THE LOEWY STRUCTURE OF THE PROJECTIVE INDECOMPOSABLE MODULES FOR $A_8$ IN CHARACTERISTIC 2

D. Benson
Matematisk Institut
Aarhus Universitet
NY Munuegade
8000 Aarhus C
DANMARU

Introduction and notation

The purpose of this paper is to establish the Loewy series for the projective modules for $A_8 \cong L_4(2)$ over a splitting field of characteristic 2.

Throughout, we shall let $F$ denote a splitting field in characteristic 2 for $A_8$ and all its subgroups, and let $(S,R,F)$ denote a splitting 2-modular system for $A_8$. We denote each simple module for a group by its dimension, together with a subscript if there is more than one simple module of that dimension. $A_8$ denotes the alternating group on 8 letters, a simple group of order $8!/2 = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. $A_7$ denotes a subgroup of index 8 stabilizing a point, and $A_6$ denotes a subgroup of $A_7$ of index 7 stabilizing a further point.

Thus the simple $FA_8$-modules are denoted $I$, $4_1$, $4_2$, $6$, $14$, $20_1$, $20_2$ and $64$. These fall into two blocks: $64$ is in a block of defect 0, while the rest are in the principal
block. Since blocks of defect 0 are easy to describe, we shall only be interested in the principal block. We denote the central idempotent for the principal block of $\Lambda_{0}$ by $e_0$.

The main result of this paper is the following theorem:

**Theorem 1.** The Loewy structures of the projective indecomposable modules for $\mathsf{FA}_8.e_0$ are as follows, where $\{i,j\} = \{1,2\}$:

\[
\begin{array}{cccc}
I & 6 & 14 & 20_1 & 20_2 \\
I & 14 & 20_1 & 20_2 & 6 \\
I & 14 & 20_1 & 20_2 & 6
\end{array}
\]
If $A$ is a group algebra over $F$ and $M$ is a finitely generated $A$-module, we write $JA$ for the Jacobson radical of $A$, and

$$L_i(M) = M/(JA)^{i-1} / M/(JA)^i$$

is the $i^{th}$ Loewy Layer of $M$. The Loewy Length of $M$ is the smallest number $i$ such that $M/(JA)^i = 0$, and the Loewy Structure for $M$ is a diagram whose $i^{th}$ layer downwards gives the simple summands of $L_i(M)$ with multiplicities (see for example Theorem 1). The Head of a module is the first Loewy layer.

Let $\text{Soc}(M)$ denote the socle of $M$, namely the sum of all the simple $A$-submodules of $M$. Let $S_1(M) = \text{Soc}(M)$ and

$$S_i(M)/S_{i-1}(M) = \text{Soc}(M/S_{i-1}(M)).$$

Then

$$0 < S_1(M) < S_2(M) < \cdots < S_{i-1}(M) < S_i(M) = M$$

is called the Socle Series of $M$. 
We shall write \((M,N)_A\) for \(\dim_F \text{Hom}_A(M,n)\), \(M^*\) for \(\text{Hom}_F(M,F)\) regarded as an \(A\)-module, and \(P_M\) for the projective cover of \(M\). Homomorphisms will usually be written on the right.

The exterior \(n^{th}\) power of \(M\) will be written \(M^{n^*}\).

Our main tools are the following lemmas, together with the easy but powerful lemmas discussed in Section 4.1.

Lemma 1 (Scott [4]). Any endomorphism of an \(FG\)-permutation module can be lifted to an endomorphism of the corresponding \(RG\)-permutation module. Thus direct summands of \(FG\)-permutation modules lift, and so their endomorphisms.

Lemma 2 (Frobenius Reciprocity). Let \(H \leq G\), \(M\) and \(FH\)-module and \(N\) an \(FG\)-module. Then \((M,N^+_FH)^+_FH = (M^+_F,G,N)_F\) and \((N^+_FH,M)^+_FH = (N,M^+_FG)_F\).

Lemma 3 (Thompson [5]). If \(M\) is an irreducible \(SG\)-module, then an \(R\)-form \(\hat{M}\) may be found such that the modular reduction \(\overline{M} = \hat{M} \otimes_R F\) has any given composition factor as its unique top factor.

Lemma 4 (Landrock [3]). Let \(M\) and \(N\) be simple \(FG\)-modules. Then the multiplicity of \(M\) in \(L_1(P_N)\) is the same as the multiplicity of \(N^*\) in \(L_1(P_{M^*})\).

Lemma 5 (Mackey Decomposition). Let \(H,K \leq G\) and \(M\) an \(FH\)-module. Then

\[
M^+_G K = \bigotimes_{H \times K} M \otimes x^+_H x^+_K,
\]

where \(x^+_H\) and \(x^+_K\) are the natural embeddings of \(H\) and \(K\) into \(G\).
where $x$ runs over a set of $H$-$K$ double coset representatives in $G$.

In Section 1 and Appendices 1-3 we collect some known results about $A_6$, $A_7$ and $A_8$. In Section 2 we examine the structure of the permutation modules for $FA_8$ on the cosets of maximal subgroups, and in Section 3 we examine the $FA_3$-modules induced up from simple $FA_7$-modules.

Section 4 is the main body of the paper, and this uses the results of the previous sections to deduce Theorem 1.

Section 1. Preliminary results on $A_6$, $A_7$ and $A_8$

1.1. Characters and subgroups of $A_8$

In this section we collect together some known facts about the group $A_8$ and some of its subgroups. In Appendix 1 we give the ordinary and 2-modular character tables of $A_8$, the decomposition matrix and the Cartan matrix. These can be extracted from James [2]. We also note the isomorphism $A_8 \simeq L_4(2)$, the group of $4 \times 4$ matrices over $GF(2)$.

We shall have cause to look at the following maximal subgroups:

<table>
<thead>
<tr>
<th>'Structure'</th>
<th>Index</th>
<th>$A_8$-name</th>
<th>$L_4(2)$-name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_7$</td>
<td>8</td>
<td>point</td>
<td></td>
</tr>
<tr>
<td>$2^3:L_3(2)$</td>
<td>15</td>
<td>-</td>
<td>point</td>
</tr>
<tr>
<td>$2^3:L_3(2)$</td>
<td>15</td>
<td>-</td>
<td>hyperplane</td>
</tr>
<tr>
<td>$S_6$</td>
<td>28</td>
<td>pair</td>
<td>symplectic form</td>
</tr>
<tr>
<td>$2^4:(S_3 \times S_3)$</td>
<td>35</td>
<td>4+4 splitting</td>
<td>2-dimensional subspace</td>
</tr>
<tr>
<td>$(A_5 \times 3).2$</td>
<td>56</td>
<td>triple</td>
<td>$GF(4)$-structure</td>
</tr>
</tbody>
</table>
The Schur multiplier of $A_8$ has order 2, so that $\dim \operatorname{Ext}^2_{A_8}(I, I) = 1$. The automorphism group of $A_8$ is exactly $S_8$, and the outer automorphism acts as the graph automorphism on $L_4(2)$ (namely transpose inverse on matrices).

