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THE LOEWY STRUCTURE OF THE PROJECTIVE INDECOMPOSABLE MODULES FOR A_8 IN CHARACTERISTIC 2

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Introduction and notation

The purpose of this paper is to establish the Loewy series for the projective modules for $A_8 \cong L_4(2)$ over a splitting field of characteristic 2.

Throughout, we shall let F denote a splitting field in characteristic 2 for A_8 and all its subgroups, and let (S,R,F) denote a splitting 2-modular system for A_8. We denote each simple module for a group by its dimension, together with a subscript if there is more than one simple module of that dimension. A_8 denotes the alternating group on 8 letters, a simple group of order $8!/2 = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. A_7 denotes a subgroup of index 8 stabilizing a point, and A_6 denotes a subgroup of A_7 of index 7 stabilizing a further point.

Thus the simple FA_8-modules are denoted I, 4_1, 4_2, 6, 14, 20_1, 20_2 and 64. These fall into two blocks: 64 is in a block of defect 0, while the rest are in the principal

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block. Since blocks of defect 0 are easy to describe, we shall only be interested in the principal block. We denote the central idempotent for the principal block of FA_8 by e_0.

The main result of this paper is the following theorem:

__Theorem 1.__ The Loewy structures of the projective indecomposable modules for $FA_8.e_0$ are as follows, where $\{i,j\}=\{1,2\}$:

$$\begin{array}{c}
I \\
6 \quad 14 \quad 20_1 \quad 20_2 \\
I \quad I \quad I \quad I \quad 4_1 \quad 4_2 \quad 6 \\
4_1 \quad 4_2 \quad 14 \quad 14 \quad 20_1 \quad 20_2 \\
I \quad I \quad I \quad 6 \quad 6 \quad 6 \quad 6 \\
4_1 \quad 4_1 \quad 4_2 \quad 4_2 \quad 14 \quad 14 \quad 20_1 \quad 20_2 \\
I \quad I \quad I \quad 6 \quad 6 \\
14 \quad 20_1 \quad 20_2 \\
I \quad I \\
14 \quad 20_1 \quad 20_2 \\
I \quad I \\
14 \quad 20_1 \quad 20_2 \\
I
\end{array}
\qquad\qquad
\begin{array}{c}
4_i \\
4_j \quad 6 \quad 20_i \\
I \quad 4_i \quad 4_j \quad 14 \\
I \quad 6 \quad 6 \\
4_i \quad 4_i \quad 14 \quad 14 \quad 20_j \\
I \quad I \quad 6 \quad 6 \\
4_j \quad 4_j \quad 14 \quad 20_i \\
6 \\
4_i
\end{array}$$

$$\begin{array}{c}
6 \\
I \quad 4_1 \quad 4_2 \quad 14 \\
I \quad 6 \quad 6 \quad 6 \\
4_1 \quad 4_1 \quad 4_2 \quad 4_2 \quad 14 \quad 14 \quad 14 \quad 20_1 \quad 20_2 \\
I \quad I \quad I \quad I \quad 6 \quad 6 \quad 6 \quad 6 \\
4_1 \quad 4_1 \quad 4_2 \quad 4_2 \quad 14 \quad 14 \quad 14 \quad 20_1 \quad 20_2 \\
I \quad I \quad 6 \quad 6 \quad 6 \\
4_1 \quad 4_2 \quad 14 \\
6
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 14 & & & \\
 & & & I \quad 6 & & & \\
 & 4_1 & 4_2 & 14 & 20_1 & 20_2 & \\
 & I & I & 6 & 6 & 6 & \\
4_1 & 4_1 & 4_2 & 4_2 & 14 & 14 & 14 \\
 & I & I & 6 & 6 & 6 & \\
 & 4_1 & 4_2 & 14 & 20_1 & 20_2 & \\
 & & & I \quad 6 & & & \\
 & & & 14 & & & \\
 & & & I & & & \\
 & & & 20_1 \quad 20_2 & & & \\
 & & & I & & & \\
 & & & 14 & & &
\end{array}
\qquad\qquad
\begin{array}{c}
20_i \\
I \quad 4_i \\
14 \\
I \quad 6 \\
4_j \quad 20_i \quad 20_j \\
I \quad 6 \\
4_i \quad 14 \\
I \\
20_i \quad 20_j \\
I \\
14 \\
I \\
20_i
\end{array}$$

If A is a group algebra over F and M is a finitely generated A-module, we write JA for the Jacobson radical of A, and

$$L_i(M) = M.(JA)^{i-1}/M.(JA)^i$$

is the i^{th} Loewy Layer of M. The Loewy Length of M is the smallest number l such that $M.(JA)^l = 0$, and the Loewy Structure for M is a diagram whose i^{th} layer downwards gives the simple summands of $L_i(M)$ with multiplicities (see for example Theorem 1). The Head of a module is the first Loewy layer.

Let $Soc(M)$ denote the socle of M, namely the sum of all the simple A-submodules of M. Let $S_1(M) = Soc(M)$ and

$$S_i(M)/S_{i-1}(M) = Soc(M/S_{i-1}(M)).$$

Then

$$0 < S_1(M) < S_2(M) < \ldots < S_{l-1}(M) < S_l(M) = M$$

is called the Socle Series of M.

We shall write $(M,N)_A$ for $\dim_F \mathrm{Hom}_A(M,n)$, M^* for $\mathrm{Hom}_F(M,F)$ regarded as an A-module, and P_M for the projective cover of M. Homomorphisms will usually be written on the right.

The exterior n^{th} power of M will be written M^{n-}.

Our main tools are the following lemmas, together with the easy but powerful lemmas discussed in Section 4.1.

Lemma 1 (Scott [4]). Any endomorphism of an FG - permutation module can be lifted to an endomorphism of the corresponding RG - permutation module. Thus direct summands of FG - permutation modules lift, and so their endomorphisms.

Lemma 2 (Frobenius Reciprocity). Let $H \leqq G$, M and FH-module and N an FG-module. Then $(M,N{\downarrow}_{FH})_{FH} = (M{\uparrow}^{FG},N)_{FG}$ and $(N{\downarrow}_{FH},M)_{FH} = (N,M{\uparrow}^{FG})_{FG}$.

Lemma 3 (Thompson [5]). If M is an irreducible SG-module, then an R-form $\hat{M}$ may be found such that the modular reduction $\bar{M} = \hat{M} \otimes_R F$ has any given composition factor as its unique top factor.

Lemma 4 (Landrock [3]). Let M and N be simple FG-modules. Then the multiplicity of M in $L_i(P_N)$ is the same as the multiplicity of N* in $L_i(P_{M*})$.

Lemma 5 (Mackey Decomposition). Let $H,K \leqq G$ and M an FH-module. Then

$$M{\uparrow}^G{\downarrow}_K = \bigoplus_{HxK} M \otimes x{\downarrow}_{H^x \cap K}{\uparrow}^K,$$

where x runs over a set of H-K double coset representatives in G.

In Section 1 and Appendices 1 - 3 we collect some known results about A_6, A_7 and A_8. In Section 2 we examine the structure of the permutation modules for FA_8 on the cosets of maximal subgroups, and in Section 3 we examine the FA_8-modules induced up from simple FA_7-modules.

Section 4 is the main body of the paper, and this uses the results of the previous sections to deduce Theorem 1.

Section 1. Preliminary results on A_6, A_7 and A_8

1.1. Characters and subgroups of A_8

In this section we collect together some known facts about the group A_8 and some of its subgroups. In Appendix 1 we give the ordinary and 2-modular character tables of A_8, the decomposition matrix and the Cartan matrix. These can be extracted from James [2]. We also note the isomorphism $A_8 \cong L_4(2)$, the group of 4 X 4 matrices over GF(2).

We shall have cause to look at the following maximal subgroups:

'Structure'	Index	A_8-name	$L_4(2)$-name
A_7	8	point	-
$2^3{:}L_3(2)$	15	-	point
$2^3{:}L_3(2)$	15	-	hyperplane
S_6	28	pair	symplectic form
$2^4{:}(S_3 \times S_3)$	35	4+4 splitting	2-dimensional subspace
$(A_5 \times 3).2$	56	triple	GF(4)-structure

The Schur multiplier of A_8 has order 2, so that $\dim \mathrm{Ext}^2_{A_8}(I,I) = 1$. The automorphism group of A_8 is exactly S_8, and the outer automorphism acts as the graph automorphism on $L_4(2)$ (namely transpose inverse on matrices).

Thus the two classes of subgroups $2^3 : L_3(2)$ are conjugate under the action of this outer automorphism.

