MODULES FOR FINITE GROUPS: REPRESENTATION RINGS, QUIVERS AND VARIETIES

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Contents

In these three lectures, I shall try to give an indication of some of the things happening at the moment in modular representation theory, by presenting three topics of current interest. The choice of topics is, of course, to a large extent an indication of my own research interests, and it should be pointed out that I am ignoring many very interesting areas. For the purpose of these talks, all groups considered will be finite, and all modules finitely generated.

Lecture 1 Uses of almost split sequences in group representation theory

Since this is a conference on the representations of algebras, it seems appropriate to begin with two applications of the almost split sequences of Auslander and Reiten, to the theory of group representations. The first of these is really something which works for all Artin algebras, but we can obtain further information in the case of a group algebra. The second application uses the finite generation of group cohomology to say that the possible shape of the Auslander-Reiten quiver for a group algebra is very restricted in comparison with what can happen for an arbitrary Artin algebra.

1.1 The first application: representation rings

Let kG be the group algebra of a (finite) group G over a field k of characteristic p. The representation ring a(G) is the free abelian group on the set of (f.g.) indecomposable kG-modules, with multiplication given by tensor product. Since we have a Krull-Schmidt theorem for kG-modules, this multiplication is well defined. We also set $A(G) = a(G) \otimes_{\mathbb{Z}} \mathbb{C}$. We introduce the bilinear form (,) given by extending $(U,V) = \dim_k \mathrm{Hom}_{kG}(U,V)$ bilinearly to a(G) and A(G).

Problem: Find elements $\tau_o(V) \in A(G)$ corresponding to the indecomposable modules V such that for U indecomposable,

$$(U, \tau_o(V)) = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{otherwise.} \end{cases}$$

Note that if such elements exist, they are necessarily unique. Note also that since A(G) is not necessarily finite dimensional, the elements $\tau_o(V)$ do not necessarily linearly span A(G).

Case 1 V = P is projective indecomposable.

In this case, if U is an indecomposable module with $U \ncong P$ then

$$(U,P) = (U, \mathrm{Rad}\, P).$$

Moreover,

$$(P,P) - (P, \mathrm{Rad}\, P) = \dim (\mathrm{End}\, P/J(\mathrm{End}\, P)).$$

<u>Definition</u>

For V indecomposable let

$d_V = \dim (\mathrm{End}\, V/J(\mathrm{End}\, V))$

(= 1 if k is algebraically closed).

Thus we may take

$$\tau_o(P) = \frac{1}{d_P}(P - \mathrm{Rad}\, P).$$

<u>Case 2</u> V non-projective indecomposable.

In this case, there is an almost split sequence

$0 \to \Omega^2 V \to X_V \to V \to 0$.

By the defining properties of almost split sequences, a homomorphism from an indecomposable U to V lifts to a homomorphism to X_V if and only if it is not an isomorphism. Hence

$$(U,X_V) = \begin{cases} (U,V) + (U,\Omega^2 V) & \text{if } U \not\cong V \\ (U,V) + (U,\Omega^2 V) - d_V & \text{if } U \cong V. \end{cases}$$

Thus we may take

$$\tau_o(V) = \frac{1}{d_V}(V + \Omega^2 V - X_V).$$

Combining the two cases we have

$$\tau_o(V) = \begin{cases} \frac{1}{d_V}(V - \mathrm{Rad}\, V) & \text{if } V \text{ projective} \\ \frac{1}{d_V}(V + \Omega^2 V - X_V) & \text{otherwise} \end{cases}$$

where $0 \to \Omega^2 V \to K_V \to V \to 0$ is almost split.

We may extend τ_o to an antilinear map on $A(G)$ by defining

$\tau_o(\Sigma\, a_i V_i) = \Sigma\, \bar{a}_i \tau_o(V_i)$.

Then for $x = \Sigma\, a_i V_i \in A(G)$, we have $(x,\tau_0(x)) = \Sigma\, |a_i|^2 \geqslant 0$ with equality if and only if $x = 0$. This can be thought of as a non-singularity statement.

<u>Theorem</u> If x is a non-zero element of $A(G)$, then there exists $y \in A(G)$ with $(x,y) \neq 0$.

<u>Corollary</u> Suppose U_1 and U_2 are kG-modules such that for all modules V, $\dim_k \mathrm{Hom}_{kG}(U_1,V) = \dim_k \mathrm{Hom}_{kG}(U_2,V)$. Then $U_1 \cong U_2$.

So far, we have not really used anything which is special to group algebras, and everything really works in greater generality. However, for group algebras we may put the above into a more symmetric form as follows. We introduce a new bilinear form by bilinearly extending to $A(G)$ the form given by

$$
\begin{aligned}
\langle U,V \rangle &= \text{dimension of space of homomorphisms from } U \text{ to } V \text{ which} \\
&\qquad \text{factor through a projective module} \\
&= \text{rank of } \sum_{g \in G} g \text{ on } U^* \otimes V \\
&= \text{number of copies of } P_1 \text{ in a direct sum decomposition} \\
&\qquad \text{of } U^* \otimes V.
\end{aligned}
$$

(Here, U^* is the vector space dual of U with the usual kG-module structure, and P_1 is the projective cover of the trivial one-dimensional kG-module 1.)

Since P_1 is self-dual, this form is symmetric, whereas $(\ ,\ )$ is not in general. The relationship between the two forms is given as follows. Let

$$u = P_1 - \mho(1)\ ,\quad v = P_1 - \Omega(1)$$

as elements of $A(G)$, where Ω is the Heller operator of taking the kernel of the projective cover, and $\mho$ is the dual operator of taking the cokernel of the injective hull.