Thus the two classes of subgroups $2^3: L_3(2)$ are conjugate under the action of this outer automorphism.

1.2. \textbf{Results on $A_7$}

In Appendix 2 we give the ordinary and 2-modular character tables of $A_7$, the decomposition matrix and Cartan matrix (see James [2]). The 6-dimensional irreducible $F A_7$-module is a direct summand of the permutation module on cosets of $A_6'$, and the 14-dimensional irreducible is a direct summand of the permutation module on the 21 coset of an $S_5$ preserving a 5+2 splitting of the 7 points; this module splits $1 \oplus 6 \oplus 14$. The permutation module on 35 cosets of an $(A_4 \times 3).2$ preserving a 4+3 splitting of the 7 points has structure:

$$6 \quad I \oplus 14 \oplus 4_1 \oplus 4_2.$$

The structures of the projective indecomposable modules in the principal block are:

$$
\begin{array}{ccc}
I & 14 & 20 \\
14 & 20 & I \\
I & \oplus & 14 \oplus 20 & 14 \\
20 & 14 & I & I \\
I & 14 & 20 \\
\end{array}
$$

and in the non-principal block:
If we take the 64-dimensional defect 0 representation of $A_8$ (which is the Steinberg representation of $L_4(2)$), the restriction to $A_7$ is exactly $P_{14}$, as can be checked by Brauer characters. Apart from this, every irreducible representation of $A_8$ remains irreducible upon restriction to $A_7$.

1.3 Results on $A_6$

In Appendix 3 we give the ordinary and 2-modular character tables of $A_6$, the decomposition matrix and the Cartan matrix (see James [2]). There are three blocks, namely the principal block and two blocks of defect 0. The structures of the projective indecomposables are as follows:

$$
\begin{array}{ccc}
4_1 & 4_2 & 6 \\
6 & 6 & 4_1 & 4_2 \\
4_2 & 4_1 & 6 & 6 \\
6 & 6 & 4_2 & 4_1 \\
4_1 & 4_2 & 6 \\
\end{array}
$$

$$
\begin{array}{ccc}
P_{4_1} & P_{4_2} & P_6 \\
\end{array}
$$

(Erdmann [1]).
1.4 **Induction and restriction between $A_6$ and $A_7$**

Brauer characters show that

1.4.1  \[(4_1)_{A_7} \downarrow A_6 = (4_2)_{A_7} \downarrow A_6 = (4_2)_{A_6}.\]

(beware!)

The composition factors of $6_{A_7} \downarrow A_6$ are $I + I + 4_1$. But

\[(I, 6_{A_7} \downarrow A_6)_{A_6} = (I, 6_{A_6} \downarrow A_7)_{A_7} = 1.\]

Since $\text{Ext}_{A_7}^1(I, I) = 0$, this means that

1.4.2  
\[
\begin{align*}
\begin{array}{c}
6_{A_7} \downarrow A_6 = 4_1. \\
I
\end{array}
\end{align*}
\]

The composition factors of $14_{A_7} \downarrow A_6$ are $I + I + 4_1 + 4_1 + 4_2$. But

\[(I, 14_{A_7} \downarrow A_6)_{A_6} = (I, 14_{A_6} \downarrow A_7)_{A_7} = 0,\]

and so the only possibility is

1.4.3  \[
\begin{align*}
\begin{array}{c}
14_{A_7} \downarrow A_6 = 4_2. \\
I
\end{array}
\end{align*}
\]

The composition factors of $20_{A_7} \downarrow A_6$ are $4_2 + 8_1 + 8_2$. Since the constituents are in different blocks, we have

1.4.4  \[
\begin{align*}
\begin{array}{c}
20_{A_7} \downarrow A_6 = 4_2 \oplus 8_1 \oplus 8_2. \\
I
\end{array}
\end{align*}
\]

Since \[(I, \left(\begin{smallmatrix} 6 \\ 4_1 \end{smallmatrix}\right)_{A_7} \downarrow A_6, A_6)_{A_7} = (I, 4_1)_{A_6} \downarrow A_7, (4_1)_{A_7} = 0,\]

we have

1.4.5  
\[
\begin{align*}
\begin{array}{c}
\left(\begin{smallmatrix} 6 \\ 4_1 \end{smallmatrix}\right)_{A_7} \downarrow A_6 = I. \\
4_2
\end{array}
\end{align*}
\]

The composition factors of $(4_2)_{A_6} \downarrow A_7$ are $4_1 + 4_2 + 20$. 
Since this module is self-dual and extends to a module for $S_6$ and $S_7$, the only possibility is

1.4.6 \[(4_2)_{A_6}^{A_7} = 4_1 \oplus 4_2 \oplus 20.\]

Since \(\left(4_1 \ 6 \ 4_2\right)_{A_7} A_6 \subseteq \left(4_1 \ 6 \ 4_2\right), \quad (4_2)_{A_6}^{A_7} A_7 = 0\)
the only possibility is

1.4.7 \[\left(4_1 \ 4_2\right)_{A_7} A_6 = 4_1^{4_1} \oplus 4_2.\]

The composition factors of \(4_1)_{A_6}^{A_7}\) are \(14 + 14\) and so,
since \((14, 4_1)_{A_6}^{A_7}) A_7 = (14, 4_1)_{A_6} A_6 A_6 = 1, \) we have

1.4.8 \[\left(4_1\right)_{A_6}^{A_7} = 14.

Finally, since \((I, (\frac{1}{20})_{A_7} A_6) A_6 = (I, A_7^{A_7}, (\frac{1}{20})) A_7 = 0, \) we
have from 1.4.4

1.4.9 \[\left(\frac{1}{20}\right)_{A_7} A_6 = I \oplus 4_1 \oplus 4_2.\]

Section 2. Some permutation modules for $A_8$

2.1 Permutations on the 8 cosets of $A_7$

Ordinary character: \(1 + 7.\)

Hence the composition factors of this $FA_8$-module $M_8$ are
\(I + I + 6.\) Frobenius reciprocity shows that $L_1(M_8) \cong S_1(M_8) \cong I,$
and so the structure is
2.2 Permutations on the 15 cosets of $2^3: L_3(2)$

Ordinary characters: $1 + 14$ for each of the two classes.

