimension function is the relevant notion.

The proposition also generalises to any coefficients where the Krull–Schmidt theorem holds for finitely generated modules, such as the $\mathfrak{p}$-localisation $\mathcal{O}_{(\mathfrak{p})}$ or the $\mathfrak{p}$-adic completion $\mathcal{O}_{\mathfrak{p}}$ of an algebraic number ring $\mathcal{O}$. The integral representation ring of a finite group in the presence of the Krull–Schmidt theorem has been studied by Reiner [**77, 79, 78**], Hannula [**50**]; see also Jensen [**60**].

The representation ring of a finite dimensional quasitriangular Hopf algebra (with mild extra conditions) is a further generalisation which fits into our framework of representation rings given by Definition 1.1.1. The quasitriangular condition ensures that for any modules M and N, the tensor products $M \otimes N$ and $N \otimes M$ are isomorphic. In this case, there are two natural isomorphisms between these modules, which may be thought of as moving M over or under N. These isomorphisms satisfy braid relations given by the Yang–Baxter equations. Finite quantum groups are examples of suitable quasitriangular Hopf algebras. Witherspoon [**95**] has investigated species of the representation ring in the particular case of the quantum double of a finite group.

It would be interesting to see how much goes through in the case of Hopf algebras with non-commutative tensor products, but that is a task for another day. The spectral theory of non-commutative Banach *-algebras is not as clean as the Gelfand theory for commutative ones.

Definition 4.2.3. Let $\mathfrak{X}$ be a collection of indecomposable kG-modules, closed under isomorphism, with the property that if M is in $\mathfrak{X}$ and N is any kG-module then $M \otimes N$ is a direct sum of modules in $\mathfrak{X}$. We also suppose that not every kG-module is in $\mathfrak{X}$. We say that $\mathfrak{X}$ is an *ideal* of indecomposables. In this situation, by abuse of notation we also write $\mathfrak{X}$ for the representation ideal of the representation ring $a(G)$ consisting of the $i \in \mathfrak{I}$ such that M_i in $\mathfrak{X}$. We write $a(G, \mathfrak{X})$ for the ideal $\langle \mathfrak{X} \rangle$, namely the linear combinations in $a(G)$ of the elements $[M_i]$ with $M_i \in \mathfrak{X}$, and $a_{\mathfrak{X}}(G)$ for the quotient $a(G)/a(G, \mathfrak{X})$. We write $a_{\mathbb{C}}(G, \mathfrak{X})$ for the ideal $\mathbb{C} \otimes_{\mathbb{Z}} a(G, \mathfrak{X})$ of $a_{\mathbb{C}}(G)$, and $a_{\mathbb{C},\mathfrak{X}}(G)$ for the quotient $a_{\mathbb{C}}(G)/a_{\mathbb{C}}(G, \mathfrak{X})$.

Example 4.2.4.
(i) There is, of course, the empty example $\mathfrak{X} = \varnothing$. In this case, we have $a(G, \mathfrak{X}) = \{0\}$, $a_{\mathbb{C}}(G, \mathfrak{X}) = \{0\}$, $a_{\mathfrak{X}}(G) = a(G)$, $a_{\mathbb{C},\mathfrak{X}}(G) = a_{\mathbb{C}}(G)$.
(ii) Let $\mathfrak{X}_{\mathsf{proj}}$ be the set of projective indecomposable modules. In this case we write $a(G, 1)$ for $a(G, \mathfrak{X}_{\mathsf{proj}})$ and $a_{\mathbb{C}}(G, 1)$ for $a_{\mathbb{C}}(G, \mathfrak{X}_{\mathsf{proj}})$. This is the minimal example, in the sense that every non-empty ideal $\mathfrak{X}$ of indecomposables contains this one. This follows from Proposition 1.3.9.
(iii) More generally, if H is a subgroup of G then the collection $\mathfrak{X}_H$ of indecomposable modules that are projective relative to H is ideal. We write $a(G, H)$ and $a_{\mathbb{C}}(G, H)$

for $a(G, \mathfrak{X}_H)$ and $a_{\mathbb{C}}(G, \mathfrak{X}_H)$. The case where H is the trivial subgroup gives the previous example.

We can do the same with any collection $\mathfrak{H}$ of subgroups $H \leqslant G$ with the property that if H is contained in a conjugate of H' and $H' \in \mathfrak{H}$ then $H \in \mathfrak{H}$. The collection of indecomposable modules that are projective relative to such a collection of subgroups $\mathfrak{H}$ is ideal. This example is discussed in further detail in Section 4.6.

(iv) If $\mathcal{V}$ is a specialisation closed subset of $\mathsf{Proj}\, H^*(G, k)$ then the indecomposable modules supported in $\mathcal{V}$ form an ideal $\mathfrak{X}_{\mathcal{V}}$. We write $a(G, \mathcal{V})$ and $a_{\mathbb{C}}(G, \mathcal{V})$ for $a(G, \mathfrak{X}_{\mathcal{V}})$ and $a_{\mathbb{C}}(G, \mathfrak{X}_{\mathcal{V}})$. For example, if $\mathcal{V}$ is the collection of all closed points, then $\mathfrak{X}_{\mathcal{V}}$ is the collection of indecomposable projective or periodic modules.

(v) It is proved in Benson and Carlson [**11**] that if k is algebraically closed, then the collection $\mathfrak{X}_p$ of indecomposable modules of dimension divisible by p is ideal. We write $a(G; p)$ and $a_{\mathbb{C}}(G; p)$ for $a(G, \mathfrak{X}_p)$ and $a_{\mathbb{C}}(G, \mathfrak{X}_p)$. This is the maximal example $\mathfrak{X}_{\max}$, in the sense that every ideal $\mathfrak{X}$ of indecomposable modules that does not consist of them all is contained in this one, see Proposition 1.3.9 (i).

Over a field which is not algebraically closed, $\mathfrak{X}_{\max}$ consists of the indecomposable modules M such that $M \otimes M^*$ has no summand isomorphic to the trivial module k. Every indecomposable module in $\mathfrak{X}_{\max}$ has dimension divisible by p, but there may be indecomposables of dimension divisible by p not in $\mathfrak{X}_{\max}$. In this case, we write $a(G; \max)$ for the ideal $\langle \mathfrak{X}_{\max} \rangle$ of $a(G)$, $a_{\max}(G)$ for the quotient $a(G)/\langle \mathfrak{X}_{\max} \rangle$, and $a_{\mathbb{C}}(G; \max)$, $a_{\mathbb{C},\max}(G)$ for their complexifications.

Definition 4.2.5. If $\mathfrak{X}$ is an ideal of indecomposable kG-modules, and M is any kG-module, we may write $M = M' \oplus M''$, where M'' is a direct sum of elements of $\mathfrak{X}$ and no summand of M' is in $\mathfrak{X}$. We define the $\mathfrak{X}$-*core*, $\mathsf{core}_{G,\mathfrak{X}}(M)$ to be M'. This is well defined up to isomorphism. If $\mathfrak{X}$ is the ideal of projective indecomposables, we just call this the *core*, denoted $\mathsf{core}_G(M)$.

4.3. The gamma invariant

By Proposition 4.2.1, we may regard $a(G)$ as a representation ring in the sense of Definition 1.1.1. In this context, we now investigate the gamma invariant defined in Section 1.4. Let $\mathfrak{X}$ be an ideal of indecomposable kG-modules. We consider the $\mathfrak{X}$-cores of tensor powers of a module. We write $\mathsf{c}_n^{G,\mathfrak{X}}(M)$ for $\dim \mathsf{core}_{G,\mathfrak{X}}(M^{\otimes n})$. If $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$ we write $\mathsf{c}_n^G(M)$. Since $\mathsf{c}_n^{G,\mathfrak{X}}(M) \leqslant \dim(M)^n$, the generating function

$$f(t) = \sum_{n=0}^{\infty} \mathsf{c}_n^{G,\mathfrak{X}}(M) t^n$$

converges in a disc of radius at least $1/\dim(M)$ around the origin. The radius of convergence r is given by the formula

$$1/r = \limsup_{n \to \infty} \sqrt[n]{\mathsf{c}_n^{G,\mathfrak{X}}(M)}.$$

We write $\gamma_{G,\mathfrak{X}}(M)$ for the value of $1/r$ given by this formula. In the case where $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, we just write $\gamma_G(M)$. This is the invariant that was investigated in [**13**].

The following properties are consequences of our investigations in Chapter 1.

THEOREM 4.3.1. *The invariant $\gamma_{G,\mathfrak{X}}(M)$ has the following properties:*

(i) *We have* $\gamma_{G,\mathfrak{X}}(M) = \lim\limits_{n\to\infty} \sqrt[n]{\mathsf{c}_n^{G,\mathfrak{X}}(M)} = \inf\limits_{n\geqslant 1} \sqrt[n]{\mathsf{c}_n^{G,\mathfrak{X}}(M)}$.

(ii) *We have* $0 \leqslant \gamma_{G,\mathfrak{X}}(M) \leqslant \dim M$.

(iii) *Some tensor power of a kG-module M has an indecomposable summand in $\mathfrak{X}$ if and only if $\gamma_{G,\mathfrak{X}}(M) < \dim M$.*

(iv) *A kG-module M is a direct sum of modules in $\mathfrak{X}$ if and only if $\gamma_{G,\mathfrak{X}}(M) = 0$; otherwise we have $\gamma_{G,\mathfrak{X}}(M) \geqslant 1$.*

(v) *If $m \in \mathbb{N}$ then $\gamma_{G,\mathfrak{X}}(mM) = m\,\gamma_{G,\mathfrak{X}}(M)$, where mM denotes a direct sum of m copies of M.*

(vi) *More generally, if a and b are non-negative integers then*

$$\gamma_{G,\mathfrak{X}}(ak \oplus bM) = a + b\gamma_{G,\mathfrak{X}}(M),$$

where k denotes the trivial kG-module.

(vii) *If $\mathfrak{X} \subseteq \mathfrak{Y}$ are ideals of indecomposable kG-modules then $\gamma_{G,\mathfrak{Y}}(M) \leqslant \gamma_{G,\mathfrak{X}}(M)$.*

(viii) *If $1 \leqslant \gamma_{G,\mathfrak{X}}(M) < \sqrt{2}$ then M is $\mathfrak{X}$-endotrivial.*

(ix) *We have* $\gamma_{G,\mathfrak{X}}(M \otimes N) \leqslant \gamma_{G,\mathfrak{X}}(M)\gamma_{G,\mathfrak{X}}(N)$,

(x) *If $m \in \mathbb{N}$ then we have $\gamma_{G,\mathfrak{X}}(M^{\otimes m}) = \gamma_{G,\mathfrak{X}}(M)^m$,*

(xi) *We have* $\gamma_{G,\mathfrak{X}}(M^*) = \gamma_{G,\mathfrak{X}}(M)$.

PROOF. By Proposition 4.2.1, $a(G)$ is a representation ring in the sense of Definition 1.1.1. So (i) follows from Theorem 1.6.4, (ii) follows from Lemma 1.4.3, (iii) follows from Lemma 1.4.9, (iv) follows from Lemma 1.4.4, (v) follows from Theorem 1.6.8, (vi) follows from Lemma 1.4.10, (vii) follows from Lemma 1.4.12, (viii) follows from Theorem 1.8.1, (ix) follows from Lemma 1.4.7, (x) follows from Lemma 1.4.8, and (xi) follows from Lemma 1.4.5. □

DEFINITION 4.3.2. If $\mathfrak{X}$ is an ideal of indecomposable kG-modules, and H is a subgroup of G, we write $\mathfrak{X}{\downarrow_H}$ for the ideal of indecomposable summands of modules of the form $M{\downarrow_H}$ with M in $\mathfrak{X}$.

THEOREM 4.3.3. *If $H \leqslant G$ and M is a kG-module then $\gamma_{H,\mathfrak{X}\downarrow_H}(M) \leqslant \gamma_{G,\mathfrak{X}}(M)$.*

PROOF. We have $\mathsf{c}_n^{H,\mathfrak{X}\downarrow_H}(M) \leqslant \mathsf{c}_n^{G,\mathfrak{X}}(M)$ for all n. □

COROLLARY 4.3.4. *If $H \leqslant G$ and M is a kG-module then $\gamma_H(M) \leqslant \gamma_G(M)$.*

PROOF. If $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$ for kG then $\mathfrak{X}{\downarrow_H} = \mathfrak{X}_{\mathsf{proj}}$ for kH. □

The following definition and theorem generalise Lemma 2.11 of [**13**] (which is Corollary 4.3.8 below).

DEFINITION 4.3.5. If K is an extension field of k, and $\mathfrak{X}$ is an ideal of indecomposable kG-modules, we write $K\mathfrak{X}$ for the ideal of indecomposable KG-modules that are summands of modules of the form $K \otimes_k M$ with M in $\mathfrak{X}$.

LEMMA 4.3.6. *Let K be an extension field of k, let $\mathfrak{X}$ be an ideal of indecomposable kG-modules, and let M be a kG-module. Then $K \otimes_k \mathsf{core}_{G,\mathfrak{X}}(M) \cong \mathsf{core}_{G,\mathfrak{X}}(K \otimes_k M)$.*

PROOF. We first note that $(K \otimes_k M){\downarrow_{kG}}$ is a direct sum of (possibly infinitely many) copies of M.

We need to show that if $K \otimes_k M$ has an indecomposable summand N in $K\mathfrak{X}$ then M has an indecomposable summand in $\mathfrak{X}$. Consider the restrictions of $K \otimes_k M$ and N from KG to kG. The restriction $N{\downarrow_{kG}}$ is a direct sum of finite dimensional indecomposable kG-modules in $\mathfrak{X}$; let N' be one of them. It is a summand of $(K \otimes_k M){\downarrow_{kG}}$, which is a direct sum of copies of M. Because it is finite dimensional, N' is a summand of a finite direct sum of copies of M, hence is a summand of M, by the Krull–Schmidt Theorem. $\square$

THEOREM 4.3.7. *If K is an extension field of k and $\mathfrak{X}$ is an ideal of indecomposable kG-modules then $\gamma_{G,K\mathfrak{X}}(K \otimes_k M) = \gamma_{G,\mathfrak{X}}(M)$.*

PROOF. This follows immediately from Lemma 4.3.6. $\square$

COROLLARY 4.3.8. *If K is an extension field of k then $\gamma_G(K \otimes_k M) = \gamma_G(M)$.*

PROOF. If P is a projective kG-module then every summand of $K \otimes_k P$ is a projective KG-module. The corollary now follows from Theorem 4.3.7. $\square$

4.4. Elementary abelian subgroups

In the case where $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, the elementary abelian subgroups of G play a crucial role, because of theorems of Chouinard and Carlson. Most of the material in this section is taken from Section 7 of [**13**].

THEOREM 4.4.1 (Chouinard [**25**]). *A kG-module M is projective if and only if its restriction to every elementary abelian p-subgroup of G is projective.* $\square$

A strengthening of Chouinard's theorem is the following theorem of Carlson.

THEOREM 4.4.2 (Carlson [**22**], Theorem 3.7). *There exists a constant B, which depends only on p and G, such that if M is a kG-module then*

$$\dim \mathsf{core}_G(M) \leqslant B \cdot \max_{E \leqslant G} \dim \mathsf{core}_E(M)$$

where the maximum is taken over the set of elementary abelian p-subgroups E of G. $\square$

LEMMA 4.4.3. *If H is a subgroup of G and M is a kG-module then $\gamma_H(M) \leqslant \gamma_G(M)$.*

PROOF. We have $\mathsf{c}_n^H(M) \leqslant \mathsf{c}_n^G(M)$ for all n. $\square$

THEOREM 4.4.4. *Let M be a kG-module. Then $\gamma_G(M) = \max_{E \leqslant G} \gamma_E(M)$, where the maximum is taken over the set of elementary abelian p-subgroups E of G.*

PROOF. By Theorem 4.4.2 and Lemma 4.4.3 we have

$$\max_{E \leqslant G} \sqrt[n]{\mathsf{c}_n^E(M)} \leqslant \sqrt[n]{\mathsf{c}_n^G(M)} \leqslant \sqrt[n]{B}. \max_{E \leqslant G} \sqrt[n]{\mathsf{c}_n^E(M)}.$$

Taking $\limsup\limits_{n\to\infty}$, the factor of $\sqrt[n]{B}$ tends to 1. $\square$

EXAMPLE 4.4.5. Let G be a generalised quaternion group and let k be a field of characteristic two. Then G has only one elementary abelian 2-subgroup $E = \langle z \rangle$, where z is the central element of order two. Let $X = 1 + z$, an element of kG satisfying $X^2 = 0$. If M is a kG-module then the restriction to kE is a direct sum of $\dim(\mathsf{Ker}(X, M)/\mathsf{Im}(X, M))$ copies of the trivial module plus a free module. It follows that

$$\gamma_G(M) = \dim(\mathsf{Ker}(X, M)/\mathsf{Im}(X, M)).$$

In particular, this is an integer.

PROPOSITION 4.4.6 (Dade [**34, 35**]). *If E is an elementary abelian p-group, then the only endotrivial kE-modules are the syzygies $\Omega^n(k)$ ($n \in \mathbb{Z}$) of the trivial module.* □

LEMMA 4.4.7. *We have $\gamma_G(\Omega^n k) = 1$ for $n \in \mathbb{Z}$, provided that p divides $|G|$.*

PROOF. First suppose that $n \geqslant 0$. We have $\mathsf{core}_G((\Omega k)^{\otimes n}) \cong \Omega^n k$, and $\dim \Omega^n k$ grows polynomially in n (see for example [**9**] §5.3). Therefore $\gamma_G(\Omega^n k) = 1$. Since $(\Omega^n k)^* \cong \Omega^{-n} k$, the lemma is also true for $n < 0$. □

For a general representation ideal $\mathfrak{X}$ in a representation ring $\mathfrak{a}$, we saw that it was possible for an $\mathfrak{X}$-endotrivial element $x \in \mathfrak{a}_{\geqslant 0}$ to have $\gamma_{\mathfrak{X}}(x) > 1$, and so we needed to introduce two different versions of the Picard group, see Section 1.7. For $\mathfrak{a} = a(G)$ and $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, this cannot happen, as shown by the following theorem.

THEOREM 4.4.8. *A kG-module M is endotrivial if and only if $\gamma_G(M) = 1$.*

PROOF. If M is neither projective nor endotrivial then it follows from Theorem 1.8.1 that $\gamma_G(M) \geqslant \sqrt{2}$. If M is projective then $\gamma_G(M) = 0$. If M is endotrivial then its restriction to every elementary abelian p-subgroup of G is endotrivial. So by Theorem 4.4.4, we may assume that $G = E$ is an elementary abelian p-group. By Proposition 4.4.6, M is a syzygy of the trivial module, so by Lemma 4.4.7 we have $\gamma_E(M) = 1$. □

QUESTION 4.4.9. For choices of representation ideals $\mathfrak{X}$ in $a(G)$ other than $\mathfrak{X} = \mathfrak{X}_{\mathsf{proj}}$, is it true that an $\mathfrak{X}$-endotrivial module M necessarily satisfies $\gamma_{G,\mathfrak{X}}(M) = 1$?

4.5. Induced modules

In view of Theorem 4.4.4, the following proposition is important in finding the gamma invariants of induced modules. For calculational purposes, this proposition should be combined with the Mackey decomposition formula for restricting induced modules to elementary abelian subgroups.

PROPOSITION 4.5.1. *If E' is a subgroup of an elementary abelian p-group E and M is a kE'-module then*

(i) $\mathsf{core}_E(M{\uparrow}^E) \cong (\mathsf{core}_{E'}(M)){\uparrow}^E$
(ii) $\gamma_E(M{\uparrow}^E) = |E : E'|\gamma_{E'}(M)$.

Proof. We have $M{\uparrow^E}{\downarrow_{E'}} \cong |E:E'|M$, a direct sum of $|E:E'|$ copies of M. So M has a projective summand if and only if $M{\uparrow^E}$ has a projective summand. This proves (i). Applying this to $M^{\otimes n}$, we have

$$\dim \mathsf{core}_E(M^{\otimes n}{\uparrow^E}) = \dim \mathsf{core}_E(M^{\otimes n}){\uparrow^E} = |E:E'| \dim \mathsf{core}_{E'}(M^{\otimes n}). \tag{4.5.2}$$

We have

$$M{\uparrow^E} \otimes M{\uparrow^E} \cong (M \otimes M{\uparrow^E}{\downarrow_{E'}}){\uparrow^E} \cong |E:E'|(M\otimes M){\uparrow^E}.$$

Applying induction, we then obtain

$$(M{\uparrow^E})^{\otimes n} \cong |E:E'|^{n-1}(M^{\otimes n}){\uparrow^E}$$

and so using (4.5.2) we have

$$\begin{aligned} \mathsf{c}_n^E(M{\uparrow^E}) &= \dim \mathsf{core}_E((M{\uparrow^E})^{\otimes n}) \\ &= |E:E'|^{n-1} \dim \mathsf{core}_E(M^{\otimes n}{\uparrow^E}) \\ &= |E:E'|^{n} \dim \mathsf{core}_{E'}(M^{\otimes n}) = |E:E'|^n \mathsf{c}_n^{E'}(M). \end{aligned}$$

Thus we have

$$\sqrt[n]{\mathsf{c}_n^E(M{\uparrow^E})} = |E:E'| \sqrt[n]{\mathsf{c}_n^{E'}(M)},$$

and taking limits as n tends to infinity, it follows that $\gamma_E(M{\uparrow^E}) = |E:E'|\gamma_{E'}(M)$, and (ii) is proved. □

4.6. Relatively projective modules and relative syzygies

In this section, we consider a generalisation of Example 4.2.4 (iii).

Definition 4.6.1. Let N be a kG-module. We say that a kG-module M is *projective relative to N* or *relatively N-projective* if M is isomorphic to a direct summand of $N \otimes U$ for some kG-module U.

Lemma 4.6.2. *For kG-modules M and N, the following are equivalent:*

(i) *M is relatively N-projective,*
(ii) *the natural map $N \otimes N^* \otimes M \to M$ given by $n \otimes f \otimes m \mapsto f(n)m$ splits,*
(iii) *the natural map $M \to N^* \otimes N \otimes M$ given by $m \mapsto \sum_i f_i \otimes n_i$ splits.*

Proof. (i) ⇒ (ii): It suffices to prove this for $M = N \otimes U$. Let n_i and f_i be dual bases of N and N^*. Then for $n \in N$ we have $\sum_i f_i(n)n_i = n$. Then the map $N\otimes N^* \otimes N \otimes U \to N \otimes U$ is split by the map sending $n \otimes u$ to $\sum_i n \otimes f_i \otimes n_i \otimes u$.

(ii) ⇒ (i): If the natural map $N \otimes N^* \otimes M \to M$ splits then M is isomorphic to a direct summand of $N \otimes U$ where $U = N^* \otimes M$.