<u>Lemma</u>

(i) $u = v^*$

(ii) $uv = 1$

(iii) $(V,W) = \langle v.V,W \rangle = \langle V,u.W \rangle$

(iv) $\langle V,W \rangle = (u.V,W) = (V,v.W)$.

Thus it is easy to pass back and forth between the two bilinear forms. In particular in order to obtain elements $\tau_1(V) \in A(G)$ such that for U,V indecomposable,

$$\langle U,\tau_1(V) \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{otherwise} \end{cases}$$

we may simply take

$$
\begin{aligned}
\tau_1(V) &= u.\tau_o(V) \\
&= \begin{cases} \mathrm{Soc}(V) \text{ if } V \text{ is projective} \\ X - \Omega(V) - \mho(V) \text{ otherwise} \end{cases}
\end{aligned}
$$

where $0 \to \Omega(V) \to X \to \mho(V) \to 0$ is an almost split sequence.

Note that not only do the elements $\tau_1(V)$ take on a more symmetric form than $\tau_o(V)$, but also the simple modules are among the list. The elements $\tau_1(V)$ are called the atoms, and in some sense they may be thought of as irreducible representations and irreducible glues. Every representation then has a formal expression as a (possibly infinite) sum of atoms, namely the composition factors and the glues holding it together. For further details see [7].

1.2 The second application: Webb's theorem

Suppose U and V are indecomposable kG-modules. A map $\lambda : U \to V$ is irreducible if λ is not an isomorphism, and wherever $\lambda = \mu\nu$ is a factorization of λ as a composite of two maps, either μ has a left inverse or ν has a right inverse.

Let Rad(U,V) be the space of non-isomorphisms from U to V, and $\mathrm{Rad}^2(U,V)$ be the space spanned by the homomorphisms of the form $\alpha\beta$ with $\alpha \in \mathrm{Rad}(U,W)$ and $\beta \in \mathrm{Rad}(W,V)$ for some indecomposable module W. Then the set of irreducible maps is precisely $\mathrm{Rad}(U,V)\backslash\mathrm{Rad}^2(U,V)$. The space $\mathrm{Irr}(U,V) = \mathrm{Rad}(U,V)/\mathrm{Rad}^2(U,V)$ is an $\mathrm{End}_{kG}(U)$ - $\mathrm{End}_{kG}(V)$ bimodule, and we write (a_{UV}, a'_{UV}) for its length as such. (Note that if k is algebraically closed then $a_{UV} = a'_{UV} = \dim_k \mathrm{Irr}(U,V)$).

The Auslander-Reiten quiver of kG-modules is the directed graph whose vertices are the indecomposable kG-modules, and with a labelled edge $U \bullet\!\xrightarrow{(a_{UV},a'_{UV})}\!\bullet V$ if $\mathrm{Irr}(U,V) \neq 0$. We write $\bullet\!\longrightarrow\!\bullet$ for $\bullet\!\xrightarrow{(1,1)}\!\bullet$. This graph is locally finite, and if U is non-injective (note that injective and projective are the same for modules over group algebras) a_{UV} is the number of copies of V as a direct summand of the middle term of the almost split sequence starting with U, while if V is non-projective then a'_{UV} is the number of copies of U as a direct summand of the middle term of the almost split sequence terminating with V. Thus the graph may be viewed as all the almost split sequences spliced together.

The stable quiver is obtained from the Auslander-Reiten quiver by removing all projective modules together with all edges attached to them. (Note that for a more general Artin algebra we must remove all preprojective and preinjective modules). To a connected component Q of the stable quiver, we may associate a tree, called its tree class, as follows. Choose a vertex x of Q, and form the tree of paths starting at x, and having no three consecutive nodes of the form $\Omega^2 y \to z \to y$

(recall that in our case Ω^2 is the Auslander-Reiten translation). Then the isomorphism type of this (undirected) tree is independent of the chosen vertex x. Moreover, if we consider a directed labelled edge $y \bullet\!\xrightarrow{(a_{UV},a'_{UV})}\!\bullet\, z$ to be equivalent to $z \bullet\!\xrightarrow{(a'_{UV},a_{UV})}\!\bullet\, y$, we obtain a well-defined undirected <u>labelled tree</u> B, with edges $y \bullet\!\frac{(a_{yz},a_{zy})}{}\!\bullet\, z$. Riedtmann [20] has shown that one may then consider Q as a quotient of a "universal quiver" $\mathbb{Z}B$ by an "admissible" group of automorphisms (admissible means that no vertex is an image of an adjacent vertex).

The tree class of a connected component of the stable quiver for an arbitrary Artin algebra is fairly unrestricted in shape. However, as we shall see, only a very restricted set of labelled trees occur in the case of a group algebra, and the fundamental reason for this seems to be the finite generation of cohomology.

If V is a kG-module, we form the Poincaré series

$$\eta_V(t) = \sum_{n=0}^{\infty} t^n \dim(P_n),$$

where

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0$$

is a minimal projective resolution of V. It follows from the finite generation of cohomology that $\eta_V(t)$ is a rational function of t of the form $p(t)/\prod_{j=1}^{s}\left(1 - t^{k_j}\right)$, where $k_1,\dots,k_s$ are the degrees of homogeneous generators of $H^*(G,k)$ (and are hence independent of V), and p(t) is a polynomial with integer coefficients.