Thus the composition factors of these $FA_3$-modules $M_{15a}$ and $M_{15b}$ are $I + 4_1 + 4_2 + 6$. Since 15 is odd, these modules have $I$ as a direct summand. Frobenius reciprocity shows that in one case the head is $I + 4_1$ and the socle is $I + 4_2$, whereas in the other case the head is $I + 4_2$ and the socle is $I + 4_1$. Thus the structures are

$$M_{15a} = I \oplus 6 \quad M_{15b} = I \oplus 6.$$ 

2.3 Permutations on the 28 cosets of $S_6$

Ordinary character: $1 + 7 + 20$.

Thus the composition factors of this $FA_6$-module $M_{28}$ are $I + I + 6 + 6 + 14$. By Scott's Lemma, the endomorphism ring has dimension 3.

Since $M_{28} = (M_6)^{2-}$, it has a submodule $I \wedge 6$ of structure $I$. By Frobenius reciprocity, $S_1(M_{28}) \cong L_1(M_{28}) \cong I \oplus 6$.

2.3.1. Lemma. $6^{2-} \cong I \oplus 14$.

Proof. The composition factors are $I + 14$, and the module is self-dual. //
Thus $M_{28}$ has $I \oplus 14$ as a subquotient. Since it is self-dual, this means the Loewy structure of $M_{28}$ is

\[
\begin{array}{c|c}
I & 6 \\
2.3.2 & I \\
 & 14 \\
 & 6 \\
\end{array}
\]

(i.e. the "diagram" for $M_{28}$ is $I \quad 14 \quad 6 \quad I$).

2.4 Permutations on the 35 cosets of $2^4(S_3 \times S_3)$

Ordinary characters: $1 + 14 + 20$.

Thus the composition factors of this $FA_8$-module $M_{35}$ are $I + 4_1 + 4_2 + 6 + 6 + 14$. Since 35 is odd, $I$ is a direct summand.

Frobenius reciprocity shows that $S_1(M_{35}) \cong L_1(M_{35}) \cong I \oplus 6$.

Since $M_{35}$ extends to a module for $S_8$, there is a subquotient $4_1 \oplus 4_2$. Since the module is self-dual, this forces the structure to be

\[
M_{35} = I \oplus 4_1 \quad 4_2 \quad 14.
\]

2.5. Permutations of the 56 cosets of $(A_5 \times 3).2$

Ordinary characters: $1 + 7 + 20 + 28$.

Thus the composition factors of this $FA_8$-module $M_{56}$ are $I + I + 4_1 + 4_2 + 6 + 6 + 6 + 14 + 14$.

2.5.1. Lemma. $M_{56}$ has a direct summand isomorphic to the module $M_8$ described in 2.1.

Proof. We construct maps $\alpha: M_8 \rightarrow M_{56}$ and $\beta: M_{56} \rightarrow M_8$ as follows:
\[ a: \text{ point } x \rightarrow \text{sum of triples containing } x \]

\[ \beta: \text{ triple } \{a, b, c\} \rightarrow a + b + c. \]

Then

\[ a\beta: \text{ point } x \rightarrow 21x + 6. \sum_{y \neq x} y = x \]

since we are in characteristic 2.

Hence \( a\beta = 1 \), and so \( \beta a \) is a projection and \( M_{56} \) splits as

\[ M_{56} = \text{Im}(\beta a) \oplus \text{Ker}(\beta a). \]

So

\[ M_{56} = M_8 \oplus M'_{56} \quad \text{where } M'_{56} = \text{Ker}(\beta a). \]

2.5.2 \( M_{56} = M_8 \oplus M'_{56} \quad \text{where } M'_{56} = \text{Ker}(\beta a). \) //

Now \( M'_{56} \) has composition factors \( 4_1 + 4_2 + 6 + 6 + 14 + 14 \).

By Frobenius reciprocity, \( S_1(M_{56}) \cong L_1(M_{56}) \cong 1 \oplus 14 \) and so

\[ S_1(M'_{56}) \cong L_1(M'_{56}) \cong 14. \]

Next we notice that \( M_{56} = (M_8)^3 \), so that it reduces at least as far as

\[ 6^2 \]

\[ 6 \oplus 6^3. \]

\[ 6^2. \]

2.5.3. Lemma. \( 6^3 \) has structure \( 4_1 6 4_2 \).

Proof. The composition factors of \( 6^3 \) are \( 4_1 + 4_2 + 6 + 6 \).

The module is self-dual and extends to a module for \( S_8 \). Hence either the lemma holds or \( 6^3 \cong 6 \oplus 6 \oplus 4_1 \oplus 4_2 \). If so, then this is still true as modules for \( A_7 \). But for \( A_7 \), \( (160)^3 \cong 6^2 \oplus 6^3 \) is a permutation module, and so \( 4_1 \) and \( 4_2 \) would be direct sum-
mands of a permutation module. But they do not lift to \( RA_7 \)-modules, contradicting Scott's lemma. //

But now this means that \( M_5^{14} \) has a subquotient isomorphic to \( 6^{3^-} \), and so it has Socle and Loewy Series

\[
\begin{array}{c}
\text{Socle} \\
14 \\
6 \\
4_1 \\
4_2 \\
6 \\
14 \\
\end{array}
\]

Hence

\[
\begin{array}{c}
\text{Socle} \\
14 \\
6 \\
4_1 \\
4_2 \\
6 \\
14 \\
\end{array}
\]

Section 3. The induced modules from simple \( A_7 \)-modules

As we have already noted, the restrictions of simple \( A_8 \)-modules to \( A_7 \) are as follows:

\[
\begin{align*}
I_{A_8} \uparrow A_7 &= I_{A_7} \\
(4_1)_{A_8} \uparrow A_7 &= (4_1)_{A_7} \\
(4_2)_{A_8} \uparrow A_7 &= (4_2)_{A_7} \\
6_{A_8} \uparrow A_7 &= 6_{A_7} \\
14_{A_8} \uparrow A_7 &= 14_{A_7} \\
(20)_{A_8} \uparrow A_7 &= (20)_{A_7} \\
64_{A_8} \uparrow A_7 &= P_{14} A_7.
\end{align*}
\]

By Frobenius reciprocity, this tells us the socle and first Loewy layer of modules induced from \( A_7 \).