The implications (i) ⇒ (iii) and (iii) ⇒ (i) are proved similarly. □

Remark 4.6.3. If $N = kG$, this is the definition of a projective kG-module. If N is the permutation module on the cosets of a subgroup $H \leqslant G$, this is the definition of a relatively H-projective module. More generally, let $\mathfrak{H}$ be a collection of subgroups of G with the property that if H is contained in a conjugate of H' and $H' \in \mathfrak{H}$ then $H \in \mathfrak{H}$. Let N be the direct sum over $H \in \mathfrak{H}$ of the permutation modules on the cosets of H. Then a module is relatively $\mathfrak{H}$-projective if and only if it is relatively N-projective.

The condition that M is injective relative to N is equivalent to the condition that it is projective relative to N, so we shall not need to use this terminology.

Definition 4.6.4. A short exact sequence

$$0 \to M_3 \to M_2 \to M_1 \to 0$$

is *relatively N-split* if the sequence

$$0 \to N \otimes M_3 \to N \otimes M_2 \to N \otimes M_1 \to 0$$

splits.

Proposition 4.6.5. *For a short exact sequence $0 \to M_3 \to M_2 \to M_1 \to 0$ the following are equivalent:*

(i) *The sequence $0 \to N \otimes M_3 \to N \otimes M_2 \to N \otimes M_1 \to 0$ splits.*
(ii) *The sequence $0 \to N^* \otimes M_3 \to N^* \otimes M_2 \to N^* \otimes M_1 \to 0$ splits.*
(iii) *The sequence $0 \to N \otimes N^* \otimes M_3 \to N \otimes N^* \otimes M_2 \to N \otimes N^* \otimes M_1 \to 0$ splits.*

Proof. This follows from the fact that N is isomorphic to a direct summand of $N \otimes N^* \otimes N$, and N^* is a direct summand of $N^* \otimes N \otimes N^*$, see Proposition 4.1.1 (i). $\square$

Corollary 4.6.6. *A short exact sequence*

$$0 \to M_3 \to M_2 \to M_1 \to 0$$

is relatively N-split if and only if the dual sequence

$$0 \to M_1^* \to M_2^* \to M_3^* \to 0$$

is relatively N-split. $\square$

Lemma 4.6.7. *If $0 \to M_3 \to M_2 \to M_1 \to 0$ is a relatively N-split short exact sequence, and either M_1 or M_3 is relatively N-projective, then the sequence splits.*

Proof. Suppose that M_1 is relatively N-projective. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & N \otimes N^* \otimes M_3 \arrow[r] \arrow[d] & N \otimes N^* \otimes M_2 \arrow[r] \arrow[d] & N \otimes N^* \otimes M_1 \arrow[r] \arrow[d] \arrow[l, bend right] & 0 \\
0 \arrow[r] & M_3 \arrow[r] & M_2 \arrow[r] & M_1 \arrow[r] \arrow[u, bend right] & 0
\end{tikzcd}$$

Composing the two splittings and the map $N \otimes N^* \otimes M_2 \to M_2$ gives a splitting for the map $M_2 \to M_1$.

The proof with M_3 relatively projective is dual (cf. Remark 4.6.3). $\square$

Definition 4.6.8. A *relative N-syzygy* of M, or *syzygy of M relative to N* is the module M_3 in a relatively N-split sequence with M_2 relatively N-projective and $M_1 = M$. This exists, because the natural epimorphism $N^* \otimes N \otimes M \to M$ is relatively split. We write $\Omega_N(M)$ for the relative N-syzygy of M.

Dually, we write $\Omega_N^{-1}(M)$ for the module M_1 in a relatively N-split sequence with M_2 relatively N-projective and $M_3 = M$. This exists because the natural map $M \to N^* \otimes N \otimes M$ is relatively split.

The following is a relative version of Schanuel's lemma. It shows that relative N-syzygies are well defined up to adding and removing relatively N-projective summands.

Lemma 4.6.9. *Let M and N be kG-modules. If*

$$0 \to A \to B \to M \to 0$$
$$0 \to A' \to B' \to M \to 0$$

are two relatively N-split sequences with B and B' relatively N-projective then $A \oplus B' \cong A' \oplus B$.

Proof. We form the pullback X of $B \to M$ and $B' \to M$. Then we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A' \arrow[r, equal] \arrow[d] & A' \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We claim that the sequence ending in B' given by the middle row of this diagram is relatively N-split. To see this, we use the fact that the bottom row is relatively N-split. Then we compose

$$N \otimes B' \to N \otimes M \to N \otimes B.$$

Since $N \otimes X$ is a pullback of $N \otimes B \to N \otimes M$ and $N \otimes B' \to N \otimes M$, we can use this composite together with the identity map on $N \otimes B'$ to obtain the required map $N \otimes B' \to N \otimes X$. Similarly, the sequence ending in B given by the middle column is relatively N-split.

Since B and B' are relatively N-projective, using Lemma 4.6.7 we see that the sequences ending in B and B' split. So we have $A \oplus B' \cong X \cong A' \oplus B$. □

Definition 4.6.10. If N is a kG-module, let $\mathfrak{X}_N$ be the collection of indecomposable modules that are projective relative to N. Then $\mathfrak{X}_N$ is an ideal of kG-modules.

Proposition 4.6.11.

(i) *The module $\mathsf{core}_{G,\mathfrak{X}_N}(\Omega_N(M))$ is well defined, and isomorphic to $\mathsf{core}_{G,\mathfrak{X}_N}(\Omega_N(k) \otimes M)$.*

(ii) $\mathsf{core}_{G,\mathfrak{X}_N}(\Omega_N^{-1}\Omega_N(M)) \cong \mathsf{core}_{G,\mathfrak{X}_N}(M)$.

(iii) *The module $\mathsf{core}_{G,\mathfrak{X}_N}(\Omega_N^{-1}(M))$ is well defined, and is isomorphic to*

$$\mathsf{core}_{G,\mathfrak{X}_N}(\Omega^{-1}(k) \otimes M).$$

(iv) $\mathsf{core}_{G,\mathfrak{X}_N}(\Omega_N\Omega_N^{-1}(M)) \cong \mathsf{core}_{G,\mathfrak{X}_N}(M)$.

Proof. The fact that $\mathsf{core}_{G,\mathfrak{x}_N}(\Omega_N(M))$ is well defined follows from Lemma 4.6.9. Examining the short exact sequence

$$0 \to \Omega_N(k) \otimes M \to N \otimes N^* \otimes M \to M \to 0,$$

we see that $\Omega_N(k) \otimes M$ is one candidate for $\Omega_N(M)$. It follows by examining the same sequence, that M is one candidate for $\Omega_N^{-1}\Omega_N(M)$. This proves parts (i) and (ii), and (iii) and (iv) are dual. □

Lemma 4.6.12. *We have $\Omega_N^{-1}(k) \cong \Omega_N(k)^*$. Thus $\gamma_{G,\mathfrak{x}_N}(\Omega_N^{-1}(k)) = \gamma_{G,\mathfrak{x}_N}(\Omega_N(k))$.*

Proof. The first statement follows by applying Corollary 4.6.6 to the sequence

$$0 \to \Omega_N(k) \to N \otimes N^* \to k \to 0.$$ □

Remark 4.6.13. It often happens that $\gamma_{G,\mathfrak{x}_N}(\Omega_N(k)) = 1$. For example, the finite generation of finite group cohomology implies that in the case $N = kG$, we obtain $\gamma_G(\Omega(k)) = 1$. More generally, a theorem of Brown [**18**] shows that if N is a permutation module then $\gamma_{G,\mathfrak{x}_N}(\Omega_N(k)) = 1$. We do not know of an example of a kG-module N for which $\gamma_{G,\mathfrak{x}_N}(\Omega_N(k)) > 1$.

Theorem 4.6.14. *We have $\dfrac{\gamma_{G,\mathfrak{x}_N}(M)}{\gamma_{G,\mathfrak{x}_N}(\Omega_N(k))} \leqslant \gamma_{G,\mathfrak{x}_N}(\Omega_N(M)) \leqslant \gamma_{G,\mathfrak{x}_N}(\Omega_N(k))\gamma_{G,\mathfrak{x}_N}(M)$.*

Proof. By Proposition 4.6.11 and Theorem 4.3.1 (ix) we have

$$\gamma_{G,\mathfrak{x}_N}(\Omega_N(M)) = \gamma_{G,\mathfrak{x}_N}(\Omega_N(k) \otimes M) \leqslant \gamma_{G,\mathfrak{x}_N}(\Omega_N(k))\gamma_{G,\mathfrak{x}_N}(M)$$

Similarly, $\gamma_{G,\mathfrak{x}_N}(M) = \gamma_{G,\mathfrak{x}_N}(\Omega_N^{-1}(k) \otimes \Omega_N(M)) \leqslant \gamma_{G,\mathfrak{x}_N}(\Omega_N^{-1}(k))\gamma_{G,\mathfrak{x}_N}(\Omega_N(M))$. □

Corollary 4.6.15. *If N is a permutation module then $\gamma_{G,\mathfrak{x}_N}(\Omega_N(M)) = \gamma_{G,\mathfrak{x}_N}(M)$.*

Proof. This follows from the theorem and the preceding remark. □

Corollary 4.6.16. *We have $\gamma_G(\Omega(M)) = \gamma_G(M)$.*

Proof. This is the case $N = kG$. □

4.7. Trivial source modules

In this section, we compute $\gamma_G(M)$ for trivial source modules M.

Definition 4.7.1. Let G be a finite group and k a field of characteristic p. A kG-module M is said to be a *trivial source module* if its restriction to a Sylow p-subgroup is a permutation module. This is equivalent to the condition that M is isomorphic to a direct summand of a permutation kG-module.

Lemma 4.7.2. *If Q is a subgroup of a finite p-group P then the permutation module P/Q is indecomposable. If M is a permutation kP-module then either M is projective or the action of P on M has a kernel.*

Lemma 4.7.3. *Let E be an elementary abelian p-group, and let $M_1, \dots, M_n$ be indecomposable permutation kE-modules. Then $M_1 \otimes \cdots \otimes M_n$ is either projective or the action of E has a kernel.*

Proof. If E_1 and E_2 are subgroups of E then by the Mackey decomposition theorem the tensor product of permutation modules $k(E/E_1) \otimes k(E/E_2)$ is a direct sum of copies of $kE/(E_1 \cap E_2)$. By induction, it follows that $k(E/E_1) \otimes \cdots \otimes k(E/E_n)$ is a direct sum of copies of $kE/(E_1 \cap \cdots \cap E_n)$. This is projective if and only if the kernel of the action, $E_1 \cap \cdots \cap E_n$ is trivial. $\square$

Proposition 4.7.4. *Let E be an elementary abelian p-group and M a permutation kE-module. Then there is a direct summand N of M such that the action of E on N is not faithful and $\gamma_E(M) = \gamma_E(N) = \dim(N)$.*

Proof. If the action of E on M is not faithful then $\gamma_E(M) = \dim(M)$ and we are done. On the other hand, if the action is faithful then $M \cong M' \oplus M_1 \oplus \cdots \oplus M_n$ with $M_i \cong k(E/E_i)$ and $E_1 \cap \cdots \cap E_n = 1$. Then by Lemma 4.7.3, $M_1 \otimes \cdots \otimes M_n$ is projective. So for $m \geqslant n$ the module $M^{\otimes m}$ is isomorphic to a summand of

$$\bigoplus_{i=1}^{n} (M' \oplus M_1 \oplus \cdots \overset{i}{\uparrow} \cdots \oplus M_n)^{\otimes m} \oplus (\mathsf{proj})$$

where the upward arrow indicates omission of the term indexed by i. It follows that

$$\gamma_E(M^{\otimes m}) \leqslant n \max_{1 \leqslant i \leqslant n} \gamma_E(M' \oplus M_1 \oplus \cdots \overset{i}{\uparrow} \cdots \oplus M_n)^{\otimes m}.$$

Taking mth roots, we have

$$\gamma_E(M) \leqslant \sqrt[m]{n} \max_{1 \leqslant i \leqslant n} \gamma_E(M' \oplus M_1 \oplus \cdots \overset{i}{\uparrow} \cdots \oplus M_n).$$

Letting m tend to infinity, we can ignore the term $\sqrt[m]{n}$. Then since each of the terms on the right hand side is less than or equal to the left hand side, we have equality:

$$\gamma_E(M) = \max_{1 \leqslant i \leqslant n} \gamma_E(M' \oplus M_1 \oplus \cdots \overset{i}{\uparrow} \cdots \oplus M_n).$$

Thus there is a non-zero summand M_i of M which may be removed from M without altering the value of $\gamma_E(M)$. Arguing by induction, there is a summand N of M which is not faithful, with $\gamma_E(M) = \gamma_E(N) = \dim(N)$. $\square$

Theorem 4.7.5. *Let M be a trivial source kG-module. Then $\gamma_G(M)$ is equal to the maximum over pairs (E, N) of $\dim(N)$, where E runs over the elementary abelian p-subgroups of G and N runs over the summands of M as kE-modules which are not faithful.*

Proof. This follows from Theorem 4.4.4 and Proposition 4.7.4. $\square$

4.8. The norms on $a(G)$

Definition 3.1.1 gives us the following.

Definition 4.8.1. The *weighted ℓ^1 norm* on $a_{\mathbb{C}}(G)$ is given by

$$\left\| \sum_{i \in \mathfrak{I}} a_i [M_i] \right\| = \sum_{i \in \mathfrak{I}} |a_i| \dim M_i.$$

We define $\hat{a}(G)$ to be the completion of $a_{\mathbb{C}}(G)$ with respect to this norm.

If $\mathfrak{X}$ is a representation ideal in $a(G)$ then by Lemma 2.2.4 (iii), the quotient $\hat{a}_{\mathfrak{X}}(G)$ of $\hat{a}(G)$ by $\langle\mathfrak{X}\rangle_{\mathbb{C}}$ is isometrically isomorphic to the completion of $a_{\mathbb{C},\mathfrak{X}}(G) = a_{\mathbb{C}}(G)/a_{\mathbb{C}}(G,\mathfrak{X})$ with respect to the quotient norm. By (3.2.2), the quotient norm on $a_{\mathbb{C},\mathfrak{X}}(G)$ is given by

$$\left\|\sum_{i\in\mathfrak{I}} a_i[M_i]\right\|_{\mathfrak{X}} = \sum_{i\in\mathfrak{I}} |a_i| \dim \mathsf{core}_{\mathfrak{X}}(M_i) = \sum_{i\in\mathfrak{I}\setminus\mathfrak{X}} \dim M_i.$$

Elements of the completion $\hat{a}_{\mathfrak{X}}(G) \cong \widehat{a_{\mathbb{C},\mathfrak{X}}(G)}$ may be regarded as infinite, but countably supported, linear combinations $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}} a_i[M_i]$ where $\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}} |a_i| \dim M_i < \infty$.

Recall that if k is algebraically closed then the modules M_i with $i \in \mathfrak{X}_{\max}$ are the indecomposables of dimension divisible by p, so that $a(G,\max) = a(G;p)$ and $a_{\max}(G) = a(G)/a(G;p)$.

In general, we have the weighted ℓ^2 norm on $a_{\mathbb{C},\max}(G) = a_{\mathbb{C}}(G)/\langle\mathfrak{X}_{\max}\rangle_{\mathbb{C}}$ described in Definition 3.7.3.

$$\left|\sum_{i\in\mathfrak{I}} a_i[M_i]\right| = \sqrt{\sum_{i\in\mathfrak{I}} n_i|a_i|^2}.$$

The completion of $a_{\mathbb{C},\max}(G)$ with respect to this norm is a Hilbert space $H(G)$. By Theorem 3.7.9, left multiplication induces a continuous map of normed $*$-algebras $a_{\mathbb{C},\max}(G) \to \mathscr{L}(H(G))$, and extends to an injective map $\hat{a}_{\max}(G) \to \mathscr{L}(H(G))$. The C^*-algebra $C^*_{\max}(G)$ is defined to be the closure of the image of this map. This is a commutative C^*-algebra, and is the completion of $a_{\max}(G)$ with respect to the sup norm

$$\|x\|_{\sup} = \sup_{|y|=1} |xy|.$$

4.9. Species of $a(G)$

We say that a species s of $a(G)$ is *$\mathfrak{X}$-core bounded* if for all kG-modules M we have

$$|s([M])| \leqslant \dim \mathsf{core}_{G,\mathfrak{X}}(M).$$

In particular, the extension of of an $\mathfrak{X}$-core bounded species to $a_{\mathbb{C}}(G)$ vanishes on $a_{\mathbb{C}}(G,\mathfrak{X})$, and so defines an algebra homomorphism $a_{\mathbb{C},\mathfrak{X}}(G) \to \mathbb{C}$. If $\mathfrak{X} = \varnothing$, we say that s is *dimension bounded*, and if $\mathfrak{X}$ is the ideal of projective indecomposable modules, we just say that s is *core bounded*.

THEOREM 4.9.1. *For a species $s\colon a_{\mathbb{C}}(G) \to \mathbb{C}$, the following are equivalent:*

(i) *s is $\mathfrak{X}$-core bounded.*
(ii) *For all $x \in a_{\mathbb{C}}(G)$ we have $|s(x)| \leqslant \|x\|_{\mathfrak{X}}$.*
(iii) *s is continuous with respect to the norm on $a_{\mathbb{C},\mathfrak{X}}(G)$.*
(iv) *s vanishes on $\langle\mathfrak{X}\rangle_{\mathbb{C}}$ and extends to an algebra homomorphism $\hat{a}_{\mathfrak{X}}(G)$.*

PROOF. This is Theorem 3.3.5 in this context. $\square$

THEOREM 4.9.2. *For $x \in a(G)$, the spectrum $\mathsf{Spec}_{\mathfrak{X}}(x)$ is the set of values of $s(x)$ as x runs over the $\mathfrak{X}$-core bounded species of $a(G)$. The spectral radius is*

$$\gamma_{\mathfrak{X}}(x) = \max_{\substack{s\colon a(G)\to\mathbb{C} \\ \mathfrak{X}-\text{core bounded}}} |s(x)|.$$

There is an $\mathfrak{X}$-core bounded species s with $\gamma_{\mathfrak{X}}(x) = s(x)$.

PROOF. This is Theorem 3.3.7 in this context. □

PROPOSITION 4.9.3. *If s is a dimension bounded species of $a(G)$ then either s is a Brauer species or s is core bounded.*

PROOF. This follows from Theorem 4.9.1. □

DEFINITION 4.9.4. If $\mathfrak{X}$ is an ideal of indecomposable kG-modules, and $A = \hat{a}_{\mathfrak{X}}(G)$, we write $\Delta_{\mathfrak{X}}(G)$ for $\Delta(\hat{a}_{\mathfrak{X}}(G))$. Thus $\Delta_{\mathfrak{X}}(G)$ may be identified with the set of $\mathfrak{X}$-core bounded species of $a(G)$, with the weak* topology. It is a compact Hausdorff topological space.

PROPOSITION 4.9.5. *If $\mathfrak{Y} \subseteq \mathfrak{X}$ are ideals of indecomposable kG-modules, then $\Delta_{\mathfrak{X}}(G)$ is homeomorphic to the subset of $\Delta_{\mathfrak{Y}}(G)$ consisting of the $\mathfrak{X}$-core bounded species.*

PROOF. Every $\mathfrak{X}$-core bounded species is $\mathfrak{Y}$-core bounded, so $\Delta_{\mathfrak{X}}(G) \subseteq \Delta_{\mathfrak{Y}}(G)$. Now apply Lemma 2.6.2. □

The corresponding notion for the weighted ℓ^2 norm is as follows.

DEFINITION 4.9.6. A species $s\colon a(G) \to \mathbb{C}$ is *sup bounded* if for all $x \in a(G)_{\geqslant 0}$ we have $|s(x)| \leqslant \|x\|_{\mathrm{sup}}$.

The results of Sections 3.10 and 3.9 give the following.

THEOREM 4.9.7. *There are no non-zero quasi-nilpotent elements in $\hat{a}_{\max}(G)$.* □

COROLLARY 4.9.8. (i) *The Jacobson radical of $\hat{a}_{\max}(G)$ is zero.*
(ii) *The Jacobson radical of $a_{\mathbb{C},\max}(G)$ is zero.* □

THEOREM 4.9.9. *If $e \in \hat{a}_{\max}(G)$ then $0 < \mathsf{Tr}(e) < 1$.* □

COROLLARY 4.9.10. *Let K be a field of characteristic zero whose only totally real subfield is $\mathbb{Q}$, and let $\mathcal{O}_K$ be its ring of integers. Then there are no idempotents in $a_{\mathcal{O}_K,\max}(G)$ other than 0 and 1.* □

4.10. Restriction and induction on $\hat{a}(G)$

If H is a subgroup of a finite group G then we have a restriction map $\mathsf{res}_{G,H}\colon a(G) \to a(H)$ and an induction map $\mathsf{ind}_{H,G}\colon a(H) \to a(G)$. The map $\mathsf{res}_{G,H}$ is a ring homomorphism, while $\mathsf{ind}_{H,G}$ is an $a(G)$-module homomorphism via restriction. These extend in an obvious way to maps $\mathsf{res}_{G,H}\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(H)$ and $\mathsf{ind}_{H,G}\colon a_{\mathbb{C}}(H) \to a_{\mathbb{C}}(G)$ with the same properties.

Proposition 4.10.1. *Let H be a subgroup of a finite group G. Then restriction*

$$\mathsf{res}_{G,H}\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(H)$$

is continuous with respect to the weighted ℓ^1 norm, and therefore extends to give a homomorphism of commutative Banach algebras $\mathsf{res}_{G,H}\colon \hat{a}(G) \to \hat{a}(H)$.

Proof. By Lemma 2.1.3, we must show that this map is bounded. If $x = \sum_{i\in\mathfrak{I}} a_i[M_i]$ with the M_i indecomposable kG-modules, then

$$\begin{aligned}
\|\mathsf{res}_{G,H}(x)\| &= \left\|\sum_{i\in\mathfrak{I}} a_i \mathsf{res}_{G,H}([M_i])\right\| \\
&\leqslant \sum_{i\in\mathfrak{I}} |a_i| \|\mathsf{res}_{G,H}([M_i])\| \\
&= \sum_{i\in\mathfrak{I}} |a_i| \dim(M_i) \\
&= \|x\|
\end{aligned}$$

and so $\mathsf{res}_{G,H}$ is bounded, and hence continuous. Since $\mathsf{res}_{G,H}$ is an algebra homomorphism from $a_{\mathbb{C}}(G)$ to $a_{\mathbb{C}}(H)$, by continuity it is an algebra homomorphism from $\hat{a}(G)$ to $\hat{a}(H)$. □

Proposition 4.10.2. *Let H be a subgroup of a finite group G. Then induction*

$$\mathsf{ind}_{H,G}\colon a(H) \to a(G)$$

is continuous with respect to the weighted ℓ^1 norm, and therefore extends to a continuous map

$$\mathsf{ind}_{H,G}\colon \hat{a}(H) \to \hat{a}(G).$$

For $x \in \hat{a}(G)$ and $y \in \hat{a}(H)$, we have

$$\mathsf{ind}_{H,G}(\mathsf{res}_{G,H}(x)y) = x\,\mathsf{ind}_{H,G}(y), \tag{4.10.3}$$

so regarding $\hat{a}(H)$ as an $\hat{a}(G)$-module via $\mathsf{res}_{G,H}$, $\mathsf{ind}_{H,G}$ is a map of Banach $\hat{a}(G)$-modules.