<u>Proposition</u>

Let f(t) be a rational function of the form $p(t)/\prod_{j=1}^{s}\left(1 - t^{k_j}\right) = \sum_{i=0}^{\infty} a_i t^i$, where p(t) is a polynomial with integer coefficients and the a_i are non-negative integers. Let c be the order of the pole of f(t) at t = 1. Then

(i) There is a positive number λ such that $a_n \leqslant \lambda n^{c-1}$ for all large enough n, but there is no positive number μ such that $a_n \leqslant \mu n^{c-2}$ for all large enough n.

(ii) $\lim_{t\to 1} (\prod_i k_i).f(t).(1 - t)^c$ is a positive integer.

If c is the order of the pole of $\eta_V(t)$ at t = 1, c is called the <u>complexity</u> of V, written $cx_G(V)$. According to part (i) of the above proposition, it measures the rate of growth of the dimensions of the minimal projective resolution. According to (ii), the number

$$\eta(V) = \lim_{t\to 1} (\prod_i k_i).\eta_V(t).(1-t)^{cx_G(V)}$$

is a positive integer.

Proposition

(i) $\eta(V) = \eta(\Omega V)$

(ii) If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of modules of the same complexity, then $\eta(V) \leqslant \eta(V') + \eta(V'')$. In particular if $0 \to \Omega^2 V \to X_V \to V \to 0$ is almost split then $\eta(X_V) \leqslant 2\eta(V)$.

It follows that η defines a function on the tree associated with any connected component of the stable quiver, and that η is subadditive in the sense that

$$2\eta(z) \geqslant \sum_{\substack{y \text{ adjacent} \\ \text{to } z}} a_{yz}\eta(y)$$

It is the existence of such a subadditive function which restricts the possible shape of the tree.

Theorem. (Vinberg, Happel, Preiser, Ringel, ...)

Let T be a connected labelled tree, and η a subadditive function on T. Then T is among the following three sets.

(i) The finite Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.

(ii) The infinite Dynkin diagrams A_∞, B_∞, C_∞, D_∞, A_∞^∞ .

(iii) The Euclidean diagrams $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\widetilde{BC}_n$, $\widetilde{BD}_n$, $\widetilde{CD}_n$, $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_{41}$, $\tilde{F}_{42}$, $\tilde{G}_{21}$, $\tilde{G}_{22}$.

(For pictures of these diagrams see [13])

Putting everything together, we obtain the following theorem.

Theorem (Webb [17])

The tree class of a connected component of the stable quiver of kG-modules is among the list given in the above theorem.

Webb then goes on to investigate the various possibilities in more detail, and finds for example that the only finite Dynkin diagram which occurs is the A_n diagram. This corresponds to a block of kG with cyclic defect.

A remarkable consequence of Webb's theorem is the following.

Corollary

If P is a (non-simple) projective indecomposable kG-module, then the maximal possible number of direct summands of Rad(P)/Soc(P) is four.

Proof

It follows from Webb's theorem that the number of non-projective direct summands of the middle term of an almost split sequence is four. Apply this result to the almost split sequence

$$0 \to \mathrm{Rad}(P) \to P \oplus \mathrm{Rad}(P)/\mathrm{Soc}(P) \to P/\mathrm{Soc}(P) \to 0.$$

Remark

Almost split sequences also exist for lattices over orders, and in particular over a p-adic group ring RG. The main difference from the modular theory is that the Auslander-Reiten translation is Ω rather than Ω^2. For the first application, it is no longer true that the bilinear form given by $\mathrm{rank}_R \mathrm{Hom}_{RG}(U,V)$ is non-singular. For example if U and U' are R-forms for the same KG-module (where K is the field of fractions of R) then $U - U'$ is in the radical of the form. The second application, however, goes through almost verbatim, see [17].

Lecture 2 Structure in representation rings

In the last lecture, we saw that the bilinear forms (,) and < , > were non-singular on A(G). In this lecture, we shall investigate further the structure of A(G), with the idea of trying to understand how much of the characteristic zero (ordinary character) theory we can mimic in characteristic p. Let us first of all summarize the main features of ordinary character theory.

It is customary when dealing with representations in characteristic zero, to work in terms of the character table. A typical entry in this table gives the trace of a group element on a representation. Why do we use the trace function? This is because the maps $V \mapsto \mathrm{tr}(g,V)$ are precisely the algebra homomorphisms from the representation ring to $\mathbb{C}$, and these homomorphisms separate representations. In particular in this case the representation ring is semisimple. This has the effect that we can compute with representations easily and effectively in terms of their characters: representations are distinguished by their characters, direct sum corresponds to addition and tensor product corresponds to multiplication.

In characteristic p, we may view the non-singularity of (,) and < , >, explained in the last lecture, as a first step in a program to mimic these concepts. How much more can we achieve?

The first problem is that Maschke's theorem no longer holds; a representation may be indecomposable without being irreducible. Thus the Grothendieck ring $A(G)/A_0(G,1)$ (where $A_0(G,1)$ is the "ideal of short exact sequences") is not the same as $A(G)$. Brauer discovered the remarkable fact that the Grothendieck ring is always semisimple, and found the set of algebra homomorphisms from this to $\mathbb{C}$, in terms of lifting eigenvalues. Thus he gets a square character table, giving information about composition factors of modules, but saying nothing about how they are glued together.

In an attempt to generalize this, we define a species of $A(G)$ to be an algebra homomorphism $A(G) \to \mathbb{C}$. If s is a species and $x \in A(G)$, we write (s,x) for the value of s on x. Even if we use the set of all species, we cannot distinguish between modules V_1 and V_2 when $V_1 - V_2$ is nilpotent as an element of $A(G)$.

In this lecture, I shall talk about nilpotent elements of $A(G)$, about various subgroups associated with species, and about the power maps on $A(G)$. Finally I shall describe how to put this information together to obtain a kind of character theory for finite summands of $A(G)$.