1.2. Results on A_7

In Appendix 2 we give the ordinary and 2-modular character tables of A_7, the decomposition matrix and Cartan matrix (see James [2]). The 6-dimensional irreducible FA_7-module is a direct summand of the permutation module on cosets of A_6, and the 14-dimensional irreducible is a direct summand of the permutation module on the 21 coset of an S_5 preserving a 5+2 splitting of the 7 points; this module splits $1 \oplus 6 \oplus 14$. The permutation module on 35 cosets of an $(A_4 \times 3).2$ preserving a 4+3 splitting of the 7 points has structure:

$$I \oplus 14 \oplus \begin{array}{ccc} & 6 & \\ 4_1 & & 4_2 \\ & 6 & \end{array}.$$

The structures of the projective indecomposable modules in the principal block are:

$$\begin{array}{ccc} \begin{array}{ccc} & I & \\ 14 & & 20 \\ I & \oplus & I \\ 20 & & 14 \\ & I & \end{array} & \qquad \begin{array}{ccc} & 14 & \\ & & I \\ 14 & \oplus & 20 \\ & & I \\ & 14 & \end{array} & \qquad \begin{array}{c} 20 \\ I \\ 14 \\ I \\ 20 \end{array} \\ \\ P_I & \qquad P_{14} & \qquad P_{20} \end{array}$$

and in the non-principal block:

$$\begin{array}{c} 4_1 \\ 6 \\ 4_2 \\ 6 \\ 4_1 \end{array} \qquad \begin{array}{c} 4_2 \\ 6 \\ 4_1 \\ 6 \\ 4_2 \end{array} \qquad \begin{array}{ccc} & 6 & \\ 4_1 & & 4_2 \\ 6 & \oplus & 6 \\ 4_2 & & 4_1 \\ & 6 & \end{array}$$

P_{4_1} P_{4_2} P_6

(Erdmann[1]).

If we take the 64-dimensional defect 0 representation of A_8 (which is the Steinberg representation of $L_4(2)$), the restriction to A_7 is exactly P_{14}, as can be checked by Brauer characters. Apart from this, every irreducible representation of A_8 remains irreducible upon restriction to A_7.

1.3 Results on A_6

In Appendix 3 we give the ordinary and 2-modular character tables of A_6, the decomposition matrix and the Cartan matrix (see James [2]). There are three blocks, namely the principal block and two blocks of defect 0. The structures of the projective indecomposables are as follows:

$$\begin{array}{c} 4_1 \\ I \\ 4_2 \\ I \\ 4_1 \\ I \\ 4_2 \\ I \\ 4_1 \end{array} \qquad \begin{array}{c} 4_2 \\ I \\ 4_1 \\ I \\ 4_2 \\ I \\ 4_1 \\ I \\ 4_2 \end{array} \qquad \begin{array}{ccc} & I & \\ 4_1 & & 4_2 \\ I & & I \\ 4_2 & & 4_1 \\ I & \oplus & I \\ 4_1 & & 4_2 \\ I & & I \\ 4_2 & & 4_1 \\ & I & \end{array}$$

P_{4_1} P_{4_2} P_I

$8_1 = P_{8_1}$ $8_2 = P_{8_2}$ (Erdmann [1])

1.4 Induction and restriction between A_6 and A_7

Brauer characters show that

1.4.1 $$(4_1)_{A_7}{\downarrow}_{A_6} = (4_2)_{A_7}{\downarrow}_{A_6} = (4_2)_{A_6}.$$

(beware!)

The composition factors of $6_{A_7}{\downarrow}_{A_6}$ are $I + I + 4_1$. But $(I, 6_{A_7}{\downarrow}_{A_6})_{A_6} = (I_{A_6}{\uparrow}^{A_7}, 6)_{A_7} = 1$. Since $\mathrm{Ext}^1_{A_7}(I,I) = 0$, this means that

1.4.2 $$6_{A_7}{\downarrow}_{A_6} = \begin{matrix} I \\ 4_1 \\ I \end{matrix}\ .$$

The composition factors of $14_{A_7}{\downarrow}_{A_6}$ are $I + I + 4_1 + 4_1 + 4_2$. But $(I, 14_{A_7}{\downarrow}_{A_6})_{A_6} = (I_{A_6}{\uparrow}^{A_7}, 14)_{A_7} = 0$, and so the only possibility is

1.4.3 $$14_{A_7}{\downarrow}_{A_6} = \begin{matrix} 4_1 \\ I \\ 4_2 \\ I \\ 4_1 \end{matrix}\ .$$

The composition factors of $20_{A_7}{\downarrow}_{A_6}$ are $4_2 + 8_1 + 8_2$. Since the constituents are in different blocks, we have

1.4.4 $$20_{A_7}{\downarrow}_{A_6} = 4_2 \oplus 8_1 \oplus 8_2.$$

Since $\left(I, \begin{pmatrix} 6 \\ 4_1 \end{pmatrix}_{A_7}{\downarrow}_{A_6}\right)_{A_6} = \left(I_{A_6}{\uparrow}^{A_7}, \begin{pmatrix} 6 \\ 4_1 \end{pmatrix}\right)_{A_7} = 0$, we have

1.4.5 $$\begin{pmatrix} 6 \\ 4_1 \end{pmatrix}_{A_7}{\downarrow}_{A_6} = \begin{matrix} I \\ 4_1 \\ I \\ 4_2 \end{matrix}\ .$$

The composition factors of $(4_2)_{A_6}{\uparrow}^{A_7}$ are $4_1 + 4_2 + 20$.

Since this module is self-dual and extends to a module for S_6 and S_7, the only possibility is

1.4.6 $$(4_2)_{A_6}\uparrow^{A_7} = 4_1 \oplus 4_2 \oplus 20.$$

Since $\left(\begin{pmatrix} & 6 & \\ 4_1 & & 4_2 \end{pmatrix}_{A_7}\downarrow_{A_6}, 4_2\right)_{A_6} = \left(\begin{pmatrix} & 6 & \\ 4_1 & & 4_2 \end{pmatrix}, (4_2)_{A_6}\uparrow^{A_7}\right)_{A_7} = 0$

the only possibility is

1.4.7 $$\begin{pmatrix} & 6 & \\ 4_1 & & 4_2 \end{pmatrix}_{A_7}\downarrow_{A_6} = \begin{matrix} I \\ 4_1 \\ I \\ 4_2 \end{matrix} \oplus 4_2.$$

The composition factors of $(4_1)_{A_6}\uparrow^{A_7}$ are $14 + 14$ and so, since $(14_{A_7}, (4_1)_{A_6}\uparrow^{A_7})_{A_7} = (14_{A_7}\downarrow_{A_6}, (4_1)_{A_6})_{A_6} = 1$, we have

1.4.8 $$(4_1)_{A_6}\uparrow^{A_7} = \begin{matrix} 14 \\ 14 \end{matrix}.$$

Finally, since $\left(I, \begin{pmatrix} I \\ 20 \end{pmatrix}_{A_7}\downarrow_{A_6}\right)_{A_6} = \left(I_{A_6}\uparrow^{A_7}, \begin{pmatrix} I \\ 20 \end{pmatrix}\right)_{A_7} = 0$, we have from 1.4.4

1.4.9 $$\begin{pmatrix} I \\ 20 \end{pmatrix}_{A_7}\downarrow_{A_6} = \begin{matrix} I \\ 4_2 \end{matrix} \oplus 8_1 \oplus 8_2.$$

Section 2. Some permutation modules for A_8

2.1 Permutations on the 8 cosets of A_7

Ordinary character: $1 + 7$.

Hence the composition factors of this FA_8-module M_8 are $I + I + 6$. Frobenius reciprocity shows that $L_1(M_8) \cong S_1(M_8) \cong I$, and so the structure is

2.1.1 $$M_8 = \begin{array}{c} I \\ 6 \\ I \end{array} .$$

2.2 Permutations on the 15 cosets of $2^3{:}L_3(2)$

Ordinary characters: $1+14$ for each of the two classes.

Thus the composition factors of these FA_8-modules M_{15a} and M_{15b} are $I+4_1+4_2+6$. Since 15 is odd, these modules have I as a direct summand. Frobenius reciprocity shows that in one case the head is $I+4_1$ and the socle is $I+4_2$, whereas in the other case the head is $I+4_2$ and the socle is $I+4_1$. Thus the structures are

2.2.1 $$M_{15a} = I \oplus \begin{array}{c} 4_1 \\ 6 \\ 4_2 \end{array} \qquad M_{15b} = I \oplus \begin{array}{c} 4_2 \\ 6 \\ 4_1 \end{array} .$$

2.3 Permutations on the 28 cosets of S_6

Ordinary character: $1+7+20$.