We dealt with \( I_{A_7} \uparrow A_8 \) in Section 2.1, and so we only consider non-trivial simple modules here.
3.1. \((4_1) \overset{A_7}{\oplus} A_8\) and \((4_2) \overset{A_7}{\oplus} A_8\)

The composition factors of \((4_1) \overset{A_7}{\oplus} A_8\) are \(4_1 + 4_1 + 4_2 + 20_1\), and \(S_1 \cong L_1 \cong (4_1) \overset{A_7}{\oplus} A_8\). Since \((4_1) \overset{A_7}{\oplus} A_8\) is the dual of \(4_2\), and also the image of \(4_2\) under the \(S_7\)-automorphism of \(A_7\), this means the Socle and Loewy Series are:

\[
\begin{align*}
3.1.1. & & (4_1) \overset{A_7}{\oplus} A_8 &= 4_2 4_1 20_1 & & (4_2) \overset{A_7}{\oplus} A_8 &= 4_1 4_2 20_2 \\
3.2. & & \text{The module } 6 \overset{A_7}{\oplus} A_8.
\end{align*}
\]

This has composition factors \(1 + 1 + 6 + 6 + 6 + 14 + 14\) and \(S_1 \cong L_1 \cong 6 \overset{A_7}{\oplus} A_8\).

\[
3.2.1. \text{Lemma. There is a homomorphism from } M_{28} \text{ to } 6 \overset{A_7}{\oplus} A_8 \text{ with one-dimensional kernel.}
\]

\textbf{Proof.} From Section 2.3 we see that since \(6 \overset{A_7}{\oplus} A_8\) is in a different block from \(I \overset{A_7}{\oplus} A_7\) and \(14 \overset{A_7}{\oplus} A_7\), \(M_{28} \overset{A_7}{\oplus} A_7\) is semisimple. Thus

\[
(M_{28} \overset{A_7}{\oplus} A_8) \overset{A_8}{\oplus} (M_{28} \overset{A_7}{\oplus} A_7) = 6 \overset{A_7}{\oplus} A_8 = 2.
\]

Thus from what we know of the structure of \(M_{28}\) since \(S_1(6 \overset{A_7}{\oplus} A_8) = 6 \overset{A_7}{\oplus} A_8\) there must be a homomorphism with kernel the trivial submodule of \(M_{28}\). //

Thus by self-duality, there is a submodule \(I \oplus 6 \overset{14}{\oplus} A_7\) and a quotient module \(6 \overset{14}{\oplus} I \oplus 14\).
3.2.2. Lemma. $6^A_{A_7}$ has exactly one copy of $I$ in its second Loewy layer.

Proof. We certainly know that there is at least one, by 3.2.1. Suppose that there is more than one. Since $6^A_{A_7}$ is not in the principal block, this means that $L_1(6^A_{A_7}, I_{A_7})$ has more than one copy of $I$ in it. However,

$$(6^A_{A_7}, I_{A_7})_{A_7} = (6^A_{A_7}, I_{A_7})_{A_7} \leq 1,$$

a contradiction. //

This forces the Loewy length to be at least 4, and since it is self-dual, we are left with only one possibility, namely the that the Loewy Series is

$$
\begin{array}{ccc}
6 & 14 \\
I & 6 \\
14 \\
6
\end{array}
$$

(i.e. the "diagram" for $6^A_{A_7}$ is).

3.2.3 The module $14^A_{A_7}$

This has composition factors $4_1 + 4_2 + 6 + 6 + 14 + 14 + 64$ and $S_1 \cong L_1 \cong 14 \oplus 64$. Since 64 is projective, this module is a direct sum of 64 and a module with $S_1 \cong L_1 \cong 14$. 

3.3.1. Lemma. \((M_{56}, 14\rightarrow A_7 + A_8) A_8 = 2.\)

Proof. \((M_{56}, 14\rightarrow A_7 + A_8) A_8 = (M_{56}, 14\rightarrow A_7) A_8.\) But \(M_{56}, 14\rightarrow A_7\) is the direct sum of the permutation module on 21 cosets of an \(S_5\) fixing a 5+2 splitting of the 7 points, and the permutation module on the 35 cosets of an \((A_4 \times 3).2\) fixing a 4+3 splitting of the 7 points. The lemma now follows from Section 1.2. //

Now from the structure of \(M_{56}\) given in 2.5.5 it follows that every such homomorphism must kill \(\text{Im}(8a)\), and some such homomorphism is an injection from \(M_{56}\) into \(14\rightarrow A_7 + A_8.\) Thus

\[
3.3.2\quad 14\rightarrow A_7 + A_8 \cong M_{56} \oplus 64 \quad (= 4_1 \oplus 4_2 \oplus 64).
\]

3.4. The module \(20\rightarrow A_8\)

This has composition factors \(1 + 1 + 1 + 1 + 4_1 + 14 + 14 + 20_1 + 20_1 + 20_2 + 20_2 + 20_2\) and \(S_1 \cong L_1 \cong 20_1 \oplus 20_2.\)

3.4.1. Lemma. \((20,20\rightarrow A_7) A_8 = 4.\)

Proof. \((20,20\rightarrow A_7) A_8 = (20,20) A_7 + (20, A_7 + A_6, 20, A_7 + A_6) A_6\) by the Mackey decomposition theorem

\[
= 1 + 3 \quad \text{by 1.4.4.}
\]

The lemma now follows easily. //

We shall complete the determination of the structure of this module in Section 4.4.
Section 4. More induced modules from $A_7$: the final assault

4.1. Induction of projective modules

By Brauer characters, we see that

$$P_{I A_7}^{A_8} = P_{I A_8} \oplus 64 \oplus 64$$
$$P_{(41) A_7}^{A_8} = P_{(41) A_8}$$
$$P_{(42) A_7}^{A_8} = P_{(42) A_8}$$
$$P_{6 A_7}^{A_8} = P_{6 A_8}$$
$$P_{14 A_7}^{A_8} = P_{14 A_8} \oplus 64 \oplus 64 \oplus 64$$
$$P_{20 A_7}^{A_8} = P_{(201) A_8} \oplus P_{(202) A_8} \oplus 64.$$

Thus the results of Section 3, together with the structure of $20 A_7^{A_8}$ which is yet to be determined, give us strong information about the structures of the projective modules for $A_8$. Namely, we are given certain filtrations for each of $P_I^{A_8}$, $P_{41}^{A_8}$, $P_{42}^{A_8}$, $P_{6}^{A_8}$, $P_{14}^{A_8}$ and $P_{201}^{A_8} \oplus P_{202}^{A_8}$, in which we know the structures of the quotient modules. We now use this to complete the determination of $\text{Ext}^1$ for simple modules, and then to get the complete Loewy structures of the projective indecomposables. All we need to know is how far certain composition factors can "slip past" each other. Our main tool will be the following observations, all of which are trivial but powerful consequences of 4.1.1:

We can identify $JFA_7^{A_8}$ as a subring of $FA_8$ via $JFA_7^{A_8} = JFA_7 \otimes FA_8 \leq FA_7 \otimes FA_8 \simeq FA_8$. 