Proof. Again by Lemma 2.1.3, we must show that the map is bounded. If $x = \sum_{i\in\mathfrak{I}} a_i[M_i]$ with the M_i indecomposable kH-modules then

$$\begin{aligned}
\|\mathsf{ind}_{H,G}(x)\| &= \left\|\sum_{i\in\mathfrak{I}} a_i \mathsf{ind}_{H,G}([M_i])\right\| \\
&\leqslant \sum_{i\in\mathfrak{I}} |a_i| \|\mathsf{ind}_{H,G}([M_i])\| \\
&= \sum_{i\in\mathfrak{I}} |a_i| |G:H| \dim(M_i) \\
&= |G:H| \|x\|,
\end{aligned}$$

so $\mathsf{ind}_{H,G}$ is bounded, and hence continuous. Frobenius reciprocity implies that (4.10.3) holds for elements $x \in a_{\mathbb{C}}(G)$ and $y \in a_{\mathbb{C}}(H)$. By the continuity of $\mathsf{ind}_{H,G}$ and $\mathsf{res}_{G,H}$ (see Proposition 4.10.1) it holds for $x \in \hat{a}(G)$ and $y \in \hat{a}(H)$. □

Theorem 4.10.4. *If H and K are subgroups of G then the Mackey decomposition formula holds for the maps $\mathsf{ind}_{K,G}\colon \hat{a}(K) \to \hat{a}(G)$ and $\mathsf{res}_{G,H}\colon \hat{a}(G) \to \hat{a}(H)$. Namely, for $x \in \hat{a}(K)$ we have*

$$\mathsf{res}_{G,H}\mathsf{ind}_{K,G}(x) = \sum_{g \in H\backslash G/K} \mathsf{ind}_{H\cap {}^gK,H}\mathsf{res}_{{}^gK,H\cap {}^gK}({}^gx).$$

Here, ${}^gK$ is the conjugate gKg^{-1} and ${}^gx$ is the image of x under the natural isomorphism $\hat{a}(K) \to \hat{a}({}^gK)$ given by conjugation by g. The sum is over a set of double coset representatives for H and K in G.

Proof. For $x \in a(K)$ this is the usual Mackey decomposition theorem. The proposition therefore follows from the continuity of restriction and induction, which hold by Propositions 4.10.1 and 4.10.2. $\square$

Theorem 4.10.5. *If H is a subgroup of G then $\hat{a}(H)$ is integral over the image of $\mathsf{res}_{G,H}\colon \hat{a}(G) \to \hat{a}(H)$.*

Proof. Proposition 5.3 of [**12**] proves this for $a(G)_{\mathbb{C}} \to a(H)_{\mathbb{C}}$, and the same proof works in this context. For convenience, we repeat the proof here.

Let $\hat{a}(H)^{N_G(H)}$ be the fixed points in $\hat{a}(H)$ under the natural conjugation action of $N_G(H)$. If $\alpha \in \hat{a}(H)$ let $\alpha_1, \dots, \alpha_n$ be the images of α under $N_G(H)$. Then coefficients of the monic polynomial $(x - \alpha_1)\dots(x - \alpha_n)$ are the symmetric functions of $\alpha_1, \dots, \alpha_n$, which are in $\hat{a}(H)^{N_G(H)}$. Since α is a root of this polynomial, it is integral over $\hat{a}(H)^{N_G(H)}$. So $\hat{a}(H)$ is integral over $\hat{a}(H)^{N_G(H)}$, and it remains to prove that $\hat{a}(H)^{N_G(H)}$ is integral over the image of $\mathsf{res}_{G,H}$.

Let $\alpha \in \hat{a}(H)^{N_G(H)}$. If K is a proper subgroup of H, we let U_K denote the subalgebra of $\hat{a}(H)$ generated by the image of $\mathsf{res}_{G,K}$ together with the elements $\mathsf{res}_{{}^gH,K}({}^g\alpha)$ with $K \leqslant {}^gH$. By induction on $|H|$, we may assume that U_K is finitely generated as a module over $\mathsf{Im}(\mathsf{res}_{G,K})$. We claim that

$$\mathsf{Im}(\mathsf{res}_{G,H}) + \sum_{K<H} \mathsf{ind}_{K,H}(U_K) \tag{4.10.6}$$

is a subalgebra of $\hat{a}(H)$, finitely generated as a module for the image of $\mathsf{res}_{G,H}$, containing α.

First we show that (4.10.6) is a subalgebra. By (4.10.3) and the transitivity of restriction, for $x, y \in \hat{a}(G)$ we have

$$\begin{aligned} \mathsf{res}_{G,H}(x)\mathsf{ind}_{K,H}(\mathsf{res}_{G,K}(y)) &= \mathsf{ind}_{K,H}(\mathsf{res}_{G,K}(xy)), \\ \mathsf{res}_{G,H}(x)\mathsf{ind}_{K,H}(\mathsf{res}_{{}^gH,K}({}^g\alpha)) &= \mathsf{ind}_{K,H}(\mathsf{res}_{G,K}(x)\mathsf{res}_{{}^gH,K}({}^g\alpha)) \end{aligned}$$

and so $\mathsf{Im}(\mathsf{res}_{G,H})U_K \subseteq U_K$. If $K < H$ and $L < H$ then using the Mackey Decomposition Theorem 4.10.4, we have

$$\begin{aligned} \mathsf{ind}_{K,H}(U_K)\mathsf{ind}_{L,H}(U_L) &\subseteq \sum_{g\in H} \mathsf{ind}_{K\cap {}^gL,H}(\mathsf{res}_{K,K\cap {}^gL}(U_K).\mathsf{res}_{{}^gL,K\cap {}^gL}(U_{{}^gL})) \\ &\subseteq \sum_{g\in H} \mathsf{ind}_{K\cap {}^gL,H}(U_{K\cap {}^gL}). \end{aligned}$$

Next we show that (4.10.6) is finitely generated as a module for $\mathsf{Im}(\mathsf{res}_{G,H})$. By induction on $|H|$, we may assume that for every $K \leqslant H$, U_K is finitely generated as a module for $\mathsf{Im}(\mathsf{res}_{G,K})$, say by elements b_i. Then using (4.10.3), $\mathsf{ind}_{K,H}(U_K)$ is finitely generated as a module for $\mathsf{Im}(\mathsf{res}_{G,H})$ by the elements $\mathsf{ind}_{K,H}(b_i)$.

Finally, we show that (4.10.6) contains α. Since α is $N_G(H)$-invariant, the Mackey Decomposition Theorem 4.10.4 gives

$$\mathsf{res}_{G,H}(\mathsf{ind}_{H,G}(\alpha)) = |N_G(H) : H|\alpha + \sum_{\substack{g\in H\backslash G/H \\ H\cap {}^gH<H}} \mathsf{ind}_{H\cap {}^gH,H}(\mathsf{res}_{{}^gH,H\cap {}^gH}(\alpha^g)). \qquad \square$$

Corollary 4.10.7. *If s is a species of $\hat{a}(G)$ which vanishes on the kernel of $\mathsf{res}_{G,H}$ then there exists a species s' of $\hat{a}(H)$ such that s is the composite*

$$\hat{a}(G) \xrightarrow{\mathsf{res}_{G,H}} \hat{a}(H) \xrightarrow{s'} \mathbb{C}.$$

Proof. This follows from Theorem 4.10.5 and the going-up theorem. $\square$

4.11. Adams psi operations

The Adams psi operations ψ^n on $a(G)$ were defined in [**7**] and further studied in [**19, 20, 21**]. In this section, we show that for any ideal of indecomposables $\mathfrak{X}$, ψ^n acts on $a_{\mathfrak{X}}(G)$ via ring endomorphisms. This gives rise to an action on $\Delta_{\mathfrak{X}}(G)$. For the purpose of this section, we assume that k is algebraically closed, although this could be avoided.

Let M be a kG-module. Then there is an action of $\mathbb{Z}/n$ on $M^{\otimes n}$ permuting the tensor factors, and this commutes with the action of G. If n is coprime to p then $M^{\otimes n}$ decomposes as a sum of eigenspaces of $\mathbb{Z}/n$. Let $\varepsilon\colon \mathbb{Z}/n \to k$ be a generator for the character group of $\mathbb{Z}/n$ over k, and write $[M^{\otimes n}]_{\varepsilon^i}$ for the eigenspace corresponding to ε^i. Let $\zeta_n = e^{2\pi \mathsf{i}/n}$ be a primitive nth root of unity in $\mathbb{C}$, and set

$$\psi^n[M] = \sum_{i=1}^{n} \zeta_n^i [M^{\otimes n}]_{\varepsilon^i} \in a_{\mathbb{C}}(G).$$

Proposition 4.11.1. *If M_1 and M_2 are kG-modules then*
(i) *$\psi^n([M_1] + [M_2]) = \psi^n[M_1] + \psi^n[M_2]$, and*
(ii) *$\psi^n([M_1 \otimes M_2] = \psi^n[M_1]\psi^n[M_2]$.*

Proof. This is proved in Proposition 1 of [**7**]. $\square$

It follows from the proposition that ψ^n can be extended to a $\mathbb{C}$-algebra homomorphism $a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(G)$.

Proposition 4.11.2. *Let n be coprime to p. For $d|n$, let $\varepsilon_d = \varepsilon^{n/d}$, a character of order d of $\mathbb{Z}/n$. Then*

$$\psi^n[M] = \sum_{d|n} \mu(d)[M^{\otimes n}]_{\varepsilon_d}.$$

Here, μ is the Möbius function.

Proof. This is proved in Proposition 2 of [**7**]. $\square$

Corollary 4.11.3. *For n coprime to p, the homomorphism ψ^n takes elements of $a(G)$ to elements of $a(G)$, and therefore induces a ring homomorphism $\psi^n\colon a(G) \to a(G)$.*

Proof. The formula given in the proposition has integer coefficients. □

Theorem 4.11.4. *If m and n are coprime to p then $\psi^m \circ \psi^n = \psi^{mn} = \psi^n \circ \psi^m$.*

Proof. This is proved in Theorem 1 of [**7**]. □

Definition 4.11.5 (Frobenius twist). For $r \geqslant 0$ we let $k^{(p^r)}$ be a k-k-bimodule, where the left action of $\lambda \in k$ is multiplication by λ, and the right action is multiplication by λ^{p^r}. If M is a kG-module, we define the rth *Frobenius twist* of M to be

$$M^{(p^r)} = k^{(p^r)} \otimes_k M.$$

Thus $\lambda^{p^r}(1 \otimes m) = \lambda^{p^r} \otimes m = 1 \otimes \lambda m$, so λ^{p^r} acts on $M^{(p^r)}$ in the way that λ acts on M. If k is perfect, then λ acts on $M^{(p^r)}$ is the way that $\lambda^{p^{-r}}$ acts on M.

Definition 4.11.6. We define $\psi^p[M] = [M^{(p)}]$, the Frobenius twist of M. This gives us ring homomorphisms $\psi^p\colon a(G) \to a(G)$ and $\psi^p\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(G)$. In general, if $n = p^a m$ with $p \nmid m$, we define $\psi^n = \psi^m \circ (\psi^p)^a = (\psi^p)^a \circ \psi^m$. Thus Theorem 4.11.4 remains true for all m and n.

If $\mathfrak{X}$ is an ideal of indecomposable kG-modules, we say that $\mathfrak{X}$ is *Frobenius stable* if $M \in \mathfrak{X} \Rightarrow M^{(p)} \in \mathfrak{X}$. In Example 4.2.4, everything is Frobenius stable except for $\mathfrak{X}_{\mathcal{V}}$, and in this case it is Frobenius stable precisely when the subset $\mathcal{V}$ is stable under the Frobenius map, namely the map induced by the pth power map on $H^*(G, k)$.

If s is a species of $a(G)$, we define $\psi^n(s)$ by the formula $\psi^n(s)(x) = s(\psi^n(x))$.

Proposition 4.11.7. *Regarding* $\dim\colon a(G) \to \mathbb{C}$ *as a species, we have* $\psi^n(\dim) = \dim$*.*

Proof. The species $\dim\colon a(G) \to \mathbb{C}$ factors through restriction to the trivial subgroup 1, $a(G) \to a(1)$. We have $a(1) = \mathbb{Z}$, with generator $[k]$. We have $[k^{\otimes n}]_1 = [k]$ and $[k^{\otimes n}]_{\varepsilon^i} = 0$ if $\varepsilon^i \neq 1$, and so we have $\psi^n[k] = [k]$. So ψ^n is the identity map on $a(1)$. □

Theorem 4.11.8. *For $x \in a_{\mathbb{C}}(G)$ we have $\|\psi^n(x)\| \leqslant \|x\|^n$.*

Proof. We first verify this with $x = [M]$ and n a prime. If $n = p$ we have $\|\psi^p[M]\| = \dim(M^{(p)}) = \dim(M) = \|[M]\| \leqslant \|[M]\|^p$. If n is a prime q not equal to p then

$$\psi^q[M] = [M^{\otimes q}]_1 - [M^{\otimes q}]_\varepsilon$$

where ε is a faithful character of $\mathbb{Z}/q$. On the other hand,

$$[M]^q = [M^{\otimes q}]_1 + (q-1)[M^{\otimes q}]_\varepsilon.$$

Thus $\|\psi^q[M]\| \leqslant (\dim M)^q = \|[M]\|^q$.

If $x = \sum_i a_i[M_i]$ and n is prime then $\psi^n(x) = \sum_i a_i \psi^n[M_i]$ and

$$\|\psi^n(x)\| \leqslant \sum_i |a_i| \|\psi^n[M_i]\| \leqslant \sum_i |a_i| \|[M_i]\|^n \leqslant \|x\|^n.$$

Finally, if n is composite, we use Theorem 4.11.4 and induction. □

Corollary 4.11.9. *ψ^n is continuous on $a_{\mathbb{C}}(G)$ with respect to the weighted ℓ^1 norm.*

Proof. By Theorem 4.11.8, if $\|x\| \leqslant 1$ then $\|\psi^n(x)\| \leqslant 1$. So ψ^n is bounded, and hence continuous by Lemma 2.1.3. □

Corollary 4.11.10. *If s is a dimension bounded species, then so is $\psi^n(s)$.*

Proof. We use Theorem 3.3.5 with $\mathfrak{X} = \varnothing$. This says that s is dimension bounded if and only if it is continuous with respect to the weighted ℓ^1 norm on $a_{\mathbb{C}}(G)$. If s is dimension bounded then it is continuous, so using Corollary 4.11.9, the composite $a_{\mathbb{C}}(G) \xrightarrow{\psi^n} a_{\mathbb{C}}(G) \xrightarrow{s} \mathbb{C}$ is continuous, and hence dimension bounded. □

Theorem 4.11.11. *Let $\mathfrak{X}$ be a Frobenius stable ideal of indecomposable kG-modules. If s is an $\mathfrak{X}$-core bounded species of $a(G)$ then $\psi^n(s)$ is also $\mathfrak{X}$-core bounded.*

Proof. It follows from the definition of ψ^n that $\psi^n\langle\mathfrak{X}\rangle_{\mathbb{C}} \leqslant \langle\mathfrak{X}\rangle_{\mathbb{C}}$. So the composite

$$a_{\mathbb{C}}(G) \xrightarrow{\psi^n} a_{\mathbb{C}}(G) \xrightarrow{s} \mathbb{C}$$

has $\langle\mathfrak{X}\rangle_{\mathbb{C}}$ in its kernel. This composite is continuous, by Corollary 4.11.9, and so it induces a continuous map $a_{\mathbb{C},\mathfrak{X}}(G) \to \mathbb{C}$. By Theorem 3.3.5, it follows that s is $\mathfrak{X}$-core bounded. □

It follows from the theorem that we have an action of ψ^n on $\Delta_{\mathfrak{X}}(G)$.

4.12. The Burnside ring

In this final section, we take a brief look at an example of a representation ring where our theory does not say much. If G is a finite group, the *Burnside ring* $b(G)$ is the Grothendieck ring of finite G-sets. It is a free abelian group whose basis elements $[G/H]$ correspond to the transitive G-sets G/H up to isomorphism. Here, G/H and G/H' are isomorphic as G-sets if and only if H and H' are conjugate subgroups. Non-negative elements of $b(G)$ are interpreted as isomorphism classes of finite G-sets, and multiplication is given by direct product of G-sets. The element ρ is the regular representation $[G/1]$, and is the only projective basis element.

It is well known that $b(G)$ is semisimple. Its species are as follows. If H is a subgroup of G then $s_H : b(G) \to \mathbb{C}$ sends a G-set X to $|X^H|$, the number of fixed points of H on X. This is obviously a ring homomorphism, and two subgroups give rise to the same species if and only if they are conjugate.

A collection $\mathfrak{X}$ of basis elements $[G/H]$ is a representation ideal if and only if it corresponds to a collection $\mathcal{H}$ of subgroups $H \leqslant G$, closed under conjugacy, closed under taking subgroups, and not containing G. Thus $\mathfrak{X}_{\max}$ corresponds to the collection of all proper subgroups, while $\mathfrak{X}_{\mathsf{proj}}$ corresponds to the collection just consisting of the trivial subgroup. Since there are as many species as there are basis elements, $b_{\mathbb{C}}(G) = \mathbb{C} \otimes_{\mathbb{Z}} b(G)$ is semisimple.

Dress [**37**] investigated idempotents in $b(G)$, and discovered that there is a one to one correspondence between the primitive idempotents and conjugacy classes of perfect subgroups of G. So the statement that the only idempotents in $b(G)$ are 0 and 1 is equivalent to the statement that G is soluble.

CHAPTER 5

Examples and Problems

5.1. The two dimensional module for $\mathbb{Z}/5$

We begin with the example in the introduction to [**13**]. Consider the two dimensional module J_2 for $G = \mathbb{Z}/5 = \langle g \mid g^5 = 1\rangle$ over a field k of characteristic five given by

$$g \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There are five indecomposable kG-modules $J_1, \dots, J_5$ corresponding to Jordan blocks of lengths between 1 and 5 and eigenvalue 1, representing the element g; the Jordan block of length two is illustrated above, and the one of length five is the projective indecomposable module. The table of tensor products is as follows.

	J_1	J_2	J_3	J_4	J_5
J_1	J_1	J_2	J_3	J_4	J_5
J_2	J_2	$J_1 \oplus J_3$	$J_2 \oplus J_4$	$J_3 \oplus J_5$	$2J_5$
J_3	J_3	$J_2 \oplus J_4$	$J_1 \oplus J_3 \oplus J_5$	$J_2 \oplus 2J_5$	$3J_5$
J_4	J_4	$J_3 \oplus J_5$	$J_2 \oplus 2J_5$	$J_1 \oplus 3J_5$	$4J_5$
J_5	J_5	$2J_5$	$3J_5$	$4J_5$	$5J_5$

It is clearly visible from this table that the only non-trivial representation ideal is the projective ideal $\{J_5\}$. The tensor powers $J_2^{\otimes n}$ are given by the columns of the following table.

$n \to$	0	1	2	3	4	5	6	7	8	9	10	11	12	$\dots$
J_1	1	0	1	0	2	0	5	0	13	0	34	0	89	
J_2	0	1	0	2	0	5	0	13	0	34	0	89	0	
J_3	0	0	1	0	3	0	8	0	21	0	55	0	144	
J_4	0	0	0	1	0	3	0	8	0	21	0	55	0	
J_5	0	0	0	0	1	2	7	14	36	72	165	330	715	$\dots$

The Fibonacci pattern in the non-projective summands is clear. It should come as no surprise that the dimension of the non-projective part of $J_2^{\otimes n}$ is given by $\mathsf{c}^G_{2n}(J_2) \sim \tau^{2n+1}$ and $\mathsf{c}^G_{2n+1}(J_2) \sim 2\tau^{2n+1}$ where

$$\tau = (1+\sqrt{5})/2 = 2\cos(\pi/5) \sim 1.618034$$

is the golden ratio. More precisely, we have

$$\mathsf{c}^G_{2n}(J_2) = F_{2n-1} + 3F_{2n} = F_{2n} + F_{2n+2} = \tau^{2n+1} + \bar\tau^{2n+1}, \qquad \mathsf{c}^G_{2n+1}(J_2) = 2\mathsf{c}^G_{2n}(J_2),$$

where F_n is the nth Fibonacci number and

$$\bar\tau = -1/\tau = 1 - \tau = (1-\sqrt{5})/2 = 2\cos(3\pi/5) \sim -0.618034.$$

Thus we have $\gamma_G(J_2) = \tau$. In the next section, we put this in a broader context.

5.2. The cyclic group of order p

The classification of modules for the cyclic group of order p in characteristic p is well known. Let $G = \langle g \mid g^p = 1$ and k be a field of characteristic p. We have $(g-1)^p = g^p - 1 = 0$ in kG, and so if M is a kG-module then $g-1$ acts nilpotently. It follows that the eigenvalues of g on M are all equal to one, and the action of g on M may be put into Jordan canonical form without extending the field. So the indecomposable kG-modules correspond to Jordan blocks with eigenvalue one and dimension between one and p:

$$g \mapsto \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & 1 & 1 \\ 0 & 0 & & \cdots & 0 & 1 \end{pmatrix}$$

We write J_j for the indecomposable kG-module of dimension j for $1 \leqslant j \leqslant p$. The projective modules are the direct sums of copies of J_p. Tensor products are determined by

$$J_2 \otimes J_j \cong \begin{cases} J_2 & j = 1 \\ J_{j+1} \oplus J_{j-1} & 2 \leqslant j \leqslant p-1 \\ J_p \oplus J_p & j = p. \end{cases}$$

It follows from these relations that every element of $a(G)$ is a polynomial in $[J_2]$. If $p = 2$, this shows that $a(G) \cong \mathbb{Z}[X]/(X^2 - 2X)$, where X corresponds to $[J_2]$. So we shall assume for the rest of this section that $p \geqslant 3$, in which case we have, for example, $[J_3] = [J_2]^2 - \mathbb{1}$.

Next, we discuss the species of $a(G)$. This discussion is based on the work of Green [**48**], Srinivasan [**88**], and involves the Chebyshev polynomials of the second kind $U_j(X)$. Background material on these polynomials can be found in Rivlin [**83**].