Thus the composition factors of this FA_8-module M_{28} are $I+I+6+6+14$. By Scott's Lemma, the endomorphism ring has dimension 3.

Since $M_{28} = (M_8)^{2-}$, it has a submodule $I \wedge \begin{array}{c} I \\ 6 \\ I \end{array}$ of structure $\begin{array}{c} I \\ 6 \end{array}$. By Frobenius reciprocity, $S_1(M_{28}) \cong L_1(M_{28}) \cong I \oplus 6$.

2.3.1. Lemma. $6^{2-} \cong I \oplus 14$.

Proof. The composition factors are $I+14$, and the module is self-dual. //

Thus M_{28} has $I \oplus 14$ as a subquotient. Since it is self-dual, this means the Loewy structure of M_{28} is

2.3.2 $$\begin{array}{cc} I & 6 \\ I & 14 \\ \multicolumn{2}{c}{6} \end{array}$$

(i.e. the "diagram" for M_{28} is $I\,{}_{\diagdown 6 \diagup}\,14\,{}^{\diagup 6 \diagdown}\,I$).

2.4 Permutations on the 35 cosets of $2^4{:}(S_3 \times S_3)$

Ordinary characters: $1 + 14 + 20$.

Thus the composition factors of this FA_8-module M_{35} are $I + 4_1 + 4_2 + 6 + 6 + 14$. Since 35 is odd, I is a direct summand. Frobenius reciprocity shows that $S_1(M_{35}) \cong L_1(M_{35}) \cong I \oplus 6$. Since M_{35} extends to a module for S_8, there is a subquotient $4_1 \oplus 4_2$. Since the module is self-dual, this forces the structure to be

2.4.1 $$M_{35} = I \oplus 4_1 \quad \begin{array}{c} 6 \\ 4_2 \\ 6 \end{array} \quad 14.$$

2.5. Permutations of the 56 cosets of $(A_5 \times 3).2$

Ordinary characters: $1 + 7 + 20 + 28$.

Thus the composition factors of this FA_8-module M_{56} are $I + I + 4_1 + 4_2 + 6 + 6 + 6 + 14 + 14$.

2.5.1. Lemma. M_{56} has a direct summand isomorphic to the module M_8 described in 2.1.

Proof. We construct maps $\alpha: M_8 \to M_{56}$ and $\beta: M_{56} \to M_8$ as follows:

α: point $x \to$ sum of triples containing x

β: triple $\{a,b,c\} \to a+b+c$.

Then

$$\alpha\beta: \text{point } x \to 21.x + 6.\sum_{y \neq x} y = x$$

since we are in characteristic 2.

Hence $\alpha\beta = 1$, and so $\beta\alpha$ is a projection and M_{56} splits as

$$M_{56} = \mathrm{Im}(\beta\alpha) \oplus \mathrm{Ker}(\beta\alpha).$$

So

2.5.2 $$M_{56} = M_8 \oplus M'_{56} \qquad \text{where } M'_{56} = \mathrm{Ker}(\beta\alpha). \quad //$$

Now M'_{56} has composition factors $4_1 + 4_2 + 6 + 6 + 14 + 14$. By Frobenius reciprocity, $S_1(M_{56}) \cong L_1(M_{56}) \cong I \oplus 14$ and so $S_1(M'_{56}) \cong L_1(M'_{56}) \cong 14$.

Next we notice that $M_{56} = (M_8)^{3-}$, so that it reduces at least as far as

$$\begin{array}{ccc} & 6^{2-} & \\ 6 & \oplus & 6^{3-}. \\ & 6^{2-} & \end{array}$$

<u>2.5.3. Lemma.</u> 6^{3-} has structure $\begin{array}{ccc} & 6 & \\ 4_1 & & 4_2. \\ & 6 & \end{array}$

<u>Proof</u>. The composition factors of 6^{3-} are $4_1 + 4_2 + 6 + 6$. The module is self-dual and extends to a module for S_8. Hence either the lemma holds or $6^{3-} \cong 6 \oplus 6 \oplus 4_1 \oplus 4_2$. If so, then this is still true as modules for A_7. But for A_7, $(1 \oplus 6)^{3-} \cong 6^{2-} \oplus 6^{3-}$ is a permutation module, and so 4_1 and 4_2 would be direct sum-

mands of a permutation module. But they do not lift to RA_7-modules, contradicting Scott's lemma. //

But now this means that M'_{56} has a subquotient isomorphic to 6^{3-}, and so it has Socle and Loewy Series

2.5.4
$$\begin{array}{ccc} & 14 & \\ & 6 & \\ 4_1 & & 4_2 \\ & 6 & \\ & 14 & \end{array} .$$

Hence

2.5.5
$$M_{56} = \begin{array}{c} I \\ 6 \\ I \end{array} \oplus \begin{array}{ccc} & 14 & \\ & 6 & \\ 4_1 & & 4_2 \\ & 6 & \\ & 14 & \end{array} .$$

Section 3. The induced modules from simple A_7-modules

As we have already noted, the restrictions of simple A_8-modules to A_7 are as follows:

$$\begin{array}{llll} I_{A_8}{\downarrow}A_7 = I_{A_7} & & (4_1)_{A_8}{\downarrow}A_7 = (4_1)_{A_7} & \\ (4_2)_{A_8}{\downarrow}A_7 = (4_2)_{A_7} & & 6_{A_8}{\downarrow}A_7 = 6_{A_7} & \\ 14_{A_8}{\downarrow}A_7 = 14_{A_7} & & (20_1)_{A_8}{\downarrow}A_7 = (20_2)_{A_8}{\downarrow}A_7 = 20_{A_7} & \end{array}$$

$$64_{A_8}{\downarrow}A_7 = P_{14_{A_7}} .$$

By Frobenius reciprocity, this tells us the socle and first Loewy layer of modules induced from A_7.

We dealt with $I_{A_7}{\uparrow}^{A_8}$ in Section 2.1, and so we only consider non-trivial simple modules here.

3.1. $(4_1)_{A_7}\uparrow^{A_8}$ and $(4_2)_{A_7}\uparrow^{A_8}$

The composition factors of $(4_1)_{A_7}\uparrow^{A_8}$ are $4_1+4_1+4_2+20_1$, and $S_1 \cong L_1 \cong (4_1)_{A_8}$. Since $(4_1)_{A_7}$ is the dual of 4_2, and also the image of 4_2 under the S_7-automorphism of A_7, this means the Socle and Loewy Series are:

3.1.1 $$(4_1)_{A_7}\uparrow^{A_8} = \begin{array}{ccc} & 4_1 & \\ 4_2 & & 20_1 \\ & 4_1 & \end{array} \qquad (4_2)_{A_7}\uparrow^{A_8} = \begin{array}{ccc} & 4_2 & \\ 4_1 & & 20_2 \\ & 4_2 & \end{array}$$

3.2. The module $6_{A_7}\uparrow^{A_8}$

This has composition factors $I+I+6+6+6+14+14$ and $S_1 \cong L_1 \cong 6_{A_8}$.

3.2.1. Lemma. There is a homomorphism from M_{28} to $6_{A_7}\uparrow^{A_8}$ with one-dimensional kernel.

Proof. From Section 2.3 we see that since 6_{A_7} is in a different block from I_{A_7} and 14_{A_7}, $M_{28}\downarrow_{A_7}$ is semisimple. Thus

$$(M_{28}, 6_{A_7}\uparrow^{A_8})_{A_8} = (M_{28}\downarrow_{A_7}, 6)_{A_7} = 2.$$

Thus from what we know of the structure of M_{28}, since $S_1(6_{A_7}\uparrow^{A_8}) = 6_{A_8}$, there must be a homorphism with kernel the trivial submodule of M_{28}. //

Thus by self-duality, there is a submodule $I \oplus \begin{array}{c} 6 \\ 14 \\ 6 \end{array}$ and a quotient module $I \oplus \begin{array}{c} 6 \\ 14 \\ 6 \end{array}$.