2.1 Nilpotent elements of A(G)

Definition

Denote by $A(G;p)$ the linear open in $A(G)$ of those modules with the property that for any extension of the ground field, every direct summand has dimension divisible by p.

Theorem (Benson - Carlson [5])

(i) $A(G;p)$ is an ideal in $A(G)$

(ii) $A(G)/A(G;p)$ has no nilpotent elements.

Let $i_{H,G} : A(H) \to A(G)$ denote the induction map, and $r_{G,H} : A(G) \to A(H)$ the restriction map.

Corollary

$A(G)/\bigcap_{H \leqslant G} r^{-1}_{G,H}(A(H;p))$ has no nilpotent elements.

If the Sylow p-subgroups of G are cyclic, it happens that $\bigcap_{H\leqslant G} r^{-1}_{G,H}(A(H;p)) = 0$, and so it follows that in this case A(G) has no nilpotent elements. More generally, if A(G,Cyc) denotes the ideal in A(G) spanned by cyclic vertex modules then A(G,Cyc) has no nilpotent elements.

For p odd, as soon as the Sylow p-subgroups are non-cyclic there are nilpotent elements in A(G), and for p = 2 the situation is more complicated. For further details, see Zemanek [18, 19] and Benson, Carlson [5].

2.2 Vertices and origins of species

The Brauer species (i.e., the species of A(G) which vanish on $A_0(G,1)$) may be evaluated by first restricting down to a cyclic subgroup of order coprime to p, and then lifting eigenvalues. The corresponding concept for a general species is the origin, namely the minimal subgroup through which the species factors.

Proposition

Let s be a species of A(G). The following conditions on a subgroup H are equivalent.

(i) Ker (s) $\geqslant$ Ker $(r_{G,H})$

(ii) Ker (s) $\not\geqslant$ Im $(i_{H,G})$

(iii) There is a species t of A(H) such that for all $x \in A(G)$, $(s,x) = (t, r_{G,H}(x))$.

Note that in (iii) the species t need not be unique. We write $t \sim s$ and say t fuses to s if (iii) is satisfied.

We say s factors through H if the equivalent conditions of the proposition are satisfied. An origin of s is a subgroup minimal among those through which s factors. A vertex of s is a subgroup D minimal with respect to the condition that there exists an indecomposable module with vertex D on which s does not vanish.

A subgroup H of G is p-hypoelementary if $H/O_p(H)$ is cyclic.

Structure Theorem (Benson-Parker [7])

Let s be a species of A(G). Then

(i) All origins of s are conjugate.

(ii) All the vertices of s are conjugate.

Let H be an origin of s. Then

(iii) H is p-hypoelementary.

(iv) $O_p(H)$ is a vertex of s.

There is also a formula for the value of a species on an induced representation.

The Induction Formula [7]

Let H be a subgroup of G and V a kH-module. Then

$$(s, i_{H,G}(V)) = \sum_{s_o \sim s} |N_G(\mathrm{Orig}(s_o)) \cap \mathrm{Stab}_G(s_o) : N_H(\mathrm{Orig}(s_o))| . (s_o, V)$$

In this expression, s_o runs over the species of H fusing to s, $\mathrm{Orig}(s_o)$ is any origin of s_o, and $\mathrm{Stab}_G(s_o)$ is the subgroup of $N_G(H)$ stabilizing the species s_o.

2.3 Power maps on A(G)

In ordinary character theory, one of the ways in which the structure of the group is reflected in the character table is via the so-called power maps, or Adams operations. Namely there are ring homomorphisms ψ^n on the character ring with the property that the character value of g on $\psi^n(V)$ is the character value of g^n on V. These are usually given in terms of the exterior power operations Λ^n, and these operations make the character ring into a special lambda-ring. It turns out that for modular representations we must first construct the ring homomorphisms $\psi^n : a(G) \to a(G)$, and then use them to construct operations λ^n, which do not agree with the exterior power operations unless $n < p$ (although they do at the level of Brauer characters), and the λ^n make $a(G) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ into a special lambda-ring. It then makes sense to use the ψ^n to define the powers of a species.

We begin by constructing the operations ψ^n in the case where n is coprime to p. Let n be a natural number coprime to p, and let $T = \langle \alpha : \alpha^n = 1 \rangle$ be a cyclic group of order n. Let ε be a primitive n^{th} root of unity in an algebraic closure of k, and let η be a primitive n^{th} root of unity in $\mathbb{C}$. If V is a kG-module, then $T \times G$ acts on $\otimes^n(V)$ by letting T permute the tensor multiplicands. Denote by $[\otimes^n(V)]_{\varepsilon^i}$ the eigenspace of α on $\otimes^n(V)$ with eigenvalue ε^i. Then $[\otimes^n(V)]_{\varepsilon^i}$ is a kG-module, and we may define

$$\psi^n(V) = \sum_{i=1}^{n} \eta^i [\otimes^n(V)]_{\varepsilon^i} \in A(G).$$

Since these operations ψ^n commute with the Frobeius map F^a, we may define for a general value of $n = n_o p^a$, $(n_o,p) = 1$,

$$\psi^n(V) = \psi^{n_o} F^a(V).$$

Properties of ψ^n

(i) $\psi^n(V \oplus W) = \psi^n(V) + \psi^n(W)$

(ii) $\psi^n(V \otimes W) = \psi^n(V) \,.\, \psi^n(W)$

Thus ψ^n may be extended linearly to give a ring homomorphism $A(G) \to A(G)$.

