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The Loewy structure of the projective indecomposable modules for A_9 in characteristic 2

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Introduction

The purpose of this paper is to establish the Loewy series for the projective indecomposable modules for A_9 over a splitting field of characteristic 2.

Since this paper depends very heavily on the results of [6], we shall number our sections as though this were a continuation of [6]. To avoid repetition we shall refer to results of [6], simply by their section number. We take (S,R,F) to be a splitting 2-modular system for A_9 and all its subgroups, and A_8 is regarded as a subgroup of A_9 stabilizing a point, A_7 a subgroup of A_8 stabilizing a further point, and so on. S_n denotes the subgroup of A_{n+2} stabilizing an unordered pair of points, and containing $A_n (n \le 7)$.

The simple FA_9-modules are denoted I, 8_1, 8_2, 8_3, 20_1, 20_2, 26, 48, 78 and 160. 8_1 and 8_2 are related by an outer automorphism of A_9, but are not dual or algebraically conjugate. 20_1 and 20_2 are related by an outer automorphism of A_9, and are dual, but are not algebraically conjugate. The ordinary and 2-modular character tables, decomposition matrix and Cartan matrix for A_9 have been extracted from James [2] and presented in Appendix 5.

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FA_9 has two blocks, whose idempotents we shall denote by f_0 (principal block containing the simple modules I, 8_1, 8_2, 20_1, 20_2, 26 and 78) and f_1 (non-principal block containing the simple modules 8_3, 48 and 160). Throughout the paper, i will denote 1 or 2, and $\{i,j\} = \{1,2\}$.

The main result of this paper is the following theorem.

Theorem 2. The Loewy structures of the projective indecomposable modules for FA_9 are as follows.

(i) Principal Block

$$\begin{array}{c}
I \\
20_1\ 20_2\ 26\ 26\ 78 \\
I\ I\ I\ I\ I\ I\ I\ 8_1\ 8_1\ 8_2\ 8_2 \\
20_1\ 20_1\ 20_1\ 20_2\ 20_2\ 20_2\ 26\ 26\ 26\ 26\ 26\ 26\ 78\ 78 \\
I\ I\ I\ I\ I\ I\ I\ I\ I\ I\ I\ I\ 8_1\ 8_1\ 8_1\ 8_1\ 8_2\ 8_2\ 8_2\ 8_2 \\
20_1\ 20_1\ 20_1\ 20_2\ 20_2\ 20_2\ 26\ 26\ 26\ 26\ 26\ 26\ 78\ 78 \\
I\ I\ I\ I\ I\ I\ I\ 8_1\ 8_1\ 8_2\ 8_2 \\
20_1\ 20_2\ 26\ 26\ 78 \\
I\ I \\
20_1\ 20_2\ 78 \\
I\ I \\
20_1\ 20_2\ 78 \\
I
\end{array}$$

$$\begin{array}{c}
8_i \\
20_i\ 26 \\
I\ I\ 8_j\ 8_j \\
20_j\ 20_j\ 26\ 26\ 78 \\
I\ I\ I\ I\ 8_i\ 8_i\ 8_i \\
20_i\ 20_i\ 26\ 26\ 78 \\
I\ I\ 8_j\ 8_j \\
20_j\ 26 \\
8_i
\end{array}$$

$$\begin{array}{c}
20_i \\
I\ 8_j \\
20_j\ 26\ 78 \\
I\ I\ I\ 8_i\ 8_i \\
20_i\ 20_i\ 20_j\ 26\ 26 \\
I\ I\ I\ 8_j\ 8_j \\
20_j\ 26\ 78 \\
I\ 8_i \\
20_i\ 20_j \\
I \\
78 \\
I \\
20_i
\end{array}$$

$$\begin{array}{c}
26 \\
I \quad I \quad 8_1 \quad 8_2 \\
20_1 \quad 20_2 \quad 26 \quad 26 \quad 26 \quad 78 \\
I \quad I \quad I \quad I \quad I \quad I \quad 8_1 \quad 8_1 \quad 8_2 \quad 8_2 \\
20_1 \quad 20_1 \quad 20_2 \quad 20_2 \quad 26 \quad 26 \quad 26 \quad 26 \quad 78 \quad 78 \\
I \quad I \quad I \quad I \quad I \quad I \quad 8_1 \quad 8_1 \quad 8_2 \quad 8_2 \\
20_1 \quad 20_2 \quad 26 \quad 26 \quad 26 \quad 78 \\
I \quad I \quad 8_1 \quad 8_2 \\
26
\end{array}$$

$$\begin{array}{c}
78 \\
I \\
20_1 \quad 20_2 \quad 26 \\
I \quad I \quad 8_1 \quad 8_2 \\
26 \quad 26 \quad 78 \\
I \quad I \quad 8_1 \quad 8_2 \\
20_1 \quad 20_2 \quad 26 \\
I \\
78 \\
I \\
20_1 \quad 20_2 \\
I \\
78
\end{array}$$

(ii) Non-principal Block

$$\begin{array}{c}
8_3 \\
48 \quad 160 \\
8_3 \quad 8_3 \\
48 \quad 160 \\
8_3
\end{array}
\qquad\qquad
\begin{array}{c}
48 \\
8_3 \quad 48 \\
160 \\
8_3 \\
48
\end{array}
\qquad\qquad
\begin{array}{c}
160 \\
8_3 \\
48 \\
8_3 \\
160
\end{array}$$

(see also section 6.1)

The notation and tools for this paper are the same as in [6], with the addition of the following.

We shall write $(M,N)^n_A$ for $\dim \operatorname{Ext}^n_A(M,N)$.