3.2.2. Lemma. $6_{A_7}\uparrow^{A_8}$ has exactly one copy of I in its second Loewy layer.

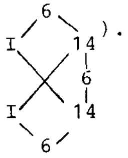
Proof. We certainly know that there is at least one, by 3.2.1. Suppose that there is more than one. Since 6_{A_7} is not in the principal block, this means that $L_1(6_{A_7}\uparrow^{A_8}\downarrow_{A_7})$ has more than one copy of I in it. However,

$$(6_{A_7}\uparrow^{A_8}\downarrow_{A_7}, I)_{A_7} = (6_{A_7}\uparrow^{A_8}, I_{A_7}\uparrow^{A_8}) \leqq 1,$$

a contradiction. //

This forces the Loewy length to be at least 4, and since it is self-dual, we are left with only one possibility, namely the that the Loewy Series is

$$\begin{array}{cc} \multicolumn{2}{c}{6} \\ I & 14 \\ I & 6 \\ \multicolumn{2}{c}{14} \\ \multicolumn{2}{c}{6} \end{array} \qquad 3.2.3$$

(i.e. the "diagram" for $6_{A_7}\uparrow^{A_8}$ is ).

3.3. The module $14_{A_7}\uparrow^{A_8}$

This has composition factors $4_1 + 4_2 + 6 + 6 + 14 + 14 + 64$ and $S_1 \cong L_1 \cong 14 \oplus 64$. Since 64 is projective, this module is a direct sum of 64 and a module with $S_1 \cong L_1 \cong 14$.

3.3.1. Lemma. $(M_{56}, 14_{A_7}\uparrow^{A_8})_{A_8} = 2.$

Proof. $(M_{56}, 14_{A_7}\uparrow^{A_8})_{A_8} = (M_{56}\downarrow_{A_7}, 14)_{A_7}$. But $M_{56}\downarrow_{A_7}$ is the direct sum of the permutation module on 21 cosets of an S_5 fixing a 5+2 splitting of the 7 points, and the permutation module on the 35 cosets of an $(A_4 \times 3).2$ fixing a 4+3 splitting of the 7 points. The lemma now follows from Section 1.2. //

Now from the structure of M_{56} given in 2.5.5 it follows that every such homomorphism must kill $\mathrm{Im}(\beta\alpha)$, and some such homomorphism is an injection from M'_{56} into $14_{A_7}\uparrow^{A_8}$. Thus

3.3.2 $$14_{A_7}\uparrow^{A_8} \cong M'_{56} \oplus 64 \qquad \left(= \begin{array}{ccc} & 14 & \\ & 6 & \\ 4_1 & & 4_2 \\ & 6 & \\ & 14 & \end{array} \oplus 64\right).$$

3.4. The module $20_{A_7}\uparrow^{A_8}$

This has composition factors $I + I + I + I + 4_1 + 14 + 14 + 20_1 + 20_1 + 20_1 + 20_2 + 20_2 + 20_2$ and $S_1 \cong L_1 \cong 20_1 \oplus 20_2$.

3.4.1. Lemma. $(20_{A_7}\uparrow^{A_8}, 20_{A_7}\uparrow^{A_8})_{A_8} = 4.$

Proof. $(20_{A_7}\uparrow^{A_8}, 20_{A_7}\uparrow^{A_8})_{A_8} = (20,20)_{A_7} + (20_{A_7}\downarrow_{A_6}, 20_{A_7}\downarrow_{A_6})_{A_6}$ by the Mackey decomposition theorem

$$= 1 + 3 \qquad \text{by } 1.4.4.$$

The lemma now follows easily. //

We shall complete the determination of the structure of this module in Section 4.4.

Section 4. More induced modules from A_7; the final assault

4.1. Induction of projective modules

By Brauer characters, we see that

$$P_{I_{A_7}}\uparrow^{A_8} = P_{I_{A_8}} \oplus 64 \oplus 64 \qquad P_{(4_1)_{A_7}}\uparrow^{A_8} = P_{(4_1)_{A_8}}$$

$$P_{(4_2)_{A_7}}\uparrow^{A_8} = P_{(4_2)_{A_8}} \qquad P_{6_{A_7}}\uparrow^{A_8} = P_{6_{A_8}}$$

$$P_{14_{A_7}}\uparrow^{A_8} = P_{14_{A_8}} \oplus 64 .\oplus 64 \oplus 64 \qquad \text{and}$$

$$P_{20_{A_7}}\uparrow^{A_8} = P_{(20_1)_{A_8}} \oplus P_{(20_2)_{A_8}} \oplus 64 .$$

Thus the results of Section 3, together with the structure of $20_{A_7}\uparrow^{A_8}$ which is yet to be determined, give us strong information about the structures of the projective modules for A_8. Namely, we are given certain filtrations for each of P_I, P_{4_1}, P_{4_2}, P_6, P_{14} and $P_{20_1} \oplus P_{20_2}$, in which we know the structures of the quotient modules. We now use this to complete the determination of Ext^1 for simple modules, and then to get the complete Loewy structures of the projective indecomposables. All we need to know is how far certain composition factors can "slip past" each other. Our main tool will be the following observations, all of which are trivial but powerful consequences of 4.1.1:

We can identify $JFA_7\uparrow^{A_8}$ as a subring of FA_8 via $JFA_7\uparrow^{A_8} = JFA_7 \underset{FA_7}{\otimes} FA_8 \leq FA_7 \underset{FA_7}{\otimes} FA_8 \cong FA_8$.

4.1.1. Lemma. $JFA_7\uparrow^{A_8}.e_0 \leqq JFA_8$.

Proof. This follows trivially from the observation that for each simple A_7-module M,

$$L_1(M\uparrow^{A_8}).e_0 \cong L_1(P_M\uparrow^{A_8}).e_0$$

By the Frobenius reciprocity theorem it is equivalent to the statement that for each simple A_8-module N in the principal block, $N\downarrow_{A_7}$ is semisimple. //

4.1.2. Lemma. $(JFA_7)^n\uparrow^{A_8}.e_0 \leqq (JFA_8)^n$ for all $n \geqq 0$.

Proof. This follows from 4.1.1. //

4.1.3. Theorem. If M is any module for A_7, then

$$\left(\frac{M\uparrow^{A_8}}{M\uparrow^{A_8}.(JFA_8)^n}\right)\cdot e_0 = \left(\frac{\left(\frac{M}{M.(JFA_7)^n}\right)\uparrow^{A_8}}{\left(\frac{M}{M.(JFA_7)^n}\right)\uparrow^{A_8}.(JFA_8)^n}\right).e_0$$

Proof. By 4.1.2 we have

$$(M(JFA_7)^n)\uparrow^{A_8}.e_0 \leqq M(JFA_8)^n.e_0.$$

Hence

$$\left(\frac{M}{M.(JFA_7)^n}\right)\uparrow^{A_8}.(JFA_8)^n.e_0 = \frac{M\uparrow^{A_8}.(JFA_8)^n}{(M.(JFA_7)^n)\uparrow^{A_8}}\cdot e_0$$

and the result follows from the third isomorphism theorem. //

4.1.4. Corollary. If M is a module for A_7, then

$$L_1(M\uparrow^{A_8}).e_0 \cong L_1((L_1(M))\uparrow^{A_8}).e_0.$$

Proof. This is just the case $n=1$ of the theorem. //

4.1.5. Corollary. If M is a module for A_7 and

$$0 \to M' \to M \to M'' \to 0$$

is a non-split short exact sequence with M' and M'' simple, then

$$L_1(M\uparrow^{A_8}).e_0 = L_1(M''\uparrow^{A_8}).e_0. \quad //$$

4.2. The Loewy structure of $\binom{4_1}{6}_{A_7}\uparrow^{A_8}$

Our filtration of $\binom{4_1}{6}_{A_7}\uparrow^{A_8}$ looks like:

$$\begin{array}{ccc} & 4_1 & \\ 4_2 & & 20_1 \\ & 4_1 & \\ \hline & 6 & \\ I & & 14 \\ I & & 6 \\ & 14 & \\ & 6 & \end{array}$$

By 4.1.5, $L_1 = 4_1$. We know from 2.5.4 that $L_2(P_{(4_1)_{A_8}})$ has a copy of 6 in it, and so applying 4.1.3 for $n=2$ to $P_{(4_1)_{A_7}}$ we see that the L_2 of both $\binom{4_1}{6}_{A_7}\uparrow^{A_8}$ and $P_{(4_1)_{A_8}}$

are $4_2 \oplus 6 \oplus 20_1$. This completes the determination of $\dim \mathrm{Ext}^1_{A_8}(4_1,M)$ and hence also of $\dim \mathrm{Ext}^1_{A_8}(4_2,M)$ for M simple.