(iii) $\psi^n(a(G)) \subseteq a(G)$

(iv) (this is the hardest part to prove) $\psi^m\psi^n = \psi^{mn}$

(v) If b_g is a Brauer species corresponding to a p'-element g, then

$$(b_g, \psi^n x) = (b_{g^n}, x).$$

(vi) If we define

$$\lambda^n(x) = \frac{1}{n!} \begin{vmatrix} \psi^1(x) & 1 & & & \bigcirc \\ \psi^2(x) & \psi^1(x) & 2 & & \\ \vdots & & & \ddots & \\ \vdots & & & & n-1 \\ \psi^n(x) & & \cdots & & \psi^1(x) \end{vmatrix}$$

then $\lambda^n(a(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p]) \subseteq a(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p]$, and these operations λ^n make $a(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p]$ into a special lambda-ring, for which the ψ^n are the psi-operations.

Definitions

If s is a species of A(G), we define its n^{th} power via

$$(s^n, x) = (s, \psi^n(x)).$$

Then s^n is again a species.

If H is a p-hypoelementary group and $n = p^a n_o$ with $(n_o,p) = 1$, we let $H^{[n]}$ be the unique subgroup of index $(|H|, n_o)$ in H.

Theorem

(i) If H is an origin of a species s, then $H^{[n]}$ is an origin of s^n.

(ii) If D is a vertex of s, then D is also a vertex of s^n.

2.4 Finite summands of $A(G)$

We now project all the information we have onto a finite dimensional summand of $A(G)$ satisfying certain natural hypotheses, and find that we obtain a type of character theory analogous to Brauer's.

Hypothesis 1

$A(G) = A \oplus B$ is an ideal direct sum decomposition, with projections $\pi_1 : A(G) \to A$ and $\pi_2 : A(G) \to B$. The summand A satisfies the following four conditions.

(i) A is finite dimensional

(ii) A is semisimple

(iii) A is freely spanned as a vector space by indecomposable modules

(iv) A is closed under taking dual modules.

Remarks

(i) Any finite dimensional semisimple ideal I is a direct summand, since

$$A(G) = I \oplus \bigcap_s \mathrm{Ker}(s),$$

where s runs over the set of species of $A(G)$ not vanishing on I.

(ii) If A satisfies (i), (ii) and (iii) of hypothesis 1, then the span in $A(G)$ of A and the duals of modules in A satisfy (i), (ii), (iii) and (iv).

(iii) If A_1 and A_2 are summands satisfying hypothesis 1, then so are $A_1 + A_2$ and $A_1 \cap A_2$.

Examples

(i) Letting $A(G,1)$ denote the linear span of the projective modules, and $A_0(G,1)$ be the linear span of elements of the form $X' - X - X''$ where $0 \to X \to X' \to X'' \to 0$ is a short exact sequence, we have

$$A(G) = A(G,1) \oplus A_0(G,1),$$

and this decomposition satisfies hypothesis 1. This case is called the Brauer case.

(ii) Letting $A(G,\mathrm{Cyc})$ denote the linear span of the cyclic vertex modules, and $A_0(G,\mathrm{Cyc})$ be the linear span of elements of the form $X' - X - X''$ where $0 \to X \to X' \to X'' \to 0$ is a short exact sequence which splits on restriction to every cyclic subgroup, we have

$$A(G) = A(G,\mathrm{Cyc}) \oplus A_0(G,\mathrm{Cyc}).$$

By the results mentioned in 2.1, this satisfies hypothesis 1. We call this the cyclic vertex case.

(iii) Let G be the Klein four group and k an algebraically closed field of characteristic two. Then $A(G)$ has infinitely many summands satisfying hypothesis 1. The quotient of $A(G)$ by the sum of all these is isomorphic to $\mathbb{C}[X,X^{-1}]$, where X and X^{-1} are the images of u and v (see the first lecture).

Lemma

Suppose $A(G) = A \oplus B$ as in hypothesis 1. Then

(i) $\langle\ ,\ \rangle$ and $(\ ,\ )$ are non-singular on A.

(ii) $A(G,1) \subseteq A$.

Thus the Brauer case is the unique minimal case of our theory.

Definitions

Let $s_1,\ldots,s_n$ be the species of A, and $V_1,\ldots,V_n$ be the indecomposable modules freely spanning A. Let $G_i = \tau_1(V_i)$ (see the first lecture).

The atom table of A is the matrix

$$T_{ij} = (s_j,G_i) = (s_j,\pi_1(G_i))$$

The representation table of A is the matrix

$$U_{ij} = (s_j,V_i).$$

The entries in these tables are algebraic integers. I don't know whether they are always cyclotomic integers; nor do I know whether $(s_j,V_i^*) = \overline{(s_j,V_i)}$ in general.

In the example of the Brauer case, T_{ij} is the table of Brauer characters of irreducible modules, and U_{ij} is the table of Brauer characters of projective indecomposable modules. In general, T_{ij} also has some rows "of degree zero":

	Brauer species	Non-Brauer species
irreducible modules	*	*
irreducible glues	0	*

T_{ij}

	Brauer species	Non-Brauer species
projective indecomposable modules	*	0
non-projective indecomposable modules	*	*

U_{ij}

One can also find uniquely defined numbers $c_j = c_G(s_j)$ (which play the rôle of centralizer orders although they need not in general be positive or rational) such that for $x \in A(G)$, $y \in A$,

$$< x,y > = \sum_j \frac{(s_j,x^*)(s_j,y)}{c_j}$$

(compare the usual formula for inner product of characters).