Definition 5.2.1. The Chebyshev polynomials of the second kind $U_j(X)$ are defined inductively for $j \geqslant 0$ by $U_0(X) = 1$, $U_1(X) = 2X$, and for $j \geqslant 2$,

$$U_j(X) = 2XU_{j-1}(X) - U_{j-2}(X).$$

The first few are as follows:

$$\begin{aligned} U_0(X) &= 1 & U_4(X) &= 16X^4 - 12X^2 + 1 \\ U_1(X) &= 2X & U_5(X) &= 32X^5 - 32X^3 + 6X \\ U_2(X) &= 4X^2 - 1 & U_6(X) &= 64X^6 - 80X^4 + 24X^2 - 1 \\ U_3(X) &= 8X^3 - 4X & U_7(X) &= 128X^7 - 192X^5 + 80X^3 - 8X. \end{aligned}$$

and in general

$$U_j(X) = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^i \binom{j-i}{i} (2X)^{j-2i}.$$

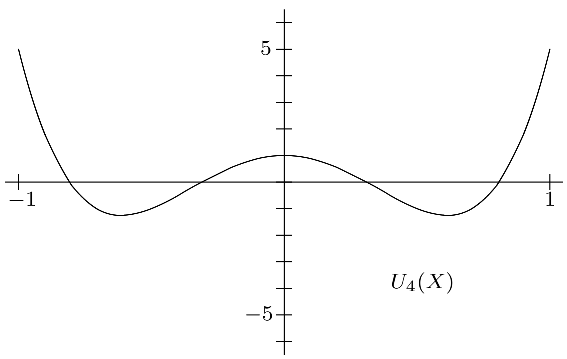
LEMMA 5.2.2. *We have*

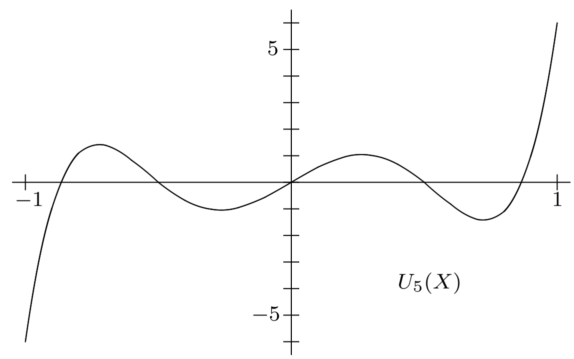
$$U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}.$$

The roots of $U_j(X)$ are real and distinct, symmetric about $X = 0$, and given by

$$X = \cos(k\pi/(j+1)), \qquad 1 \leqslant k \leqslant j.$$

PROOF. The verification of the formula for $U_j(\cos\theta)$ is an easy inductive argument using standard trigonometric identities. The vanishing for these values of X then follows from the fact that $\sin(k\pi) = 0$ while $\sin(k\pi/(j+1)) \neq 0$. Since the number of such roots is j, which is the degree of $U_j(X)$, these are the only roots. $\square$





DEFINITION 5.2.3. We define $f_j(X) = U_{j-1}(X/2)$. So we have $f_1(X) = 1$, $f_2(X) = X$ and for $j \geqslant 2$,

$$Xf_j(X) = f_{j+1}(X) + f_{j-1}(X). \tag{5.2.4}$$

The first few are as follows:

$$\begin{aligned} f_1(X) &= 1 & f_6(X) &= X^5 - 4X^3 + 3X \\ f_2(X) &= X & f_7(X) &= X^6 - 5X^4 + 6X^2 - 1 \\ f_3(X) &= X^2 - 1 & f_8(X) &= X^7 - 6X^5 + 10X^3 - 4X \\ f_4(X) &= X^3 - 2X & f_9(X) &= X^8 - 7X^6 + 15X^4 - 10X^2 + 1 \\ f_5(X) &= X^4 - 3X^2 + 1 & f_{10}(X) &= X^9 - 8X^7 + 21X^5 - 20X^3 + 5X. \end{aligned}$$

By Lemma 5.2.2 we have

$$f_j\left(\frac{\sin 2\theta}{\sin\theta}\right) = \frac{\sin j\theta}{\sin\theta} \tag{5.2.5}$$

(note that $\sin 2\theta/\sin\theta = 2\cos\theta$).

We write ζ_n for $e^{2\pi \mathrm{i}/n}$, a primitive nth root of unity in $\mathbb{C}$. For the purposes of this section, we do not need part (vii) of the following lemma, only part (vi). But part (vii) will be used in Section 5.8.

LEMMA 5.2.6.

(i) *$f_j(X)$ is a polynomial of degree $j-1$ in X with integer coefficients.*

(ii) *If j is odd then $f_j(X)$ is a polynomial in X^2 of degree $(j-1)/2$.*
(iii) *If j is even then $f_j(X)$ is X times a polynomial in X^2 of degree $(j-2)/2$.*
(iv) *The roots of $f_j(X)$ are $\zeta_{2j}^k + \zeta_{2j}^{-k} = 2\cos(k\pi/j)$ with $1 \leqslant k \leqslant j-1$, each with multiplicity one.*
(v) *If j_1 divides j_2 then $f_{j_1}(X)$ divides $f_{j_2}(X)$.*
(vi) *If $p > 2$ is a prime then $f_p(X)$ is an irreducible polynomial in X^2 of degree $(p-1)/2$, whose roots are $\zeta_{2p}^k + \zeta_{2p}^{-k}$ with $1 \leqslant k \leqslant p-1$. The splitting field of $f_p(X)$ is the real subfield of the cyclotomic field of pth roots of unity.*
(vii) *More generally, if $p^m > 2$ is a prime power then $f_{p^m}(X)/f_{p^{m-1}}(X)$ is an irreducible polynomial in X^2 of degree $p^{m-1}(p-1)/2$, whose roots are $\zeta_{2p^m}^k + \zeta_{2p^m}^{-k}$ with $1 \leqslant k \leqslant p^m - 1$ not divisible by p. The splitting field of $f_{p^m}(X)/f_{p^{m-1}}(X)$ is the real subfield of the cyclotomic field of $2p^m$th roots of unity.*

PROOF. Parts (i)–(iii) follow from the definition and induction on j. Part (iv) follows from Lemma 5.2.2. It follows from (iv) that if j_1 divides j_2 then the roots of f_{j_1} are among the roots of f_{j_2}, which proves (v). Since (vi) is a special case of (vii), it remains to prove (vii). This follows from the fact that the real subfield of the cyclotomic field of $2p^m$th roots of unity is generated by the element $\zeta_{2p^m} + \zeta_{2p^m}^{-1}$, whose conjugates are the $\zeta_{2p^m}^k + \zeta_{2p^m}^{-k}$ with $1 \leqslant k \leqslant p^m - 1$ not divisible by p, and the degree of this real subfield is $p^{m-1}(p-1)/2$ provided $p^m > 2$. Note that if p is odd then the cyclotomic field of $2p^m$th roots of unity is the same as that of p^mth roots of unity. $\square$

THEOREM 5.2.7. *For $1 \leqslant j \leqslant p$ we have $f_j([J_2]) = [J_j]$ in $a(G)$. We have*

$$a(G)/a(G,1) \cong \mathbb{Z}[X]/(f_p(X)),$$

where X corresponds to the element $[J_2]$.

PROOF. We have $[J_2][J_j] = [J_{j-1}] + [J_{j+1}]$ for $1 \leqslant j \leqslant p-1$, and we also have $f_2(X)f_j(X) = f_{j-1}(X) + f_{j+1}(X)$. So by induction on j we deduce that $f_j([J_2]) = [J_j]$ for $1 \leqslant j \leqslant p$. The ideal $a(G,1)$ is generated by $[J_p]$, so we have a surjective map $\mathbb{Z}[X] \to a(G)/a(G,1)$ sending X to $[J_2]$ and $f_j(X)$ to $[J_j]$ for $1 \leqslant j \leqslant p-1$, and whose kernel contains $f_p(X)$. Comparing ranks, we see that this map induces an isomorphism $\mathbb{Z}[X]/(f_p(X)) \to a(G)/a(G,1)$. $\square$

THEOREM 5.2.8. *The species of $a_{\mathsf{proj}}(G) = a(G)/a(G,1)$ are given by*

$$s_k[J_2] = 2\cos(k\pi/p)$$

for $1 \leqslant k \leqslant p-1$. We have

$$s_k[J_j] = \frac{\sin(jk\pi/p)}{\sin(k\pi/p)} = f_j(2\cos(k\pi/p)).$$

PROOF. By Theorem 5.2.7 we have $a(G)/a(G,1) \cong \mathbb{Z}[X]/(f_p(X))$ with X corresponding to J_2. By Lemma 5.2.6 with $m = 1$, $f_p(X)$ is irreducible, and its roots are $X = 2\cos(k\pi/p)$ with $1 \leqslant k \leqslant p-1$. So the ring homomorphisms $a(G)/a(G,1) \to \mathbb{C}$ are given by $s_k[J_2] = 2\cos(k\pi/p)$. Still referring to Theorem 5.2.7, we have $[J_j] = f_j[J_2]$, and so

$$s_k[J_j] = f_j(s_k[J_2]) = f_j(2\cos(k\pi/p)),$$

which by (5.2.5) is equal to $\sin(jk\pi/p)/\sin(k\pi/p)$. $\square$

COROLLARY 5.2.9. *The species of $a(G)$ are the core bounded ones, which are given in Theorem 5.2.8, together with $s_0 = \dim\colon a(G) \to \mathbb{C}$.* $\square$

REMARK 5.2.10. Since $[J_2][J_p] = 2[J_p]$, the proof of Theorem 5.2.7 shows that

$$a(G) \cong \mathbb{Z}[X]/((X-2)f_p(X)),$$

with X corresponding to $[J_2]$.

EXAMPLE 5.2.11. For $p = 3$ and $p = 5$ the species table of $\mathbb{Z}/p$ is as follows:

$p=3$	s_0	s_1	s_2
$[J_1]$	1	1	1
$[J_2]$	2	1	-1
$[J_3]$	3	0	0

$p=5$	s_0	s_1	s_2	s_3	s_4
$[J_1]$	1	1	1	1	1
$[J_2]$	2	τ	$-\bar{\tau}$	$\bar{\tau}$	$-\tau$
$[J_3]$	3	τ	$\bar{\tau}$	$\bar{\tau}$	τ
$[J_4]$	4	1	-1	1	-1
$[J_5]$	5	0	0	0	0

THEOREM 5.2.12. *We have $\gamma_G(J_2) = 2\cos(\pi/p)$, and more generally*

$$\gamma_G(J_j) = \sin(j\pi/p)/\sin(\pi/p).$$

PROOF. It follows from Theorem 4.9.2 that $\gamma_G(J_2)$ is the largest value of a core bounded species on $[J_2]$. The core bounded species are described in Theorem 5.2.8 and its corollary. The largest of the numbers $s_k[J_2] = 2\cos(k\pi/p)$ for $1 \leqslant k \leqslant p-1$ is $s_1[J_2] = 2\cos(\pi/p)$, and so $\gamma_G(J_2) = 2\cos(\pi/p)$. By Theorem 3.5.1, for all kG-modules M we have $\gamma_G(M) = s_1[M]$. In particular, $\gamma_G(J_j) = s_1[J_j] = \sin(j\pi/p)/\sin(\pi/p)$. $\square$

Approximate values of $\gamma_G(J_j)$

p	3	5	7	11	13	17	19	23
J_1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
J_2	1.00000	1.61803	1.80194	1.91899	1.94188	1.96595	1.97272	1.98137
J_3	0.00000	1.61803	2.24698	2.68251	2.77091	2.86494	2.89163	2.92583
J_4		1.00000	2.24698	3.22871	3.43891	3.66638	3.73167	3.81579
J_5		0.00000	1.80194	3.51334	3.90704	4.34296	4.46992	4.63467
J_6			1.00000	3.51334	4.14811	4.87165	5.08623	5.36722
J_7			0.00000	3.22871	4.14811	5.23444	5.56381	5.99978
J_8				2.68251	3.90704	5.41898	5.88962	6.52058

The Adams operations ψ^n on the representation rings of cyclic groups were investigated by Almkvist [**2**], Kouwenhoven [**63**], Bryant and Johnson [**20, 21**], Nam and Oh [**71, 72**].

THEOREM 5.2.13. *Suppose that p does not divide n. Then we have the following.*

(i) *For $1 \leqslant n \leqslant p-1$, $\psi^n[J_2] = [J_{n+1}] - [J_{n-1}]$.*
(ii) *$\psi^{2p-n} = \psi^{2p+n} = \psi^n = \psi^{p^m n}$ on $a(G)$ for all $m \geqslant 0$.*

(iii) $\psi^n(s_k)[J_2] = 2\cos nk\pi/p$.
(iv) *If* $nk \equiv \pm j \pmod{2p}$ *with* $1 \leqslant j \leqslant p-1$ *then* $\psi^n(s_k) = s_j$.

Proof. (i) This is proved in Section 5 of [**2**], see also Theorem 5.1 of [**20**].

(ii) The first two equalities are proved in Theorem 3.3 of [**20**]. The third follows from the fact that the modules J_i are defined over $\mathbb{F}_p$, so the Frobenius map ψ^{p^m} acts as the identity map on $a(G)$ for all $m \geqslant 0$.

(iii) Using (i), we have

$$\begin{aligned}\psi^n(s_k)[J_2] &= s_k(\psi^n[J_2])\\ &= s_k[J_{n+1}] - s_k[J_{n-1}]\\ &= (\sin((n+1)k\pi/p) - \sin((n-1)k\pi/p))/\sin(k\pi/p)\\ &= (2\cos nk\pi/p \sin k\pi/p)/\sin k\pi/p\\ &= 2\cos nk\pi/p.\end{aligned}$$

(iv) A species s_j is determined by its value on $[J_2]$. So this follows from (ii), together with the fact that if $m \equiv \pm j \pmod{2p}$ then $\cos m\pi/p = \cos j\pi/p$. □

5.3. Some Frobenius groups

Let G be a Frobenius group with cyclic normal Sylow p-subgroup and cyclic quotient. The structure of $a(G)$ was investigated by O'Reilly [**73, 74**], Lam [**64**], Lam and Reiner [**65**], Benson and Carlson [**11**]. In this section we shall concentrate on the case $\mathbb{Z}/p \rtimes \mathbb{Z}/m$ where the Sylow p-subgroup has order p. We allow the action to have a kernel, so we do not assume that m divides $p-1$. In a later section we shall examine $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, so we set up the notation in that generality, and then impose the assumption that $n = 1$.

Let p^n be a prime power, let m be a positive integer not divisible by p, and let q be a positive integer satisfying $q^m \equiv 1 \pmod{p^n}$. Let

$$G = \langle g, h \mid g^{p^n} = 1,\ h^{2m} = 1, hgh^{-1} = g^q\rangle \cong \mathbb{Z}/p \rtimes \mathbb{Z}/2m,$$

a Frobenius group of order $2p^n m$, and let $P = \langle g\rangle$, $H = \langle h\rangle$ as subgroups of G. We remark that the representation ring of $G/\langle h^m\rangle \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ is contained in $a(G)$, and we shall identify the image. But it turns out to be convenient to have this extra central involution. This only really matters when m is even, but we do it in all cases for uniformity. Let k be a field of characteristic p containing a primitive $2m$th root of unity η. Let d be chosen so that $\eta^{2d} \equiv q \pmod{p}$.

The element

$$x = \sum_{\substack{1\leqslant j\leqslant p^n-1\\ (p,j)=1}} g^j/j$$

spans an H-invariant complement to $J^2(kP)$ in $J(kP)$, and $hxh^{-1} = qx$. We have the presentation

$$kG = k\langle h, x \mid x^{p^n} = 0, h^{2m} = 1, hx = qxh\rangle.$$

There are $2m$ isomorphism classes of simple modules S_i, $i \in \mathbb{Z}/2m$, all one dimensional, corresponding to the characters of H. Letting v_i be a basis vector for S_i, we have $hv_i = \eta^i v_i$

and $xv_i = 0$. The space $\mathsf{Ext}^1_{kG}(S_i, S_j)$ is one dimensional if $j = i + 2d$ and zero dimensional otherwise. The projective indecomposable modules are uniserial of length p^n, with composition factors (from the top) of P_i being S_i, S_{i+2d}, S_{i+4d}, $\dots$, S_i. Every indecomposable module is a quotient of a projective indecomposable module. We write J_j $(1 \leqslant j \leqslant p^n)$ for the indecomposable module of length j with composition factors $S_{-d(j-1)}$, $S_{-d(j-3)}, \dots, S_{d(j-1)}$. Note that it is only because of the extra involution that we are able to do this symmetrically about the middle. A complete list of the $2mp$ isomorphism classes of indecomposable kG-modules is given by the modules $J_j \otimes S_i$ with $1 \leqslant j \leqslant p^n$, $0 \leqslant i < 2m$.

Now assume that $n = 1$. We have

$$J_2 \otimes J_j \cong \begin{cases} J_2 & j = 1, \\ J_{j+1} \oplus J_{j-1} & 2 \leqslant j \leqslant p-1 \\ (J_p \otimes S_d) \oplus (J_p \otimes S_{-d}) & j = p. \end{cases}$$

This should be compared with the relations given in Section 5.2, where the modules J_j are the restrictions to P of the ones here.

Theorem 5.3.1. *Let $G = \mathbb{Z}/p \rtimes \mathbb{Z}/2m$ as above. We have*

$$a(G) \cong \mathbb{Z}[X, Y]/(Y^{2m} - 1, (X - Y^d - Y^{-d})f_p(X)),$$

where X corresponds to J_2 and Y corresponds to S_1, and the polynomials f_i are described in Definition 5.2.3.

Proof. This is proved in essentially the same way as Theorem 5.2.7 and Remark 5.2.10, with $f_j(X)$ corresponding to $[J_j]$ for $1 \leqslant j \leqslant p$. $\square$

Corollary 5.3.2. *The ring $a(G)$ is semisimple. Its $2pm$ species $s_{i,j}$ $(0 \leqslant i < p,\ 0 \leqslant j < 2m)$ are given by*

$$X \mapsto \begin{cases} \zeta_{2m}^{dj} + \zeta_{2m}^{-dj} = 2\cos(dj\pi/m) & i = 0, \\ \zeta_{2p}^{i} + \zeta_{2p}^{-i} = 2\cos(i\pi/p) & 0 < i < p. \end{cases}$$
$$Y \mapsto \zeta_{2m}^{j}.$$

The Brauer species are the ones with $i = 0$.

Proof. The assignment $Y \mapsto \zeta_{2m}^j$ satisfies the relation $Y^{2m} - 1 = 0$. If $i = 0$ then $X \mapsto \zeta_{2m}^{dj} + \zeta_{2m}^{-dj}$ makes the factor $(X - Y^d - Y^{-d})$ vanish. If $0 < i < p-1$ then $X \mapsto \zeta_{2p}^i + \zeta_{2p}^{-i}$ makes the factor $f_p(X)$ vanish by Lemma 5.2.6.

There are $2mp$ species $s_{i,j}$, and the $\mathbb{Z}$-rank of $a(G)$ is $2mp$, so there can be no more species, and $a_{\mathbb{C}}(G)$ is semisimple. $\square$

To identify the subring $a(G)/\langle h^m \rangle$, we define some polynomials ϕ_i of two variables as follows.

Definition 5.3.3. Define polynomials $\phi_i(y, z)$ inductively as follows: $\phi_1(y, z) = 1$, $\phi_2(y, z) = z$, and

$$\phi_i(y, z) = z\phi_{i-1}(y, z) - y\phi_{i-2}(y, z) \qquad (i \geqslant 3),$$

an inhomogeneous polynomial of degree $i-1$. These are related to the Chebyshev polynomials of the second kind $U_j(X)$ introduced in Section 5.2 via

$$\phi_i(y,z) = y^{\frac{i-1}{2}} f_i\big(\tfrac{z}{y^{1/2}}\big) = y^{\frac{i-1}{2}} U_{i-1}\big(\tfrac{z}{2y^{1/2}}\big).$$

There is an illusory choice of square root of y here: as long as we take $y^{\frac{i-1}{2}}$ to be the $(i-1)$st power of $y^{1/2}$, there is no ambiguity in the answer.

THEOREM 5.3.4. $a(\mathbb{Z}/p \rtimes \mathbb{Z}/m) = a(G/\langle h^m\rangle) \cong \mathbb{Z}[y,z]/(y^m-1, (z-y^d-1)\phi_p(y^d,z))$ *with* $y = Y^2$ *and* $z = XY^d$. *This is a complete intersection of* $\mathbb{Z}$*-rank* pm*, with a* $\mathbb{Z}$*-basis consisting of the monomials* y^iz^j *with* $0 \leqslant i < m$, $0 \leqslant j < p$.

PROOF. The modules S_2 and $J_2 \otimes S_d$ generate the representation ring of $G/\langle h^m\rangle$, and correspond to the elements Y^2 and XY^d. We have $\phi_p(y^d,z) = \phi_p(Y^{2d}, XY^d) = Y^{(p-1)d}f_p(X)$, and so $(z-y^d-1)\phi_p(y,z) = (X-Y^d-Y^{-d})Y^{pd}f_p(X)$. Since Y^{pd} is invertible, this is equivalent to the relation given in $a(G)$. $\square$

COROLLARY 5.3.5. *The ring* $a_{\mathbb{C}}(\mathbb{Z}/p \rtimes \mathbb{Z}/m)$ *is semisimple. Its* pm *species* $s_{i,j}$ $(0 \leqslant i < p,\ 0 \leqslant j < m)$ *are given by*

$$\begin{aligned} y &\mapsto \zeta_{2m}^{2j} \\ z &\mapsto \begin{cases} \zeta_{2m}^{2dj}+1 & i=0, \\ (\zeta_{2p}^{i}+\zeta_{2p}^{-i})\zeta_{2m}^{dj} & 0<i<p. \end{cases} \end{aligned}$$

The Brauer species are the ones with $i = 0$. $\square$

EXAMPLE 5.3.6. The smallest example is the symmetric group of degree three, which is a semidirect product $\mathbb{Z}/3 \rtimes \mathbb{Z}/2$, in characteristic three. The species table is as follows.

	$s_{0,0}$	$s_{0,1}$	$s_{1,0}$	$s_{1,1}$	$s_{2,0}$	$s_{2,1}$
$[S_0]$	1	1	1	1	1	1
$y = [S_2]$	1	-1	1	-1	1	-1
$z = [J_2 \otimes S_1]$	2	0	1	i	-1	$-\mathsf{i}$
$yz = [J_2 \otimes S_3]$	2	0	1	$-\mathsf{i}$	-1	i
$[J_3]$	3	1	0	0	0	0
$[J_3 \otimes S_2]$	3	-1	0	0	0	0

Example 5.3.7. The species table for $\mathbb{Z}/5 \rtimes \mathbb{Z}/4$ in characteristic five is as follows, with $\tau = (1+\sqrt{5})/2$ and $\zeta = \zeta_8 = e^{\pi\mathsf{i}/4} = (1+\mathsf{i})/\sqrt{2}$.