Thus, $FA_7 \\ FA_7$
4.1.1. Lemma. \( JFA_7^+ \cdot e_0 \leq JFA_8 \).

Proof. This follows trivially from the observation that for each simple \( A_7 \)-module \( M \),

\[
L_1(M^+ \cdot A_8).e_0 \cong L_1(\text{pr}_M^+ \cdot A_8).e_0
\]

By the Frobenius reciprocity theorem it is equivalent to the statement that for each simple \( A_8 \)-module \( N \) in the principal block, \( N^+_{A_7} \) is semisimple. //

4.1.2. Lemma. \( (JFA_7)^n^+ \cdot A_8\cdot e_0 \leq (JFA_8)^n \) for all \( n \geq 0 \).

Proof. This follows from 4.1.1. //

4.1.3. Theorem. If \( M \) is any module for \( A_7 \), then

\[
\left( \frac{A_8}{M^+ \cdot (JFA_8)^n} \right)^+ \cdot e_0 = \left( \frac{M}{M \cdot (JFA_7)^n} \right)^+ \cdot \left( \frac{A_8}{(JFA_8)^n} \right)^+ \cdot e_0
\]

Proof. By 4.1.2 we have

\[
(M(JFA_7)^n)^+ \cdot A_8 \cdot e_0 \leq M(JFA_8)^n \cdot e_0.
\]

Hence

\[
\left( \frac{M}{M \cdot (JFA_7)^n} \right)^+ \cdot (JFA_8)^n \cdot e_0 = \frac{M^+ \cdot (JFA_8)^n}{(M \cdot (JFA_7)^n)^+} \cdot A_8 \cdot e_0
\]

and the result follows from the third isomorphism theorem. //
4.1.4. **Corollary.** If $M$ is a module for $A_7$, then

$$L_1(M^A_{A_7})e_0 \cong L_1((L_1(M))^A_{A_7})e_0.$$ 

**Proof.** This is just the case $n=1$ of the theorem. //

4.1.5. **Corollary.** If $M$ is a module for $A_7$ and

$$0 \to M' \to M \to M'' \to 0$$

is a non-split short exact sequence with $M'$ and $M''$ simple, then

$$L_1(M^A_{A_8})e_0 = L_1(M''^A_{A_8})e_0.$$ //

4.2. **The Loewy structure of** \( \begin{pmatrix} 4 \\ 6 \end{pmatrix}^{A_{A_7}} \)

Our filtration of \( \begin{pmatrix} 4 \\ 6 \end{pmatrix}^{A_{A_7}} \) looks like:

\[ \begin{array}{c}
4_1 \\
4_2 \quad 20_1 \\
4_1 \\
6 \\
1 \quad 14 \\
1 \quad 6 \\
14 \\
6 \\
\end{array} \]

By 4.1.5, $L_1 = 4_1$. We know from 2.5.4 that $L_2(P(4_1)^A_{A_7})$ has a copy of 6 in it, and so applying 4.1.3 for $n=2$ to $P(4_1)^A_{A_7}$ we see that the $L_2$ of both \( \begin{pmatrix} 4 \\ 6 \end{pmatrix}^{A_{A_7}} \) and $P(4_1)^A_{A_8}$
are $4_2 \otimes 6 \otimes 20_1$. This completes the determination of $\dim \text{Ext}^1_{A_8}(4_1, M)$ and hence also of $\dim \text{Ext}^1_{A_8}(4_2, M)$ for $M$ simple.

Also from 2.5.4 we see that $L_3(P(4_1)_{A_8})$ has a copy of 14 in it, so that again applying 4.1.3 for $n = 3$ we see that $L_3((4_1)_{A_8})$ has a copy of 14 in it. Now since $\dim \text{Ext}^1_{A_8}(4_1, I) = 0$, it follows that the Loewy series for $((4_1)_{A_8})$ is as follows:

$$
\begin{array}{cccc}
4_1 & 4_2 & 6 & 20_1 \\
I & 4_1 & 14 \\
I & 6 \\
14 \\
6 \\
\end{array}
$$

4.3. The Loewy structure of $(4_1 4_2)_{A_7}$

Our filtration of this module looks like:

$$
\begin{array}{cccc}
6 \\
I & 14 \\
I & 6 \\
14 \\
6 \\
\end{array}
$$

From 2.5.5 we see that $L_2(P_6)_{A_8}$ has a copy of 4_1 and of 4_2 in it. Thus applying 4.1.3 for $n = 2$ to $P_6_{A_8}$ we see that $L_2$ of both $(4_1 4_2)_{A_7}$ and $P_6_{A_8}$ are $I \oplus 4_1 \oplus 4_2 \oplus 14$. This completes the determination of $\dim \text{Ext}^1_{A_8}(6, M)$ for $M$ simple.
4.3.1. Lemma. L₃(〈6 4₁ A₇ 〉A₈) does not contain copies of 2₀₁ or 4₂.

Proof. (〈6 4₁ A₇ 〉A₈, 2₀₁ A₇ = (6 4₁ 2₀₁ A₇ ) + (〈6 4₁ A₇ 〉A₆, (2₀₁ A₇ )A₆)

by the Mackey decomposition theorem

I

I

= 0 + (4₁ I 4₂ 8₁ 8₂ A₆

4₂

by 1.4.5 and 1.4.9

Also,

I

I

I

I

= 0 + (4₁ 4₂ 8₁ 8₂ A₆

4₂

by 1.4.4 and 1.4.5

= 0.

However, if L₃(〈6 4₁ A₇ 〉A₈) has a copy of 2₀₁ in it, then our knowledge of Ext₁ₐ₇ shows that there would be a map in one of the above sets.