Also, letting $p_j = (s_j,u)$ we have

$$(x,y) = \sum_j \frac{p_j(s_j,x^*)(s_j,y)}{c_j} \quad .$$

If s is a species of A which factors through H, and t is a species of $A' \supseteq r_{G,H}(A)$ which fuses to s, then

$$c_G(s) = |N_G(\mathrm{Orig}(t)) \cap \mathrm{Stab}_G(t) : N_H(\mathrm{Orig}(t))| \cdot c_H(t),$$

and we can rewrite the induction formula in the form

$$(s,i_{H,G}(V)) = \sum_{s_o \sim s} \frac{c_G(s)}{c_H(s_o)} \; (s_o,V)$$

where s_o runs over those species of A' fusing to s.

Just as in ordinary character theory, when we give tables T_{ij} and U_{ij} as above, we usually also mark in information about the "centralizer orders" c_j. If the summand is closed under the operations ψ^n described in 2.3, we may also mark on information about powers of species.

<u>Example</u>

$G = S_3$, $A = A(G)$, $k = \bar{\mathbb{F}}_3$

	s_1	s_2	s_3	s_4	s_5	s_6
$c_j =$	6	2	12	-4	-4	-4
G_1	1	1	1	1	1	1
G_2	1	-1	1	1	-1	1
G_3	0	0	3	-1	-1	-1
G_4	0	0	3	-1	1	1
G_5	0	0	-3	-1	i	-i
G_6	0	0	-3	-1	-i	i

T_{ij}

	s_1	s_2	s_3	s_4	s_5	s_6
V_1	3	1	0	0	0	0
V_2	3	-1	0	0	0	0
V_3	1	1	1	1	1	1
V_4	1	-1	1	1	-1	-1
V_5	2	0	-1	1	-i	i
V_6	2	0	-1	1	i	-i

U_{ij}

$s_1^2 = s_2^2 = s_1 \qquad s_3^2 = s_4^2 = s_5^2 = s_6^2 = s_3$

For p odd, $s_i^p = s_i$

$\mathrm{Orig}(s_1) = 1 \qquad \mathrm{Orig}\,(s_2) = C_2$

$\mathrm{Orig}(s_3) = \mathrm{Orig}(s_4) = C_3 \qquad \mathrm{Orig}(s_5) = \mathrm{Orig}(s_6) = S_3.$

See the appendix of [4] for further examples.

Lecture 3 Varieties for modules and a problem of Steenrod

Having spent some time looking at the structure of A(G), we are led to a desire to understand better the behaviour of tensor products and direct sums of modules (or you could say that this was the motivation for studying A(G) in the first place). One of the most interesting recent ideas in this area is Carlson's idea of associating varieties to modules. To each module V we associate a homogeneous subvariety $X_G(V)$ of Spec $H^{ev}(G,k)$, the spectrum of the even cohomology ring of G. These varieties have the properties that $X_G(V \oplus W) = X_G(V) \cup X_G(W)$, $X_G(V \otimes W) = X_G(V) \cap X_G(W)$, and if V is indecomposable then the projective variety $\overline{X}_G(V)$ corresponding to $X_G(V)$ is connected. It also turns out that the dimension of $X_G(V)$ is equal to the complexity of V, as defined in the first lecture. Thus at the level of representation rings, if X is a homogeneous subset of Spec $H^{ev}(G,k)$, then the linear span A(G,X) of the modules V with $X_G(V) \subseteq X$ is an ideal in A(G).

Finally, we give an application of these varieties to obtain information about a problem of Steenrod in algebraic topology.

3.1 Definition and properties of the varieties

This subject started off with some work of Quillen [14, 15] describing the structure of the set of prime ideals of the even equivariant cohomology ring of a compact Lie group $H_G^{ev}(X)$ with coefficients in a permutation representation X. His main results, when interpreted for finite groups, give a description of Spec $H^{ev}(G, \mathbb{Z}/p\,\mathbb{Z})$ in terms of the elementary abelian p-subgroups and their normalizers (the Quillen stratification theorem). In particular he was able to prove the following theorem.

Theorem

An element x of $H^{ev}(G, \mathbb{Z}/p\,\mathbb{Z})$ is nilpotent if and only if $\mathrm{res}_{G,E}(x)$ is nilpotent for all elementary abelian p-subgroups E of G. At the level of varieties, this means that

$$\mathrm{Spec}\ H^{ev}(G, \mathbb{Z}/p\,\mathbb{Z}) = \bigcup_{\substack{E \leqslant G \\ \text{elementary} \\ \text{abelian}}} \mathrm{res}^{*}_{G,E}\ \mathrm{Spec}\ H^{ev}(E, \mathbb{Z}/p\,\mathbb{Z}).$$

Corollary

The (Krull) dimension of $\mathrm{Spec}\ H^{ev}(G, \mathbb{Z}/p\,\mathbb{Z})$ is equal to the maximum rank of an elementary abelian p-subgroup of G.

The next step was taken by Chouinard [11], who showed that an arbitrary module in characteristic p is projective if and only if its restriction to every elementary abelian p-subgroup is projective.

The connection between the above two results was unclear, until Alperin and Evens [1] found the appropriate common generalization. They formulated the concept of complexity which I defined in the first lecture, and showed that the complexity of an arbitrary module in characteristic p is equal to the maximal complexity of its restrictions to the elementary abelian p-subgroups. Chouinard's result is the case of complexity zero, while the corollary above of Quillen's results is the case of the trivial one-dimensional module (after a bit of reinterpretation).

The final (?) stage of generalization was Carlson's notion of varieties for modules, which we now describe.