Lemma 6. (Ext Reciprocity). Let $H \le G$, M an FH-module and N an FG-module. Then

$$(M, N\downarrow_{FH})^n_{FH} = (M\uparrow^{FG}, N)^n_{FG} \quad .$$

Proof. This follows from Lemma 2 by dimension shifting and induction on n. //

Lemma 7. Let M_1, M_2 and M_3 be FG-modules. Then

$$(M_1, M_2 \otimes M_3)_{FG} = (M_1 \otimes M_2^*, M_3)_{FG}$$

Section 5.

Restriction and induction between A_8 and A_9; calculation of $\dim \mathrm{Ext}^1_{A_9}$ for simple modules

5.1. Restriction and induction of simple modules.

Using Brauer characters, we see that

5.1.1. $I_{A_9}\downarrow_{A_8} = I$

5.1.2. $(20_i)_{A_9}\downarrow_{A_8} = 20_i$

By block theory,

5.1.3. $78_{A_9}\downarrow_{A_8} = 14 \oplus 64$

5.1.4. $14_{A_8}\uparrow^{A_9} = 48 \oplus 78$

5.1.5. $(20_i)_{A_8}\uparrow^{A_9} = 20_i \oplus 160$

5.1.6. $I_{A_8}\uparrow^{A_9} = I \oplus 8_3$

Now $(4_1)_{A_8}\uparrow^{A_9}$ has composition factors $8_1 + 8_2 + 20_2$, and

$$(20_2,(4_1)_{A_8}\uparrow^{A_9})_{A_9} = (20_2\downarrow_{A_8},4_1)_{A_8} = 0$$

$$((4_1)_{A_8}\uparrow^{A_9},20_2)_{A_9} = (4_1,20_2\downarrow_{A_8})_{A_8} = 0$$

Thus $(4_1)_{A_8}\uparrow^{A_9}$ is uniserial, with 20_2 in the middle. Since we haven't yet chosen which is which of the conjugacy classes 9A and 9B for A_9, we may choose whichever we like of 8_1 and 8_2 to be the bottom composition factor of $(4_1)_{A_8}\uparrow^{A_9}$. We choose that

5.1.7 $$(4_1)_{A_8}\uparrow^{A_9} = \begin{matrix} 8_2 \\ 20_2 \\ 8_1 \end{matrix}$$

Dualizing this, we get

5.1.8 $$(4_2)_{A_8}\uparrow^{A_9} = \begin{matrix} 8_1 \\ 20_1 \\ 8_2 \end{matrix}$$

Thus by Frobenius reciprocity

5.1.9 $$(8_i)_{A_9}{\downarrow}_{A_8} = \begin{matrix} 4_i \\ 4_j \end{matrix}$$

By 5.1.6 and 2.1.1 we have

5.1.10 $$(8_3)_{A_9}{\downarrow}_{A_8} = I_{A_8}{\uparrow}^{A_9}{\downarrow}_{A_8} - I = I_{A_7}{\uparrow}^{A_8} = \begin{matrix} I \\ 6 \\ I \end{matrix}$$

By Lemma 7, $(I, 8_3^{2-})_{A_9} \le (I, 8_3 \otimes 8_3)_{A_9} = 1$. Since the composition factors of 8_3^{2-} are $I + I + 26$, and the module is self-dual, we have

5.1.11 $$8_3^{2-} = \begin{matrix} I \\ 26 \\ I \end{matrix} \quad .$$

Thus by 5.1.10 and 2.3 we have

5.1.12 $$26_{A_9}{\downarrow}_{A_8} = \begin{matrix} 6 \\ 14 \\ 6 \end{matrix}$$

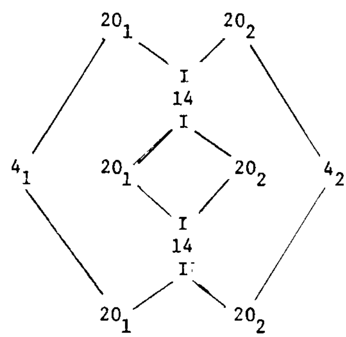
By 5.1.4, 5.1.3, Lemma 5 and 3.3.2,

$$\begin{aligned} 48_{A_9}{\downarrow}_{A_8} &= 14_{A_8}{\uparrow}^{A_9}{\downarrow}_{A_8} - 78{\downarrow}_{A_8} \\ &= 14 + 14_{A_7}{\uparrow}^{A_8} - 14 - 64 \qquad (5.1.13) \\ &= M'_{56} \\ &= \begin{matrix} 14 \\ 6 \\ 4_1 \;\; 4_2 \\ 6 \\ 14 \end{matrix} \end{aligned}$$

Similarly, by 5.1.5 and 4.6,

$$160_{A_9}{\downarrow}_{A_8} = (20_1)_{A_8}{\uparrow}^{A_9}{\downarrow}_{A_8} - 20_1 = 20_{A_7}{\uparrow}^{A_8}$$

=



(5.1.14)

Now $6_{A_8}\uparrow^{A_9}$ has composition factors $I + I + 26 + 26$. Thus by 5.1.1, Frobenius reciprocity and the fact that $(I,I)^1_{A_9} = 0$ (since A_9 is simple), we have

5.1.15 $$6_{A_8}\uparrow^{A_9} = \begin{matrix} & 26 & \\ I & & I \\ & 26 & \end{matrix}$$

Also, by Brauer characters we have

5.1.16 $$64_{A_8}\uparrow^{A_9} = P_{78} .$$

5.2 Calculation of $\dim \mathrm{Ext}^1_{A_9}$

In this section we work out $(M,N)^1_{A_9}$ for M and N simple. This information is displayed in Appendix 6.