Similarly, if L₃(〈6 4₁ A₇ 〉A₈) has a copy of 4₂ in it, there would be a map

〈4₁ 6 4₂ A₇ 〉A₈ 4₂ A₇ → (4₂ A₈)
for on restriction to \( A_7 \), we have

\[
\dim \text{Ext}^1_{A_7}(14, 4_2) = \dim \text{Ext}^1_{A_7}(I, 4_2) = \dim \text{Ext}^1_{A_7}(4_1, 4_2) = 0
\]

and hence \( L_2((6)_{A_7}^{A_8} + A_7) \) would have 4_2 in it, and hence so would \( L_1((4_1, 4_2)_{A_7}^{A_8} + A_7) \) since \( \dim \text{Ext}^1_{A_7}(6, 4_2) = 1 \).

However,

\[
(4_1, 4_2)_{A_7}^{A_8} + A_7 = (4_1, 4_2)_{A_7}^{A_8} + A_7
\]

by Mackey decomposition

\[
= 0 + (4_1, 4_2, 4_2)_{A_6}
\]

by 1.4.7

= 0, a contradiction. //

Thus with the results of Section 4.2 and the fact that \((6)_{A_7}^{A_8}\) has a submodule \((14)_{A_7}^{A_6}\) (see 2.5.4), we see that the Loewy series of \((4_1, 4_1)_{A_7}^{A_8}\) is as follows:

\[
\begin{array}{cccc}
6 & 14 & 4_1 & 4_2 \\
6 & 14 & 20_1 & 20_2 \\
4_1 & 4_2 & 6 & \\
4_1 & 4_2 & \\
\end{array}
\]
4.4. The Loewy structures of $P_{(4_1)^A_8}$ and $P_{(4_2)^A_8}$.

4.4.1. Lemma. There is no uniserial module $4_1$.

Proof. Applying 4.1.3. with $n=3$ to $P_{I_{A_8}}$ we see that any copy of $4_1$ in $L_3(P_{I_{A_8}})$ is stuck underneath a $14$, a $20_1$, or a $20_2$. \\

4.4.2. Corollary. There is a non-split group extension $2^4A_8$.

Proof. By 4.4.1. the image of the cup-product map

$$\text{Ext}^1_{A_8}(I,6) \otimes \text{Ext}^1_{A_8}(6,4_1) + \text{Ext}^2_{A_8}(I,4_1) \cong H^2(A_8,4_1)$$

is non-zero. \\

We first examine the structure of $P_{(4_1)^A_8}$. From 2.5.3 we see that $P_{6_{A_8}}$ has a quotient module

$\begin{array}{c}
6 \\
4_1 \\
4_2 \\
6 \\
14 \\
\end{array}$
Hence \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) has a quotient module \( 4_1 \). Now from 3.2 we also know that it has a quotient 14.

Hence \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) has a quotient module. Hence \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) has a quotient module. Hence \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) has a quotient module. Hence \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) has a quotient module. Hence \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) has a quotient module.

\((JFA_8)^3\) has Loewy series

\[ 6 \]

\[ I \quad 4_1 \quad 14 \]

\[ I \quad 6 \quad 6 \]

and socle series

\[ 6 \]

\[ I \quad 4_1 \quad 14 \]

\[ I \quad I \quad 6 \quad 6 \]

If there were a copy of \( I \) in \( L_4 \) \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) then there would be a uniserial module 6 contradicting lemma 4.4.1. Hence there are two copies of \( I \) in \( L_5 \) \( \begin{pmatrix} 6 \\ 4_1 \end{pmatrix} \) + \( A_8 \) and hence the complete Loewy series is:

\[ 6 \]

\[ I \quad 4_1 \quad 14 \]

\[ I \quad 6 \quad 6 \]

\[ 4_2 \quad 14 \quad 14 \quad 20_1 \]

\[ I \quad I \quad 6 \quad 6 \]

\[ 4_1 \quad 14 \]

\[ 6 \]
Now we attack \( \left( \begin{array}{c} 4_1 \\ 6 \\ 4_2 \end{array} \right) \uparrow \mathbb{A}_8 \). We know from Section 2.2 that \( P_1(4_1)_3 \mathbb{A}_8 \) has quotient module \( 6 \), and hence by 4.1.3 with \( n = 3 \) we see that \( L_3\left( \left( \begin{array}{c} 4_1 \\ 6 \\ 4_2 \end{array} \right) \uparrow \mathbb{A}_8 \right) \) has \( 4_2 \) in it. Now our knowledge of \( \dim \text{Ext}^1_\mathbb{A}_8 \) together with the results of Sections 4.2 and 4.3 tell us that the Loewy series of \( \left( \begin{array}{c} 4_1 \\ 6 \\ 4_2 \end{array} \right) \uparrow \mathbb{A}_8 \) is as follows:

\[
\begin{array}{cccc}
4_1 & 6 & 20_1 \\
I & 4_1 & 4_2 & 14 \\
I & 6 \\
4_1 & 14 & 20_2 \\
6 \\
4_2
\end{array}
\]

Thus the Loewy series for \( P_{4_1} \) and \( P_{4_2} \) are as in Theorem 1. We can demonstrate our filtrations diagramatically as follows:

\[
\begin{array}{cccccc}
4_1 & 20_1 \\
I & 14 & 4_1 \\
I & 6 \\
14 & 4_1 & 20_1 & 14 \\
6 & I & I & 6 \\
4_3 & 14 \\
6 \\
4_1
\end{array}
\]

Figure 1

Now, since \( L_4(P_{4_1}) \) has 2 copies of 6 in it, \( L_4(P_{4_2}) \) has two copies of each \( 4_1 \) in it by Landrock's lemma. Thus the Loewy series of \( P_6 \) is as in Theorem 1, and our filtrations can be shown diagramatically as in Figure 2.
4.5. The Loewy structure of \( \begin{pmatrix} 1 \\ 14 \end{pmatrix}_{A_7} \uparrow^A_8 \)

Our filtration of this module looks like:

\[
\begin{array}{c}
I \\
6 \\
I \\
14 \\
6 \\
4_1 \\
4_2 \\
6 \\
14
\end{array}
\]

Now from 3.2 we know that \( P_1 \) has a quotient module \( \frac{I}{6} \).