Definitions

Suppose k is algebraically closed. Denote by X_G the affine variety $\mathrm{Spec}\ H^{\bullet}(G,k)$, where $H^{\bullet}(G,k) = H^{*}(G,k)$ in case $p = 2$, and $H^{ev}(G,k)$, the even cohomology ring, in case $p \neq 2$ (to ensure commutativity). Then X_G is a union of lines through the origin, so we may form a projective variety $\bar{X}_G = \mathrm{Proj}\ H^{\bullet}(G,k)$ of one smaller dimension.

Denote by $\mathrm{Ann}_G(V)$ the ideal of $H^{\bullet}(G,k)$ consisting of those elements annihilating $H^{*}(G,V)$ (note that $H^{*}(G,V)$ is an $H^{\bullet}(G,k)$-module via cup-product). The <u>support</u> of a module V, written $X_G(V)$, is the subvariety of X_G consisting of those prime ideals which contain $\mathrm{Ann}_G(V \otimes S)$ for some module S (it turns out to be sufficient to restrict our attention to the cases where S is simple, and it also turns out to be sufficient to take $S = V^*$). Denote by $I_G(V)$ the ideal of $H^\bullet(G,k)$ consisting of those elements x such that for all modules S, there exists a positive integer j with $H^*(G, V \otimes S).x^j = 0$ (again the same remarks apply to choices of S). Then $X_G(V) = \mathrm{Spec}(H^\bullet(G,k)/I_G(V)) \subseteq X_G$, and $\bar{X}_G(V) = \mathrm{Proj}(H^\bullet(G,k)/I_G(V))$ is a projective (closed) subvariety of $\bar{X}_G$.

If H is a subgroup of G, denote by $t_{H,G}$ the map from X_H to X_G induced by $\mathrm{res}_{G,H} : H^\bullet(G,k) \to H^\bullet(H,k)$. The following theorem summarizes some of the main properties of these cohomology varieties.

<u>Theorem</u> (Properties of $X_G(V)$)

Let $H \leqslant G$, and V be a kG-module and W a kH-module.

(i) $\dim(X_G(V)) = cx_G(V)$

(ii) $X_G(V) = X_G(V^*) = X_G(V \otimes V^*) = X_G(\Omega V)$

(iii) $X_H(V{\downarrow_H}) = t_{H,G}^{-1}(X_G(V))$

(iv) $X_G(W{\uparrow^G}) = t_{H,G}(X_H(W))$

(here, we have used $\downarrow_H$ and $\uparrow^G$ to denote restriction and induction)

(v) If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of kG-modules then

$X_G(V_i) \subseteq X_G(V_j) \cup X_G(V_k)$, $\{i,j,k\} = \{1,2,3\}$

(vi) $X_G(V \oplus V') = X_G(V) \cup X_G(V')$

(vii) (Avrunin, Scott [2]) $X_G(V \otimes V') = X_G(V) \cap X_G(V')$

(viii) $X_G(V) = \{0\}$ if and only if V is projective

(ix) $X_G(V) = \bigcup_E t_{E,G}(X_E(V{\downarrow_E}))$ as E ranges over the elementary abelian p-subgroups of G.

(x) If V and V' are indecomposable modules in the same connected component of the stable quiver (see the first lecture), then $X_G(V) = X_G(V')$.

(xi) Given a closed homogeneous subvariety $X \subseteq X_G$ there is a module V with $X_G(V) = X$.

(xii) Given a closed homogeneous rational subvariety (<u>rational</u> means stable under the Frobenius morphism) $X \subseteq X_G$ there is a $\mathbb{Z}$-free $\mathbb{Z}G$-module U with $X_G(U \otimes_{\mathbb{Z}} k) = X$.

(xiii) If $X_G(V) \cap X_G(V') = \{0\}$ then $\mathrm{Ext}^i_{kG}(V,V') = 0$ for all $i > 0$.

(xiv) (Carlson, [9]) If $X_G(V) \subseteq X_1 \cup X_2$, where X_1 and X_2 are closed homogeneous subvarieties of X_G with $X_1 \cap X_2 = \{0\}$, then we may write $V = V_1 \oplus V_2$ with $X_G(V_1) \subseteq X_1$ and $X_G(V_2) \subseteq X_2$. In particular if V is indecomposable then $\bar{X}_G(V)$ is topologically connected.

3.2 Elementary abelian subgroups and rank varieties

Avrunin and Scott [2] have shown that $X_G(V)$ has a stratification similar to Quillen's stratification of X_G, in terms of the restrictions of V to the elementary abelian subgroups. If we define

$$X^+_E(V) = X_E(V)\setminus \bigcup_{E'<E} t_{E',E}(X_{E'}(V))$$

$$X^+_{G,E}(V) = t_{E,G}(X^+_E(V))$$

then we have the following theorem.

Theorem (Quillen stratification for modules)

$X_G(V)$ is a disjoint union of the locally closed subvarieties $X^+_{G,E}(V)$ as E runs over a set of representatives of conjugacy classes of elementary abelian p-subgroups of G. The group $W_G(E) = N_G(E)/C_G(E)$ acts freely on $X^+_E(V)$, and $t_{E,G}$ induces a finite homeomorphism

$$X^+_E(V)/W_G(E) \to X^+_{G,E}(V)$$

(i.e., homeomorphism in the Zariski topology; Quillen calls this map an 'inseparable isogeny').

The natural map

$$\varinjlim_{E} X_E(V) \to X_G(V)$$

is a bijective finite morphism.

This means that in order to be able to calculate effectively the varieties $X_G(V)$, we may restrict our attention to the case where G = E is elementary abelian. In this case, $X_E(V)$ turns out to be naturally isomorphic to another variety $Y_E(V)$ defined as follows. Let $J = J(kE)$ be the Jacobson radical of the group ring (of codimension 1 in kE). If $x \in J$ then $1 + x$ is an invertible element of kE of order p.