Also from 2.5.4 we wee that $L_3(P_{(4_1)_{A_8}})$ has a copy of 14 in it, so that again applying 4.1.3 for $n=3$ we see that $L_3\left(\begin{pmatrix}4_1\\6\end{pmatrix}_{A_7}\uparrow^{A_8}\right)$ has a copy of 14 in it. Now since $\dim \mathrm{Ext}^1_{A_8}(4_1,I)=0$, it follows that the Loewy series for $\begin{pmatrix}4_1\\6\end{pmatrix}_{A_7}\uparrow^{A_8}$ is as follows:

$$\begin{array}{ccccc}
 & & 4_1 & & \\
4_2 & & 6 & & 20_1 \\
I & & 4_1 & & 14 \\
 & I & & 6 & \\
 & & 14 & & \\
 & & 6 & &
\end{array}$$

4.3. The Loewy structure of $\begin{pmatrix} & 6 & \\ 4_1 & & 4_2\end{pmatrix}_{A_7}\uparrow^{A_8}$

Our filtration of this module looks like:

$$\begin{array}{c}
\begin{array}{ccc}
 & 6 & \\
I & & 14 \\
I & & 6 \\
 & 14 & \\
 & 6 &
\end{array} \\
\hline
\begin{array}{ccc}
 & 4_1 & \\
4_2 & & 20_1 \\
 & 4_1 &
\end{array}
\quad \oplus \quad
\begin{array}{ccc}
 & 4_2 & \\
4_1 & & 20_2 \\
 & 4_2 &
\end{array}
\end{array}$$

From 2.5.5 we see that $L_2(P_{6_{A_8}})$ has a copy of 4_1 and of 4_2 in it. Thus applying 4.1.3 for $n=2$ to $P_{6_{A_7}}$ we see that L_2 of both $\begin{pmatrix} & 6 & \\ 4_1 & & 4_2\end{pmatrix}_{A_7}\uparrow^{A_8}$ and $P_{6_{A_8}}$ are $I \oplus 4_1 \oplus 4_2 \oplus 14$. This completes the determination of $\dim \mathrm{Ext}^1_{A_8}(6,M)$ for M simple.

4.3.1. Lemma. $L_3\left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\uparrow^{A_8}\right)$ does not contain copies of 20_1 or 4_2.

Proof. $$\left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\uparrow^{A_8}\downarrow_{A_7},\ \begin{matrix}I\\20\end{matrix}\right)_{A_7} = \left(\begin{matrix}6\\4_1\end{matrix},\ \begin{matrix}I\\20\end{matrix}\right)_{A_7}$$
$$+ \left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\downarrow_{A_6},\ \begin{pmatrix}I\\20\end{pmatrix}_{A_7}\downarrow_{A_6}\right)_{A_6}$$

by the Mackey decomposition theorem

$$= 0 + \left(\begin{matrix}I\\4_1\\I\\4_2\end{matrix},\ \begin{matrix}I\\4_2\end{matrix}\oplus 8_1\oplus 8_2\right)_{A_6}$$

by 1.4.5 and 1.4.9

$$= 0.$$

Also,

$$\left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\uparrow^{A_8}\downarrow_{A_7},\ 20\right)_{A_7} = \left(\begin{matrix}6\\4_1\end{matrix},\ 20\right)_{A_7}$$
$$+ \left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\downarrow_{A_6},\ \begin{pmatrix}I\\20\end{pmatrix}_{A_7}\downarrow_{A_6}\right)_{A_6}$$
$$= 0 + \left(\begin{matrix}I\\4_1\\I\\4_2\end{matrix},\ 4_2\oplus 8_1\oplus 8_2\right)_{A_6}$$

by 1.4.4 and 1.4.5

$$= 0.$$

However, if $L_3\left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\uparrow^{A_8}\right)$ has a copy of 20_1 in it, then our knowledge of $\mathrm{Ext}^1_{A_7}$ shows that there would be a map in one of the above sets.

Similarly, if $L_3\left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\uparrow^{A_8}\right)$ has a copy of 4_2 in it, there would be a map

$$\begin{pmatrix} & 6 & \\ 4_1 & & 4_2\end{pmatrix}_{A_7}\uparrow^{A_8}\downarrow_{A_7} \to (4_2)_{A_7}$$

for on restriction to A_7, we have

$$\dim \mathrm{Ext}^1_{A_7}(14,4_2) = \dim \mathrm{Ext}^1_{A_7}(I,4_2) = \dim \mathrm{Ext}^1_{A_7}(4_1,4_2) = 0$$

and hence $L_2\left(\begin{pmatrix}6\\4_1\end{pmatrix}_{A7}\uparrow^{A_8}\downarrow_{A7}\right)$ would have 4_2 in it, and hence so would $L_1\left(\begin{pmatrix} & 6 & \\ 4_1 & & 4_2\end{pmatrix}_{A_7}\uparrow^{A_8}\downarrow_{A_7}\right)$ since $\dim \mathrm{Ext}^1_{A_7}(6,4_2) = 1$. However,

$$\left(\begin{pmatrix} & 6 & \\ 4_1 & & 4_2\end{pmatrix}_{A_7}\uparrow^{A_8}\downarrow_{A_7}, 4_2\right)_{A_7} = \left(\begin{matrix} & 6 & \\ 4_1 & & 4_2\end{matrix}, 4_2\right)_{A_7}$$

$$+ \left(\begin{pmatrix} & 6 & \\ 4_1 & & 4_2\end{pmatrix}_{A_7}\downarrow_{A_6}, (4_2)_{A_7}\downarrow_{A_6}\right)_{A_6}$$

by Mackey decomposition

$$= 0 + \left(\begin{matrix} & I & \\ 4_1 & & 4_2 \\ I & & \\ 4_2 & & \end{matrix}, 4_2\right)_{A_6}$$

by 1.4.7

$$= 0, \quad \text{a contradiction.} \quad //$$

Thus with the results of Section 4.2 and the fact that $\begin{pmatrix}6\\4_1\end{pmatrix}_{A_7}\uparrow^{A_8}$ has a submodule $\begin{pmatrix}14\\6\\4_1\end{pmatrix}$ (see 2.5.4), we see that the Loewy series of $\begin{pmatrix} & 6 & \\ 4_1 & & 4_1\end{pmatrix}_{A_7}\uparrow^{A_8}$ is as follows:

$$\begin{array}{c} 6 \\ I \quad 4_1 \quad 4_2 \quad 14 \\ I \quad 6 \\ 4_1 \quad 4_2 \quad 14 \quad 20_1 \quad 20_2 \\ 6 \\ 4_1 \quad 4_2 \end{array}$$

4.4. The Loewy structures of $\begin{pmatrix}6\\4_1\\6\end{pmatrix}_{A_7}^{\uparrow A_8}$, $\begin{pmatrix}4_1\\6\\4_2\end{pmatrix}_{A_7}^{\uparrow A_8}$, $P_{6_{A_8}}$, $P_{(4_1)_{A_8}}$ and $P_{(4_2)_{A_8}}$.

4.4.1. Lemma. There is no uniserial module $\begin{matrix}I\\6\\4_1\end{matrix}$.

Proof. Applying 4.1.3. with $n=3$ to $P_{I_{A_7}}$ we see that any copy of 4_1 in $L_3(P_{I_{A_8}})$ is stuck underneath a 14, a 20_1 or a 20_2. //

4.4.2. Corollary. There is a non-split group extension 2^4A_8.

Proof. By 4.4.1. the image of the cup-product map

$$\mathrm{Ext}^1_{A_8}(I,6)\otimes\mathrm{Ext}^1_{A_8}(6,4_1)\to\mathrm{Ext}^2_{A_8}(I,4_1)\cong H^2(A_8,4_1)$$

is non-zero. //

We first examine the structure of $\begin{pmatrix}6\\4_1\\6\end{pmatrix}_{A_7}^{\uparrow A_8}$. From 2.5.3 we see that $P_{6_{A_8}}$ has a quotient module

$$\begin{array}{ccc} & 6 & \\ 4_1 & & 4_2 \\ & 6 & \\ & 14 & \end{array}$$

Hence $\begin{pmatrix}6\\4_1\\6\end{pmatrix}\uparrow_{A_7}^{A_8}$ has a quotient module $\begin{matrix}6\\4_1\\6\\14\end{matrix}$. Now from 3.2 we also know that it has a quotient $\begin{matrix}6\\14\end{matrix}$.