	$s_{0,0}$	$s_{0,1}$	$s_{0,2}$	$s_{0,3}$	$s_{1,0}$	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{2,0}$	$s_{2,1}$	$s_{2,2}$	$s_{2,3}$	$s_{3,0}$	$s_{3,1}$	$s_{3,2}$	$s_{3,2}$	$s_{4.0}$	$s_{4,1}$	$s_{4,2}$	$s_{4,3}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
y	1	i	-1	$-\mathsf{i}$	1	i	-1	$-\mathsf{i}$	1	i	-1	$-\mathsf{i}$	1	i	-1	$-\mathsf{i}$	1	i	-1	$-\mathsf{i}$
y^2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
y^3	1	$-\mathsf{i}$	-1	i	1	$-\mathsf{i}$	-1	i	1	$-\mathsf{i}$	-1	i	1	$-\mathsf{i}$	-1	i	1	$-\mathsf{i}$	-1	i
z	2	$1+\mathsf{i}$	0	$1-\mathsf{i}$	τ	$\zeta\tau$	$\mathsf{i}\tau$	$-\bar\zeta\tau$	$-\bar\tau$	$-\zeta\bar\tau$	$-\mathsf{i}\bar\tau$	$\bar\zeta\bar\tau$	$\bar\tau$	$\zeta\bar\tau$	$\mathsf{i}\bar\tau$	$-\bar\zeta\bar\tau$	$-\tau$	$-\zeta\tau$	$-\mathsf{i}\tau$	$\bar\zeta\tau$
	2	$-1+\mathsf{i}$	0	$-1-\mathsf{i}$	τ	$-\bar\zeta\tau$	$-\mathsf{i}\tau$	$\zeta\tau$	$-\bar\tau$	$\bar\zeta\bar\tau$	$\mathsf{i}\bar\tau$	$-\zeta\bar\tau$	$\bar\tau$	$-\bar\zeta\bar\tau$	$-\mathsf{i}\bar\tau$	$\zeta\bar\tau$	$-\tau$	$\bar\zeta\tau$	$\mathsf{i}\tau$	$-\zeta\tau$
	2	$-1-\mathsf{i}$	0	$-1+\mathsf{i}$	τ	$-\zeta\tau$	$\mathsf{i}\tau$	$\bar\zeta\tau$	$-\bar\tau$	$\zeta\bar\tau$	$-\mathsf{i}\bar\tau$	$-\bar\zeta\bar\tau$	$\bar\tau$	$-\zeta\bar\tau$	$\mathsf{i}\bar\tau$	$\bar\zeta\bar\tau$	$-\tau$	$\zeta\tau$	$-\mathsf{i}\tau$	$-\bar\zeta\tau$
	2	$1-\mathsf{i}$	0	$1+\mathsf{i}$	τ	$\bar\zeta\tau$	$-\mathsf{i}\tau$	$-\zeta\tau$	$-\bar\tau$	$-\bar\zeta\bar\tau$	$\mathsf{i}\bar\tau$	$\zeta\bar\tau$	$\bar\tau$	$\bar\zeta\bar\tau$	$-\mathsf{i}\bar\tau$	$-\zeta\bar\tau$	$-\tau$	$-\bar\zeta\tau$	$\mathsf{i}\tau$	$\zeta\tau$
	3	i	1	$-\mathsf{i}$	τ	$\mathsf{i}\tau$	$-\tau$	$-\mathsf{i}\tau$	$\bar\tau$	$\mathsf{i}\bar\tau$	$-\bar\tau$	$-\mathsf{i}\bar\tau$	$\bar\tau$	$\mathsf{i}\bar\tau$	$-\bar\tau$	$-\mathsf{i}\bar\tau$	τ	$\mathsf{i}\tau$	$-\tau$	$-\mathsf{i}\tau$
	3	-1	-1	-1	τ	$-\tau$	τ	$-\tau$	$\bar\tau$	$-\bar\tau$	$\bar\tau$	$-\bar\tau$	$\bar\tau$	$-\bar\tau$	$\bar\tau$	$-\bar\tau$	τ	$-\tau$	τ	$-\tau$
	3	$-\mathsf{i}$	1	i	τ	$-\mathsf{i}\tau$	$-\tau$	$\mathsf{i}\tau$	$\bar\tau$	$-\mathsf{i}\bar\tau$	$-\bar\tau$	$\mathsf{i}\bar\tau$	$\bar\tau$	$-\mathsf{i}\bar\tau$	$-\bar\tau$	$\mathsf{i}\bar\tau$	τ	$-\mathsf{i}\tau$	$-\tau$	$\mathsf{i}\tau$
	3	1	-1	1	τ	τ	τ	τ	$\bar\tau$	$\bar\tau$	$\bar\tau$	$\bar\tau$	$\bar\tau$	$\bar\tau$	$\bar\tau$	$\bar\tau$	τ	τ	τ	τ
	4	0	0	0	1	$-\bar\zeta$	$-\mathsf{i}$	ζ	-1	$\bar\zeta$	i	$-\zeta$	1	$-\bar\zeta$	$-\mathsf{i}$	ζ	-1	$\bar\zeta$	i	$-\zeta$
	4	0	0	0	1	$-\zeta$	i	$\bar\zeta$	-1	ζ	$-\mathsf{i}$	$-\bar\zeta$	1	$-\zeta$	i	$\bar\zeta$	-1	ζ	$-\mathsf{i}$	$-\bar\zeta$
	4	0	0	0	1	$\bar\zeta$	$-\mathsf{i}$	$-\zeta$	-1	$-\bar\zeta$	i	ζ	1	$\bar\zeta$	$-\mathsf{i}$	$-\zeta$	-1	$-\bar\zeta$	i	ζ
	4	0	0	0	1	ζ	i	$-\bar\zeta$	-1	$-\zeta$	$-\mathsf{i}$	$\bar\zeta$	1	ζ	i	$-\bar\zeta$	-1	$-\zeta$	$-\mathsf{i}$	$\bar\zeta$
	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	i	-1	$-\mathsf{i}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	$-\mathsf{i}$	-1	i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

5.4. Cyclic central subgroups

Let g be a central element of a group G of order p, and $Z = \langle g\rangle \leqslant G$ with $Z \cong \mathbb{Z}/p$. In this section we describe $p-1$ algebra homomorphisms $\hat{s}_\ell \colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(G/Z)$ $(1 \leqslant \ell \leqslant p-1)$. We show that these extend to Banach algebra homomorphisms $\hat{s}_\ell \colon \hat{a}(G) \to \hat{a}(G/Z)$. In the next section we use this construction to examine the species of $a(\mathbb{Z}/p^n)$.

We write X for $g - 1 \in kG$, so $X^p = 0$. If M is a kG-module, and $1 \leqslant i \leqslant p$, we let

$$F_i(M) = \frac{\mathsf{Ker}(X) \cap \mathsf{Im}(X^{i-1})}{\mathsf{Ker}(X) \cap \mathsf{Im}(X^i)}.$$

This is annihilated by X, so g acts as the identity, and $F_i(M)$ therefore has the structure of a kG/Z-module.

As in Section 5.2, we write $[J_i]$, $1 \leqslant i \leqslant p$, for the indecomposable kZ-modules, and we write $s_j \colon a(Z) \to \mathbb{C}$ for the species, with $0 \leqslant j \leqslant p-1$, described in Theorem 5.2.8 and its Corollary. The interpretation of F_i is that it picks out the socles of the Jordan blocks of X on M of length exactly i.

Lemma 5.4.1. *The dimension of $F_i(M)$ is equal to $[\mathsf{res}_{G,Z}(M) : J_i]$.*

Proof. Applying $\mathsf{Ker}(X) \cap \mathsf{Im}(X^{i-1})$ to J_j, we get zero if $j < i$ and $\mathsf{Soc}(J_j)$ if $j \geqslant i$. So the quotient $(\mathsf{Ker}(X) \cap \mathsf{Im}(X^{i-1}))/(\mathsf{Ker}(X) \cap \mathsf{Im}(X^i))$ is isomorphic to $\mathsf{Soc}(J_i)$ if $j = i$ and zero otherwise. $\square$

PROPOSITION 5.4.2. *If $J_i \otimes J_j = \bigoplus_k c_{i,j,k} J_k$ then for $1 \leqslant k \leqslant p-1$ we have*

$$F_k(M \otimes N) \cong \bigoplus_{i,j} c_{i,j,k} F_i(M) \otimes F_j(N).$$

PROOF. This amounts to the semisimplicity of the coproduct on the simple functors $F_k \colon \mathsf{mod}(k\mathbb{Z}/p) \to \mathsf{vect}(k)$ for $1 \leqslant k \leqslant p-1$:

$$\Delta F_k \cong \bigoplus_{i,j} c_{i,j,k} F_i \otimes F_j \colon \mathsf{mod}(k\mathbb{Z}/p) \times \mathsf{mod}(k\mathbb{Z}/p) \to \mathsf{vect}(k).$$

The easiest proof of semisimplicity is to notice that this functor is isomorphic to the contragredient dual functor sending the pair (M, N) to $(F_k(M^* \otimes N^*))^*$, and all the multiplicities $c_{i,j,k}$ with $1 \leqslant k \leqslant p-1$ are zero or one. More precisely, if $i \leqslant j$ and $i+j \leqslant p$ then (Feit [**40**], Theorem VIII.2.7) $J_i \otimes J_j \cong \bigoplus_{s=1}^{i} J_{j-i+2s-1}$ and $J_{p-i} \otimes J_{p-j} \cong (p-i-j)J_p \oplus (J_i \otimes J_j)$. Using the commutativity of tensor product, all cases are covered in these two statements, since either $i+j \leqslant p$ or $(p-i)+(p-j) \leqslant p$. Note that some of the multiplicities $c_{i,j,p}$ are greater than one, and ΔF_p is not semisimple. □

THEOREM 5.4.3. *For $1 \leqslant \ell \leqslant p-1$ the map*

$$\hat{s}_\ell \colon [M] \mapsto \sum_{k=1}^{p-1} s_\ell([J_k])[F_k(M)]$$

defines an algebra homomorphism $\hat{s}_\ell \colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(G/Z)$, which is the identity on the subalgebra $a_{\mathbb{C}}(G/Z) \subseteq a_{\mathbb{C}}(G)$. This is continuous with respect to the norm, and so it extends to give a map of Banach algebras $\hat{s}_j \colon \hat{a}(G) \to \hat{a}(G/Z)$.

PROOF. To see that $\hat{s}_j$ is a ring homomorphism, we just have to check multiplicativity. Using Proposition 5.4.2 we have

$$\begin{aligned}
\hat{s}_\ell([M])\hat{s}_\ell([N]) &= \sum_{i=1}^{p-1} s_\ell([J_i])[F_i(M)] \sum_{j=1}^{p-1} s_\ell([J_j])[F_j(N)] \\
&= \sum_{i,j=1}^{p-1} s_\ell([J_i \otimes J_j])[F_i(M)][F_j(N)] \\
&= \sum_{k=1}^{p-1} \sum_{i,j} c_{i,j,k} s_\ell([J_k])[F_i(M) \otimes F_j(N)] \\
&= \sum_{k=1}^{p-1} s_\ell([J_k])[F_k(M \otimes N)] \\
&= \hat{s}_\ell([M \otimes N]).
\end{aligned}$$

To see that $\hat{s}_\ell$ is the identity on $a_{\mathbb{C}}(G/Z)$, we note that if M is a $k(G/Z)$-module, regarded as a kG-module via inflation, then $F_1(M) = M$ and $F_k(M) = 0$ for $2 \leqslant k \leqslant p$. Thus $\hat{s}_\ell([M]) = s_\ell([J_1])[M] = [M]$.

To prove continuity, we prove boundedness and use Lemma 2.1.3. If $x = \sum_{i\in\mathfrak{I}} a_i[M_i]$ then

$$\begin{aligned}\|\hat{s}_\ell(x)\| &\leqslant \sum_{k=1}^{p-1}\sum_{i\in\mathfrak{I}} |a_i||s_\ell(J_k)|\dim F_k(M_i)\\ &\leqslant \sum_{i\in\mathfrak{I}}|a_i|\sum_{k=1}^{p-1} k\dim F_k(M_i)\\ &= \sum_{i\in\mathfrak{I}}|a_i|\dim M_i = \|x\|. \qquad\square\end{aligned}$$

Remark 5.4.4. When $p = 2$, there is only one relevant value of ℓ in Proposition 5.4.2 and Theorem 5.4.3. In this case, they both amount to the statement that $F_1(M\otimes N)\cong F_1(M)\otimes F_1(N)$. This is known as the Künneth theorem.

If $s\colon a_{\mathbb{C}}(G/Z)\to\mathbb{C}$ is a species, then we compose to give $p-1$ species $s\hat{s}_\ell\colon a_{\mathbb{C}}(G)\to\mathbb{C}$ $(1\leqslant\ell\leqslant p-1)$.

5.5. Cyclic p-groups

The structure of $a(G)$ with G a cyclic group of order a power of a prime was investigated by Almkvist and Fossum [**3**], Green [**48**], Renaud [**80**], Srinivasan [**88**]. An abstract proof of semisimplicity of $a(G)$ avoiding computations was given in Benson and Carlson [**11**]. In this section we provide a new computational approach to the description of the species of $a_{\mathbb{C}}(\mathbb{Z}/p^n)$, and prove that it is semisimple.

It turns out that there is one more ring homomorphism $\hat{s}_0\colon a(\mathbb{Z}/p^{n+1})\to a(\mathbb{Z}/p^n)$ that we need to use. It would be nice to have a more "functorial" way to describe this, but that does not seem so easy.

Lemma 5.5.1. *The linear map defined by*

$$\hat{s}_0\colon [J_{2bp^n\pm r}]\mapsto 2b[J_{p^n}]\pm[J_r]$$

is a ring homomorphism $a(\mathbb{Z}/p^{n+1})\to a(\mathbb{Z}/p^n)$, which is the identity on the subring $a(\mathbb{Z}/p^n)\subseteq a(\mathbb{Z}/p^{n+1})$. Here, $0\leqslant b\leqslant p/2$ and $0\leqslant r\leqslant p^n$.

This formula covers all possible lengths of Jordan blocks, and if $r = p^n$ the two different values of b describing the same module give the same answer.

Proof. The ring $a(\mathbb{Z}/p^{n+1})$ is generated by the subring $a(\mathbb{Z}/p^n)$ together with one more element $[J_{p^n+1}]$. The effect of multiplying $[J_{p^n+1}]$ with a basis element $[J_j]$ (copied from (2.8c)–(2.8e) of Green [**48**], see also Section I.1 of [**3**]) is as follows. We write $j = j_0p^n + j_1$ with $0\leqslant j_1\leqslant p^n-1$. Then

$$[J_{p^n+1}][J_j] = \begin{cases}[J_{p^n+j_1}] + (j_1-1)[J_{p^n}] & j\leqslant p^n\\ [J_{(j_0+1)p^n+j_1}] + (j_1-1)[J_{(j_0+1)p^n}] + [J_{(j_0+1)p^n-j_1}] & \\ \qquad +(p^n-j_1-1)[J_{j_0p^n}] + [J_{(j_0-1)p^n+j_1}] & p^n<j\leqslant(p-1)p^n\\ (j_1+1)[J_{p^{n+1}}] + (p^n-j_1-1)[J_{(p-1)p^n}] + [J_{(p-2)p^n+j_1}] & (p-1)p^n<j\leqslant p^{n+1}\end{cases}$$

Now the map $\hat{s}_0$ preserves dimension, and takes $[J_{j_0p^n}]$ to $j_0[J_{p^n}]$ for $1 \leqslant j_0 \leqslant p$, so it suffices to work modulo the basis elements of the form $[J_{j_0p^n}]$. This makes the above relations easier to read:

$$(5.5.2)\quad [J_{p^n+1}][J_j] \equiv \begin{cases} [J_{p^n+j_1}] & j \leqslant p^n \\ [J_{(j_0+1)p^n+j_1}] + [J_{(j_0+1)p^n-j_1}] + [J_{(j_0-1)p^n+j_1}] & p^n < j \leqslant (p-1)p^n \\ [J_{(p-2)p^n+j_1}] & (p-1)p^n < j \leqslant p^{n+1} \end{cases}$$

Under the map $\hat{s}_0$, we must divide into two cases according to the parity of j_0. If j_0 is even, both sides go to $-[J_{p^n-j_1}]$, whereas if j_0 is odd, both sides go to $[J_{j_1}]$. □

THEOREM 5.5.3. *The species $s_{\ell_0}\hat{s}_{\ell_1}\dots\hat{s}_{\ell_{n-1}}\colon a_{\mathbb{C}}(\mathbb{Z}/p^n) \to \mathbb{C}$ with $0 \leqslant \ell_i \leqslant p-1$ for $0 \leqslant i \leqslant n-1$ are all distinct.*

PROOF. Consider the elements $\chi_0, \dots, \chi_{n-1}$ of $a_{\mathbb{C}}(\mathbb{Z}/p^n)$ given by

$$\chi_i = \begin{cases} [J_2] & i = 0, \\ [J_{p^i+1}] - [J_{p^i-1}] & 1 \leqslant i \leqslant n-1. \end{cases}$$

These generate $a_{\mathbb{C}}(\mathbb{Z}/p^n)$ as an algebra. The element χ_i is in $a_{\mathbb{C}}(\mathbb{Z}/p^{i+1}) \subseteq a_{\mathbb{C}}(\mathbb{Z}/p^n)$, and the map $\hat{s}_\ell\colon a_{\mathbb{C}}(\mathbb{Z}/p^{i+1}) \to a_{\mathbb{C}}(\mathbb{Z}/p^i)$ evaluated at χ_i gives

$$\hat{s}_\ell(\chi_i) = \begin{cases} 2([J_{p^i}] - [J_{p^i-1}]) & \ell = 0 \\ 2\cos \ell\pi/p\,[J_1] & 1 \leqslant \ell \leqslant p-1. \end{cases}$$

Furthermore, we have

$$\hat{s}_\ell([J_{p^{i+1}}] - [J_{p^{i+1}-1}]) = \begin{cases} [J_{p^i}] - [J_{p^i-1}] & \ell = 0 \\ (-1)^\ell [J_1] & 1 \leqslant \ell \leqslant p-1. \end{cases}$$

Applying these formulas inductively, we have

$$s_{\ell_0}\hat{s}_{\ell_1}\dots\hat{s}_{\ell_{n-1}}(\chi_i) = \begin{cases} \pm 2 & \ell_i = 0 \\ 2\cos \ell_i\pi/p & 1 \leqslant \ell_i \leqslant p-1. \end{cases}$$

If $1 \leqslant \ell_i \leqslant p-1$ then $|2\cos \ell_i\pi/p| < 2$, and since $\cos\colon [0,\pi] \to \mathbb{R}$ is injective, $2\cos \ell_i\pi/p$ determines ℓ_i. Thus ℓ_i is determined by the value of $s_{\ell_0}\hat{s}_{\ell_1}\dots\hat{s}_{\ell_{n-1}}$ on χ_i. □

COROLLARY 5.5.4. *The algebra $a_{\mathbb{C}}(\mathbb{Z}/p^n)$ is semisimple.*

PROOF. By Lemma 1.1.10, the p^n distinct species of the algebra $a_{\mathbb{C}}(\mathbb{Z}/p^n)$ described in Theorem 5.5.3 are linearly independent. This algebra has dimension p^n. These species therefore give us an isomorphism between $a_{\mathbb{C}}(\mathbb{Z}/p^n)$ and a product of p^n copies of $\mathbb{C}$. □

EXAMPLE 5.5.5. Let $G = \mathbb{Z}/4$. The tensor product table is as follows, with the obvious abbreviated notation.

1	2	3	4
2	2^2	42	4^2
3	42	$4^2 1$	4^3
4	4^2	4^3	4^4

Then $F_j\colon a(G) \to a(\mathbb{Z}/2)$, the $\hat{s}_j\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(\mathbb{Z}/2)$ and the $s_i\hat{s}_j$ are as follows.

	F_1	F_2	$\hat{s}_0$	$\hat{s}_1$	$s_0\hat{s}_0$	$s_1\hat{s}_0$	$s_0\hat{s}_1$	$s_1\hat{s}_1$
$[J_1]$	$[J_1]$	0	$[J_1]$	$[J_1]$	1	1	1	1
$[J_2]$	$[J_2]$	0	$[J_2]$	$[J_2]$	2	0	2	0
$[J_3]$	$[J_1]$	$[J_1]$	$2[J_2]-[J_1]$	$[J_1]$	3	-1	1	1
$[J_4]$	0	$[J_2]$	$2[J_2]$	0	4	0	0	0

The last four columns of this table give the species table for $a(\mathbb{Z}/4)$.

Example 5.5.6. Let $G = \mathbb{Z}/8$. The tensor product table is as follows.

1	2	3	4	5	6	7	8
2	2^2	42	4^2	64	6^2	86	8^2
3	42	4^21	4^3	74^2	864	8^25	8^3
4	4^2	4^3	4^4	84^3	8^24^2	8^34	8^4
5	64	74^2	84^3	8^24^21	8^342	8^43	8^5
6	6^2	864	8^24^2	8^342	8^42^2	8^52	8^6
7	86	8^25	8^34	8^43	8^52	8^61	8^7
8	8^2	8^3	8^4	8^5	8^6	8^7	8^8

Then the $F_j\colon a(G) \to a(\mathbb{Z}/4)$, the $\hat{s}_j\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(\mathbb{Z}/4)$ and the $s_i\hat{s}_j\hat{s}_k$ are as follows.

	F_1	F_2	$\hat{s}_0$	$\hat{s}_1$	$s_0\hat{s}_0\hat{s}_0$	$s_1\hat{s}_0\hat{s}_0$	$s_0\hat{s}_1\hat{s}_0$	$s_1\hat{s}_1\hat{s}_0$	$s_0\hat{s}_0\hat{s}_1$	$s_1\hat{s}_0\hat{s}_1$	$s_0\hat{s}_1\hat{s}_1$	$s_1\hat{s}_1\hat{s}_1$
$[J_1]$	$[J_1]$	0	$[J_1]$	$[J_1]$	1	1	1	1	1	1	1	1
$[J_2]$	$[J_2]$	0	$[J_2]$	$[J_2]$	2	0	2	0	2	0	2	0
$[J_3]$	$[J_3]$	0	$[J_3]$	$[J_3]$	3	-1	1	1	3	-1	1	1
$[J_4]$	$[J_4]$	0	$[J_4]$	$[J_4]$	4	0	0	0	4	0	0	0
$[J_5]$	$[J_3]$	$[J_1]$	$2[J_4]-[J_3]$	$[J_3]$	5	1	-1	-1	3	-1	1	1
$[J_6]$	$[J_2]$	$[J_2]$	$2[J_4]-[J_2]$	$[J_2]$	6	0	-2	0	2	0	2	0
$[J_7]$	$[J_1]$	$[J_3]$	$2[J_4]-[J_1]$	$[J_1]$	7	-1	-1	-1	1	1	1	1
$[J_8]$	0	$[J_4]$	$[2J_4]$	0	8	0	0	0	0	0	0	0

The last eight columns of this table give the species table for $a(\mathbb{Z}/8)$.

Example 5.5.7. Let $G = \mathbb{Z}/9$. The tensor product table is as follows.

1	2	3	4	5	6	7	8	9
2	31	3^2	53	64	6^2	86	97	9^2
3	3^2	3^3	63^2	6^23	6^3	96^2	9^26	9^3
4	53	63^2	7531	8642	96^23	9^264	9^35	9^4
5	64	6^23	8642	97531	9^263^2	9^353	9^44	9^5
6	6^2	6^3	96^23	9^263^2	9^33^3	9^43^2	9^53	9^6
7	86	96^2	9^264	9^353	9^43^2	9^531	9^62	9^7
8	97	9^26	9^35	9^44	9^53	9^62	9^71	9^8
9	9^2	9^3	9^4	9^5	9^6	9^7	9^8	9^9

Then the $F_j\colon a(G) \to a(\mathbb{Z}/3)$, the $\hat{s}_j\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(\mathbb{Z}/3)$ and the $s_i\hat{s}_j$ are as follows.