First we attack the non-principal block.

5.2.1 Lemma. For M a simple FA_9-module

$$(8_3,M)^1_{A_9} = \begin{cases} 1 & \text{if } M = 48 \text{ or } 160 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For M a simple $FA_9.f_1$-module, by 5.1.6 and Lemma 6 we have

$$(8_3,M)^1_{A_9} = (I_{A_8}\uparrow^{A_9},M)^1_{A_9} = (I,M\downarrow_{A_8})^1_{A_8} .$$

By 5.1.10,

$$(I,8_3\downarrow_{A_8})^1_{A_8} = (I,I_{A_7}\uparrow^{A_8})^1_{A_8} = (I,I)^1_{A_7} = 0.$$

By 5.1.13 and 2.5,

$$(I,48\downarrow_{A_8})^1_{A_8} = (I,M'_{56})^1_{A_8} = (I,M_{56})^1_{A_8} - (I,M_8)^1_{A_8}$$

$$= (I,I)^1_{(A_5\times 3)2} - (I,I)^1_{A_7} = 1.$$

By 5.1.14 and 1.2,

$$(I,160\downarrow_{A_8})^1_{A_8} = (I,20_{A_7}\uparrow^{A_8})^1_{A_8} = (I,20)^1_{A_7} = 1. \qquad //$$

5.2.2 <u>Lemma</u>. For M a simple FA_9-module,

$$(48,M)^1_{A_9} = \begin{cases} 1 & \text{if } M = 8_3 \text{ or } 48 \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u>. For M a simple $FA_9.f_1$-module, by 5.1.4 we have

$$(48,M)^1_{A_9} = (14_{A_8}\uparrow^{A_9},M)^1_{A_9} = (14,M\downarrow_{A_8})^1_{A_8}.$$

By 5.1.13 and 2.5,

$$\begin{aligned}(14,48\downarrow_{A_8})^1_{A_8} &= (14,M'_{56})^1_{A_8} = (14,M_{56})^1_{A_8} - (14,M_8)^1_{A_8} \\ &= (14\downarrow_{(A_5\times 3)2},I)^1_{(A_5\times 3)2} - (14\downarrow_{A_7},I)_{A_7} \\ &= (14\downarrow_{S_5},I)^1_{S_5} - 1 \\ &= (I_{S_5}\uparrow^{A_7}\downarrow_{S_5} - I_{A_6}\uparrow^{A_7}\downarrow_{S_5},I)^1_{S_5} - 1 \\ &= (I,I)^1_{S_5} + (I,I)^1_{A_4} + (I,I)^1_{S_3\times 2} - (I,I)^1_{A_5} - (I,I)^1_{S_4} - 1 \\ &= 1 + 0 + 2 - 0 - 1 - 1 = 1.\end{aligned}$$

By 5.1.14 and 1.2,

$$(14,160\downarrow_{A_8})^1_{A_8} = (14,20_{A_7}\uparrow^{A_8})^1_{A_8} = (14,20)^1_{A_7} = 0. \qquad //$$

5.2.3 <u>Lemma</u>. For M a simple FA_9-module,

$$(160,M)^1_{A_9} = \begin{cases} 1 & \text{if } M = 8_3 \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u>. The only case left to consider here is $(160,160)^1_{A_9}$. By 5.1.4, 5.1.5 and 1.2,

$$(160,160)^1_{A_9} = ((20_1)_{A_8}\uparrow^{A_9},160)^1_{A_9} = (20_1,160\downarrow_{A_8})^1_{A_8}$$

$$= (20_1,20_{A_7}\uparrow^{A_8})^1_{A_8} = (20,20)^1_{A_7} = 0. \qquad //$$

Now we turn to the principal block.

5.2.4 Lemma. For M a simple FA_9-module,

$$(I,M)^1_{A_9} = \begin{cases} 2 & \text{if } M = 26 \\ 1 & \text{if } M = 20_1 \text{ or } 78 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By 5.1.6, if M is a simple $FA_9.f_0$-module,

$$(I,M)^1_{A_9} = (I_{A_8}\uparrow^{A_9},M)^1_{A_9} = (I,M\downarrow_{A_8})^1_{A_8}.$$

By 5.1.2 and Appendix 4,

$$(I,20_1\downarrow_{A_8})^1_{A_8} = (I,20_1)^1_{A_8} = 1.$$

By 5.1.12 and Appendix 4,

$$(I,26\downarrow_{A_8})^1_{A_8} = \left(I,\begin{matrix}6\\14\\6\end{matrix}\right)^1_{A_8} = 2$$

since by 3.2, there is an A_8-module

$$\begin{matrix} I & & 6 \\ & \diagdown & | \\ & & 14 \\ I & \diagdown & | \\ & & 6 \end{matrix}$$

but by Theorem 1 there is no A_8-module

$$\begin{matrix} I \\ 6 \\ 14 \\ 6 \end{matrix} .$$

By 5.1.9 and Appendix 4,

$$(I,8_1\downarrow_{A_8})^1_{A_8} = \left(I,\begin{matrix}4_1\\4_1\end{matrix}\right)^1_{A_8} = 0.$$

By 5.1.3 and Appendix 4,

$$(I,78{\downarrow}_{A_8})^1_{A_8} = (I,14 \oplus 64)^1_{A_8} = 1. \qquad //$$

5.2.5 __Lemma.__ For M a simple FA_9-module

$$(78,M)^1_{A_9} = \begin{cases} 1 & \text{if } M = I \\ 0 & \text{otherwise} \end{cases}$$