Hence $\begin{pmatrix}6\\4_1\\6\end{pmatrix}\uparrow_{A_7}^{A_8}\Big/\begin{pmatrix}6\\4_1\\6\end{pmatrix}\uparrow_{A_7}^{A_8} \cdot (JFA_8)^3$ $\overset{6}{\text{has}}$ Loewy series

$$\begin{array}{ccc} & 6 & \\ I & 4_1 & 14 \\ I & 6 & 6 \end{array}$$

and socle series

$$\begin{array}{ccccc} & & 6 & & \\ & 4_1 & & 14 & \\ I & I & & 6 & 6 \end{array}$$

If there were a copy of I in $L_4\left(\begin{pmatrix}6\\4_1\\6\end{pmatrix}\uparrow_{A_7}^{A_8}\right)$ then there would be a uniserial module $\begin{matrix}4_1\\6\\I\end{matrix}$ contradicting lemma 4.4.1. Hence there are two copies of I in $L_5\left(\begin{pmatrix}6\\4_1\\6\end{pmatrix}\uparrow_{A_7}^{A_8}\right)$ and hence the complete Loewy series is:

$$\begin{array}{ccccccc} & & & 6 & & & \\ & I & & 4_1 & & 14 & \\ & I & & 6 & & 6 & \\ 4_2 & & 14 & & 14 & & 20_1 \\ I & & I & & 6 & & 6 \\ & & 4_1 & & 14 & & \\ & & & 6 & & & \end{array}$$

Now we attack $\begin{pmatrix}4_1\\6\\4_2\end{pmatrix}_{A_7}\uparrow^{A_8}$. We know from Section 2.2 that $P_{(4_1)_{A_8}}$ has quotient module $\begin{matrix}4_1\\6\\4_2\end{matrix}$, and hence by 4.1.3 with $n=3$ we see that $L_3\left(\begin{pmatrix}4_1\\6\\4_2\end{pmatrix}_{A_7}\uparrow^{A_8}\right)$ has 4_2 in it. Now our knowledge of $\dim \mathrm{Ext}^1_{A_8}(4_1,-)$ together with the results of Sections 4.2 and 4.3 tell us that the Loewy series of $\begin{pmatrix}4_1\\6\\4_2\end{pmatrix}_{A_7}\uparrow^{A_8}$ is as follows:

$$\begin{array}{ccccc} & & 4_1 & & \\ & 4_2 & 6 & 20_1 & \\ I & 4_1 & 4_2 & 14 & \\ & I & 6 & & \\ & 4_1 & 14 & 20_2 & \\ & & 6 & & \\ & & 4_2 & & \end{array}.$$

Thus the Loewy series for P_{4_1} and P_{4_2} are as in Theorem 1. We can demonstrate our filtrations diagramatically as follows:

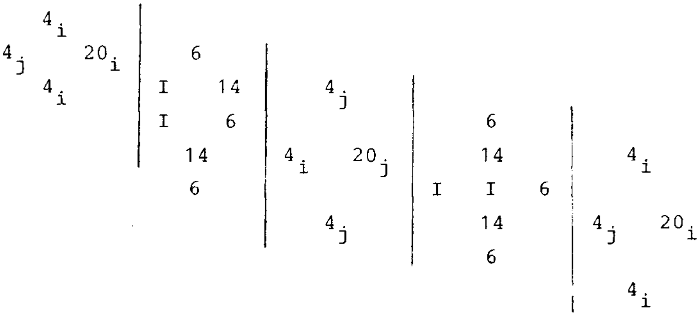


Figure 1

Now, since $L_4(P_{4_i})$ has 2 copies of 6 in it, $L_4(P_6)$ has two copies of each 4_i in it by Landrock's lemma. Thus the Loewy series of P_6 is as in Theorem 1, and our filtrations can be shown diagramatically as in Figure 2.

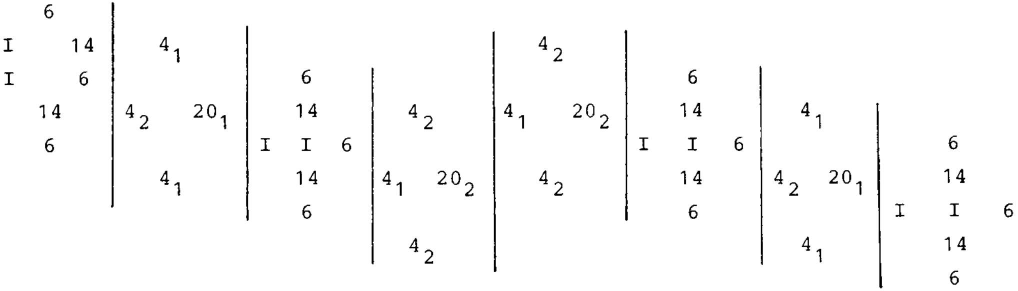

6
I 14
I 6
14
6
4₁
4₂ 20₁
4₁
6
14
I I 6
14
6
4₂
4₁ 20₂
4₂
4₂
4₁ 20₂
4₂
6
14
I I 6
14
6
4₁
4₂ 20₁
4₁
6
14
I I 6
14
6

Figure 2

4.5. The Loewy structure of $\begin{pmatrix} 1 \\ 14 \end{pmatrix}_{A_7}\uparrow^{A_8}$

Our filtration of this module looks like:

$$\begin{array}{c} \qquad\qquad I \\ \qquad\qquad 6 \\ \qquad\qquad I \\ \hline \begin{array}{ccc} 14 & & \\ 6 & & \\ 4_1 \quad 4_2 & \oplus & 64 \\ 6 & & \\ 14 & & \end{array} \end{array}$$

Now from 3.2 we know that P_I has a quotient module $\begin{matrix} I \\ 14 \\ 6 \end{matrix}$. Since $L_1(20_{A_7}\uparrow^{A_8}) = 20_1 \oplus 20_2$, an application of 4.1.3 to $P_{I_{A_7}}$ shows that $\begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7}\uparrow^{A_8}$ must have a quotient module $\begin{matrix} I \\ 14 \\ 6 \end{matrix}$. Thus $\begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7}\uparrow^{A_8}.e_0$ has Loewy series

$$\begin{array}{ccc} & I & \\ 6 & & 14 \\ I & & 6 \\ 4_1 & & 4_2 \\ & 6 & \\ & 14 & \end{array}$$

and $\begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7}\uparrow^{A_3} = \begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7}\uparrow^{A_8}.e_0 \oplus 64.$

By Thompson's Lemma on the ordinary characters of dimension 21, we see that

$$\dim \mathrm{Ext}^1_{A_8}(I, 20_i) \geqq 1, \qquad i = 1, 2.$$

Thus our argument also shows that the L_2 of $P_{I_{A_7}}\uparrow^{A_8}.e_0 = P_{I_{A_8}}$ is exactly $6 \oplus 14 \oplus 20_1 \oplus 20_2$. This completes the determination of $\dim \mathrm{Ext}^1_{A_8}(I, M)$ for M simple.

4.6. The Loewy structure of $20_{A_7}{\uparrow}^{A_8}$

We are now ready to complete the work of Section 3.4.

4.6.1. Lemma. $\left(\begin{matrix} I \\ 20_i \end{matrix},\ 20_{A_7}{\uparrow}^{A_8}\right)_{A_8} = 0, \quad i = 1,2.$

Proof. Since $P_{20_i}{\downarrow}_{A_7} = P_{20} \oplus P_{20} \oplus P_{20} \oplus P_{4_i}$ we see that $\begin{matrix} I \\ 20_i \end{matrix}$ remains indecomposable on restriction to A_7. Thus

$$\left(\begin{matrix} I \\ 20_i \end{matrix},\ 20_{A_7}{\uparrow}^{A_3}\right)_{A_8} = \left(\begin{matrix} I \\ 20 \end{matrix},\ 20\right)_{A_7} = 0. \quad //$$

4.6.2. Lemma. $\left(\begin{matrix} & I & \\ 20_1 & & 20_2 \end{matrix},\ 20_{A_7}{\uparrow}^{A_8}\right)_{A_8} = 1.$

Proof. Since $\dim \mathrm{Ext}^1_{A_7}(I,20) = 1$, we have

$$\left(\begin{matrix} & I & \\ 20_1 & & 20_2 \end{matrix}\right)_{A_8}{\downarrow}_{A_7} = \begin{matrix} I \\ 20 \end{matrix} \oplus 20.$$

Hence

$$\left(\begin{matrix} & I & \\ 20_1 & & 20_2 \end{matrix},\ 20_{A_7}{\uparrow}^{A_8}\right)_{A_8} = \left(\begin{matrix} I \\ 20 \end{matrix} \oplus 20,\ 20\right)_{A_7} = 1. \quad //$$

4.6.3. Lemma. $\mathrm{Rad}(20_{A_7}{\uparrow}^{A_8})/\mathrm{Soc}(20_{A_7}{\uparrow}^{A_8}) = 4_1 \oplus 4_2 \oplus X$, where X has composition factors $I + I + I + I + 14 + 14 + 20_1 + 20_2$.