Since \( L_1(20, \uparrow^A_8) = 20_1 \oplus 20_2 \), an application of 4.1.3 to \( P_{IA_7} \) shows that \( \begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7} \uparrow^A_8 \) must have a quotient module \( \frac{I}{6} \). Thus \( \begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7} \uparrow^A_8 \cdot e_0 \) has Loewy series

\[
\begin{array}{c}
I \\
6 \\
14 \\
I \\
6 \\
4_1 \\
4_2 \\
6 \\
14
\end{array}
\]

and \( \begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7} \uparrow^A_3 = \begin{pmatrix} I \\ 14 \end{pmatrix}_{A_7} \uparrow^A_8 \cdot e_0 \oplus 64. \)

By Thompson's Lemma on the ordinary characters of dimension 21, we see that

\[ \dim \text{Ext}^1_{A_8}(I, 20_i) \geq 1, \quad i = 1, 2. \]

Thus our argument also shows that the \( L_2 \) of \( P_{IA_7} \uparrow^A_8 \cdot e_0 = P_{IA_8} \) is exactly \( 6 \oplus 14 \oplus 20_1 \oplus 20_2 \). This completes the determination of \( \dim \text{Ext}^1_{A_8}(I, M) \) for \( M \) simple.
4.6. The Loewy structure of $20_{A_7}^A$

We are now ready to complete the work of Section 3.4.

4.6.1. Lemma. $(20_1', 20_{A_7}^A, A_8) = 0, \quad i = 1, 2.$

Proof. Since $P_{20_1 A_7} = P_{20 \oplus 20 \oplus 20 \oplus 4}$, we see that $20_i$ remains indecomposable on restriction to $A_7$. Thus

$(20_1', 20_{A_7}^A, A_8) = (I, 20, 20)_{A_7} = 0.$

4.6.2. Lemma. $(20_1, 20_2, 20_{A_7}^A, A_8) = 1.$

Proof. Since $\dim \text{Ext}_{A_7}^1 (I, 20) = 1$, we have

$\left(\begin{array}{c} I \\ 20_1 \\ 20_2 \end{array}\right)_{A_8} = I \oplus 20.$

Hence

$(20_1, 20_2, 20_{A_7}^A, A_8) = (I \oplus 20, 20)_{A_7} = 1.$

4.6.3. Lemma. $\text{Rad}(20_{A_7}^A)/\text{Soc}(20_{A_7}^A) = 4_1 \oplus 4_2 \oplus X$, where $X$ has composition factors $I + I + I + 14 + 14 + 20_1 + 20_2$.

Proof. Since $20_{A_7}^A$ extends to a module for $S_8$, there is a subquotient $4_1 \oplus 4_2$. By self-duality and since $\dim \text{Ext}_{A_8}^1 (4_1, I) =$
= \dim \text{Ext}^1_{A_8}(4, 14) = 0 \quad \text{(Section 4.1)}, \text{ this means that } 4 \oplus 4_{2}

is a direct summand of \( \text{Rad}(20_{A_7} \uparrow_{A_3})/\text{Soc}(20_{A_7} \uparrow_{A_3}) \). //

4.6.4. Lemma. \( \dim \text{Ext}^1_{A_8}(14, 20_i) = 0 \).

Proof. Apply 4.1.3 to \( P_{14_{A_7}} \) with \( n = 2 \). //

4.6.5. Lemma. \( \text{Soc}(X) = I \).

Proof. Lemmas 4.6.1 and 4.6.2 show that there is exactly one copy of \( I \) in \( \text{Soc}(X) \). There can be no copies of \( 20_i \) in \( \text{Soc}(X) \) since \( \dim \text{End}_{A_8}(20_{A_7} \uparrow_{A_8}) = 4 \) (Lemma 3.4.1). There can be no copies of \( 14 \) in \( \text{Soc}(X) \) by Lemma 4.6.4. //

Thus \( X \) has the form \( Y \) where \( Y \) has composition factors \( I + I + 14 + 14 + 20_i + 20_2 \).

4.6.6. Lemma. \( \text{Soc}(Y) = 14 \)

Proof. Since \( \dim \text{Ext}^1_{A_8}(I, I) = 0 \), \( \text{Soc}(Y) \) can contain no copies of \( I \). Since \( Y \) is self-dual and extends to a module for \( S_8 \), if \( \text{Soc}(Y) \) contains a copy of \( 20_i \), then \( 20_i \) is a summand of \( Y \) for \( i = 1, 2 \). But then the other direct summand would have to have Loewy series \( I^{14} \) whereas \( \dim \text{Ext}^1_{A_8}(I, 14) = 1 \) from Section 4.5. //
Thus by 4.6.4 the structure of $Y$ is:

$$
\begin{array}{c}
14 \\
I \\
20_1 & 20_2 \\
I \\
14 \\
\end{array}
$$

Hence the Loewy series for $20_{A_7}$ is

$$
\begin{array}{c}
20_1 & 20_2 \\
I & 4_1 & 4_2 \\
14 \\
I \\
20_1 & 20_2 \\
I \\
14 \\
I \\
20_1 & 20_2 \\
\end{array}
$$

(i.e. the "diagram" is

4.7. The remaining projective indecomposable modules

From Section 4.6 we have a quotient of $P_{20_1}$ with Loewy series
LOEWY SERIES FOR THE PROJECTIVE MODULES FOR $A_8$

Since this accounts for all the copies $20_1$ and $20_2$ in $P_{20_1}$, this means that the Loewy length is at least 13.

By Landrock's Lemma we see that $P_{20_1}$ has a copy of 6 in its $L_4$ and $L_6$, and a copy of $4_2$ in its $L_5$, and $4_1$ in its $L_7$.

Thus all the composition factors are accounted for and the Loewy structure of $P_{20_1}$ is as given in Theorem 1.

Hence the appropriate diagram for our filtration of $P_{20_1} \oplus P_{20_2}$ is as follows:

![DIAGRAM](Figure 3)

Now we have enough information to see that $P_I$ has the Loewy series given in Theorem 1, and the appropriate diagram for our filtration is as given in figure 4.
4.7.1. Lemma. \((14)_{A_7} \times A_8 e_0\) has Loewy series
\[
\begin{array}{cccc}
14 & 6 \\
14 & 6 \\
4_1 & 4_2 & 14 \\
6 & 6 \\
4_1 & 4_2 & 14 \\
6 & 6 \\
14 & 14 \\
\end{array}
\]

Proof. From Section 3.2 we know there is a module \(14\) has \(P_{14} A_7\) we see that Thus it also has \(6\) as submodule, and so since \(\dim Ext_{A_8}^1 (6,6) = 0\) from Section 4.3
the result follows. //

This now gives us enough information to see that the Loewy series for \(P_{14}\) given in Theorem 1 is correct, and the appropriate diagram for our filtration is as in Figure 5.
This completes the proof of Theorem 1, and the determination of $\dim \text{Ext}_A^1(M,N)$ for $M$ and $N$ simple. This information is displayed in Appendix 4.

