Proceedings of Symposia in Pure Mathematics
Volume 47 (1987)

Some Remarks on the Decomposition Numbers for the Symmetric Groups

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1. Introduction. One of the main techniques for analyzing the irreducible representations of finite Chevalley groups in their own characteristic is the sum formula of Jantzen [**14**–**16**]. Recently, Schaper, a student of Jantzen, wrote a thesis [**20**] in which he develops a similar formula for the symmetric groups in arbitrary characteristic. The main purpose of this paper is to present some calculations which I have been performing using this formula in characteristic two and to formulate some conjectures and questions based on the results.

Two particular outcomes of these calculations are worth pointing out in this introduction. One is that it seems to be worth pursuing the 2-modular reductions of the spin representations of the symmetric groups to obtain information about questions not easily answered by other methods. Some explicit conjectures appear in §4. The other outcome worth mentioning concerns an old conjecture of Brauer. The conjecture states that the power of p dividing the degree of a p-modular irreducible character should be at most the power of p dividing the group order [**3**, p. 166]. The first known counterexample to this conjecture appeared in J. Thackray's thesis [**22**] in 1981; namely he used a digital computer to construct an irreducible 2-modular character of MacLaughlin's simple group of degree $7 \cdot 2^9$, whereas the order of the group is only divisible by 2^7. We present here the second known counterexample, a 2-modular irreducible character of $\mathfrak{S}_{15}$ of degree $4096 = 2^{12}$ (corresponding to the partition $(7,6,2)$), whereas the order of $\mathfrak{S}_{15}$ is only divisible by 2^{11}. One advantage of our counterexample is that the calculations can be checked (and were made) entirely by hand (some think this is a disadvantage!). It seems plausible that the symmetric groups should provide further counterexamples to the conjecture. In particular, the characters coming from 2-modular reductions of spin representations are all divisible by high powers of two, and the character of degree 4096 may be written in terms of spin

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1980 *Mathematics Subject Classification* (1985 *Revision*). Primary: 20C30.
This paper is in final form, and no version of it will be submitted for publication elsewhere.

characters as $\langle\overline{13,2}\rangle - \langle\overline{14,1}\rangle$ (i.e., $\langle\overline{14,1}\rangle$ is equal to the 2-modular irreducible $D^{(8,6,1)}$ of degree 832, while $\langle\overline{13,2}\rangle = D^{(8,6,1)} + D^{(7,6,2)}$).

2. Schaper's thesis. Since Schaper's version of the Jantzen sum formula is not readily available in the literature (I have not seen his thesis myself; I simply guessed the formula and was informed by several sources that it was to be found there!), I think it is worth saying a few words to indicate the nature of this formula.

Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of n (written $\lambda \vdash n$), i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ and $\sum_{i=1}^{s} \lambda_i = n$. Denote by S^λ the Specht module [**7**, p. 13], defined over the integers, corresponding to λ. This is given as a submodule of the permutation module M^λ on the cosets of the Young subgroup $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_s}$ corresponding to λ. The natural symmetric bilinear form on M^λ given by making the permutation basis orthonormal (i.e., length one and inner product zero) gives rise to a nonsingular bilinear form $\langle\ ,\ \rangle$ on S^λ.

Let p be a prime number, and denote by $\overline{S^\lambda}$ the reduction modulo p of S^λ. Following Jantzen [**14–16**], we define a filtration on $\overline{S^\lambda}$ as follows. Denote by $S^\lambda(r)$ the submodule of S^λ consisting of those elements $x \in S^\lambda$ with the property that for all $y \in S^\lambda$, $\langle x, y\rangle$ is divisible by p^r. Denote by $\overline{S^\lambda(r)}$ the image of $S^\lambda(r)$ in $\overline{S^\lambda}$ under reduction modulo p. This gives a filtration

$$\overline{S^\lambda} = \overline{S^\lambda(0)} \geq \overline{S^\lambda(1)} \geq \overline{S^\lambda(2)} \geq \cdots$$

with $\overline{S^\lambda(r)} = 0$ for sufficiently large r. Note that by [**7**, Theorems 4.9 and 11.1], the module $D^\lambda = \overline{S^\lambda}/\overline{S^\lambda(1)}$ is always zero or irreducible, and is zero if and only if λ is p-singular (i.e., has a part repeated p or more times). Every irreducible module for the symmetric group over $\mathbf{F}_p$ arises exactly once this way and is absolutely irreducible. Schaper's formula gives the value of the sum of the Brauer characters $\chi(\overline{S^\lambda(r)})$ of the modules in the above sequence in terms of things we can compute.