Proof. Since $20_{A_7}{\uparrow}^{A_8}$ extends to a module for S_8, there is a subquotient $4_1 \oplus 4_2$. By self-duality and since $\dim \mathrm{Ext}^1_{A_8}(4_i, I) =$

$= \dim \mathrm{Ext}^1_{A_8}(4_i,14) = 0$ (Section 4.1), this means that $4_1 \oplus 4_2$ is a direct summand of $\mathrm{Rad}(20_{A_7}\uparrow^{A_8})/\mathrm{Soc}(20_{A_7}\uparrow^{A_8})$. //

4.6.4. Lemma. $\dim \mathrm{Ext}^1_{A_8}(14,20_i) = 0$.

Proof. Apply 4.1.3 to $P_{14_{A_7}}$ with $n=2$. //

4.6.5. Lemma. $\mathrm{Soc}(X) = I$.

Proof. Lemmas 4.6.1 and 4.6.2 show that there is exactly one copy of I in $\mathrm{Soc}(X)$. There can be no copies of 20_i in $\mathrm{Soc}(X)$ since $\dim \mathrm{End}_{A_8}(20_{A_7}\uparrow^{A_8}) = 4$ (Lemma 3.4.1). There can be no copies of 14 in $\mathrm{Soc}(X)$ by Lemma 4.6.4. //

Thus X has the form $\begin{matrix} I \\ Y \\ I \end{matrix}$ where Y has composition factors $I + I + 14 + 14 + 20_1 + 20_2$.

4.6.6. Lemma. $\mathrm{Soc}(Y) = 14$

Proof. Since $\dim \mathrm{Ext}^1_{A_8}(I,I) = 0$, $\mathrm{Soc}(Y)$ can contain no copies of I. Since Y is self-dual and extends to a module for S_8, if $\mathrm{Soc}(Y)$ contains a copy of 20_i, then 20_i is a summand of Y for $i=1,2$. But then the other direct summand would have to have Loewy series $\begin{matrix} & 14 & \\ I & & I \\ & 14 & \end{matrix}$ whereas $\dim \mathrm{Ext}^1_{A_8}(I,14) = 1$ from Section 4.5. //

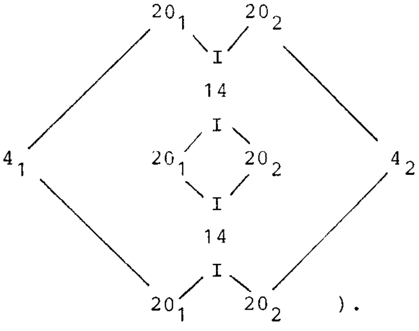
Thus by 4.6.4 the structure of Y is:

$$\begin{array}{c} 14 \\ I \\ 20_1 \quad 20_2 \\ I \\ 14 \end{array} .$$

Hence the Loewy series for $20_{A_7}\uparrow^{A_8}$ is

$$\begin{array}{c} 20_1 \quad 20_2 \\ I \quad 4_1 \quad 4_2 \\ 14 \\ I \\ 20_1 \quad 20_2 \\ I \\ 14 \\ I \\ 20_1 \quad 20_2 \end{array}$$

(i.e. the "diagram" is



).

4.7. The remaining projective indecomposable modules

From Section 4.6 we have a quotient of P_{20_1} with Loewy series

20_1

I 4_1

14

I

20_1 20_2

I

14

I

20_1 20_2 .

Since this accounts for all the copies 20_1 and 20_2 in P_{20_1}, this means that the Loewy length is at least 13.

By Landrock's Lemma we see that P_{20_1} has a copy of 6 in its L_4 and L_6, and a copy of 4_2 in its L_5, and 4_1 in its L_7.

Thus all the composition factors are accounted for and the Loewy structure of P_{20_i} is as given in Theorem 1.

Hence the appropriate diagram for our filtration of $P_{20_1} \oplus P_{20_2}$ is as follows:

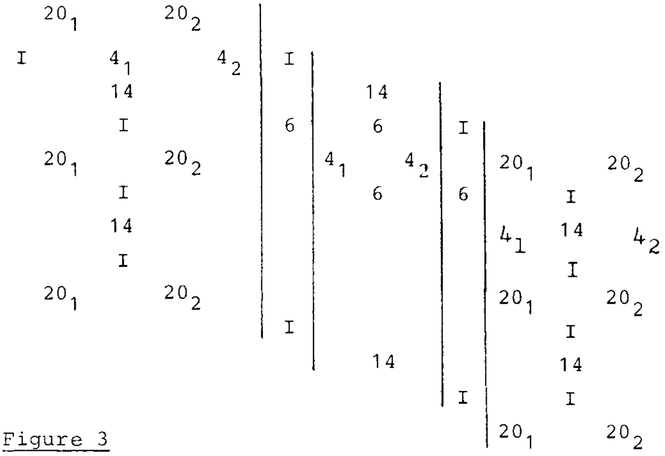


Figure 3

Now we have enough information to see that P_I has the Loewy series given in Theorem 1, and the appropriate diagram for our filtration is as given in figure 4.

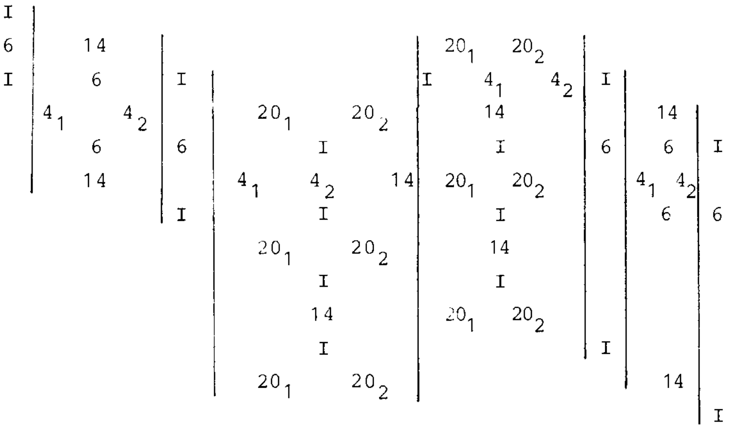


Figure 4

<u>4.7.1. Lemma.</u> $\begin{pmatrix}14\\14\end{pmatrix}_{A_7}\uparrow^{A_8}.e_0$ has Loewy series

$$\begin{array}{ccccc} & & 14 & & \\ & & 6 & & \\ 4_1 & & 4_2 & & 14 \\ & 6 & & 6 & \\ 4_1 & & 4_2 & & 14 \\ & & 6 & & \\ & & 14 & & \end{array}$$

<u>Proof</u>. From Section 3.2 we know there is a module $\begin{smallmatrix}14\\6\\14\end{smallmatrix}$. Thus applying 4.1.3 with $n=3$ to $P_{14_{A_7}}$ we see that $\begin{pmatrix}14\\14\end{pmatrix}_{A_7}\uparrow^{A_8}.e_0$ has $\begin{smallmatrix}14\\6\\14\end{smallmatrix}$ as quotient. Thus it also has $\begin{smallmatrix}14\\6\\14\end{smallmatrix}$ as a submodule, and so since $\dim \mathrm{Ext}^1_{A_8}(6,6) = 0$ from Section 4.3 the result follows. //

This now gives us enough information to see that the Loewy series for P_{14} given in Theorem 1 is correct, and the appropriate diagram for our filtration is as in Figure 5.

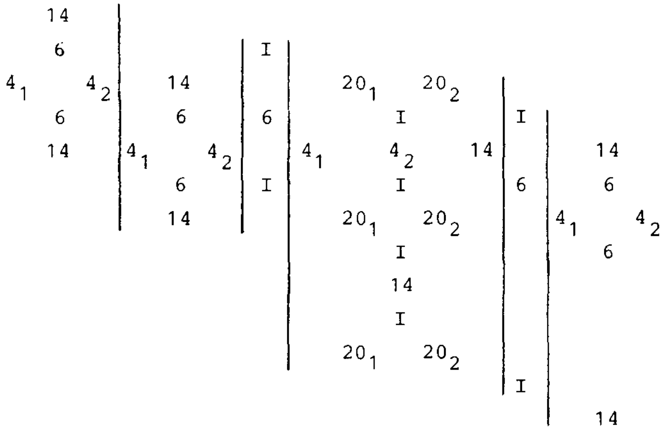


Figure 5

This completes the proof of Theorem 1, and the determination of $\dim \mathrm{Ext}^1_{A_8}(M,N)$ for M and N simple. This information is displayed in Appendix 4.