**Notation for character tables**

The only irrationalities we come across in our character tables are:

$$bn = \begin{cases} \frac{1}{2}(-1+i\sqrt{n}) & \text{if } n \equiv 1 \pmod{4} \\ \frac{1}{2}(-1+\sqrt{n}) & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

i.e. the "Gauss sum" of half the primitive $n$th roots of unity.

Under the column headed "ind" is given the Frobenius-Schur indicator of the representation, namely:

- + if the representation is orthogonal
- - if the representation is symplectic but not orthogonal
0 if the representation is neither symplectic nor orthogonal.

(In characteristic 0 this is \( \frac{1}{|G|} \sum_{g \in G} \chi(g^2) \).)

The top row carries the centralizer orders.

Acknowledgement

I would like to thank Dr. P. Landrock and the Matematisk Institut of Aarhus Universitet for their generous help, and the Royal Society, Great Britain, through whom I have been financially supported whilst undertaking this work.

REFERENCES

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3. P. Landrock, The Cartan matrix of a group algebra modulo any power of its radical
5. J. G. Thompson, Vertices and sources. J. Algebra 6 (1967), 1-6
Appendix 1. Characters of $A_8$

(i) Ordinary characters

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<td>A</td>
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<td>B</td>
<td>A</td>
<td>A</td>
<td>AA</td>
<td>AA</td>
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<td>A</td>
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<td>B</td>
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<td>7A</td>
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(ii) 2 - modular characters

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(iv) Cartan Matrix

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Appendix 2. Characters of $A_7$

(i) Ordinary characters

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$fusion$
(ii) \(2 - \) Modular characters

\[
\begin{array}{ccccccc}
2520 & 36 & 9 & 5 & 7 & 7 & \\
\text{ind} & 1A & 3A & 3B & 5A & 7A & B**
\end{array}
\]

\[
\begin{array}{cccccccc}
+ & 1 & 1 & 1 & 1 & 1 & 1 & \\
0 & 4_1 & -2 & 1 & -1 & -b7 & ** & \\
0 & 4_2 & -2 & 1 & -1 & -b7 & ** & -b7 & \\
+ & 6 & 3 & 0 & 1 & -1 & -1 & \\
+ & 14 & 2 & -1 & -1 & 0 & 0 & \\
- & 20 & -4 & -1 & 0 & -1 & -1 & \\
\end{array}
\]

(iii) Decomposition Matrix

\[
\begin{array}{cccccc}
1 & 1 & 14 & 20 & 4_1 & 4_2 \\
15 & 1 & 1 & . & . & . \\
21 & 1 & 1 & . & . & . \\
35 & 1 & 1 & 1 & . & . \\
14 & . & 1 & . & 1 & . \\
6 & . & . & 1 & . & . \\
10 & . & . & 1 & . & . \\
10 & 1 & 1 & 1 & . & . \\
14 & 1 & 1 & 1 & . & . \\
\end{array}
\]

(iv) Cartan Matrix

\[
\begin{array}{cccccc}
1 & 14 & 20 & 4_1 & 4_2 & 6 \\
1 & 10 & 1 & 2 & 1 & 2 \\
14 & 1 & 2 & 1 & 2 & \\
20 & 1 & 2 & 1 & 2 & \\
4_1 & 2 & 1 & 2 & . & . \\
4_2 & 1 & 2 & 2 & . & . \\
6 & 2 & 2 & 4 & . & . \\
\end{array}
\]

Appendix 3. Characters of \(A_6\)

(i) Ordinary characters

\[
\begin{array}{ccccccccc}
360 & 8 & 9 & 9 & 4 & 5 & 5 & \\
\text{ind} & 1A & 2A & 3A & 3B & 4A & 5A & B* & \\
\text{fusion} & + & 1 & 1 & 1 & 1 & 1 & 1 & \\
& + & 5 & 1 & 2 & -1 & -1 & 0 & 0 & \\
& + & 5 & 1 & -1 & 2 & -1 & 0 & 0 & \\
& + & 8 & 0 & -1 & -1 & 0 & -b5 & * & \\
& + & 8 & 0 & -1 & -1 & 0 & * & -b5 & \\
& + & 9 & 1 & 0 & 0 & 1 & -1 & -1 & \\
& + & 10 & -2 & 1 & 1 & 0 & 0 & 0 & \\
\end{array}
\]
(ii) **Modular characters**

\[
\begin{array}{cccccc}
360 & 9 & 9 & 5 & 5 & 5 \\
\text{ind} & 1A & 3A & 3B & 5A & B^* \\
+ & 1 & 1 & 1 & 1 & 1 \\
- & 4_1 & 1 & -2 & -1 & -1 \\
- & 4_2 & -2 & 1 & -1 & -1 \\
+ & 8_1 & -1 & -1 & -b_5 & * \\
+ & 8_2 & -1 & -1 & * & -b_5 \\
\end{array}
\]

(iii) **Decomposition Matrix**

\[
\begin{bmatrix}
1 & 4_1 & 4_2 & 8_1 & 8_2 \\
1 & . & . & . & . \\
5 & 1 & 1 & . & . \\
5 & 1 & . & 1 & . \\
9 & 1 & 1 & 1 & . \\
10 & 2 & 1 & 1 & . \\
8 & . & 1 & . & . \\
8 & . & . & 1 & . \\
\end{bmatrix}
\]

(iv) **Cartan Matrix**

\[
\begin{bmatrix}
1 & 4_1 & 4_2 & 8_1 & 8_2 \\
1 & 8 & 4 & 4 & . \\
4_1 & 4 & 3 & 2 & . \\
4_2 & 4 & 2 & 3 & . \\
8_1 & & & & 1 \\
8_2 & & & & . \\
\end{bmatrix}
\]

Appendix 4

\[\dim \text{Ext}^1_{A_B} (M,N) \text{ for } M,N \text{ simple.}\]

\[
\begin{bmatrix}
1 & 4_1 & 4_2 & 6 & 14 & 20_1 & 20_2 & 64 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
4_1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
4_2 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
6 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
14 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
20_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
20_2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
64 & & & & & & & 0 \\
\end{bmatrix}
\]

Received: February 1982
Revised: June 1982