Now recall from James and Kerber [**12**, p. 77 ff.] the definition and elementary properties of β-numbers, and their relationship to hooks and skew-hooks (= rim-hooks). In particular, recall that if $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition, then any sequence of the form $(\lambda_1 - 1 + t, \lambda_2 - 2 + t, \dots, \lambda_s - s + t, -s - 1 + t, \dots, -t + t)$ $(t \geq s)$ is a sequence of β-numbers corresponding to λ. If $(\beta_1, \dots, \beta_t)$ is a sequence of integers, we define $\chi(\beta_1, \dots, \beta_t)$ to be zero if two of the β_i are equal or if any of the β_i are negative, and to equal plus or minus the ordinary character of the representation of the symmetric group corresponding to the partition whose β-numbers are $\{\beta_1, \dots, \beta_t\}$ otherwise. The sign is equal to the signature of the permutation π of $\{1, 2, \dots, t\}$ for which $\beta_{\pi(1)} > \beta_{\pi(2)} > \cdots > \beta_{\pi(t)}$.

If (a, b) is a node in the Young diagram corresponding to λ, we denote by h^λ_{ab} its hook length [**12**, p. 56]. We denote by ν_p the p-adic valuation on integers, namely $\nu_p(p^\alpha g) = \alpha$ if $p \nmid q$.

We may now state Schaper's formula.

THEOREM (SCHAPER [20]). *The Brauer character* $\sum_{r>0} \chi(\overline{S^\lambda(r)})$ *agrees on p-regular conjugacy classes with the ordinary character*

$$\sum_{1\le a\le b\le s} \sum_{c=1}^{\lambda_b} (\nu_p(h^\lambda_{ac}) - \nu_p(h^\lambda_{bc})) \cdot \chi(h^\lambda_{11}, h^\lambda_{21}, \ldots, h^\lambda_{a-1,1}, h^\lambda_{a1} + h^\lambda_{bc}, h^\lambda_{a+1,1}, \ldots, h^\lambda_{b1} - h^\lambda_{bc}, \ldots, h^\lambda_{s1}).$$

Note that the formula obtained by taking dimensions on both sides of Schaper's formula may be found in James and Murphy [13], where some examples and consequences are also given.

The way the formula is used is as follows. If we know the decomposition numbers for partitions dominated by a given partition (in the sense of James [7, p. 8]), then the formula tells us whether or not the decomposition numbers for the given partition are zero and also gives an upper bound. This may be applied to p-singular as well as p-regular partitions, and indeed often more delicate information is obtained from the p-singular ones. This corresponds to using Jantzen's formula outside the restricted fundamental region.

REMARK. One can show that $\overline{S^\lambda(r)/S^\lambda(r+1)}$ is a self-dual module (in Jantzen's case the corresponding subquotient supports a "contravariant form"). However, these subquotients are not always semisimple. For example if $p = 2$, $\lambda = (4, 1^2)$, $n = 6$ then $\overline{S^\lambda(2)/S^\lambda(3)}$ is a uniserial module with composition factors $D^{(6)}$, $D^{(4,2)}$, $D^{(6)}$.

3. Decomposition numbers for $\mathfrak{S}_{14}$ and $\mathfrak{S}_{15}$ mod 2. Using Schaper's formula together with the following techniques, we established the decomposition matrices given in the appendix. An explanation of these tables may be found in §5.

(i) *Restriction and induction.* As well as the obvious information given by the fact that restriction and induction coefficients must be nonnegative, we also use the information given by the Frobenius reciprocity statement

$$\dim_k \operatorname{Hom}_{kG}(M, N\uparrow^G) = \dim_k \operatorname{Hom}_{kH}(M\downarrow_H, N).$$

This, together with the self-duality of the irreducible modules, can often be used to eliminate possibilities. The idea is that the restriction of a large irreducible $\mathfrak{S}_n$-module to $\mathfrak{S}_{n-1}$ cannot have small modules in the socle, since this would imply that the induced module from the small module would map onto the large module. The same technique may be used with the roles of restriction and induction reversed.

(ii) *Spin representations.* Schur [21] and Morris [18, 19] have investigated the ordinary characters of the proper double covers of $\mathfrak{S}_n$ (there are two isoclinic proper double covers). Of course, these must be expressible as positive linear combinations of the 2-modular irreducible characters of $\mathfrak{S}_n$. This often gives strong information about partitions not easily accessible by other means. See also the conjectures in §4.

(iii) *First row and first column removal.* James [**9**] has shown that the coefficients of $D^{(\lambda_1,\dots,\lambda_s)}$ in $\overline{S}^{(\lambda_1,\mu_2,\dots,\mu_t)}$ is the same as the coefficient of $D^{(\lambda_2,\dots,\lambda_s)}$ in $\overline{S}^{(\mu_2,\dots,\mu_t)}$. This may be seen as saying that the first rows of the partitions may be removed if they are equal. He also shows that the same is true for first column removal. In fact, Donkin [**2**] has recently found a pleasing generalization of this to the removal of arbitrary rectangles in the upper left-hand corner of the partition.

(iv) We have also made use of James's results [**6**] on partitions of the form (λ_1, λ_2) and $(\lambda_1, \lambda_2, 1)$.

