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Brauer Trees for $12M_{22}$

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The purpose of this note is to correct and extend some of the results in Humphreys [1]. Our main result is Theorem 1, which corrects three (or two and a half!) of Humphreys' trees, and determines the planar embeddings and Green correspondence. These results would not really be very interesting by themselves were it not for the fact that quite an interesting phenomenon is revealed in the behaviour in characteristic eleven. There are four algebraically conjugate 11-blocks of faithful ordinary characters of $12M_{22}$, which come in two complex conjugate pairs, and in fact the trees for the two pairs are totally different. This corresponds to the fact that there is more than one prime ideal lying above eleven in the field of twelfth roots of unity. Two of the blocks have a 24-dimensional irreducible, while the other two do not. This then gives us strong enough information to investigate the planar embeddings of the trees.

LEMMA 1. *There are exactly two isomorphism classes of groups of shape $12M_{22}2$, with the full covering group $12M_{22}$ as a normal subgroup of index two, and $\mathrm{Aut}(M_{22})$ as a quotient.*

Proof. The group $\mathrm{Aut}(12M_{22})$ has shape $M_{22}2$ and has involutions in the outer half. Thus we may form the split extension of $12M_{22}$ by such an involution. The fact that there are two isomorphism classes follows from isoclinism theory. ∎

Notation. We let G denote the full covering group $12M_{22}$, and we choose a group $\hat{G}$ of shape $12M_{22}2$ as in Lemma 1. It does not matter which we choose since the two groups have isomorphic Brauer trees. We

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shall also make use of the subgroup H of index 22, of shape $12M_{21} \cong 12L_3(4)$, whose order is coprime to eleven.

Let (F, R, S) be an 11-modular splitting system for $\hat{G}$ and all its subgroups. Namely, R is a $\mathscr{P}$-adic completion of an algebraic number field, $\mathscr{P}$ is a prime ideal lying above 11, S is the field of fractions of R, and $F = R/\mathscr{P}$ is a finite field of characteristic eleven.

We shall be making use of Green correspondence in characteristic eleven. The Sylow 11-normalizer N in G has shape $12 \times N_0$, with N_0 a Frobenius group of order 55. If V is an indecomposable module for FG, we let $f(V)$ denote the restriction of the Green correspondent of V to N_0. Similarly if W is an indecomposable module for $F\hat{G}$, we obtain a module $f(W)$ for the corresponding Frobenius group $\hat{N}_0$ of order 110.

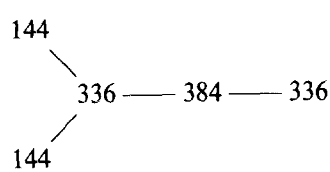
Let $X_1(\theta)$ be the simple FN_0-module with $\dim \operatorname{Ext}^1_{N_0}(I, X_1(\theta)) = 1$. Then the simple FN_0-modules are denoted $X_1(\theta^r)$, and the indecomposable FN_0-module of dimension n with socle $X_1(\theta^r)$ is denoted $X_n(\theta^r)$. It is uniserial with Loewy layers $L_i(X_n(\theta^r)) \cong X_1(\theta^{r-n+i})$.

Ordinary and modular irreducible characters are named by their degree, with a subscript if there is more than one of the same degree. A bar denotes complex conjugation.

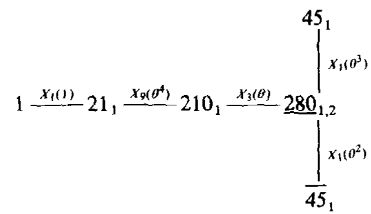
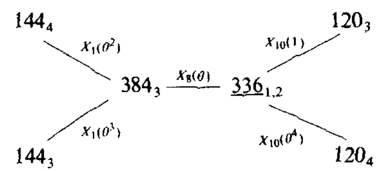
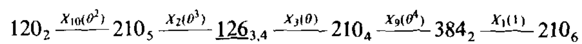
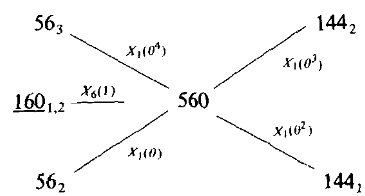
Denote by α a faithful ordinary irreducible representation of $Z(G)$, so that $\{1, \alpha, ..., \alpha^{11}\}$ is the set of all ordinary irreducibles for $Z(G)$.