__Proof.__ By 5.1.4, for M a simple $FA_9.f_0$-module,

$$(78,M)^1_{A_9} = (14_{A_8}{\uparrow}^{A_9},M)^1_{A_9} = (14,M{\downarrow}_{A_8})^1_{A_8} .$$

By 5.1.2, 5.1.3, 5.1.9, 3.2, 1.2 and Appendix 4, we have

$$(14,26{\downarrow}_{A_8})^1_{A_8} = (14,\begin{smallmatrix}6\\14\\6\end{smallmatrix})^1_{A_8} \le (14,6_{A_7}{\uparrow}^{A_8})^1_{A_8} = (14,6)^1_{A_7} = 0.$$

$$(14,78{\downarrow}_{A_8})^1_{A_8} = (14,14 \oplus 64)^1_{A_8} = 0.$$

$$(14,8_i{\downarrow}_{A_8})^1_{A_8} = (14,\begin{smallmatrix}4_i\\4_j\end{smallmatrix})^1_{A_8} = 0. \qquad //$$

5.2.6 __Lemma.__ For M a simple FA_9-module,

$$(20_i,M)^1_{A_9} = \begin{cases} 1 & \text{if } M = I,\ 8_j \\ 0 & \text{otherwise} \end{cases}$$

__Proof.__ By 5.1.5, for M a simple $FA_9.f_0$-module,

$$(20_i,M)^1_{A_9} = ((20_i)_{A_8}{\uparrow}^{A_9},M)^1_{A_9} = (20_i,M{\downarrow}_{A_8})^1_{A_8}.$$

By 5.1.5, 5.1.9, 5.1.12 and Theorem 1, we have

$$(20_i,20_i{\downarrow}_{A_8})^1_{A_8} = (20_i,20_i)^1_{A_8} = 0$$

$$(20_i,8_i{\downarrow}_{A_8})^1_{A_8} = (20_i,\begin{smallmatrix}4_i\\4_j\end{smallmatrix})^1_{A_8} = 0$$

$$(20_i,8_j{\downarrow}_{A_8})^1_{A_8} = (20_i,\begin{smallmatrix}4_j\\4_i\end{smallmatrix})^1_{A_8} = 1$$

$$(20_i,26{\downarrow}_{A_8})^1_{A_8} = (20_i,\begin{smallmatrix}6\\14\\6\end{smallmatrix})^1_{A_8} = 0 \qquad //$$

5.2.7 Lemma. For M a simple FA_9-module,

$$(26,M)^1_{A_9} = \begin{cases} 2 & \text{if } M = I \\ 1 & \text{if } M = 8_i \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$$(26,26)^1_{A_9} \le (6_{A_8}\uparrow^{A_9},26)^1_{A_9} = (6,\overset{6}{\underset{6}{14}})^1_{A_8} = 0 \quad \text{by Theorem 1}$$

$$(26,8_i)^1_{A_9} \le (26,(4_i)_{A_8}\uparrow^{A_9})^1_{A_9} \quad \text{by 5.1.7 and 5.1.8}$$

$$= (\overset{6}{\underset{6}{14}},4_i)^1_{A_8} \le (6_{A_7}\uparrow^{A_8},4_i)^1_{A_8} \quad \text{by 3.2}$$

$$= (6,4_i)^1_{A_7} = 1 \quad \text{by 1.2.}$$

But $(26,8_i)^1_{A_9} \geq 1$ by applying Lemma 3 and 5.2.4 to the 35 dimensional ordinary character of A_9. //

5.2.8 Lemma. For M a simple FA_9-module

$$(8_i,M) = \begin{cases} 1 & \text{if } M = 20_i \text{ or } 26 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By 5.2.6 and duality,

$$(8_i,20_i)^1_{A_9} = (20_j,8_i)^1_{A_9} = 1$$

$$(8_i,20_j)^1_{A_9} = (20_i,8_i)^1_{A_9} = 0$$

By 5.1.7, 5.1.8 and Appendix 4,

$$(8_i,8_j)^1_{A_9} \le (8_i,(4_j)_{A_8}\uparrow^{A_9})^1_{A_9} = (\overset{4_i}{4_j},4_j)^1_{A_8} = 0$$

$$(8_i,8_i)^1_{A_9} \le (8_i,(4_i)_{A_8}\uparrow^{A_9})^1_{A_9} = (\overset{4_i}{4_j},4_i)^1_{A_8} = 0$$

since there is only one copy of 4_i in $L_3(P_{4_i})$, and there is a module

$$\begin{array}{ccc} & 4_i & \\ 4_j & & 20_i \\ & 4_i & \end{array}$$

for A_8 by 3.1. //

This completes the determination of $\dim \mathrm{Ext}^1_{A_9}(M,N)$ for M and N simple modules.

5.2.9 Comment

It can be seen from Appendix 6 that we may divide the simple $FA_9.f_0$-modules into two sets $S = \{I, 8_1, 8_2\}$ and $T = \{20_1, 20_2, 26, 78\}$ in such a way that elements of each set only extend elements of the other set. Thus in a projective indecomposable module, any particular Loewy layer will be a direct sum of modules from just one of these sets.

5.3 Induction of two-step FA_8-modules

In this section we shall induce up to A_9 each of the non-trivial extensions of a simple module by a simple module for A_8.