(v) Occasionally in desperation, we use inner tensor products and the associated reciprocity laws. Thus, for example, the last question to resolve for the decomposition matrix of $\mathfrak{S}_{14}$ mod 2 was: "How many copies of $D^{(13,1)}$ are there in $\overline{S}^{(7,4,2,1)}$?" It is easy to show that the answer is $5 + \gamma$ with $\gamma = 0, 1$, or 2. But then we see that $D^{(13,1)} \otimes D^{(8,4,2)}$ has composition factors $D^{(7,4,3)} + D^{(7,4,2,1)} + \gamma \,.\, D^{(13,1)}$. If $\gamma \neq 0$, then by self-duality there is a nonzero homomorphism $D^{(13,1)} \to D^{(13,1)} \otimes D^{(8,4,2)}$ and hence there is a nonzero homomorphism $D^{(13,1)} \otimes D^{(13,1)} \to D^{(8,4,2)}$. This is absurd since $\dim D^{(13,1)} = 12$ and $\dim D^{(8,4,2)} = 2510$.

4. Some conjectures and questions. In some sense, the most important reason for performing extensive calculations is in order to see general patterns and make sensible conjectures. I do not claim that all of the following conjectures are sensible. Some are much more likely to be right than others.

For the purpose of this section, we only work in characteristic two. Most of our conjectures are either meaningless or false in odd characteristic, although some may have analogues.

First, we treat the subject of restriction and induction. The following conjecture seems quite likely to be true.

CONJECTURE 1. *Let $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$. Then $D^\lambda \downarrow_{\mathfrak{S}_{n-1}}$ is irreducible if and only if all the λ_i are congruent modulo two. In this case the restriction is $D^{(\lambda_1-1,\lambda_2,\dots,\lambda_s)}$.*

Thus, for example, the module $D^{(7,6,2)}$ for $\mathfrak{S}_{15}$, of dimension 4096, is the restriction of the module $D^{(8,6,2)}$ for $\mathfrak{S}_{16}$, also of dimension 4096.

Reducing to a minimal counterexample by the method of first column removal (note that first column removal holds for restriction coefficients, while first row removal doesn't), and by using the induction-restriction techniques described in §3, one can see that Conjecture 1 follows from the following conjecture.

CONJECTURE 2. *Let $n = \lambda_1 + \cdots + \lambda_s + 1$. Suppose $\lambda_1, \dots, \lambda_s$ are all congruent modulo two. Then $D^{(\lambda_1,\dots,\lambda_s)}$ appears in the socle of $D^{(\lambda_1,\dots,\lambda_s,1)} \downarrow_{\mathfrak{S}_{n-1}}$ if and only if the λ_i are even.*

There are more complicated conjectures akin to Conjecture 1. For example, we may conjecture the following.

CONJECTURE 3. Let $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$. Suppose $\lambda_1 \equiv \lambda_2 \equiv \cdots \equiv \lambda_{2r} \not\equiv \lambda_{2r+1} \equiv \cdots \equiv \lambda_s \pmod 2$ for some value of r. Then

$$D^\lambda \downarrow_{\mathfrak{S}_{n-1}} \cong D^{(\lambda_1 - 1, \lambda_2, \ldots, \lambda_s)} \oplus D^{(\lambda_1, \ldots, \lambda_{2r+1} - 1, \ldots, \lambda_s)}.$$

Or more boldly

CONJECTURE 4. Let $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$. Then $D^\lambda \downarrow_{\mathfrak{S}_{n-1}}$ is completely reducible if and only if there are values $0 < r_1 < \cdots < r_k$ (possibly $k = 0$) such that $\lambda_j \equiv \lambda_{j+1} \pmod 2$ unless and only unless $j \in \{2r_1, \ldots, 2r_k\}$. In this case, we have

$$\begin{aligned} D^\lambda \downarrow_{\mathfrak{S}_{n-1}} &\cong D^{(\lambda_1 - 1, \ldots, \lambda_s)} \oplus D^{(\lambda_1, \ldots, \lambda_{2r_1+1} - 1, \ldots, \lambda_s)} \\ &\oplus \cdots \oplus D^{(\lambda_1, \ldots, \lambda_{2r_k+1} - 1, \ldots, \lambda_s)}. \end{aligned}$$

Thus, for example, according to this conjecture

$$D^{(7,5,4,2,1)} \downarrow_{\mathfrak{S}_{18}} \cong D^{(6,5,4,2,1)} \oplus D^{(7,5,3,2,1)} \oplus D^{(7,5,4,2)}.$$

On the other hand, for induction we make the following conjecture.

CONJECTURE 5. Let $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$. Then $D^\lambda \uparrow^{\mathfrak{S}_{n+1}}$ is completely reducible if and only if all the λ_i are even. In this case,

$$D^\lambda \uparrow^{\mathfrak{S}_{n+1}} \cong D^{(\lambda_1, \ldots, \lambda_s + 1)} \oplus D^{(\lambda_1, \ldots, \lambda_s, 1)}.$$

We may form analogous conjectures for the extended decomposition matrices of James [**8**]. It is shown in James [**10**] that the extended character table of $\mathfrak{S}_n$ is in duality with the table of indecomposable direct summands of permutation modules on Young subgroups. It follows that if we restrict or induce an extended modular character of $\mathfrak{S}_n$, we get a positive integral combination of extended modular characters of $\mathfrak{S}_{n-1}$ and $\mathfrak{S}_{n+1}$ respectively. The analogue of Conjecture 1, for example, comes out as follows.