Notation for character tables

The only irrationalities we come across in our character tables are:

$$bn = \begin{cases} \frac{1}{2}(-1+\sqrt{n}) & \text{if} \quad n \equiv 1 \pmod 4 \\ \frac{1}{2}(-1+i\sqrt{n}) & \text{if} \quad n \equiv 3 \pmod 4 \end{cases}$$

i.e. the "Gauss sum" of half the primitive n^{th} roots of unity.

Under the column headed "ind" is given the Frobenius-Schur indicator of the representation, namely

- \+ if the representation is orthogonal
- \- if the representation is symplectic but not orthogonal

0 if the representation is neither symplectic nor orthogonal.

(In characteristic 0 this is $\frac{1}{|G|} \sum_{g\in G} \chi(g^2)$.)

The top row carries the centralizer orders.

Acknowledgement

I would like to thank Dr. P. Landrock and the Matematisk Institut of Aarhus Universitet for their generous help, and the Royal Society, Great Britain, through whom I have been financially supported whilst undertaking this work.

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Appendix 1. Characters of A_8

(i) Ordinary characters

	20160	192	96	180	18	16	8	15	12	6	7	7	15	15	
p	power	A	A	A	A	A	B	A	AB	BA	A	A	AA	AA	
p'	part	A	A	A	A	A	A	A	AB	BA	A	A	AA	AA	S8
ind	1A	2A	2B	3A	3B	4A	4B	5A	6A	6B	7A	B**	15A	B**	fusion
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:
+	7	-1	3	4	1	-1	1	2	0	-1	0	0	-1	-1	:
+	14	6	2	-1	2	2	0	-1	-1	0	0	0	-1	-1	:
+	20	4	4	5	-1	0	0	0	1	1	-1	-1	0	0	:
+	21	-3	1	6	0	1	-1	1	-2	0	0	0	1	1	:
0	21	-3	1	-3	0	1	-1	1	1	0	0	0	b15	**	.
0	21	-3	1	-3	0	1	-1	1	1	0	0	0	**	b15	|.
+	28	-4	4	1	1	0	0	-2	1	-1	0	0	1	1	:
+	35	3	-5	5	2	-1	-1	0	1	0	0	0	0	0	:
0	45	-3	-3	0	0	1	1	0	0	0	b7	**	0	0	.
0	45	-3	-3	0	0	1	1	0	0	0	**	b7	0	0	|.
+	56	8	0	-4	-1	0	0	1	0	-1	0	0	1	1	:
+	64	0	0	4	-2	0	0	-1	0	0	1	1	-1	-1	:
+	70	-2	2	-5	1	-2	0	0	-1	1	0	0	0	0	:

(ii) 2 - modular characters

	20160	180	18	15	7	7	15	15	
p	power	A	A	A	A	A	AA	AA	
p'	part	A	A	A	A	A	AA	AA	S8
ind	1A	3A	3B	5A	7A	B**	15A	B**	fusion
+	1	1	1	1	1	1	1	1	:
0	4_1	-2	1	-1	-b7	**	-b15	**	.
0	4_2	-2	1	-1	**	-b7	**	-b15	|.
+	6	3	0	1	-1	-1	-2	-2	:
+	14	2	-1	-1	0	0	2	2	:
0	20_1	-4	-1	0	-1	-1	b15-1	**	.
0	20_2	-4	-1	0	-1	-1	**	b15-1	|.
+	64	4	-2	-1	1	1	-1	-1	:

(iii) Decomposition Matrix

$$\begin{array}{r|cccccccc|}
 & 1 & 4_1 & 4_2 & 6 & 14 & 20_1 & 20_2 & 64 \\
\hline
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
7 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \\
14 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \\
20 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \\
21 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \\
21 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \\
21 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \\
28 & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \\
35 & 1 & 1 & 1 & 2 & 1 & \cdot & \cdot & \\
45 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & \\
45 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \\
56 & 2 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \\
70 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & \\
64 & & & & & & & & 1 \\
\hline
\end{array}$$

(iv) Cartan Matrix

$$\begin{array}{r|cccccccc|}
 & 1 & 4_1 & 4_2 & 6 & 14 & 20_1 & 20_2 & 64 \\
\hline
1 & 16 & 4 & 4 & 8 & 8 & 6 & 6 & \\
4_1 & 4 & 5 & 4 & 6 & 4 & 2 & 1 & \\
4_2 & 4 & 4 & 5 & 6 & 4 & 1 & 2 & \\
6 & 8 & 6 & 6 & 12 & 8 & 2 & 2 & \\
14 & 8 & 4 & 4 & 8 & 8 & 3 & 3 & \\
20_1 & 6 & 2 & 1 & 2 & 3 & 4 & 2 & \\
20_2 & 6 & 1 & 2 & 2 & 3 & 2 & 4 & \\
64 & & & & & & & & 1 \\
\hline
\end{array}$$

Appendix 2. Characters of A_7

(i) Ordinary characters

	2520	24	36	9	4	5	12	7	7	
p	power	A	A	A	A	A	AA	A	A	
p'	part	A	A	A	A	A	AA	A	A	S7
ind	1A	2A	3A	3B	4A	5A	6A	7A	B**	fusion
+	1	1	1	1	1	1	1	1	1	:
+	6	2	3	0	0	1	-1	-1	-1	:
0	10	-2	1	1	0	0	1	b7	**	.
0	10	-2	1	1	0	0	1	**	b7	|
+	14	2	2	-1	0	-1	2	0	0	:
+	14	2	-1	2	0	-1	-1	0	0	:
+	15	-1	3	0	-1	0	-1	1	1	:
+	21	1	-3	0	-1	1	1	0	0	:
+	35	-1	-1	-1	1	0	-1	0	0	:

(ii) 2 - Modular characters

	2520	36	9	5	7	7	
p	power	A	A	A	A	A	
p'	part	A	A	A	A	A	S7
ind	1A	3A	3B	5A	7A	B**	fusion
+	1	1	1	1	1	1	:
0	4_1	-2	1	-1	-b7	**	.
0	4_2	-2	1	-1	**	-b7	.
+	6	3	0	1	-1	-1	:
+	14	2	-1	-1	0	0	:
-	20	-4	-1	0	-1	-1	:

(iii) Decomposition Matrix

	1	14	20	4_1	4_2	6
1	1	.	.			
15	1	1	.			
21	1	.	1			
35	1	1	1			
14	.	1	.			
6				.	.	1
10				.	1	1
10				1	.	1
14				1	1	1

(iv) Cartan Matrix

	1	14	20	4_1	4_2	6
1	4	2	2			
14	2	3	1			
20	2	1	2			
4_1				2	1	2
4_2				1	2	2
6				2	2	4

Appendix 3. Characters of A_6

(i) Ordinary characters

	360	8	9	9	4	5	5	
p	power	A	A	A	A	A	A	
p'	part	A	A	A	A	A	A	S6
ind	1A	2A	3A	3B	4A	5A	B*	fusion
+	1	1	1	1	1	1	1	:
+	5	1	2	-1	-1	0	0	:
+	5	1	-1	2	-1	0	0	:
+	8	0	-1	-1	0	-b5	*	.
+	8	0	-1	-1	0	*	-b5	.
+	9	1	0	0	1	-1	-1	:
+	10	-2	1	1	0	0	0	:

(ii) 2 - Modular characters

	360	9	9	5	5
p	power	A	A	A	A
p'	part	A	A	A	A
ind	1A	3A	3B	5A	B*
+	1	1	1	1	1
-	4_1	1	-2	-1	-1
-	4_2	-2	1	-1	-1
+	8_1	-1	-1	-b5	*
+	8_2	-1	-1	*	-b5

(iii) Decomposition Matrix

	1	4_1	4_2	8_1	8_2
1	1	.	.		
5	1	1	.		
5	1	.	1		
9	1	1	1		
10	2	1	1		
8				1	
8					1

(iv) Cartan Matrix

	1	4_1	4_2	8_1	8_2
1	8	4	4		
4_1	4	3	2		
4_2	4	2	3		
8_1				1	
8_2					1

Appendix 4

$\dim \mathrm{Ext}^1_{A_8}(M,N)$ for M,N simple.

	1	4_1	4_2	6	14	20_1	20_2	64
1	0	0	0	1	1	1	1	
4_1	0	0	1	1	0	1	0	
4_2	0	1	0	1	0	0	1	
6	1	1	1	0	1	0	0	
14	1	0	0	1	0	0	0	
20_1	1	1	0	0	0	0	0	
20_2	1	0	1	0	0	0	0	
64								0

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