By Frobenius reciprocity, we have

5.3.1 $\binom{I}{14}_{A_8}\uparrow^{A_9} = \begin{smallmatrix} I \\ 78 \end{smallmatrix} \oplus \begin{smallmatrix} 8_3 \\ 48 \end{smallmatrix}$

5.3.2 $\binom{I}{20_i}_{A_8}\uparrow^{A_9} = \begin{smallmatrix} I \\ 20_i \end{smallmatrix} \oplus \begin{smallmatrix} 8_3 \\ 160 \end{smallmatrix}$

5.3.3 $\binom{I}{6}_{A_8}\uparrow^{A_9} = \begin{array}{ccc} & I & \\ & 26 & \\ I & & I \\ & 26 & \end{array} \oplus 8_3$

Now $(26, \binom{6}{4_i}_{A_8}\uparrow^{A_9})_{A_9} = \left(14, \begin{smallmatrix} 6 \\ 4_i \\ 6 \end{smallmatrix}\right)_{A_8}$ ...

Similarly $L_1(\binom{6}{4_i}_{A_8}\uparrow^{A_9}) = 26$. However, the proof of 5.2.7 shows that the only non-trivial extension of 26 by $(4_i)_{A_8}\uparrow^{A_9}$ is

$$\begin{array}{ccccccc} & & & & & & 8_j \\ & & & & & / & \\ 26 & & & & 20_j & & \\ & \diagdown & & / & & & \\ & & 8_i & & & & \end{array} \quad .$$

Thus $\binom{6}{4_i}_{A_8}\uparrow^{A_9}$ has socle and Loewy series

$$5.3.4 \qquad \begin{array}{ccc} & 26 & \\ I & I & 8_j \\ 26 & 20_j & \\ & 8_i & \end{array}$$

Since A_8 contains a vertex of any A_9-module, 8_i is a direct summand of $8_i\downarrow_{A_8}\uparrow^{A_9} = \binom{4_i}{4_j}_{A_8}\uparrow^{A_9}$. By Frobenius reciprocity, we have $S_1((\binom{4_i}{4_j}_{A_8}\uparrow^{A_9}) = L_1((\binom{4_i}{4_j}_{A_8}\uparrow^{A_9}) = 8_i \oplus 8_j$. Hence

$$5.3.5 \qquad \binom{4_i}{4_j}_{A_8}\uparrow^{A_9} = \begin{array}{c} 8_j \\ 20_j \\ 8_i \\ 20_i \\ 8_j \end{array} \oplus 8_i \quad .$$

Again by Frobenius reciprocity, we have $S_1((\binom{4_i}{20_i}_{A_8}\uparrow^{A_9})_{A_9} = 20_i \oplus 160$, and so

$$5.3.6 \qquad \binom{4_i}{20_i}_{A_8}\uparrow^{A_9} = \begin{array}{c} 8_j \\ 20_j \\ 8_i \\ 20_i \end{array} \oplus 160.$$

Finally, we have $S_1((\binom{6}{14}_{A_8}\uparrow^{A_9}) = 26 \oplus 48 \oplus 78$ and $L_1((\binom{6}{14}_{A_8}\uparrow^{A_9}) = 26 \oplus 48$. Thus

$$5.3.7 \qquad \binom{6}{14}_{A_8}\uparrow^{A_9} = \begin{array}{ccccc} & & 26 & & \\ & / & & \diagdown & \\ I & & & & I \\ & \diagdown & & / & & \diagdown \\ & & 26 & & & & 78 \end{array} \oplus 48$$

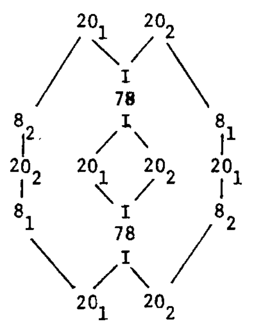
5.4 Induced modules from A_7 to A_9

The results of sections 3, 4.6 and 5.3 give us the following.

5.4.1 $$I_{A_7}\uparrow^{A_9} = \begin{array}{c} I \\ 26 \\ I\ \ I \\ 26 \\ I \end{array} \oplus 8_3 \oplus 8_3$$

5.4.2 $$(14_{A_7}\uparrow^{A_8}.e_0)\uparrow^{A_9}.f_0 = \left(\begin{array}{ccc} & 14 & \\ & 6 & \\ 4_1 & & 4_2 \\ & 6 & \\ & 14 & \end{array}\right) \qquad A_8\uparrow^{A_9}.f_0 = \begin{array}{cccc} & 26 & 78 & \\ I & I & 8_1 & 8_2 \\ 20_1 & 20_2 & 26 & 26 \\ I & I & 8_1 & 8_2 \\ & 26 & 78 & \end{array}$$

5.4.3 $(20_{A_7}\uparrow^{A_8}.e_0)\uparrow^{A_9}.f_0 =$



Now $(4_1)_{A_7}\uparrow^{A_9} = 8_{S_7}\uparrow^{A_9}$, and so $(4_1)_{A_7}\uparrow^{A_9} \cong (4_2)_{A_7}\uparrow^{A_9}$. By Frobenius reciprocity, the socle and head are $8_1 \oplus 8_2$. Thus by 5.3.5, 5.3.6 and Appendix 6, there is only one possibility.

5.4.4. $$(4_1)_{A_7}\uparrow^{A_9}.f_0 = \begin{array}{c} 8_1 \\ 20_1 \\ 8_2 \\ 20_2 \\ 8_1 \end{array} \oplus \begin{array}{c} 8_2 \\ 20_2 \\ 8_1 \\ 20_1 \\ 8_2 \end{array}$$

We shall postpone discussion of $6_{A_7}\uparrow^{A_9}.f_0$ until Section 6.4.

Section 6.