CONJECTURE 6. Let $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash n$. If we restrict the irreducible extended modular character of $\mathfrak{S}_n$ corresponding to λ to $\mathfrak{S}_{n-1}$, then the result is

(a) zero if and only if every part of λ occurs with even multiplicity,

(b) an irreducible extended modular character of $\mathfrak{S}_{n-1}$ if and only if the parts of λ occurring with odd multiplicity are congruent modulo two.

In this case, the restriction is the character corresponding to the partition obtained by decreasing by one the last occurrence of the first part to appear with odd multiplicity.

Thus, for example, according to this conjecture, the restriction of the character corresponding to $(8^2 7^2 6^3 4\ 3^2 2)$ is the character corresponding to $(8^2 7^2 6^2 5\ 4\ 3^2 2)$.

Let us turn now to the spin representations. The basic references here are Schur [**21**] and Morris [**18, 19**]. Let Γ_n be one of the two isoclinic proper double covers of $\mathfrak{S}_n$. Then corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of n into s unequal parts, we have either one faithful irreducible representation of Γ_n if $(n - s)$ is even, or two algebraically conjugate irreducible representations of Γ_n if $(n - s)$ is odd. In the latter case, the characters only differ on elements of even order, and so the reductions modulo two will have the same composition factors. Thus we shall not distinguish them for our present purposes, and in each case we shall write $\langle\lambda\rangle = \langle\lambda_1,\dots,\lambda_s\rangle$ for the character of Γ_n corresponding to $(\lambda_1,\dots,\lambda_s)$.

We now describe a certain doubling process on partitions, in terms of which our conjectures are formulated. This is a process by which each part λ_i is replaced by two parts to make a new partition. If $\lambda_i = 2k+1$ is odd, then we replace λ_i by parts of lengths $k+1$ and k, while if $\lambda_i = 2k$ is even, then we replace λ_i by parts of lengths $k+1$ and $k-1$. We call a partition "spin regular" if the resulting parts form a partition of n into distinct parts in decreasing order. Thus, for example, $\mathrm{dbl}\langle 11,6,1\rangle = (6,5,4,2,1)$, while $\langle 11,6,2\rangle$ is not spin regular because $(6,5,4,2,2)$ are not distinct, and $\langle 11,6,4\rangle$ is not spin regular because $(6,5,4,2,3,1)$ are not in decreasing order.

Now if $\langle\lambda_1,\dots,\lambda_s\rangle$ is a representation of Γ_n, then each composition factor of $\overline{\langle\lambda_1,\dots,\lambda_s\rangle}$, its reduction modulo two, has the center of Γ_n in its kernel and is hence a representation of $\mathfrak{S}_n$. Thus the composition factors are of the form D^λ. For example, for the basic spin character $\langle n\rangle$, we have $\langle\overline{n}\rangle = D^{\mathrm{dbl}\langle n\rangle}$. The following two theorems were stated as conjectures in the first draft of this paper. Proofs will appear elsewhere.

THEOREM 7. *Let $\langle\lambda\rangle = \langle\lambda_1,\dots,\lambda_s\rangle$ be a spin regular partition of n. Then the composition factors of $\overline{\langle\lambda_1,\dots,\lambda_s\rangle}$ consist of $D^{\mathrm{dbl}\langle\lambda\rangle}$ with multiplicity a power of two, and possibly some other composition factors, each of which is of the form D^μ with $\mathrm{dbl}\langle\lambda\rangle \rhd \mu$ (for notation see James* [**7**, *p.* 8]).

THEOREM 8. *The module D^λ for $\mathfrak{S}_n$ splits on restriction to $\mathfrak{A}_n$ if and only if λ is of the form $\mathrm{dbl}\langle\mu\rangle$, with μ a spin regular partition with no part congruent to* 2 (mod 4). *Otherwise it remains irreducible.*

Thus for example, for $n = 17$, the modules which split are $D^{(9,8)}$, $D^{(9,7,1)}$, $D^{(7,6,3,1)}$, $D^{(7,5,3,2)}$, and $D^{(6,5,3,2,1)}$, corresponding to the doubles of $\langle 17\rangle$, $\langle 16,1\rangle$, $\langle 13,4\rangle$, $\langle 12,5\rangle$, and $\langle 11,5,1\rangle$.

It is probably slightly more tricky to see exactly what happens, say, for two-part partitions. In order to make our conjecture, we first define some functions.