	F_1	F_2	F_3	$\hat{s}_0$	$\hat{s}_1$	$\hat{s}_2$	$s_0\hat{s}_0$	$s_1\hat{s}_0$	$s_2\hat{s}_0$	$s_0\hat{s}_1$	$s_1\hat{s}_1$	$s_2\hat{s}_1$	$s_0\hat{s}_2$	$s_1\hat{s}_2$	$s_2\hat{s}_2$
$[J_1]$	$[J_1]$	0	0	$[J_1]$	$[J_1]$	$[J_1]$	1	1	1	1	1	1	1	1	1
$[J_2]$	$[J_2]$	0	0	$[J_2]$	$[J_2]$	$[J_2]$	2	1	-1	2	1	-1	2	1	-1
$[J_3]$	$[J_3]$	0	0	$[J_3]$	$[J_3]$	$[J_3]$	3	0	0	3	0	0	3	0	0
$[J_4]$	$[J_2]$	$[J_1]$	0	$2[J_3]-[J_2]$	$[J_2]+[J_1]$	$[J_2]-[J_1]$	4	-1	1	3	2	0	1	0	-2
$[J_5]$	$[J_1]$	$[J_2]$	0	$2[J_3]-[J_1]$	$[J_1]+[J_2]$	$[J_1]-[J_2]$	5	-1	-1	3	2	0	-1	0	2
$[J_6]$	0	$[J_3]$	0	$2[J_3]$	$[J_3]$	$-[J_3]$	6	0	0	3	0	0	-3	0	0
$[J_7]$	0	$[J_2]$	$[J_1]$	$2[J_3]+[J_1]$	$[J_2]$	$-[J_2]$	7	1	1	2	1	-1	-2	-1	1
$[J_8]$	0	$[J_1]$	$[J_2]$	$2[J_3]+[J_2]$	$[J_1]$	$-[J_1]$	8	1	-1	1	1	1	-1	-1	-1
$[J_9]$	0	0	$[J_3]$	$3[J_3]$	0	0	9	0	0	0	0	0	0	0	0

The last nine columns of this table give the species table for $a(\mathbb{Z}/9)$.

EXAMPLE 5.5.8. Let $G = \mathbb{Z}/25$. A quarter of the tensor product table is as follows; the rest may be deduced using $\Omega(M) \otimes N \cong \Omega(M \otimes N) \oplus (\text{projective})$ (Schanuel's lemma), and so on.

1	2	3	4	5	6	7	8
2	3 1	4 2	5 3	5^2	7 5	8 6	9 7
3	4 2	5 3 1	5^2 2	5^3	8 5^2	9 7 5	10 8 6
4	5 3	5^2 2	5^3 1	5^4	9 5^3	10 8 5^2	10^2 7 5
5	5^2	5^3	5^4	5^5	10 5^4	10^2 5^3	10^3 5^2
6	7 5	8 5^2	9 5^3	10 5^4	11 9 5^3 1	12 10 8 5^2 2	13 10^2 7 5 3
7	8 6	9 7 5	10 8 5^2	10^2 5^3	12 10 8 5^2 2	13 11 9 7 5 3 1	14 12 10 8 6 4 2
8	9 7	10 8 6	10^2 7 5	10^3 5^2	13 10^2 7 5 3	14 12 10 8 6 4 2	15 13 11 9 7 5 3 1
9	10 8	10^2 7	10^3 6	10^4 5	14 10^3 6 4	15 13 10^2 7 5 3	15^2 12 10 8 5^2 2
10	10^2	10^3	10^4	10^5	15 10^4 5	15^2 10^3 5^2	15^3 10^2 5^3
11	12 10	13 10^2	14 10^3	15 10^4	16 14 10^3 6	17 15 13 10^2 7 5	18 15^2 12 10 8 5^2
12	13 11	14 12 10	15 13 10^2	15^2 10^3	17 15 13 10^2 7 5	18 16 14 12 10 8 6 4	19 17 15 13 11 9 7 5

1	9	10	11	12
2	10 8	10^2	12 10	13 11
3	10^2 7	10^3	13 10^2	14 12 10
4	10^3 6	10^4	14 10^3	15 13 10^2
5	10^4 5	10^5	15 10^4	15^2 10^3
6	14 10^3 6 4	15 10^4 5	16 14 10^3 6	17 15 13 10^2 7
7	15 13 10^2 7 5 3	15^2 10^3 5^2	17 15 13 10^2 7 5	18 16 14 12 10 8 6
8	15^2 12 10 8 5^2 2	15^3 10^2 5^3	18 15^2 12 10 8 5^2	19 17 15 13 11 9 7 5
9	15^3 11 9 5^3 1	15^4 10 5^4	19 15^3 11 9 5^3	20 18 15^2 12 10 8 5^2
10	15^4 10 5^4	15^5 5^5	20 15^4 10 5^4	20^2 15^3 10^2 5^3
11	19 15^3 11 9 5^3	20 15^4 10 5^4	21 19 15^3 11 9 5^3 1	22 20 18 15^2 12 10 8 5^2 2
12	20 18 15^2 12 10 8 5^2	20^2 15^3 10^2 5^3	22 20 18 15^2 12 10 8 5^2 2	23 21 19 17 15 13 11 9 7 5 3 1

Then the $F_j\colon a(G) \to a(\mathbb{Z}/5)$, the $\hat{s}_j\colon a_{\mathbb{C}}(G) \to a_{\mathbb{C}}(\mathbb{Z}/5)$ and the $s_i\hat{s}_j$ are as follows.

	F_1	F_2	F_3	F_4	F_5	$\hat{s}_0$	$\hat{s}_1$	$\hat{s}_2$	$\hat{s}_3$	$\hat{s}_4$	$s_0\hat{s}_0$	$s_1\hat{s}_0$	$s_2\hat{s}_0$	$s_3\hat{s}_0$	$s_4\hat{s}_0$
$[J_1]$	$[J_1]$	0	0	0	0	$[J_1]$	$[J_1]$	$[J_1]$	$[J_1]$	$[J_1]$	1	1	1	1	1
$[J_2]$	$[J_2]$	0	0	0	0	$[J_2]$	$[J_2]$	$[J_2]$	$[J_2]$	$[J_2]$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$
$[J_3]$	$[J_3]$	0	0	0	0	$[J_3]$	$[J_3]$	$[J_3]$	$[J_3]$	$[J_3]$	3	τ	$\bar\tau$	$\bar\tau$	τ
$[J_4]$	$[J_4]$	0	0	0	0	$[J_4]$	$[J_4]$	$[J_4]$	$[J_4]$	$[J_4]$	4	1	-1	1	-1
$[J_5]$	$[J_5]$	0	0	0	0	$[J_5]$	$[J_5]$	$[J_5]$	$[J_5]$	$[J_5]$	5	0	0	0	0
$[J_6]$	$[J_4]$	$[J_1]$	0	0	0	$2[J_5]-[J_4]$	$[J_4]+\tau[J_1]$	$[J_4]-\bar\tau[J_1]$	$[J_4]+\bar\tau[J_1]$	$[J_4]-\tau[J_1]$	6	-1	1	-1	1
$[J_7]$	$[J_3]$	$[J_2]$	0	0	0	$2[J_5]-[J_3]$	$[J_3]+\tau[J_2]$	$[J_3]-\bar\tau[J_2]$	$[J_3]+\bar\tau[J_2]$	$[J_3]-\tau[J_2]$	7	$-\tau$	$-\bar\tau$	$-\bar\tau$	$-\tau$
$[J_8]$	$[J_2]$	$[J_3]$	0	0	0	$2[J_5]-[J_2]$	$[J_2]+\tau[J_3]$	$[J_2]-\bar\tau[J_3]$	$[J_2]+\bar\tau[J_3]$	$[J_2]-\tau[J_3]$	8	$-\tau$	$\bar\tau$	$-\bar\tau$	τ
$[J_9]$	$[J_1]$	$[J_4]$	0	0	0	$2[J_5]-[J_1]$	$[J_1]+\tau[J_4]$	$[J_1]-\bar\tau[J_4]$	$[J_1]+\bar\tau[J_4]$	$[J_1]-\tau[J_4]$	9	-1	-1	-1	-1
$[J_{10}]$	0	$[J_5]$	0	0	0	$2[J_5]$	$\tau[J_5]$	$-\bar\tau[J_5]$	$\bar\tau[J_5]$	$-\tau[J_5]$	10	0	0	0	0
$[J_{11}]$	0	$[J_4]$	$[J_1]$	0	0	$2[J_5]+[J_1]$	$\tau([J_4]+[J_1])$	$\bar\tau([J_1]-[J_4])$	$\bar\tau([J_4]+[J_1])$	$\tau([J_1]-[J_4])$	11	1	1	1	1
$[J_{12}]$	0	$[J_3]$	$[J_2]$	0	0	$2[J_5]+[J_2]$	$\tau[J_3]+\tau[J_2]$	$\bar\tau([J_2]-[J_3])$	$\bar\tau([J_3]+[J_2])$	$\tau([J_2]-[J_3])$	12	τ	$-\bar\tau$	$\bar\tau$	$-\tau$
$[J_{13}]$	0	$[J_2]$	$[J_3]$	0	0	$2[J_5]+[J_3]$	$\tau[J_2]+\tau[J_3]$	$\bar\tau([J_3]-[J_2])$	$\bar\tau([J_2]+[J_3])$	$\tau([J_3]-[J_2])$	13	τ	$\bar\tau$	$\bar\tau$	τ
$[J_{14}]$	0	$[J_1]$	$[J_4]$	0	0	$2[J_5]+[J_4]$	$\tau[J_1]+\tau[J_4]$	$\bar\tau([J_4]-[J_1])$	$\bar\tau([J_1]+[J_4])$	$\tau([J_4]-[J_1])$	14	1	-1	1	-1
$[J_{15}]$	0	0	$[J_5]$	0	0	$3[J_5]$	$\tau[J_5]$	$\bar\tau[J_5]$	$\bar\tau[J_5]$	$\tau[J_5]$	15	0	0	0	0
$[J_{16}]$	0	0	$[J_4]$	$[J_1]$	0	$4[J_5]-[J_4]$	$\tau[J_4]+[J_1]$	$\bar\tau[J_4]-[J_1]$	$\bar\tau[J_4]+[J_1]$	$\tau[J_4]-[J_1]$	16	-1	1	-1	1
$[J_{17}]$	0	0	$[J_3]$	$[J_2]$	0	$4[J_5]-[J_3]$	$\tau[J_3]+[J_2]$	$\bar\tau[J_3]-[J_2]$	$\bar\tau[J_3]+[J_2]$	$\tau[J_3]-[J_2]$	17	$-\tau$	$-\bar\tau$	$-\bar\tau$	$-\tau$
$[J_{18}]$	0	0	$[J_2]$	$[J_3]$	0	$4[J_5]-[J_2]$	$\tau[J_2]+[J_3]$	$\bar\tau[J_2]-[J_3]$	$\bar\tau[J_2]+[J_3]$	$\tau[J_2]-[J_3]$	18	$-\tau$	$\bar\tau$	$-\bar\tau$	τ
$[J_{19}]$	0	0	$[J_1]$	$[J_4]$	0	$4[J_5]-[J_1]$	$\tau[J_1]+[J_4]$	$\bar\tau[J_1]-[J_4]$	$\bar\tau[J_1]+[J_4]$	$\tau[J_1]-[J_4]$	19	-1	-1	-1	-1
$[J_{20}]$	0	0	0	$[J_5]$	0	$4[J_5]$	$[J_5]$	$-[J_5]$	$[J_5]$	$-[J_5]$	20	0	0	0	0
$[J_{21}]$	0	0	0	$[J_4]$	$[J_1]$	$4[J_5]+[J_1]$	$[J_4]$	$-[J_4]$	$[J_4]$	$-[J_4]$	21	1	1	1	1
$[J_{22}]$	0	0	0	$[J_3]$	$[J_2]$	$4[J_5]+[J_2]$	$[J_3]$	$-[J_3]$	$[J_3]$	$-[J_3]$	22	τ	$-\bar\tau$	$\bar\tau$	$-\tau$
$[J_{23}]$	0	0	0	$[J_2]$	$[J_3]$	$4[J_5]+[J_3]$	$[J_2]$	$-[J_2]$	$[J_2]$	$-[J_2]$	23	τ	$\bar\tau$	$\bar\tau$	τ
$[J_{24}]$	0	0	0	$[J_1]$	$[J_4]$	$4[J_5]+[J_4]$	$[J_1]$	$-[J_1]$	$[J_1]$	$-[J_1]$	24	1	-1	1	-1
$[J_{25}]$	0	0	0	0	$[J_5]$	$5[J_5]$	0	0	0	0	25	0	0	0	0

	$s_0\hat{s}_1$	$s_1\hat{s}_1$	$s_2\hat{s}_1$	$s_3\hat{s}_1$	$s_4\hat{s}_1$	$s_0\hat{s}_2$	$s_1\hat{s}_2$	$s_2\hat{s}_2$	$s_3\hat{s}_2$	$s_4\hat{s}_2$
$[J_1]$	1	1	1	1	1	1	1	1	1	1
$[J_2]$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$
$[J_3]$	3	τ	$\bar\tau$	$\bar\tau$	τ	3	τ	$\bar\tau$	$\bar\tau$	τ
$[J_4]$	4	1	-1	1	-1	4	1	-1	1	-1
$[J_5]$	5	0	0	0	0	5	0	0	0	0
$[J_6]$	$4+\tau$	τ^2	$-\bar\tau$	τ^2	$-\bar\tau$	$4-\bar\tau$	τ	$-\bar\tau^2$	τ	$-\bar\tau^2$
$[J_7]$	$3+2\tau$	τ^3	$\bar\tau^2$	$-\tau$	-1	$3-2\bar\tau$	τ^2	$\bar\tau^3$	-1	$-\bar\tau$
$[J_8]$	$2+3\tau$	τ^3	$-\bar\tau^2$	$-\tau$	1	$2-3\bar\tau$	τ^2	$-\bar\tau^3$	-1	$\bar\tau$
$[J_9]$	$1+4\tau$	τ^2	$\bar\tau$	τ^2	$\bar\tau$	$1-4\bar\tau$	τ	$\bar\tau^2$	τ	$\bar\tau^2$
$[J_{10}]$	5τ	0	0	0	0	$-5\bar\tau$	0	0	0	0
$[J_{11}]$	5τ	2τ	0	2τ	0	$-3\bar\tau$	0	$2\bar\tau$	0	$2\bar\tau$
$[J_{12}]$	5τ	$2\tau^2$	0	-2	0	$-\bar\tau$	0	$-2\bar\tau^2$	0	2
$[J_{13}]$	5τ	$2\tau^2$	0	-2	0	$\bar\tau$	0	$2\bar\tau^2$	0	-2
$[J_{14}]$	5τ	2τ	0	2τ	0	$3\bar\tau$	0	$-2\bar\tau$	0	$-2\bar\tau$
$[J_{15}]$	5τ	0	0	0	0	$5\bar\tau$	0	0	0	0
$[J_{16}]$	$1+4\tau$	τ^2	$\bar\tau$	τ^2	$\bar\tau$	$4\bar\tau-1$	$-\tau$	$-\bar\tau^2$	$-\tau$	$-\bar\tau^2$
$[J_{17}]$	$2+3\tau$	τ^3	$-\bar\tau^2$	$-\tau$	1	$3\bar\tau-2$	$-\tau^2$	$\bar\tau^3$	1	$-\bar\tau$
$[J_{18}]$	$3+2\tau$	τ^3	$\bar\tau^2$	$-\tau$	-1	$2\bar\tau-3$	$-\tau^2$	$-\bar\tau^3$	1	$\bar\tau$
$[J_{19}]$	$4+\tau$	τ^2	$-\bar\tau$	τ^2	$-\bar\tau$	$\bar\tau-4$	$-\tau$	$\bar\tau^2$	$-\tau$	$\bar\tau^2$
$[J_{20}]$	5	0	0	0	0	-5	0	0	0	0
$[J_{21}]$	4	1	-1	1	-1	-4	-1	1	-1	1
$[J_{22}]$	3	τ	$\bar\tau$	$\bar\tau$	τ	-3	$-\tau$	$-\bar\tau$	$-\bar\tau$	$-\tau$
$[J_{23}]$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$	-2	$-\tau$	$\bar\tau$	$-\bar\tau$	τ
$[J_{24}]$	1	1	1	1	1	-1	-1	-1	-1	-1
$[J_{25}]$	0	0	0	0	0	0	0	0	0	0

	$s_0\hat{s}_3$	$s_1\hat{s}_3$	$s_2\hat{s}_3$	$s_3\hat{s}_3$	$s_4\hat{s}_3$	$s_0\hat{s}_4$	$s_1\hat{s}_4$	$s_2\hat{s}_4$	$s_3\hat{s}_4$	$s_4\hat{s}_4$
$[J_1]$	1	1	1	1	1	1	1	1	1	1
$[J_2]$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$
$[J_3]$	3	τ	$\bar\tau$	$\bar\tau$	τ	3	τ	$\bar\tau$	$\bar\tau$	τ
$[J_4]$	4	1	-1	1	-1	4	1	-1	1	-1
$[J_5]$	5	0	0	0	0	5	0	0	0	0
$[J_6]$	$4+\bar\tau$	$\bar\tau^2$	$-\tau$	$\bar\tau^2$	$-\tau$	$4-\tau$	$\bar\tau$	$-\tau^2$	$\bar\tau$	$-\tau^2$
$[J_7]$	$3+2\bar\tau$	$-\bar\tau$	-1	$\bar\tau^3$	τ^2	$3-2\tau$	-1	$-\tau$	$\bar\tau^2$	τ^3
$[J_8]$	$2+3\bar\tau$	$-\bar\tau$	1	$\bar\tau^3$	$-\tau^2$	$2-3\tau$	-1	τ	$\bar\tau^2$	$-\tau^3$
$[J_9]$	$1+4\bar\tau$	$\bar\tau^2$	τ	$\bar\tau^2$	τ	$1-4\tau$	$\bar\tau$	τ^2	$\bar\tau$	τ^2
$[J_{10}]$	$5\bar\tau$	0	0	0	0	-5τ	0	0	0	0
$[J_{11}]$	$5\bar\tau$	$2\bar\tau$	0	$2\bar\tau$	0	-3τ	0	2τ	0	2τ
$[J_{12}]$	$5\bar\tau$	-2	0	$2\bar\tau^2$	0	$-\tau$	0	2	0	$-2\tau^2$
$[J_{13}]$	$5\bar\tau$	-2	0	$2\bar\tau^2$	0	τ	0	-2	0	$2\tau^2$
$[J_{14}]$	$5\bar\tau$	$2\bar\tau$	0	$2\bar\tau$	0	3τ	0	-2τ	0	-2τ
$[J_{15}]$	$5\bar\tau$	0	0	0	0	5τ	0	0	0	0
$[J_{16}]$	$1+4\bar\tau$	$\bar\tau^2$	τ	$\bar\tau^2$	τ	$4\tau-1$	$-\bar\tau$	$-\tau^2$	$-\bar\tau$	$-\tau^2$
$[J_{17}]$	$2+3\bar\tau$	$-\bar\tau$	1	$\bar\tau^3$	$-\tau^2$	$3\tau-2$	1	$-\tau$	$-\bar\tau^2$	τ^3
$[J_{18}]$	$3+2\bar\tau$	$-\bar\tau$	-1	$\bar\tau^3$	τ^2	$2\tau-3$	1	τ	$-\bar\tau^2$	$-\tau^3$
$[J_{19}]$	$4+\bar\tau$	$\bar\tau^2$	$-\tau$	$\bar\tau^2$	$-\tau$	$\tau-4$	$-\bar\tau$	τ^2	$-\bar\tau$	τ^2
$[J_{20}]$	5	0	0	0	0	-5	0	0	0	0
$[J_{21}]$	4	1	-1	1	-1	-4	-1	1	-1	1
$[J_{22}]$	3	τ	$\bar\tau$	$\bar\tau$	τ	-3	$-\tau$	$-\bar\tau$	$-\bar\tau$	$-\tau$
$[J_{23}]$	2	τ	$-\bar\tau$	$\bar\tau$	$-\tau$	-2	$-\tau$	$\bar\tau$	$-\bar\tau$	τ
$[J_{24}]$	1	1	1	1	1	-1	-1	-1	-1	-1
$[J_{25}]$	0	0	0	0	0	0	0	0	0	0

The last 5 columns of the first table above together with the second table give the species table for $a(\mathbb{Z}/25)$.

5.6. Cyclic normal subgroups

The methods of Sections 5.3 and 5.4 can be combined to deal with cyclic normal subgroups, taking into account the action of the normaliser. Let $P = \langle g \rangle \trianglelefteq G$ be a normal subgroup of order p, and let $C = C_G(P)$, a normal subgroup of G of index m a divisor of $p-1$. Thus we have $G/C \cong \mathbb{Z}/m$. Let k be a field of characteristic p containing a primitive $2m$th root of unity. And as in Section 5.3, it is convenient to make a central extension of G by an element of order two. In order to do this, we form the pullback $\tilde{G}$ of $G \to \mathbb{Z}/m$ and $\mathbb{Z}/2m \to \mathbb{Z}/m$:

$$\begin{tikzcd} & & 1 \arrow[d] & 1 \arrow[d] & \\ & & \mathbb{Z}/2 \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \arrow[d] & \\ 1 \arrow[r] & C \arrow[r] \arrow[d, equal] & \tilde{G} \arrow[r] \arrow[d] & \mathbb{Z}/2m \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & C \arrow[r] & G \arrow[r] \arrow[d] & \mathbb{Z}/m \arrow[r] \arrow[d] & 1 \\ & & 1 & 1 & \end{tikzcd}$$

Let $h \in G$ be an element mapping to a generator of G/C, let $hgh^{-1} = g^q$, and let

$$x = \sum_{1 \leqslant j \leqslant p-1} g^j \in kP \leqslant kG.$$

Then we have $x^p = 0$, $h^{2m} \in C \leqslant \tilde{G}$, $hx = qxh$. Let η be a square root of q in k, and let S_i ($i \in \mathbb{Z}/2m$) be the simple kG-module with a basis vector v_i such that C acts trivially, and $hv_i = \eta^i v_i$. The functors F_i of Section 5.4 is designed to pick out the socles of the Jordan blocks of length i of the action of kP. Tensoring with S_2 moves us down one radical layer of these Jordan blocks, so to obtain a suitably symmetric definition, we should define

$$F_i(M) = S_{-i} \otimes \frac{\mathsf{Ker}(x) \cap \mathsf{Im}(x^{i-1})}{\mathsf{Ker}(x) \cap \mathsf{Im}(x^i)}.$$

With this definition, as in Proposition 5.4.2, for $1 \leqslant k \leqslant p-1$ we have

$$F_k(M \otimes N) \cong \bigoplus_{i,j} c_{i,j,k} F_i(M) \otimes F_j(N).$$

with the same coefficients $c_{i,j,k}$ as before.

THEOREM 5.6.1. *We have $p-1$ algebra homomorphisms $\hat{s}_i\colon a_{\mathbb{C}}(\tilde{G}) \to a_{\mathbb{C}}(\tilde{G}/P)$*

$$\hat{s}_i\colon [M] \mapsto \sum_{k=1}^{p-1} 2\cos(ik\pi/p)[F_k(M)]$$

with $0 < i < p$. They are continuous with respect to the norm, and extend to give maps of Banach algebras $\hat{s}_i\colon \hat{a}(\tilde{G}) \to \hat{a}(\tilde{G}/P)$.