We shall make free use of the description in [2] of the indecomposable modules in a block of cyclic defect.

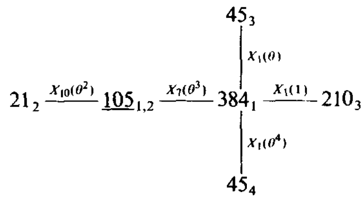
THEOREM 1. (i) *There are four algebraically conjugate blocks of defect one consisting of faithful characters of G over a splitting field of characteristic five. All four trees have the following shape.*



(ii) *The following are the Brauer trees for G over our splitting system (F, R, S) of characteristic eleven, with the planar embedding given correctly (i.e., consistently—what we show is that we may choose our prime ideal $\mathscr{P}$ lying above 11 so that the trees are embedded as shown), and with $f(V)$ marked in for each modular irreducible V. The exceptional ordinary characters are underlined.*

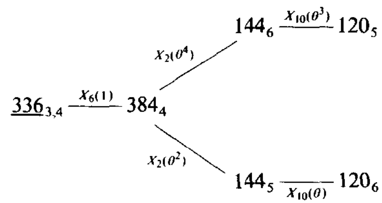
<table>
<tr><th></th><th>Representation of Z(G)</th><th>Size of centre represented</th><th>Brauer trees</th><th>Characters of defect zero</th></tr>
<tr><td>(a)</td><td>1</td><td>1</td><td>
</td><td>$55, 99_1, 154_1, 231_1,$ 385</td></tr>
<tr><td>(b)</td><td>α</td><td>12</td><td>
</td><td>—</td></tr>
<tr><td>(c)</td><td>α^2</td><td>6</td><td>
</td><td>$66_1, 66_2, 330_3$</td></tr>
<tr><td>(d)</td><td>α^3</td><td>4</td><td>
</td><td>176</td></tr>
</table>

(e) α^4 3



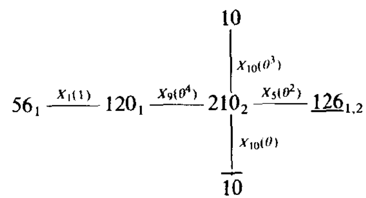
$99_2, 231_2, 231_3, 330_2$

(f) α^5 12



—

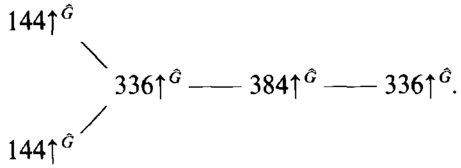
(g) α^6 2



$154_2, \overline{154}_2, 330_1, 440$

The blocks corresponding to α^{12-n} are dual to those for α^n, and the trees are obtained by reflection.

Proof. (i) From the ordinary character table we see that the real stems for the corresponding (induced) blocks of $\hat{G}$ consist of $336\uparrow^{\hat{G}}$, $336\uparrow^{\hat{G}}$ and $384\uparrow^{\hat{G}}$. Thus the trees for $\hat{G}$ are both of the form



(ii)(a) The real stem consists of 1, 21_1, 210_1 and $280_{1,2}$. The permutation module on 22 cosets of H is projective, so the real stem is as shown. Thus applying the Galois automorphism $i\sqrt{7} \leftrightarrow -i\sqrt{7}$ to $\mathscr{P}$ if necessary, the tree and planar embedding are as shown.

(g) The real stem consists of 56_1, 120_1, 210_2 and $126_{1,2}$. Now $126_{1,2}\downarrow_H = 90 \oplus 36$ and $56_1\downarrow_H = 28 \oplus \overline{28}$. Hence 56_1 is modular irreducible and is not a modular constituent of $126_{1,2}$. Thus the real stem is as shown. $10_H\uparrow^G = 210_2 + 10$ and $\overline{10}_H\uparrow^G = 210_2 + \overline{10}$, and so 10 and $\overline{10}$ are joined to 210_2. Finally, $10 \otimes 45_1 = 440 \oplus \overline{10}$, while $X_{10}(\theta) \otimes X_1(\theta^3) \neq X_{10}(\theta_3) \oplus$ projective, and so $f(10) \neq X_{10}(\theta)$. Thus the planar embedding is as shown.