We define quarter infinite matrices of types V, VI, and VII, analogous to those of types I–IV in James [**6**], as follows.

$$\begin{array}{ccccccccccc|}
1 & & & & & & & & & & \\
1 & 1 & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & & 1 & 1 & & & & & & & \\
1 & 1 & & & 1 & & & & & & \\
1 & 1 & & & 1 & 1 & & & & & \\
 & & 1 & 1 & & & 1 & & & & \\
 & & 1 & 1 & & & 1 & 1 & & & \\
1 & 1 & & & 1 & 1 & & & 1 & & \\
1 & 1 & & & 1 & 1 & & & 1 & 1 & \\
 & & 1 & 1 & & & 1 & 1 & & & 1 \\
\hline
\end{array}$$

type V

$$
\begin{array}{ccccccccccc|}
1 & & & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & & 1 & & & & & & & & \\
 & 1 & & 1 & & & & & & & \\
1 & & & & 1 & & & & & & \\
 & 1 & & 1 & & 1 & & & & & \\
 & & 1 & & & & 1 & & & & \\
 & 1 & & 1 & & 1 & & 1 & & & \\
1 & & & & 1 & & & & 1 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & 1 & & & & 1 & & & & 1 \\
\hline
\end{array}
$$

type VI

$$
\begin{array}{ccccccccccc|}
1 & & & & & & & & & & \\
2 & 1 & & & & & & & & & \\
1 & 1 & 1 & & & & & & & & \\
 & & 2 & 1 & & & & & & & \\
1 & 1 & 1 & 1 & 1 & & & & & & \\
2 & 2 & & & 2 & 1 & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & 2 & 2 & & & 2 & 1 & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
2 & 2 & & & 2 & 2 & & & 2 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
$$

type VII

The matrix of type V is formed by tiling $\mathbf{N}^2$ with alternate blocks of four zeros and four ones, and then cutting off everything above the diagonal. The matrix of type VI is formed by filling every fourth leading and trailing diagonal of $\mathbf{N}^2$ with ones, and then cutting off everything above the diagonal. The matrix of type VII is formed from the matrix of type V by adding to each entry the entry immediately above it. A $k \times k$ type V, VI, or VII matrix is given by the intersection of the bottom k rows and the right-hand k columns of the corresponding infinite matrix.

We then define functions ε, ς, and η analogous to James' α, β, γ, and δ as follows. $\varepsilon(x, m, j)$ is the jth entry in the mth row of an x by x type V matrix. $\varsigma(x, m, j)$ and $\eta(x, m, j)$ are defined in the same way, replacing "type V" by "type VI" or "type VII", respectively.

Thus we have

$$\varepsilon(x,m,j)=\begin{cases}1 & \text{if } 0<j\le m\le x \text{ and either } 4\mid(m-j)\\ & \qquad\qquad\text{or } 4\mid(2x+1-m-j),\\ 0 & \text{otherwise;}\end{cases}$$

$$\varsigma(x,m,j)=\begin{cases}1 & \text{if } 0<j\le m\le x \text{ and either } 4\mid(m-j)\\ & \qquad\qquad\text{or } 4\mid(2x-m-j),\\ 0 & \text{otherwise;}\end{cases}$$

$$\eta(x,m,j)=\begin{cases}\varepsilon(x,m,j)+\varepsilon(x,m-1,j) & \text{if } 0<j\le m\le x,\\ 0 & \text{otherwise.}\end{cases}$$

CONJECTURE 9. Let $\lambda=\langle\lambda_1,\lambda_2\rangle$ be a spin regular partition of n into two parts. Then the composition factors of $\langle\overline{\lambda_1,\lambda_2}\rangle$ are all of the form $D^{\mathrm{dbl}\langle\mu\rangle}$ with $\lambda\trianglerighteq\mu=\langle\mu_1,\mu_2\rangle$ (note that this is not true for partitions into more than two parts). The multiplicity of $D^{\mathrm{dbl}\langle\mu\rangle}$ in $\langle\overline{\lambda_1,\lambda_2}\rangle$ is given as follows.

(a) If $n\equiv 1 \pmod 2$ then the multiplicity is $\varepsilon(\frac{n-3}{2},\lambda_2+1,\mu_2+1)$.

(b) If $n\equiv 2 \pmod 4$ then the multiplicity is

$$\begin{cases}\varsigma\left(\dfrac{n-2}{2},\lambda_2+1,\mu_2+1\right) & \text{if } \lambda_2=0 \text{ or } \lambda_2 \text{ is odd,}\\ 2\varsigma\left(\dfrac{n-2}{2},\lambda_2+1,\mu_2+1\right) & \text{otherwise.}\end{cases}$$

(c) If $n\equiv 0 \pmod 4$ then the multiplicity is

$$\begin{cases}\eta\left(\dfrac{n-4}{2},\lambda_2+1,\mu_2+1\right) & \text{if } \mu_2=0 \text{ or } \mu_2 \text{ is odd,}\\ 2\eta\left(\dfrac{n-4}{2},\lambda_2+1,\mu_2+1\right) & \text{otherwise.}\end{cases}$$

We now turn to the possibility of finding further formulas like those of Jantzen and Schaper. From the work of Fong and Srinivasan [**4**], one knows that the block structure of the classical Chevalley groups away from the natural characteristic may be described in a way strongly analogous to the Nakayama conjecture for the symmetric groups. James [**11**] has carried this analogy further for the unipotent representations of the general linear groups. Thus we may consider the problem of writing down a formula analogous to the Jantzen-Schaper formulas, for representations of general linear groups away from the natural characteristic. One should probably restrict one's attention to the unipotent representations, which are the ones that behave the most like Specht modules.