PROOF. As in the proof of Theorem 5.4.3, if s is a non-Brauer species of $a(\mathbb{Z}/p \rtimes \mathbb{Z}/2m)$, the map

$$\hat{s}\colon [M] \mapsto \sum_{k=1}^{p-1} s([J_k])[F_k(M)]$$

defines an algebra homomorphism $\hat{s}\colon a_{\mathbb{C}}(\tilde{G}) \to a_{\mathbb{C}}(\tilde{G}/P)$, which is the identity on the subalgebra $a_{\mathbb{C}}(\tilde{G}/P) \subseteq a_{\mathbb{C}}(\tilde{G})$. This is continuous with respect to the norm, and so it extends to give a map of Banach algebras $\hat{s}\colon \hat{a}(\tilde{G}) \to \hat{a}(\tilde{G}/P)$.

If s is a non-Brauer species of $a(\mathbb{Z}/p \rtimes \mathbb{Z}/2m)$, then the map $\hat{s}$ only depends on the value of s on the elements $[J_k]$. We are in the situation where $d = 1$ in Theorem 5.3.1, and so we have

$$a(\mathbb{Z}/p \rtimes \mathbb{Z}/2m) \cong \mathbb{Z}[X,Y]/(Y^{2m}-1, (X-Y-Y^{-1})f_p(X)).$$

The element $[J_k]$ corresponds to $f_k(X)$, which is in the subring generated by X. The $2(p-1)m$ non-Brauer species $s_{i,j}$ are given by $X \mapsto \zeta_{2p}^i + \zeta_{2p}^{-i} = 2\cos(i\pi/p)$, $Y \mapsto \zeta_{2m}^j$, with $0 < i < p$, $0 \leqslant j < 2m$. The value of $s_{i,j}$ on the elements $[J_k]$ therefore only depends on i, and we write $\hat{s}_i$ for the common value of the $\hat{s}_{i,j}$. □

This theorem may be used in order to construct all the species of the Frobenius group $a(\mathbb{Z}/p^n \rtimes \mathbb{Z}/m)$ with m coprime to p, and show that it is semisimple. As in that case, we do need one more ring homomorphism

$$\hat{s}_0\colon a(\mathbb{Z}/p^{n+1} \rtimes \mathbb{Z}/2m) \to a(\mathbb{Z}/p^n \rtimes \mathbb{Z}/2m)$$

as in Lemma 5.5.1, constructed in a similar way. Even though the tensor products are more complicated than in the cyclic case, working modulo the ideal spanned by the modules $J_{p^n} \otimes S_i$, the tensor product relations (5.5.2) still hold. In the proof, instead of preserving dimension, we have to preserve Brauer species, and so the map is given by

$$\begin{aligned}\hat{s}_0\colon [J_{2bp^n\pm r}] \to [&S_{-d((2b-1)p^n\pm r)} \oplus S_{-d((2b-3)p^n\pm r)} \oplus \cdots \oplus S_{-d(3p^n\pm r)} \oplus S_{-d(p^n\pm r)} \\ &\oplus S_{d(p^n\pm r)} \oplus S_{d(3p^n\pm r)} \oplus \cdots \oplus S_{d((2b-3)p^n\pm r)} \oplus S_{d((2b-1)p^n\pm r)}][J_{p^n}] \pm [J_r]\end{aligned}$$

and $\hat{s}_0\colon [S_i] \mapsto [S_i]$. The $2mp^n$ species of $a(\mathbb{Z}/p^n \rtimes \mathbb{Z}/2m)$ are then given by

$$s_{\ell_0,j}\hat{s}_{\ell_1}\ldots\hat{s}_{\ell_{n-1}}$$

with $0 \leqslant \ell_i \leqslant p-1$ for $0 \leqslant i \leqslant n-1$, and with $0 \leqslant j < 2m$. Restricting to the range $0 \leqslant j < m$ gives the species for $a(\mathbb{Z}/p^n \rtimes \mathbb{Z}/m)$.

REMARK 5.6.2. All these species satisfy $s([M^*]) = \overline{s([M])}$ for all modules M. It follows that $a(\mathbb{Z}/p^n \rtimes \mathbb{Z}/m)$ is a symmetric Banach $*$-algebra, see Section 3.4.

5.7. An integral example

In modular representation theory of finite groups, finite representation type implies that the representation ring is semisimple. Here we give an example to show that this no longer holds in integral representation theory. Let $\mathbb{Z}_2$ denote the ring of 2-adic integers, and consider the group ring $\mathbb{Z}_2G$, where $G = \mathbb{Z}/4$, the cyclic group of order four. Troy [**92**], Roiter [**84**] showed that there are nine isomorphism classes of indecomposable finitely generated $\mathbb{Z}_2$-free $\mathbb{Z}_2G$-modules. Reiner [**77**] denotes the basis elements of the representation ring corresponding to these indecomposable modules $c_1, \dots, c_9$, and computes the tensor products, which are as in the following table.

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
c_2	c_1	c_3	c_4	c_6	c_5	c_7	c_8	c_9
c_3	c_3	$2c_4$	$2c_3$	c_4+c_9	c_4+c_9	$c_3+c_4+c_9$	$c_3+c_4+c_9$	$2c_9$
c_4	c_4	$2c_3$	$2c_4$	c_3+c_9	c_3+c_9	$c_3+c_4+c_9$	$c_3+c_4+c_9$	$2c_9$
c_5	c_6	c_4+c_9	c_3+c_9	c_1+2c_9	c_2+2c_9	c_8+2c_9	c_7+2c_9	$3c_9$
c_6	c_5	c_4+c_9	c_3+c_9	c_2+2c_9	c_1+2c_9	c_8+2c_9	c_7+2c_9	$3c_9$
c_7	c_7	$c_3+c_4+c_9$	$c_3+c_4+c_9$	c_8+2c_9	c_8+2c_9	$c_7+c_8+2c_9$	$c_7+c_8+2c_9$	$4c_9$
c_8	c_8	$c_3+c_4+c_9$	$c_3+c_4+c_9$	c_7+2c_9	c_7+2c_9	$c_7+c_8+2c_9$	$c_7+c_8+2c_9$	$4c_9$
c_9	c_9	$2c_9$	$2c_9$	$3c_9$	$3c_9$	$4c_9$	$4c_9$	$4c_9$

The representation ring $a(\mathbb{Z}_2G)$ and its representation ideals are displayed in the following diagram:

$$\begin{tikzcd} a(\mathbb{Z}_2G) \arrow[d, dash] \\ a_{\max}(\mathbb{Z}_2G) = \langle c_3, c_4, c_7, c_8, c_9\rangle \arrow[d, dash, "\leftarrow\text{The nil radical lives here}"] \\ \langle c_3, c_4, c_9\rangle \arrow[d, dash] \\ a_{\mathsf{proj}}(\mathbb{Z}_2G) = \langle c_9\rangle \end{tikzcd}$$

The element $c_7 - c_8$ squares to zero, and generates the nil radical. The quotient is semisimple, with eight species given by the following table, where we have reordered the indecomposables to reflect the structure of the representation ideals.

c_1	1	1	1	1	1	1	1	1
c_2	1	1	1	1	1	1	-1	-1
c_5	3	1	-1	1	1	-1	1	-1
c_6	3	1	-1	1	1	-1	-1	1
c_7	4	2	0	2	0	0	0	0
c_8	4	2	0	2	0	0	0	0
c_3	2	2	-2	0	0	0	0	0
c_4	2	2	2	0	0	0	0	0
c_9	4	0	0	0	0	0	0	0

It is fairly easy to see that the role of the element ρ is played by c_9, which is the only projective module. The dimension function is the first column of numbers in the table, and is the only Brauer species. Every module is self-dual, and since all the entries in the table above are real, it follows that $a(\mathbb{Z}_2G)$ is a symmetric representation ring, see Section 3.4.

REMARK 5.7.1. In the same paper, Reiner [**77**] shows that whenever G is a cyclic group of order p^n with $n \geqslant 2$, there is a non-zero nilpotent element in $a(\mathbb{Z}_pG)$. This result is extended in Reiner [**79**].

The representation type of $\mathbb{Z}_pG$ is finite if and only if the Sylow p-subgroups of G are trivial, or cyclic of order p or p^2, see Heller and Reiner [**52, 53**]. Integral representations of the dihedral group of order $2p$ are described in Lee [**67**]. It would be interesting to know the tensor products of integral representations in cyclic and dihedral cases of finite representation type.

5.8. The group $SL(2,q)$

Let $q = p^m$ be a power of a prime p, and let k be a field containing $\mathbb{F}_q$, and let $G = SL(2,q)$. In this section, we examine the two dimensional natural module M for $SL(2,q)$ over k. The goal is to show that $\gamma_G(M) = 2\cos(\pi/q)$. On the way to this, we shall show that the subring of $a(G)$ generated by summands of tensor powers of M is isomorphic to a ring of algebraic integers $\mathbb{Z}[2\cos(\pi/q)]$, which in turn is the real subring of the cyclotomic integers $\mathbb{Z}[\zeta_q]$ where $\zeta_q = e^{2\pi \mathrm{i}/q}$.

We shall use the theory of tilting modules $T(n)$ for $SL(2,\bar{k})$, which turn out to be the direct summands of the natural module $L(1)$. A general discussion of tilting modules for reductive groups may be found in Donkin [**36**], to which we refer for general background. There are also relevant discussions of summands of tensor powers of the natural $SL(2,q)$-modules in Alperin [**4**] and Craven [**32**].

The simple $SL(2,\bar{k})$-modules $L(n)$ are indexed by their highest weight, which in this case is an integer $n \geqslant 0$. In particular, $L(0)$ is the trivial module, $L(1)$ is the natural two dimensional module, and for $0 \leqslant n \leqslant p-1$ we have $L(n) \cong S^n(L(1))$, the symmetric powers of the natural module.

Steinberg's tensor product theorem states that if $n = \sum_{j=0}^{m-1} n_j p^j$ with $0 \leqslant n_j \leqslant p-1$ then

$$L(n) \cong \bigotimes_{j=0}^{m-1} F^j(L(n_j)).$$

The restriction of the modules $L(n)$ to $SL(2,q)$ for $0 \leqslant n < q$, which we continue to denote $L(n)$, form a complete set of irreducible modules for $SL(2,q)$.

From [**36**], we know that the tilting module $T(n)$ (Donkin's notation is $M(\lambda)$) is the unique indecomposable summand of $L(1)^{\otimes n}$ with n as a highest weight. A module is a direct sum of tilting modules if and only if it is a direct summand of a direct sum of tensor powers of $L(1)$. Tilting modules are determined by their weights.

THEOREM 5.8.1. *Let $L(1)$ be the natural two dimensional module for $SL(2,q)$ as above. Then $L(1)$ is algebraic, and we have*

$$\gamma(L(1)) = 2\cos\pi/q.$$

More generally, if $1 \leqslant j \leqslant p-1$ then

$$\gamma(L(j)) = \sin((j+1)\pi/q)/\sin(\pi/q).$$

PROOF. Let $a_{\mathsf{tilt}}(SL(2,\bar{k}))$ be the subring of the representation ring of rational $SL(2,\bar{k})$-modules generated by the tilting modules. Then $a_{\mathsf{tilt}}(SL(2,\bar{k}))$ is isomorphic to the subring of $\mathbb{Z}[t,t^{-1}]$ generated by $t+t^{-1}$, with the powers of t representing the non-negative weights, and $L(1)$ corresponding to $t+t^{-1}$.

Let $f_j(t)$ be the polynomials defined in Definition 5.2.3. Then we have

$$f_j(t+t^{-1}) = t^{j-1} + t^{j-3} + \cdots + t^{-j+3} + t^{-j+1}$$

and

$$(t+t^{-1})f_j(t+t^{-1}) = f_{j+1}(t+t^{-1}) + f_{j-1}(t+t^{-1}).$$

In particular,

$$\begin{aligned} f_q(t+t^{-1}) &= t^{q-1} + t^{q-3} + \cdots + t^{-q+3} + t^{-q+1} \\ &= \prod_{j=1}^{m} (t^{p^{j-1}(p-1)} + t^{p^{j-1}(p-3)} + \cdots + t^{-p^{j-1}(p-1)}) \\ &= \prod_{j=1}^{m} f_p(t^{p^j} + t^{-p^j}) \end{aligned}$$

is the character of the Steinberg module $L(q-1)$ for $SL(2,q)$. This a projective module of dimension q. It follows that in $a(G)/a(G,1)$, we have $f_q[L(1)] = 0$. In particular, by Lemma 1.9.3, $[L(1)]$ is algebraic in $a(G)$. Now by Lemma 5.2.6, the irreducible factors of $f_q(X)$ exactly correspond to the Steinberg tensor product factors of

$$L(q-1) = \bigotimes_{j=0}^{m-1} F^j(L(p-1)).$$

No smaller tensor product of these modules is projective, so f_q is the minimal polynomial of $L(1)$ in $a(G)/a(G,1)$. Again using Lemma 5.2.6, the largest of the roots of $f_q(X)$ is $2\cos(\pi/q)$. Applying Theorem 3.5.1, it follows that $\gamma(L(1)) = 2\cos\pi/q$. If $1 \leqslant j \leqslant p-1$ then $[L(j)] = f_{j+1}[L(1)]$ and so

$$\gamma(L(j)) = f_{j+1}(2\cos\pi/q) = \sin((j+1)\pi/q)/\sin(\pi/q). \qquad \square$$

CONJECTURE 5.8.2. *If M is a kG-module with $\gamma(M) < 2$ then for some integer $q \geqslant 2$ we have $\gamma(M) = 2\cos(\pi/q)$.*

It is even plausible that if $\gamma(M) = 2\cos(\pi/q)$ then q is a power of the characteristic p of the coefficient field k. We know of no counterexamples to this statement.

5.9. The Klein four group

Let $G = \mathbb{Z}/2\times\mathbb{Z}/2$ and k an algebraically closed field of characteristic two. It was shown in Conlon [**27**] that elements of he representation ring $a_{\mathbb{C}}(G)$ are separated by species (which he calls G-characters) $s\colon a_{\mathbb{C}}(G) \to \mathbb{C}$, and therefore $a_{\mathbb{C}}(G)$ is semisimple. The species are described there, and more explicitly in Benson and Parker [**12**], and we repeat the description here.

The set of species for $a_{\mathbb{C}}(G)$ falls naturally into three subsets:

(i) The dimension.
(ii) A continuous set of species parametrised by the non-zero complex numbers $z \in \mathbb{C}\setminus\{0\}$.
(iii) A discrete set of species parametrised by the set of ordered pairs (N,λ) with $N > 0$ in $\mathbb{Z}$ and $\lambda \in \mathbb{P}^1(k)$, the projective line over k.

The set of indecomposable kG-modules also falls naturally into three subsets:

(i) The projective indecomposable module of dimension four.
(ii) The syzygies of the trivial module $\Omega^m(k)$, $m \in \mathbb{Z}$, of dimension $2|m|+1$.
(iii) A set of representations parametrised by the set of ordered pairs (n,λ) with $n > 0$ in $\mathbb{Z}$ and $\lambda \in \mathbb{P}^1(k)$, of dimension $2n$.

Define infinite matrices A and B as follows.

$$\begin{array}{cc|ccccccc}
A & & N & \to & & & & \\
\hline
 & & 1 & 2 & 3 & 4 & 5 & \cdots \\
\hline
n & 1 & 2 & 0 & 0 & 0 & 0 & \\
\downarrow & 2 & 2 & 2 & 0 & 0 & 0 & \\
 & 3 & 2 & 2 & 2 & 0 & 0 & \\
 & 4 & 2 & 2 & 2 & 2 & 0 & \\
 & 5 & 2 & 2 & 2 & 2 & 2 & \\
 & \vdots & & & & & & \ddots
\end{array}
\qquad
\begin{array}{cc|ccccccc}
B & & N & \to & & & & \\
\hline
 & & 1 & 2 & 3 & 4 & 5 & \cdots \\
\hline
n & 1 & \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & \\
\downarrow & 2 & 2 & 2 & 0 & 0 & 0 & \\
 & 3 & 2 & 2 & 2 & 0 & 0 & \\
 & 4 & 2 & 2 & 2 & 2 & 0 & \\
 & 5 & 2 & 2 & 2 & 2 & 2 & \\
 & \vdots & & & & & & \ddots
\end{array}$$

Then the representation table is as follows.

Parameters	dim	z	(N,∞)	$(N,0)$	$(N,1)$	(N,λ_1)	(N,λ_2)	(N,λ_3)	$\cdots$
(Projective)	4	0	0	0	0	0	0	0	
m	$2\|m\|+1$	z^m	1	1	1	1	1	1	
(n,∞)	$2n$	0	A	0	0	0	0	0	
$(n,0)$	$2n$	0	0	A	0	0	0	0	
$(n,1)$	$2n$	0	0	0	A	0	0	0	
(n,λ_1)	$2n$	0	0	0	0	B	0	0	
(n,λ_2)	$2n$	0	0	0	0	0	B	0	
(n,λ_3)	$2n$	0	0	0	0	0	0	B	
$\vdots$									

Thus there are three special points $\infty, 0, 1 \in \mathbb{P}^1$ where the matrix A is used, and for the rest of the points the matrix B is used. The members of the continuous family of species with $|z| \neq 1$ are not dimension bounded; the rest of the species are. There is a single Brauer species dim, which is dimension bounded but not core bounded; the rest of the dimension bounded species are core bounded. So the set of core bounded species is $S^1 \cup (\mathbb{P}^1(k) \times \mathbb{Z}_{>0})$. The weak* topology on this may be described as follows. The subset S^1 has the usual topology inherited from $\mathbb{C}$. The subset $\mathbb{P}^1(k) \times \mathbb{Z}_{>0}$ is discrete, but its closure is the one point compactification, using the point $1 \in S^1$. So the space $\Delta(G)$ is a wedge of a circle with the one point compactification of the discrete space $\mathbb{P}^1(k) \times \mathbb{Z}_{>0}$.

5.10. The alternating group A_4

Let k be an algebraically closed field of characteristic two, and let G be the alternating group A_4. Let V_4 be the normal subgroup of G of index three, isomorphic to the Klein four group. The indecomposable kG-modules are described in Conlon [**27**] in terms of those of kV_4; see also Conlon [**28**] and the appendix to Benson [**8**]. Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\} \subseteq k$, and write k, ω and $\bar{\omega}$ for the three one dimensional representations where a generator h for $G/V_4 \cong \mathbb{Z}/3$ goes to 1, ω, $\bar{\omega}$ respectively. We also have an action of G/V_4 on $\mathbb{P}^1(k)$, in which h sends λ to $\lambda^h = 1/(1+\lambda)$. The fixed points of this action are ω and $\bar{\omega}$.

The set of indecomposable kG-modules falls naturally into three subsets:

(i) The projective indecomposables P_k, P_ω and $P_{\bar{\omega}}$, each of dimension four.
(ii) The syzygies of the simple modules $\Omega^n(k)$, $\Omega^n(\omega)$ and $\Omega^n(\bar{\omega})$, each of which restrict to $\Omega^n(k)$ as a kV_4-module.
(iii) For each orbit of G/V_4 on $\mathbb{P}^1(k) \setminus \{\omega, \bar{\omega}\}$ and each $n \in \mathbb{Z}_{>0}$ there is an indecomposable kG-module of dimension $6n$ which restricts to the sum of the indecomposable kV_4-modules corresponding to (n,λ), $(n, 1+1/\lambda)$, $(1, 1/(1+\lambda))$; for each $\lambda \in \{\omega, \bar{\omega}\}$ and each $n \in \mathbb{Z}_{>0}$ there are three kG-modules of dimension $2n$, restricting to the kV_4-module corresponding to (n,λ).

We define one more infinite matrix C as follows.

$$
\begin{array}{cc|cccccc}
C & & N & \rightarrow & & & & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \cdots \\
\hline
n & 1 & \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
\downarrow & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
 & 3 & -1 & -1 & -1 & 0 & 0 & 0 \\
 & 4 & -1 & -1 & -1 & -1 & 0 & 0 \\
 & 5 & 2 & 2 & 2 & 2 & 2 & 0 \\
 & 6 & -1 & -1 & -1 & -1 & -1 & -1 \\
 & \vdots & & & & & & \ddots
\end{array}
$$

After the first row, each column repeats with period three where it is non-zero. Then the representation table is as follows. As in the appendix to [**8**], we have used the Atlas conventions [**30**] to illustrate the relationship with the tables for the Klein four group.

Params	dim	z	(N,∞)	$(N,0)$	$(N,1)$	(N,ω)	$(N,\bar\omega)$	(N,λ)	(N,λ^h)	(N,λ^{h^2})	fus	h	z	(N,ω)	$(N,\bar\omega)$
(Proj)	4	0	0	0	0	0	0	0	0	0	:	1	0	0	0
m	$2\|m\|+1$	z^m	1	1	1	1	1	1	1	1	:	ε	z^m	1	1
(n,∞)	$2n$	0	A									0	0	0	0
$(n,0)$	$2n$	0		A					0						
$(n,1)$	$2n$	0			A										
(n,ω)	$2n$	0				B					:	ε	0	C	0
$(n,\bar\omega)$	$2n$	0					B				:	ε	0	0	C
(n,μ)	$2n$	0						$B\delta_{\lambda,\mu}$				0	0	0	0
(n,μ^h)	$2n$	0		0					$B\delta_{\lambda,\mu}$						
(n,μ^{h^2})	$2n$	0								$B\delta_{\lambda,\mu}$					

Here, $\varepsilon \in \{-1,0,1\}$ is congruent to the dimension modulo three. Just as in the case of the Klein four group, the structure space $\Delta(G)$ has a discrete part and a continuous part. The continuous part is $\Delta_{\max}(G)$, and consists of three disjoint circles, corresponding to the columns headed "z" (the one on the right side is really two, one for each non-trivial character of the quotient G/V_4). All but the second row of this table (which is really an infinite set of rows) represent elements of $a(G,\max)$, on which these columns take the value zero. The closure of the discrete part is again its one point compactification, attached at the basepoint of one of the three circles.

Expanding out the table for the quotient $a_{\max}(G)$ from Atlas format to full notation, we obtain the following table:

$\Omega^m(k)$	z^m	z^m	z^m
$\Omega^m(\omega)$	z^m	ωz^m	$\bar\omega z^m$
$\Omega^m(\bar\omega)$	z^m	$\bar\omega z^m$	ωz^m

This is exactly the character table for $\mathbb{Z}\times\mathbb{Z}/3$. This is because $a_{\max}(G)$ is isomorphic to the group ring of $\mathsf{Pic}_{\max}(G) \cong \mathbb{Z}\times\mathbb{Z}/3$, a group with generators $\Omega(k)$ and ω.

5.11. Dihedral 2-groups

The indecomposable modules for the dihedral groups D_{2^n} $(n \geqslant 3)$ were classified by Ringel [**82**]. Let k be a field of characteristic two, let

$$G = D_{2^n} = \langle x, y \mid x^2 = 1,\ y^2 = 1,\ (xy)^{2^{n-1}} = 1 \rangle,$$

and let $X = x - 1$, $Y = y - 1$ as elements of kG. Then

$$kG = k\langle X, Y \rangle / (X^2, Y^2, (XY)^{2^{n-2}} - (YX)^{2^{n-2}}).$$

The modules come in two types, called *strings* and *bands*. The ones of odd dimension are string modules.