(c) All characters in this block become real when induced to $\hat{G}$, so the tree is a straight line. For the moment ignoring the subscripts on the characters of degree 210, we have exactly two possibilities, namely, the given one and

$$\underline{126}_{3,4} \text{——} 210 \text{——} 120_2 \text{——} 210 \text{——} 384_2 \text{——} 210.$$

Suppose the latter were true. We have $\underline{126}_{3,4}\downarrow_H = 90 + 36$, $210_4\downarrow_H = 90 + 42 + 42' + 36$ and $210_5\downarrow_H = 210_6\downarrow_H = 90 + 60 + 60'$, and so $\underline{126}_{3,4}$ is joined to 210_4, leaving a modular irreducible of degree 84. However, $120_2\downarrow_H = 60 + 60'$, giving a contradiction. Thus the first possibility is true, and 120_2 is joined to 210_5 or 210_6. By applying the Galois automorphism $i \leftrightarrow -i$ (but fixing $i\sqrt{7}$) to $\mathscr{P}$ if necessary, without loss of generality 120_2 is joined to 210_5, leaving a modular irreducible of dimension 90. Subtracting this from $\underline{126}_{3,4}$ we obtain a 36-dimensional modular irreducible, and hence 210_4 is joined on the other side of $\underline{126}_{3,4}$ rather than 210_6.

(e) The real stem for the induced block for $\hat{G}$ consists of $21_2\uparrow^{\hat{G}}$, $\underline{105}_{1,2}\uparrow^{\hat{G}}$, $384_1\uparrow^{\hat{G}}$ and $210_3\uparrow^{\hat{G}}$. Now 21_2 is modular irreducible, and $210_3\downarrow_H = 84 + 63 + 63'$, so that 210_3 does not contain 21_2 as a modular constituent. Thus the real stem is as shown, and the two 45's are joined to 384_1. Now $45_3 \otimes \overline{10} = \overline{120}_2 \oplus \overline{330}_3$, while $X_1(\theta^4) \otimes X_{10}(\theta) \neq X_{10}(\theta^2) \oplus$ projective, so the planar embedding is as shown.

(b) and (f) For each of (b) and (f), the real stem for $\hat{G}$ consists of $\underline{336}$ and 384. Thus each block has either

(A) two 144's joined to 384 and two 120's joined to 336, or

(B) two 144's joined to 384 and a 120 joined to each 144.

Thus we have three possible situations, namely:

(1) four blocks of type A,

(2) four blocks of type B,

(3) two blocks of each type.

We shall derive a contradiction from each of the first two possibilities.

1. We have a 96-dimensional modular irreducible in each of the four blocks, denoted 96_1, $\overline{96}_1$, 96_2 and $\overline{96}_2$. The character of 96_1 is $384_3-(144_3+144_4)$. Now $f(96_1\uparrow^{\hat{G}})=f(96_1)\uparrow^{\hat{N}_0}$ is self-dual, and hence so is $f(96_1)$, which is hence $X_8(\theta)$. Now choose $U\in\{144_3, 144_4\}$ with $\dim_F \operatorname{Ext}^1_{FG}(96_1, U)=1$. Then $f(U)=X_1(\theta^2)$, and continuing in this way, we obtain the Green correspondence for this block. Thus

$$\begin{aligned} f(96_1\otimes 10)\oplus \text{projective} &= X_8(\theta)\otimes X_{10}(\theta^3) \\ &= X_3(\theta^2)\oplus \text{projective} \end{aligned}$$

and so $f(96_1\otimes 10)=X_3(\theta^2)$. Hence

$$96_1\otimes 10 = \begin{array}{ccc} & \overline{96}_2 & \\ / & & \backslash \\ \overline{144}_5 & & \overline{120}_5 \\ & & | \\ & & \overline{120}_6 \\ & & \overline{96}_2 \\ & & | \\ & & \overline{120}_5 \end{array} \oplus \text{projective}$$

with possibly the subscripts 5 and 6 interchanged (the diagram indicates how the composition factors are glued together), and so $\overline{120}_6+\overline{144}_5$ is a projective character, providing us with a contradiction.