CONJECTURE 10. There is an analogue of the Jantzen-Schaper formulae for representations of $\mathrm{GL}_n(q)$ in characteristic $p\nmid q$.

In [**17**], Lusztig writes down a conjecture for the decomposition numbers of Weyl modules in terms of irreducible modules for semisimple algebraic groups in characteristic p. The coefficients are given in terms of the Kazhdan-Lusztig polynomials $P_{y,w}$, evaluated at 1. One should be able to write down a corresponding conjecture for the decomposition numbers of the symmetric groups,

			(14) 1	(13,1) 12	(12,2) 64	(11,3) 208	(10,4) 364	(9,5) 560	(8,6) 64	(11,2,1) 560	(10,3,1) 1300	(9,4,1) 3808	(8,5,1) 4576	(7,6,1) 768	(9,3,2) 3418	(8,4,2) 2510	(7,5,2) 2016	(7,4,3) 10880	(6,5,3) 4704	(8,3,2,1) 23296	(7,4,2,1) 19240	(6,5,2,1) 13312	(6,4,3,1) 11648	(5,4,3,2) 35840
1	(14)	(1^{14})	1																					
13	(131)	(21^{12})	1	1																				
77	(122)	$(2^2 1^{10})$	1	1	1																			
273	(113)	$(2^3 1^8)$	1		1	1																		
637	(104)	$(2^4 1^6)$	1		1	1	1																	
1001	(95)	$(2^5 1^4)$	1	1	1		1	1																
1001	(86)	$(2^6 1^2)$	1	1			1	1	1															
560	(1121)	(321^9)								1														
2002	(1031)	$(32^2 1^7)$	2		2	1	1				1													
4368	(941)	$(32^3 1^5)$								1		1												
6006	(851)	$(32^4 1^3)$	2	1	1		2	1	1				1											
4576	(761)	$(32^5 1)$										1		1										
5733	(932)	$(3^2 21^6)$	3	2	1		1	1			1				1									
12012	(842)	$(3^2 2^2 1^4)$	4	2	2		2	1	1				1		1	1								
14014	(752)	$(3^2 2^3 1^2)$	2	1	1		2	1	2				1		1	1	1							
16016	(743)	$(3^3 21^3)$								1		1		1				1						
15015	(653)	$(3^3 2^2 1)$	3	1	1		1	1	1		1				1	1	1		1					
23296	(8321)	(4321^5)																		1				
42042	(7421)	$(432^2 1^3)$	6	5	3	1	4	4	3				1		2	2	1				1			
36608	(6521)	$(432^3 1)$																		1		1		
50050	(6431)	$(43^2 21^2)$	8	4	4	2	2	2	2		1				1	2	1		1		1		1	
48048	(5432)	$(4^2 321)$								1				1				1						1

The decomposition matrix of $\mathfrak{S}_{14}$ for the prime 2.

and maybe even, following the above ideas, for the decomposition numbers for the classical Chevalley groups away from the natural characteristic. This would give us a good hold on all decomposition numbers for all simple groups modulo all primes. Of course, the sporadic groups would have to be dealt with separately, either on an ad hoc basis using a lot of computer work, or using some sort of geometric theory like that of Smith and Ronan.

5. Explanation of the tables. Our format for decomposition matrices of $\mathfrak{S}_n$ is the same as in James [**7**], except that since the rows corresponding to p-regular partitions determine the rest by a simple algorithm, I only provide these rows of the decomposition matrices. Complete decomposition matrices would take up too much space.

In the decomposition matrix for $\mathfrak{S}_{15}$ mod 2, I have not been able to eliminate one last ambiguity, and so the answers are given in terms of a parameter α, which takes on the value zero or one. It may well be that an extensive check of inner tensor products and symmetrizations will resolve this ambiguity.