The string modules $M(C)$ correspond to words $C = w_1 w_2 \dots w_m$ where the w_i alternate between $X^{\pm 1}$ and $Y^{\pm 1}$. The dimension of the module is $m+1$. Thus for example the word $X^{-1}YXYX^{-1}Y^{-1}$ gives a module with schema

$$\bullet \xrightarrow{X} \bullet \xleftarrow{Y} \bullet \xleftarrow{X} \bullet \xleftarrow{Y} \bullet \xrightarrow{X} \bullet \xrightarrow{Y} \bullet$$

For a particular order of dihedral group, there is also a restriction on the number of consecutive letters which are all direct or all inverse. The module corresponding to a given word $w_1 w_2 \dots w_m$ has a k-basis $v_0, v_1, v_2, \dots, v_m$. The elements X and Y in kG act in the manner indicated by the schema, sending each basis either to an adjacent basis element or to zero. In the example, we have

$$\begin{array}{llllllll} X\colon & v_0 \mapsto v_1, & v_1 \mapsto 0, & v_2 \mapsto 0, & v_3 \mapsto v_2, & v_4 \mapsto v_5, & v_5 \mapsto 0, & v_6 \mapsto 0, \\ Y\colon & v_0 \mapsto 0, & v_1 \mapsto 0, & v_2 \mapsto v_1, & v_3 \mapsto 0, & v_4 \mapsto v_3, & v_5 \mapsto v_6, & v_6 \mapsto 0. \end{array}$$

Modules coming from two different words are isomorphic if and only if one word is the inverse of the other. To invert a word, reverse the letters and invert each one. So for example the inverse of the word above is $YXY^{-1}X^{-1}Y^{-1}X$.

The band modules $M(C, \phi)$ are similar, except that the word has to have even length, and the beginning and end of the word are linked to make a cycle. Instead of putting one basis element at each vertex, we take a vector space V and an indecomposable automorphism $\phi\colon V \to V$, and we put a copy of V at each vertex. The arrows are identity maps, but the two end vertices are identified using ϕ. So for example the word above gives us a schema

$$\begin{tikzcd} V \arrow[r, "X"] & V & V \arrow[l, "Y"'] & V \arrow[l, "X"'] & V \arrow[l, "Y"'] \arrow[r, "X"] & V \arrow[lllll, bend left, "Y"', "\phi"] \end{tikzcd}$$

The word C is not allowed to be a power of a smaller word, as this would be absorbed into making the vector space V larger. Modules $M(C, \phi)$ and $M(C', \phi')$ are isomorphic if and only if either C and C' differ by a rotation and $\phi = \phi'$, or C^{-1} and C' differ by a rotation and $\phi^{-1} = \phi'$.

The band modules all have even dimension. So the odd dimensional modules are string modules for words of even length. Inverting the word if necessary, we may assume that it starts with $X^{\pm 1}$ and ends with $Y^{\pm 1}$, and then we don't need to bother about equivalent words. Thus the odd dimensional modules are of the form $M(C)$ with $C = X^{\pm 1} \dots Y^{\pm 1}$. This includes the empty word, which we take to corrspond to the trivial module.

LEMMA 5.11.1. *If M is an odd dimensional indecomposable kD_{2^n}-module then $M{\downarrow_{\langle x\rangle}}$ is a direct sum of a one dimensional trivial module and a projective module.*

PROOF. This follows immediately from the description above. The one dimensional summand corresponds to the right hand vertex. The remaining pairs of vertices give free summands as modules for $k\langle x\rangle = k\langle X\rangle$. □

The next two theorems come from Archer [**5**].

THEOREM 5.11.2. *If M and N are odd dimensional indecomposable kD_{2^n}-modules then $M \otimes N$ has a unique odd dimensional indecomposable summand.*

PROOF. This follows by restricting to $\langle x\rangle$ and using Lemma 5.11.1. □

It follows from this theorem that the isomorphism classes of odd dimensional indecomposable kD_{2^n}-modules form an abelian group, equal to $\mathbb{P}\mathsf{ic}_{\max}(a(kD_{2^n}))$ (see Section 1.7). The product of $[M]$ and $[N]$ in this group is the isomorphism class of the unique odd dimensional summand of $M \otimes N$. The inverse of $[M]$ is $[M^*]$.

THEOREM 5.11.3. *This group is torsion free.*

REMARK 5.11.4. Zemanek [**98**] showed that there are non-zero nilpotent elements in $a(D_{2^n})$ ($n \geqslant 3$); see also Benson and Carlson [**11**], Heldner [**51**]. So we cannot hope to separate elements of $a(D_{2^n})$ using species, as we did in the case of the Klein four group.

The papers of Herschend [**54, 55**] study a different tensor product on representations of dihedral group algebras.

5.12. Semidihedral 2-groups

Let k be a field of characteristic two, and let

$$G = SD_{2^n} = \langle x, y \mid x^2 = 1,\, y^{2^{n-1}} = 1,\, yx = xy^{2^{n-2}-1}\rangle.$$

In Section 3 of Bondarenko and Drozd [**17**], an explicit isomorphism is given between the quotient by the one dimensional socle, $kG/\mathsf{Soc}(kG)$, and the algebra $\Lambda_{2^{n-1}-1}$ where

$$\Lambda_m = k\langle X, Y\rangle/(X^3, Y^2, X^2 - (YX)^m Y).$$

Since every non-projective indecomposable kG-module has $\mathsf{Soc}(kG)$ in the kernel, classification of the indecomposable kG-modules amounts to classification of the indecomposable $\Lambda_{2^{n-1}-1}$-modules. The indecomposable Λ_m-modules for $m \geqslant 1$ were classified by Crawley-Boevey [**33**]; see also Geiß [**44**]. They have a description in terms similar to the strings and bands described in the last section, but more complicated. There are four types, called asymmetric strings, symmetric strings, asymmetric bands, and symmetric bands. The asymmetric and symmetric bands have even dimension. If M is an odd dimensional indecomposable kG-module then M is an asymmetric or symmetric string module corresponding to a word of even length. In particular, just as in the dihedral case, the restriction of an odd dimensional module to the two dimensional subalgebra generated by Y is trivial plus free. So we get the following theorem.

THEOREM 5.12.1. *If M and N are odd dimensional indecomposable kSD_{2^n}-modules then $M \otimes N$ has a unique odd dimensional indecomposable summand.* $\square$

Again we deduce that the isomorphism classes of odd dimensional kSD_{2n}-modules form an abelian group, equal to $\mathbb{P}\mathsf{ic}_{\max}(a(kSD_{2^n}))$. But this time it is not torsion free. There is a self-dual module $M = \Lambda_{2^{n-1}-1}/\langle YX\rangle$ of dimension $2^{n-1}+1$, with simple socle, such that $M \otimes M \cong k \oplus kG$. Thus $[M]^2 = \mathbb{1}$ in this group.

5.13. Finite 2-groups

The following conjecture appears in [**10**].

CONJECTURE 5.13.1. *Let G be a finite 2-group and k an algebraically closed field of characteristic 2. If M is an indecomposable kG-module of odd dimension then $M \otimes M^*$ is a direct sum of k and indecomposable modules of dimension divisible by four.*

The conjecture is true in the case of cyclic groups, dihedral groups, and semidihedral groups.

Given this conjecture, the isomorphism classes of indecomposables of odd dimension form a discrete abelian group, equal to $\mathbb{P}\mathsf{ic}(a(G))$, and $a_{\max}(G)$ is its group algebra, weighted with the function $M \mapsto \dim \mathsf{core}_{\max}(M)$.

5.14. Some Hopf algebras

Whenever the modular representation ring of a finite group is finite dimensional, it is semisimple. We saw in Section 5.7 that this is not the case for integral representation rings. We shall see in this section that it is also not true for finite dimensional Hopf algebras over a field. To illustrate this, we examine some generalisations of Hopf algebras introduced by Earl Taft [**90**], which are neither commutative nor cocommutative. They are similar to the group algebras of the Frobenius groups studied in Section 5.3, but sufficiently different that we find it worthwhile to spell out the details. The end result is that the radical is contained in the ideal of projective modules, but is non-zero.

We begin with the smallest case, which is Sweedler's four dimensional Hopf algebra; see page 89–90 of Sweedler's book, as well as Remark 5.8 in Cibils [**26**] and Remark 1.5.6 in Montgomery [**70**]. Let k be a field of characteristic not equal to two, and consider the k-algebra with a vector space basis consisting of elements 1, g, x and gx. The multiplication is given by $g^2 = 1$, $x^2 = 0$, $xg = -gx$. The comultiplication is given by $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$. The counit is given by $\varepsilon(g) = 1$, $\varepsilon(x) = 0$, and the antipode is given by $S(g) = g$, $S(x) = gx$, $S(gx) = -x$, with $S^4 = 1$.

This Hopf algebra has four isomorphism classes of indecomposables. There are two simples, S_0 and S_1, and their projective covers P_0 and P_1. The tensor products are given by the following table:

	$[S_0]$	$[S_1]$	$[P_0]$	$[P_1]$
$[S_0]$	$[S_0]$	$[S_1]$	$[P_0]$	$[P_1]$
$[S_1]$	$[S_1]$	$[S_0]$	$[P_1]$	$[P_0]$
$[P_0]$	$[P_0]$	$[P_1]$	$[P_0]+[P_1]$	$[P_0]+[P_1]$
$[P_1]$	$[P_1]$	$[P_0]$	$[P_0]+[P_1]$	$[P_0]+[P_1]$

Note, in particular, that the representation ring is commutative. Its radical is generated by $[P_0]-[P_1]$. There are three species, given by the following table.

	s_1	s_2	s_3
$[S_0]$	1	1	1
$[S_1]$	1	-1	1
$[P_0]$	0	0	$\sqrt{2}$
$[P_1]$	0	0	$\sqrt{2}$

The representation ideals $\mathfrak{X}_{\max}$ and $\mathfrak{X}_{\mathsf{proj}}$ are equal, and consist of the projectives $[P_0]$ and $[P_1]$. So there is only one possible non-zero choice for a representation ideal $\mathfrak{X}$, namely $\mathfrak{X}=\mathfrak{X}_{\max}=\mathfrak{X}_{\mathsf{proj}}$. We have $\gamma(S_0)=\gamma(S_1)=1$, $\gamma(P_0)=\gamma(P_1)=0$.

The family of generalised Taft algebras $H_{m,n}(q)$ [**90**] has the Sweedler four dimensional algebra as the case $H_{2,2}(-1)$. Their representation rings were studied in [**24, 56, 68, 76, 96**]. They were constructed as examples of Hopf algebras whose antipode has arbitrarily large finite (even) order. Let k be a field having a primitive mth root of unity η, with m an integer at least two; in particular, we assume that k has characteristic coprime to m. Let $n\geqslant 2$ be a divisor of m, and let $q=\eta^d$, a primitive nth root of unity, where $d=m/n$. The algebra $H_{m,n}(q)$ is generated over k by elements h and x satisfying $h^m=1$, $x^n=0$, $hx=qxh$. The comultiplication is given by $\Delta(h)=h\otimes h$, $\Delta(x)=1\otimes x+x\otimes h$, the counit is given by $\varepsilon(h)=1$, $\varepsilon(x)=0$. The antipode is given by $S(h)=h^{-1}$, $S(x)=-xh^{-1}$, and has order $2n$. For reasons similar to those given in Section 5.3, we consider the double cover $H_{2m,n}(q)$ of $H_{m,n}(q)$ first and then identify $a(H_{m,n}(q))$ as a subring of $a(H_{2m,n}(q))$.

So we now suppose that the field k has a primitive $2m$th root of unity η, the integer $n\geqslant 2$ divides m, and $q=\eta^{2d}$ is a primitive nth root of unity, where $d=m/n$. The Hopf algebra $H_{2m,n}(q)$ over k has generators h and x satisfying $h^{2m}=1$, $x^n=0$, $hx=qxh$. The comultiplication, counit and antipode are as before.

The algebra $H_{2m,n}(q)$ has $2m$ isomorphism classes of simple modules S_i, $i\in\mathbb{Z}/2m$, all one dimensional, corresponding to the characters of the subgroup generated by h. Letting v_i be a basis element for S_i, we have $hv_i=\eta^i v_i$ and $xv_i=0$. The space $\mathsf{Ext}^1_{H_{2m,n}(q)}(S_i,S_j)$ is one dimensional if $j=i+2d$ and zero dimensional otherwise. The projective indecomposable modules are uniserial of length n, with composition factors (from top to bottom) of P_i being S_i, S_{i+2d}, S_{i+4d}, $\dots$, S_{i-2d}. So $H_{2m,n}(q)$ is a Frobenius algebra, but not a symmetric algebra. Every indecomposable module is a quotient of a projective indecomposable module. We write J_j $(1\leqslant j\leqslant n)$ for the indecomposable module of length j with composition factors $S_{-d(j-1)}$, $S_{-d(j-3)}$, $\dots$, $S_{d(j-1)}$. A complete list of the $2mn$ isomorphism classes of indecomposable $\tilde{H}_{m,n}(q)$-modules is given by the modules $J_j \otimes S_i$ with $1 \leqslant j \leqslant n$, $0 \leqslant i < 2m$. The representation ring $a(H_{m,n}(q))$ is commutative, even though this Hopf algebra is not quasitriangular. As in Section 5.3, we have

$$J_2 \otimes J_j \cong \begin{cases} J_2 & j = 1 \\ J_{j+1} \oplus J_{j-1} & 2 \leqslant j \leqslant n-1 \\ (J_n \otimes S_d) \oplus (J_n \otimes S_{-d}) & j = n. \end{cases}$$

THEOREM 5.14.1. *We have*

$$a(H_{2m,n}(q)) \cong \mathbb{Z}[X,Y]/(Y^{2m}-1, (X - Y^d - Y^{-d})f_n(X))$$

where X corresponds to J_2 and Y corresponds to S_1, and the polynomials f_i are described in Definition 5.2.3. This ring has a basis consisting of the X^iY^j with $0 \leqslant i < n$ and $0 \leqslant j < 2m$.

PROOF. This follows from the above relations, as in Theorem 5.2.7, Remark 5.2.10 and Theorem 5.3.1. The element $f_j(X)$ again corresponds to $[J_j]$. □

Our next task is to identify the species and radical of this representation ring.

LEMMA 5.14.2. *In $\mathbb{Z}[Y, Y^{-1}]$, for $j \geqslant 0$ we have $(Y - Y^{-1})f_j(Y + Y^{-1}) = Y^j - Y^{-j}$.*

PROOF. We prove this by induction on j, the cases $j = 0$ and $j = 1$ being trivial to verify. For the inductive step, with $j \geqslant 2$, we have

$$\begin{aligned}(Y - Y^{-1})f_{j+1}(Y + Y^{-1}) &= (Y - Y^{-1})((Y + Y^{-1})f_j(Y + Y^{-1}) - f_{j-1}(Y + Y^{-1})) \\ &= (Y + Y^{-1})(Y^j - Y^{-j}) - (Y^{j-1} - Y^{-j+1}) \\ &= Y^{j+1} - Y^{-j-1}.\end{aligned}$$

□

LEMMA 5.14.3. *In $\mathbb{Z}[X, Y, Y^{-1}]$ the element $(Y^d - Y^{-d})f_n(X) - (Y^m - Y^{-m})$ is divisible by $X - Y^d - Y^{-d}$.*

PROOF. It follows from Lemma 5.14.2 that $(Y^d - Y^{-d})f_n(Y^d + Y^{-d}) = Y^m - Y^{-m}$. Now use the factor theorem. □

PROPOSITION 5.14.4. *The element $(Y^d - Y^{-d})f_n(X)$ squares to zero in $a(H_{2n,m}(q))$.*

PROOF. Since $Y^m - Y^{-m}$ is zero in $a(H_{2n,m}(q))$, it follows from Lemma 5.14.3 that the element $(Y^d - Y^{-d})f_n(X)$ is divisible by $(X - Y^d - Y^{-d})$. Hence its square is divisible by $(X - Y^d - Y^{-d})f_n(X)$, which is zero in $a(H_{2n,m}(q))$. □

THEOREM 5.14.5. *The nil radical of $a(H_{2m,n}(q))$ is generated by $(Y^d - Y^{-d})f_n(X)$, and has $\mathbb{Z}$-rank $2(m-d)$. There are $2(mn-m+d)$ species $s_{i,j}$ of $a(H_{2m,n}(q))$, and they are given by*

$$\begin{aligned} X &\mapsto \zeta_{2n}^i + \zeta_{2n}^{-i} = 2\cos(i\pi/n) \\ Y &\mapsto \zeta_{2m}^j. \end{aligned}$$

where $0 \leqslant i \leqslant n$, $0 \leqslant j < 2m$, and if j is divisible by n then $i \equiv j \pmod{2n}$.

The ideal of projective modules is generated by $f_n(X)$, and the $2d$ Brauer species are the ones with j divisible by n. The quotient $a_{\mathsf{proj}}(H_{2m,n}(q)) = a(H_{2m,n}(q))/(f_n(X))$ is semisimple.

PROOF. By Proposition 5.14.4, $(Y^d - Y^{-d})f_n(X)$ is in the radical. Consider the quotient $a(H_{2m,n}(q))/((Y^d - Y^{-d})f_n(X))$. Since $X - Y^d - Y^{-d}$ annihilates $(Y^d - Y^{-d})f_n(X)$, we have

$$X(Y^d - Y^{-d})f_n(X) = (Y^d + Y^{-d})(Y^d - Y^{-d})f_n(X).$$

We also have

$$(Y^{2(m-d)} + Y^{2(m-2d)} + \cdots + Y^{2d} + 1)(Y^d - Y^{-d})f_n(X) = Y^{-d}(Y^{2m} - 1)f_n(X) = 0.$$

So the ideal generated by $(Y^d - Y^{-d})f_n(X)$ is the $\mathbb{Z}$-span of the elements $Y^i(Y^d - Y^{-d})f_n(X)$ with $0 \leqslant i < 2(m-d)$. The quotient therefore has rank $2(mn - m + d)$, and has a $\mathbb{Z}$-basis consisting of the X^iY^j with $0 \leqslant i < n$, $0 \leqslant j < 2m$, such that if $i = n-1$ then $0 \leqslant j < 2d$.

For the species, we must satisfy the two relations $Y^{2m} = 1$ and $(X - Y^d - Y^{-d})f_n(X) = 0$. The first implies that $Y \mapsto \zeta_{2m}^j$ with $0 \leqslant j < 2m$. Then the second relation becomes $(X - \zeta_{2n}^j - \zeta_{2n}^{-j})f_n(X) = 0$. The roots of $f_n(X) = 0$ are $X \mapsto \zeta_{2n}^i + \zeta_{2n}^{-i}$ with $0 < i < n$. So the product has a repeated root unless $\zeta_{2n}^j = \pm 1$, namely j is divisible by n. In that case, $X \mapsto 2$ if j is divisible by $2n$ and $X \mapsto -2$ otherwise. This accounts for the cases $i = 0$ and $i = n$. The element $f_n(X) = [J_n]$ generates the projectives, and so the Brauer species are the ones where $f_n(X)$ does not go to zero. This is the case where j is divisible by n. $\square$

To go down from $H_{2m,n}(q)$ to $H_{m,n}(q)$, we use the polynomials $\phi_i(y,z)$ given in Definition 5.3.3.

THEOREM 5.14.6. *$a(H_{m,n}(q)) = a(H_{2m,n}(q)/\langle h^m \rangle) \cong \mathbb{Z}[y,z]/(y^m - 1, (z - y^d - 1)\phi_n(y^d, z))$ with $y = Y^2$ and $z = XY^d$. This is a complete intersection of $\mathbb{Z}$-rank mn, with a $\mathbb{Z}$-basis consisting of the monomials y^iz^j with $0 \leqslant i < m$, $0 \leqslant j < n$.*

The nil radical of $a(H_{m,n}(q))$ is generated by the element $(y^d - 1)\phi_n(y,z)$, which squares to zero. There are $mn - m + d$ species $s_{i,j}$ of $a(H_{m,n}(q))$, and they are given by

$$\begin{aligned} y &\mapsto \zeta_m^j \\ z &\mapsto \zeta_{2n}^{j+i} + \zeta_{2n}^{j-i} \end{aligned}$$

where $0 \leqslant i \leqslant n$, $0 \leqslant j < m$, and if j is divisible by n then $i \equiv j \pmod{2n}$.

The ideal of projective modules is generated by $\phi_n(y^d, z)$, and the d Brauer species are the one with j divisible by n. The quotient $a_{\mathsf{proj}}(H_{m,n}(q)) = a(H_{m,n}(q))/(\phi_n(y^d, z))$ is semisimple.

PROOF. The proof of the first part is the same as the proof of Theorem 5.3.4. The second part follows from Theorem 5.14.5. $\square$

REMARK 5.14.7. The Taft algebras $H_n(q)$ are the case of the generalised Taft algebras $H_{m,n}(q)$ where $m = n$ and $d = 1$.

5.15. Some open problems

Throughout this section, G is a finite group, k is a field of characteristic p, and M is a kG-module. Some of the following questions come from [**10, 13**]. Others have been around for a while.

QUESTION 5.15.1. If M is a kG-module, is $\gamma_G(M \otimes M^*) = \gamma_G(M)^2$?

QUESTION 5.15.2. Is $\hat{a}(G)$ a symmetric Banach $*$-algebra? If so, then by Proposition 3.4.2, we can deduce that Question 5.15.1 has a positive answer. More generally, develop a good way of testing whether a representation ring is symmetric.

QUESTION 5.15.3. If $1 < \gamma_G(M) < 2$, is $\gamma_G(M) = 2\cos\pi/n$ for some integer $n \geqslant 4$? If this holds, is n a power of p?

QUESTION 5.15.4. Is $\gamma_G(M)$ an algebraic integer?

QUESTION 5.15.5. Do the numbers $c_n^G(M)$ satisfy a linear recurrence relation with constant coefficients, for all sufficiently large values of n? If so, then Question 5.15.4 has a positive answer.

QUESTION 5.15.6. Let G be a finite 2-group and k an algebraically closed field of characteristic 2. If M is an indecomposable kG-module of odd dimension then Conjecture 5.13.1 states that $M \otimes M^*$ is a direct sum of k and indecomposable modules of dimension divisible by four. Is this true?

QUESTION 5.15.7. The indecomposable modules for the dihedral and semidihedral groups in characteristic two are known. How do their tensor products decompose? Compute the radical of the representation ring. Is it nilpotent?

QUESTION 5.15.8. What are the indecomposable modules for the quaternion groups in characteristic two? We know that the group algebra has tame representation type, but a classification of the modules is not known.

QUESTION 5.15.9. Are there nilpotent elements in the representation ring of an elementary abelian 2-group $(\mathbb{Z}/2)^r$ in characteristic two, when $r \geqslant 3$? The same question may be asked of the representation ring of an exterior algebra of rank at least three in any characteristic, regarded as a finite supergroup scheme.

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