Structure of the projective modules for FA_9

6.1 The non-principal block

By Lemmas 5.2.1, 5.2.2 and 5.2.3, and the Cartan matrix given in

Appendix 5, we see that the only possible structures for the projective indecomposable modules in the non-principal block are:

$$\begin{array}{ccc} & 8_3 & \\ 48 & & 160 \\ 8_3 & \oplus & 8_3 \\ 160 & & 48 \\ & 8_3 & \end{array}$$

P_{8_3}

$$\begin{array}{ccc} & 48 & \\ & & 8_3 \\ 48 & \oplus & 160 \\ & & 8_3 \\ & 48 & \end{array}$$

P_{48}

$$\begin{array}{c} 160 \\ 8_3 \\ 48 \\ 8_3 \\ 160 \end{array}$$

P_{160}

This should be compared with the principal block of A_7, displayed in 1.2.

6.2 The structure of $P_{(20_i)A_9}$

It may be deduced from the Loewy structure given in Theorem 1 and from the diagram in Section 4.6 that $P_{(20_i)A_8}$ has a diagram

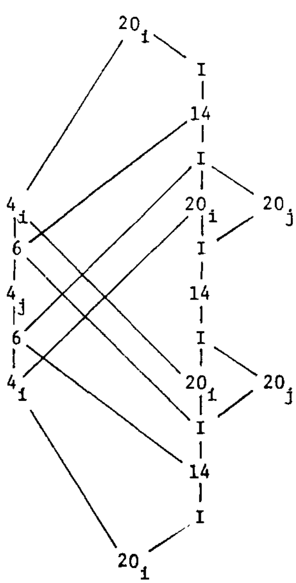


FIGURE 6

This, together with the results of section 5, immediately gives the Loewy series shown in Theorem 2 for $P_{(20_1)_{A_9}} = P_{(20_1)_{A_8}} \uparrow^{A_9} .f_0$.

Hence the appropriate diagram for our filtration of $P_{(20_1)_{A_9}} \oplus P_{(20_2)_{A_9}}$ induced from Figure 3 is as follows.

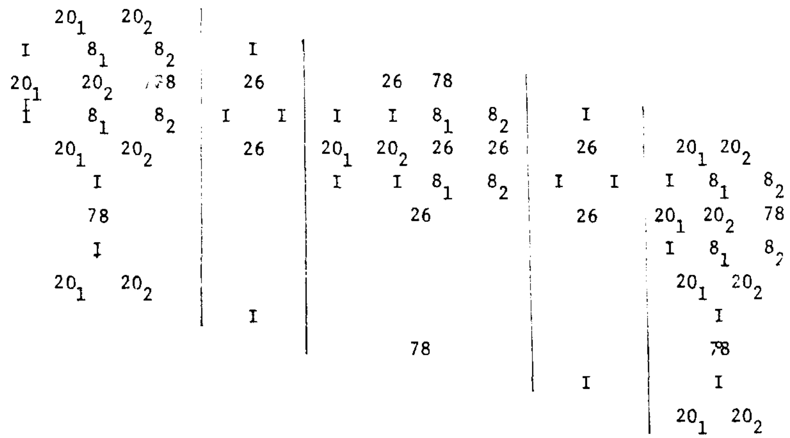


FIGURE 7

6.3 The Structure of $P_{I_{A_9}}$

Since $P_{I_{A_9}} = P_{I_{A_8}} \uparrow^{A_9} .f_0$, we may deduce from Figures 3, 4 and 7 that we have the Loewy series given in Theorem 2, and that the appropriate diagram for the induced filtration is

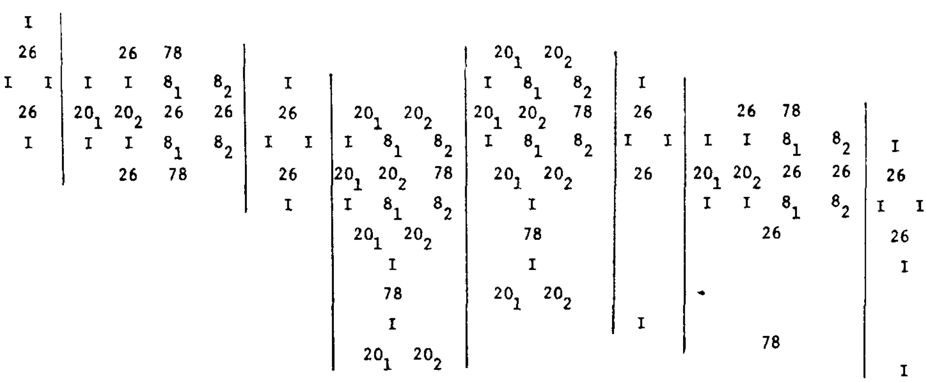


FIGURE 8

6.4 The structures of $P_{26_{A_9}}$, $P_{(8_1)_{A_9}}$ and $6_{A_7}\uparrow^{A_9}$

By Brauer characters, we have

$$P_{6_{A_8}}\uparrow^{A_9}.f_0 = P_{26_{A_9}} \oplus P_{26_{A_9}}$$

$$P_{(4_1)_{A_8}}\uparrow^{A_9}.f_0 = P_{(4_2)_{A_8}}\uparrow^{A_9}.f_0 = P_{(8_1)_{A_9}} \oplus P_{(8_2)_{A_9}} .$$

Landrock's lemma and Figure 8 tell us that $P_{26} \oplus P_{26}$ has 4 copies of I in each of L_2 and L_8 and 12 copies in each of L_4 and L_6. Examining Figure 2 and Section 5.3, this makes it clear that $6_{A_7}\uparrow^{A_9}$ has 4 copies of I in each of L_2 and L_4.* Thus $6_{A_7}\uparrow^{A_9}.f_0$ has socle and Loewy series

6.4.1

	26	26	
I	I	I	I
26	26	78	78
I	I	I	I
	26	26	

Similarly Figure 7 tells us that $P_{26} \oplus P_{26}$ has 2 copies of each 20_i in each of L_3 and L_7, and 4 copies of each 20_i in L_5. This and 5.4.4 now force the Loewy series for $P_{26} \oplus P_{26}$ to be as displayed in Figure 9, and so the Loewy series for P_{26} is as given in Theorem 2.