Proposition (Carlson [8])

Suppose V is a kG-module. If $x, y \in J$ and $x - y \in J^2$ then $V{\downarrow}_{k\langle 1+x\rangle}$ is free if and only if $V{\downarrow}_{k\langle 1+y\rangle}$ is free.

For $V \neq 0$, we now define $Y_E(V)$ to be the subset of $Y_E = J/J^2$ consisting of zero together with the image in Y_E of the set of $x \in J$ such that $V{\downarrow}_{k\langle 1+x\rangle}$ is not free. For $V = 0$, we define $Y_E(V) = \emptyset$. Since $x \in Y_E(V)$ if and only if the rank of the matrix representing x is less than $\dim(V)/p$, $Y_E(V)$ is defined by polynomial equations (namely the vanishing of certain minors), and is hence a subvariety of Y_E.

The following theorem was conjectured by Carlson [8] and proved by Avrunin and Scott [2].

Theorem

There is a natural isomorphism $Y_E \cong X_E$ which has the property that for every module V, the image of $Y_E(V)$ is $X_E(V)$.

The following corollary was first proved by Dade [12] using completely different techniques.

Corollary

Let V be a kE-module. Then V is free if and only if for every $x \in J\backslash J^2$, $V{\downarrow}_{k<1+x>}$ is free.

The following example demonstrates how effective a computational device the above theorem gives.

Carlson's Favourite Example

Let E be an elementary abelian group of order 8, and k an algebraically closed field of characteristic two. Consider the following family of representations $V_{a,b,c}$.

$$E = < x_1, x_2, x_3 : x_i^2 = [x_i, x_j] = 1 >$$

$$x_1 \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix} \quad x_2 \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ a & & 1 & \\ & b & & 1 \end{pmatrix} \quad x_3 \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ & c & 1 & \\ 1 & & & 1 \end{pmatrix}$$

The elements $x_1 - 1$, $x_2 - 1$, $x_3 - 1$ form a basis for J/J^2, and we see that $V_{a,b,c}{\downarrow}_{k<1+\lambda_1(x_1-1)+\lambda_2(x_2-1)+\lambda_3(x_3-1)>}$ is free if and only if the 2×2 minor $\begin{vmatrix} \lambda_1 + \lambda_2 a & \lambda_3 c \\ \lambda_3 & \lambda_1 + \lambda_2 b \end{vmatrix} \neq 0$.

Thus $Y_E(V_{a,b,c})$ (and hence $X_E(V_{a,b,c})$) is the variety in affine 3-space defined by the (homogeneous) equation

$$(X_1 + aX_2)(X_1 + bX_2) = cX_3^2.$$

3.3 Application to a problem of Steenrod

Let G be a finite group, R a commutative ring, and V an RG-module. An RG-Moore space of type (V,n) is a topological space X with an action of G defined on it (i.e., a G-space) with

$$\tilde{H}_i(X;R) = \begin{cases} V & i = n \\ 0 & \text{otherwise} \end{cases}$$

as RG-modules. Here, $\tilde{H}_i$ denotes reduced singular homology. We say V is realizable if an RG-Moore space of type (V,n) exists for some n.

Steenrod's problem: Which modules are realizable?

Remark By the universal coefficient theorem, if V is a $\mathbb{Z}$-free realizable $\mathbb{Z}G$-module, then $V \otimes_{\mathbb{Z}} R$ is a realizable RG-module for all R.

For a brief historical introduction to this problem, see Vogel [16]. In this paper he also shows that every $\mathbb{Z}G$-module is realizable provided $|G|$ is square-free. The first example of a non-realizable representation was provided by G. Carlsson [10] for an elementary abelian group of order p^2. To do this, he showed that the annihilator in $H^*(G,\mathbb{F}_p)$ of $H^*(G,V^*)$ is invariant under the Steenrod algebra for any realizable $\mathbb{F}_pG$-module V. He then produced a $\mathbb{Z}$-free $\mathbb{Z}G$-module such that the reduction modulo p did not have this property. This technique may be extended to show the following. Let T denote the total Steenrod operation $\sum_{i\geq 0} P^i$ (or $\sum_{i\geq 0} Sq^i$ if p = 2). Since T acts as an algebra endomorphism on $H^{\bullet}(G,\mathbb{F}_p)$, we have an induced map on X_G.

Theorem (Benson, Habegger [6])

Suppose V is a realizable $\mathbb{F}_p G$-module. Then for every direct summand W of V, $X_G(W)$ is a T-invariant subset of X_G.

Such subsets are in fact not very thick on the ground.

Proposition

If X is a T-invariant subset of X_G then X is of the form $\bigcup_E t_{E,G}(X_E)$ for some collection of elementary abelian subgroups E of G.

Note that by property (xii) of cohomology varieties (section 3.1), every closed homogeneous rational subvariety of X_G is of the form $X_G(U \otimes_{\mathbb{Z}} k)$ for some $\mathbb{Z}$-free $\mathbb{Z}G$-module U, and so we have a plentiful supply of non-realizable $\mathbb{Z}$-free $\mathbb{Z}G$-modules as soon as G has a non-cyclic elementary abelian subgroup.

For the elementary abelian group of order four, it turns out that the above theorem gives a necessary and sufficient condition for $\mathbb{F}_2G$-modules to be realizable. This is not true in general, as there exist non-realizable modules for $\mathbb{F}_2Q_8$, where Q_8 is the quaternion group of order 8.

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