			(15)	(14,1)	(13,2)	(12,3)	(11,4)	(10,5)	(9,6)	(8,7)	(1221)	(1131)	(1041)	(951)	(861)	(1032)	(942)	(852)	(762)	(843)	(753)	(654)	(9321)	(8421)	(7521)	(7431)	(6531)	(6432)	(54321)
			1	14	90	336	910	1288	1912	128	624	1300	4172	4576	832	7448	9346	32032	4096	13390	4704	21152	31276	24260	17344	$76828+2\alpha$	23296	59136	292864
1	(15)	(1^{15})	1																										
14	(141)	(21^{13})		1																									
90	(132)	$(2^2 1^{11})$			1																								
350	(123)	$(2^3 1^9)$		1		1																							
910	(114)	$(2^4 1^7)$					1																						
1638	(105)	$(2^5 1^5)$		1		1		1																					
2002	(96)	$(2^6 1^3)$			1				1																				
1430	(87)	$(2^7 1)$		1				1		1																			
715	(1221)	(321^{10})	1		1						1																		
2835	(1131)	$(32^2 1^8)$	1				1				1	1																	
7007	(1041)	$(32^3 1^6)$	1				1				1	1	1																
11375	(951)	$(32^4 1^4)$	1		1				1		1		1	1															
11583	(861)	$(32^5 1^2)$	1		1				1				1	1	1														
9100	(1032)	$(3^2 21^7)$		2		1		1								1													
22113	(942)	$(3^2 2^2 1^5)$	3		2				1		1	1	1	1			1												
32032	(852)	$(3^2 2^3 1^3)$																1											
25025	(762)	$(3^2 2^4 1)$	1		1				1				1	1	1		1		1										
35035	(843)	$(3^3 21^4)$	3		2				1		1		1	1	1		1			1									
45045	(753)	$(3^3 2^2 1^2)$	3		1				1		1	1	1	1	1		1		1	1	1								
30030	(654)	$(3^4 21)$		1				1		1						1						1							
42042	(9321)	(4321^6)		5		2		2								1							1						
91000	(8421)	$(432^2 1^4)$		4		2		2		1								1					1	1					
108108	(7521)	$(432^3 1^2)$		2		1		2		2								1					1	1	1				
135135	(7431)	$(43^2 21^3)$	$7-2\alpha$		3		2		2		1	1	1	1	1		2		1	1	1					1			
128700	(6531)	$(43^2 2^2 1)$		6		3		2		2						1						1	1	1	1		1		
175175	(6432)	$(4^2 321)$	$7-2\alpha$		3		3		1		1	1			1		1		1	1	1					1		1	
292864	* (54321)																												1
	Block number:		1	2	1	2	1	2	1	2	1	1	1	1	1	2	1	2	1	1	1	2	2	2	2	1	2	1	3

$\boxed{\alpha = 0 \text{ or } 1}$

The decomposition matrix of $\mathfrak{S}_{15}$ for the prime 2.

Γ_4

		(4) 1	(31) 2
2	<4>		1
4	<31>	2	1

Γ_5

		(5) 1	(41) 4	(32) 4
4	<5>			1
6	<41>	2		1
4	<32>		1	

Γ_6

		(6) 1	(51) 4	(42) 4	(321) 16
4	<6>			1	
16	<51>				1
20	<42>	4	2	2	
4	<321>		1		

Γ_7

		(7) 1	(61) 6	(52) 14	(43) 8	(421) 20
8	<7>				1	
20	<61>					1
36	<52>	2		1		1
20	<43>		2		1	
28	<421>			1		

Γ_8

		(8) 1	(71) 6	(62) 14	(53) 8	(521) 64	(431) 40
8	<8>				1		
48	<71>				1		1
112	<62>	4		2			2
112	<53>	4	4	2	2		1
64	<521>					1	
48	<431>		2	2	1		

Γ_9

		(9) 1	(81) 8	(72) 26	(63) 48	(54) 16	(621) 78	(531) 40	(432) 160
16	<9>					1			
56	<81>					1		1	
160	<72>								1
224	<63>		2		1				1
112	<54>	4		2		1		1	
240	<621>	4					2	2	
336	<531>	4		4		2	2	1	
96	<432>				2				

Γ_{10}

		(10) 1	(91) 8	(82) 26	(73) 48	(64) 16	(721) 160	(631) 198	(541) 128	(532) 200	(4321) 768
16	<10>					1					
128	<91>								1		
432	<82>					2				2	
768	<73>										1
672	<64>	8	4	4	2	2				2	
400	<721>	2						1		1	
800	<631>	4	2	4	1	2		2		1	
448	<541>						2		1		
432	<532>	2		4	2	2		1			
96	<4321>				2						

Γ_{11}

		(11) 1	(101) 10	(92) 44	(83) 100	(74) 164	(65) 32	(821) 186	(731) 198	(641) 144	(632) 848	(542) 416	(5321) 1168
32	<11>						1						
144	<101>									1			
560	<92>									1		1	
1200	<83>						1						1
1440	<74>		4		2		1						1
672	<65>	4		2		1						1	
1232	<821>	4							2			2	
3168	<731>		4		2		2				2		1
3168	<641>	8		4		2		4	4	2		2	
2464	<632>		4		6		4				2		
1760	<542>	4				2		4	2	2			
528	<5321>	2				2			1				

Reductions of spin modules modulo two.