*This needs further clarification. If the Loewy length of $6_{A_7}\uparrow^{A_9}.f_0$ were bigger than 5, there would be copies of I in its L_6 and hence in $L_{10}(P_{26} \oplus P_{26})$ (using 5.3 and 5.4.4). Thus the Loewy length is 5, and the Loewy series is as shown. In fact it is a direct sum of two isomorphic modules with socle and Loewy series

	26	
I		I
26		78
I		I
	26	

(see figure 9).

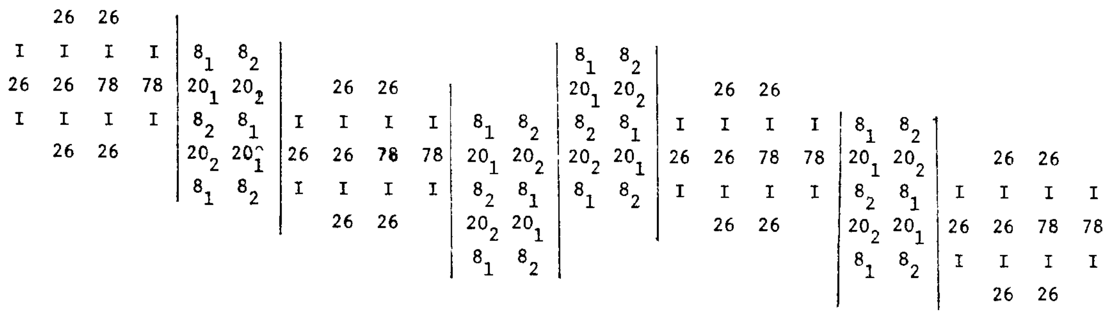

26 26
I I I I
26 26 78 78
I I I I
26 26
8_1 8_2
20_1 20_2
8_2 8_1
20_2 20_1
8_1 8_2
26 26
I I I I
26 26 78 78
I I I I
26 26
8_1 8_2
20_1 20_2
8_2 8_1
20_2 20_1
8_1 8_2
8_1 8_2
20_1 20_2
8_2 8_1
20_2 20_1
8_1 8_2
26 26
I I I I
26 26 78 78
I I I I
26 26
8_1 8_2
20_1 20_2
8_2 8_1
20_2 20_1
8_1 8_2
26 26
I I I I
26 26 78 78
I I I I
26 26

FIGURE 9

It immediately follows from this and Figure 1 that our diagram for $P_{8_1} \oplus P_{8_2}$ is as follows.

8_1	8_2												
20_1	20_2		26	26									
8_2	8_1	I	I	I	I	8_1	8_2						
20_2	20_1	26	26	78	78	20_1	20_2		26	26			
8_1	8_2	I	I	I	I	8_2	8_1	I	I	I	I	8_1	8_2
			26	26		20_2	20_1	26	26	78	78	20_1	20_2
						8_1	8_2	I	I	I	I	8_2	8_1
									26	26		20_2	20_1
												8_1	8_2

FIGURE 10

Thus apart from the distribution of i's and j's, the Loewy series given in Theorem 2 for P_{8_i} follows. Using Landrock's lemma and Figure 7, we see that the distribution of 20_i's and 20_j's in P_{8_i} is as given. But now every copy of 8_i in Figure 10 is glued either above a 20_i or below a 20_j. Hence the Loewy structure of P_{8_i} is as in Theorem 2.

6.5 The structure of $P_{78_{A_9}}$

By Brauer characters,

$$P_{14_{A_8}} \uparrow^{A_9} . f_0 = P_{26} \oplus P_{78} \quad .$$

Using Landrock's lemma and Figures 5, 7 and 8, we see immediately that our diagram for $P_{26} \oplus P_{78}$ is as follows.

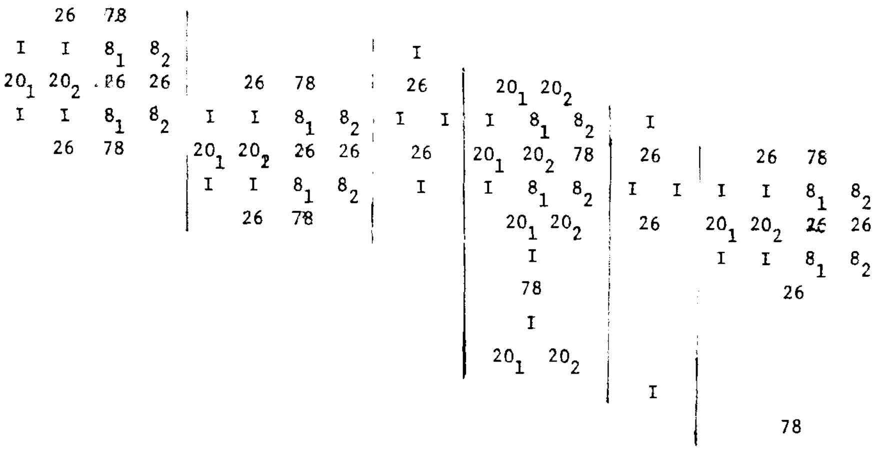


FIGURE 11

Now subtracting out the Loewy structure of P_{26}, we obtain the Loewy structure of P_{78} given in Theorem 2. This completes the proof of Theorem 2.