Γ_{12}		1	10	44	100	164	32	320	570	1408	288	1046	416	1792	5632	2368
		(12)	(111)	(102)	(93)	(84)	(75)	(921)	(831)	(741)	(651)	(732)	(642)	(543)	(6321)	(5421)
32	<12>						1									
320	<111>						1				1					
1408	<102>										2		2			
3520	<93>						1				1		2			1
5280	<84>		8		4		2									2
4224	<75>	8	8	4	4	2	1						2			1
1792	<921>													1		
5600	<831>	4	4		2		2					2	2			1
7776	<741>	8	8	4	4	2	2		2		1	2	2			1
3840	<651>							2		1				1		
5632	<732>														1	
7392	<642>	12	4	4	6	4	4		4		2	2	2			
1760	<543>	4				2			2		1					
3520	<6321>	4	4		6	4	4					2				
2816	<5421>									2						

Γ_{13}		1	12	64	208	364	560	64	364	570	1572	288	2848	2510	1728	2208	8008	3200	8448
		(13)	(121)	(112)	(103)	(94)	(85)	(76)	(1021)	(931)	(841)	(751)	(832)	(742)	(652)	(643)	(7321)	(6421)	(5431)
64	<13>							1											
352	<121>							1				1							
1728	<112>														1				
4928	<103>														1			1	
8800	<94>							1				1							1
9504	<85>	8		4		2		1											1
4224	<76>		4		2		1											1	
4992	<1021>											2				2			
18304	<931>	4						2				1		2		2			1
32032	<841>	16		8		4		4		4		2		4					2
27456	<751>	20		8		4		2	4	4	2	2		2		2			1
22464	<832>		4														2	2	
41600	<742>		12		4		6						4		2		2	2	
22464	<652>	20		4		6		4	4	8	2	4		2		2			
18304	<643>		8				6						4		2				
9152	<7321>		2				2										1		
13728	<6421>	12		4		6		4		4	2	2		2					
4576	<5431>	4								2	2	1							

Reductions of spin modules modulo two (cont.)

Γ_{14}		(14) 1	(131) 12	(122) 64	(113) 208	(104) 364	(95) 560	(86) 64	(1121) 560	(1031) 1300	(941) 3808	(851) 4576	(761) 768	(932) 3418	(842) 2510	(752) 2016	(743) 10880	(653) 4704	(8321) 23296	(7421) 19240	(6521) 13312	(6431) 11648	(5432) 35840
64	<14>							1															
768	<131>												1										
4160	<122>							2								2							
13312	<113>																				1		
27456	<104>							2								2						2	
36608	<95>												1										1
27456	<86>	16	8	8	4	4	2	2														2	
6720	<1121>															1		1					
28224	<1031>	4						2							2	1		2				1	
59136	<941>												2				2						1
68992	<851>								4		2		2				2						1
31680	<761>	12	4	8	2	4	1	2		2		1			2			1				1	
40768	<932>	4	2					2							2			1		1		1	
96096	<842>	16	12	8	4	4	6	4						4	4	2				2		2	
91520	<752>	24	10	12	4	10	6	6		4		2		4	4	2		2		1		1	
59904	<743>																		2		1		
40768	<653>	12	8	4		6	6	4		2		1		4	2	2		1					
40768	<8321>		4				4													2			
87360	<7421>	16	12	8	4	12	10	8				2		4	4	2				2			
49918	<6521>								4		6		4				2						
54912	<6431>	16	8	8		12	6	8				4		4	4	2							
9152	<5432>											2											

Γ_{15}		(15) 1	(141) 14	(132) 90	(123) 336	(114) 910	(105) 1288	(96) 1912	(87) 128	(1221) 624	(1131) 1300	(1041) 4172	(951) 4576	(861) 832	(1032) 7448	(942) 9346	(852) 32032	(762) 4096	(843) 13390	(753) 4704	(654) 21152	(9321) 31276	(8421) 24260	(7521) 17344	(7431) $76828 + 2\alpha$	(6531) 23296	(6432) 59136	(54321) 292864
128	<15>								1																			
832	<141>													1														
4928	<132>													1				1										
17472	<123>								1															1				
40768	<114>								1															1		1		
64064	<105>													1				1									1	
64064	<96>	4		4		2		1						1													1	
27456	<87>		8		4		2		1																	1		
17600	<1221>																	2		2								
83200	<1131>								2												2			1		1		
202176	<1041>	8												4				2	4	4							2	
292864	<951>																											1
228800	<861>	16		8		4		2		8	4	4	2	4					4	2							2	
137984	<1032>		4						4												2		2			2		
392000	<942>	$16-4\alpha$		8		4		2						4		4		2	4	2					2		2	
256608	<852>	$32-4\alpha$		16		8		8		8	8	4	4	4		8		4	4	4					2		2	
246400	<762>		20		12		8		4						4		2				2		2			2		
312000	<843>		16		8		4		2													4	4	2		2		
384384	<753>		20		8		10		6						4		2				2	4	2	2		1		
81536	<654>	8		8				6			4		2			4		2		2								
81536	<9321>	$4-2\alpha$																		1					1			
224224	<8421>	$8-4\alpha$		8		4		6					2			4		2							2			
228800	<7521>	$16-2\alpha$		8		4		6		4	4	6	4	4		4		2	2	2					1			
202176	<7431>		8		4		6		4								2					2	2	1				
145598	<6531>	12		8				6		4	2	6	5	4		4		2	2	1								
64064	<6432>																2											
9152	<54321>												2															

Reductions of spin modules modulo two (cont.)

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