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Appendix 5. Characters of A_9

(i) Ordinary characters

181440		480	192	1080	81	54	24	16	60	24	6	7	9	9	20	12	15	15	
p power		A	A	A	A	A	A	B	A	AA	CB	A	B	B	AA	AA	AA	AA	
p'part		A	A	A	A	A	A	A	A	AA	CB	A	A	A	AA	AA	AA	AA	S9
ind	1A	2A	2B	3A	3B	3C	4A	4B	5A	6A	6B	7A	9A	9B	10A	12A	15A	B**	fusion
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:
+	8	4	0	5	-1	2	2	0	3	1	0	1	-1	-1	-1	-1	0	0	:
o	21	1	-3	-3	3	0	-1	1	1	1	0	0	0	0	1	-1	b15	**	[
o	21	1	-3	-3	3	0	-1	1	1	1	0	0	0	0	1	-1	**	b15	]
+	27	7	3	9	0	0	1	-1	2	1	0	-1	0	0	2	1	-1	-1	:
+	28	4	-4	10	1	1	0	0	3	-2	-1	0	1	1	-1	0	0	0	:
+	35	-5	3	5	-1	2	-1	-1	0	1	0	0	2	-1	0	-1	0	0	[
+	35	-5	3	5	-1	2	-1	-1	0	1	0	0	-1	2	0	-1	0	0	]
+	42	6	2	0	-3	3	0	2	-3	0	-1	0	0	0	1	0	0	0	:
+	48	8	0	6	3	0	0	0	-2	2	0	-1	0	0	-2	0	1	1	:
+	56	-4	0	11	2	2	-2	0	1	-1	0	0	-1	-1	1	1	1	1	:
+	84	4	4	-6	3	3	0	0	-1	-2	1	0	0	0	-1	0	-1	-1	:
+	105	5	1	15	-3	-3	-1	1	0	-1	1	0	0	0	0	-1	0	0	:
+	120	0	8	0	3	-3	0	0	0	0	-1	1	0	0	0	0	0	0	:
+	162	6	-6	0	0	0	0	-2	-3	0	0	1	0	0	1	0	0	0	:
+	168	4	0	-15	-3	0	-2	0	3	1	0	0	0	0	-1	1	0	0	:
+	189	-11	-3	9	0	0	1	1	-1	1	0	0	0	0	-1	1	-1	-1	:
+	216	-4	0	-9	0	0	2	0	1	-1	0	-1	0	0	1	-1	1	1	:

(ii) 2-modular characters

181440		1080	81	54	60	7	9	9	15	15	
p power		A	A	A	A	A	A	B	AA	AA	
p'part		A	A	A	A	A	A	A	AA	AA	S9
ind	1A	3A	3B	3C	5A	7A	9A	9B	15A	B**	fusion
+	1	1	1	1	1	1	1	1	1	1	:
+	8_1	-4	-1	2	-2	1	2	-1	1	1	[
+	8_2	-4	-1	2	-2	1	-1	2	1	1	]
+	8_3	5	-1	2	3	1	-1	-1	0	0	:
o	20_1	-4	2	-1	0	-1	-1	-1	b15-1	**	[
o	20_2	-4	2	-1	0	-1	-1	-1	**	b15-1	]
+	26	8	-1	-1	1	-2	-1	-1	-2	-2	:
+	48	6	3	0	-2	-1	0	0	1	1	:
+	78	6	-3	-3	-2	1	0	0	1	1	:
+	160	-20	-2	-2	0	-1	1	1	0	0	:

(iii) Decomposition Matrix

	I	8_1	8_2	20_1	20_2	26	78	8_3	48	160
1	1	.	.	.	.	.	.			
21	1	.	.	1	.	.	.			
21	1	.	.	.	1	.	.			
27	1	.	.	.	.	1	.			
28	2	.	.	.	.	1	.			
35	1	1	.	.	.	1	.			
35	1	.	1	.	.	1	.			
42	.	1	1	.	.	1	.			
84	2	1	1	1	1	1	.			
105	1	.	.	.	.	1	1			
120	2	.	.	1	1	.	1			
162	2	1	1	1	1	1	1			
189	3	1	1	1	1	2	1			
8								1	.	.
48								.	1	.
56								1	1	.
168								1	.	1
216								1	1	1

(iv) Cartan Matrix

	I	8_1	8_2	20_1	20_2	26	78	8_3	48	160
I	32	8	8	10	10	16	8			
8_1	8	5	4	3	3	6	2			
8_2	8	4	5	3	3	6	2			
20_1	10	3	3	5	4	4	3			
20_2	10	3	3	4	5	4	3			
26	16	6	6	4	4	12	4			
78	8	2	2	3	3	3	4			
8_3								4	2	2
48								2	3	1
160								2	1	2

Appendix 6. dim $\mathrm{Ext}^1_{A_9}(M,N)$ for M,N simple

M	I	8_1	8_2	20_1	20_2	26	78	8_3	48	160
I	0	0	0	1	1	2	1			
8_1	0	0	0	1	0	1	0			
8_2	0	0	0	0	1	1	0			
20_1	1	0	1	0	0	0	0			
20_2	1	1	0	0	0	0	0			
26	2	1	1	0	0	0	0			
78	1	0	0	0	0	0	0			
8_3								0	1	1
48								1	1	0
160								1	0	0

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