2. In this case, there is an irreducible representation of dimension 24 in each block. Choosing one and tensoring with an appropriate representation of dimension 10 (see (g)), we obtain a 240-dimensional representation with irrational character values on the elements of order 7. However, the character of this module will not decompose as a positive sum of the modular characters given by tree type B.

Thus the third possibility holds, and by choosing which is which of α and α^5, we may assume that the shapes of the trees are as shown.

By applying a Galois automorphism which sends $\sqrt{2} \leftrightarrow -\sqrt{2}$ while fixing $\mathbb{Q}(i, \sqrt{7})$ if necessary, we may assume 120_3 and 120_4 are as shown.

If 144_3 and 144_4 are not as shown, then

$$\begin{aligned} f(144_3 \otimes \overline{10}) \oplus \text{projective} &= X_1(\theta^2) \otimes X_{10}(\theta) \\ &= X_{10}(\theta^3) \oplus \text{projective} \end{aligned}$$

and so $f(144_3 \otimes \overline{10}) = f(\overline{120}_5)$ or $f(\overline{120}_6)$. But $144_3 \otimes \overline{10} = \overline{384}_4 + \overline{384}_4 + \overline{336}_3 + \overline{336}_4$ giving a contradiction. Thus they are as shown.

Next we deal with the tree for α^5. We have

$$f(144_{5/6} - 120_{5/6}) = X_2(\theta^4)$$

for some choice of indices; let the corresponding modular irreducible be V, so that $f(V) = X_2(\theta^4)$. Then

$$\begin{aligned} f(V \otimes 10) \oplus \text{projective} &= X_2(\theta^4) \otimes X_{10}(\theta^3) \\ &= X_9(\theta) \oplus \text{projective} \end{aligned}$$

and so

$$V \otimes 10 = \begin{matrix} \overline{144}_4 \\ \overline{96}_1 \end{matrix}$$

Comparing characters, we find $V = 144_6 - 120_{5/6}$. Similarly,

$$\begin{aligned} f(V \otimes \overline{21}_2) \oplus \text{projective} &= X_2(\theta^4) \otimes X_{10}(\theta^2) \\ &= X_9(1) \oplus \text{projective} \end{aligned}$$

and so

$$V \otimes \overline{21}_2 = \begin{array}{ccc} & & 144_4 \\ & & | \\ 120_3 & & 144_3 \\ & \diagdown \quad \diagup & \\ & 96_1 & \end{array}$$

Comparing characters, we find that $V = 144_6 - 120_5$.

(d) Apart from the planar embedding, the tree is clearly as shown. We have

$$\begin{aligned} f(V \otimes 21_2) \oplus \text{projective} &= X_2(\theta^4) \otimes X_{10}(\theta^2) \\ &= X_9(1) \oplus \text{projective}. \end{aligned}$$

APPENDIX: ORDINARY CHARACTERS OF $12M_{22}$ IN "ATLAS" FORMAT[1]

443520		384	36	32	16	5	12	7	7	8	11	11			1344	320	48	32	6	8	5	6	7	7
p power		A	A	A	A	A	AA	A	A	A	A	A			A	A	A	A	AB	A	AC	AC	AB	BB
p′ part		A	A	A	A	A	AA	A	A	A	A	A			A	A	A	A	AB	A	AC	AC	AB	BB
ind	1A	2A	3A	4A	4B	5A	6A	7A	B**	8A	11A	B**	fus	ind	2B	2C	4C	4D	6B	8B	10A	12A	14A	B**
+	1	1	1	1	1	1	1	1	1	1	1	1	:	++	1	1	1	1	1	1	1	1	1	1
+	21_1	5	3	1	1	1	-1	0	0	-1	1	1	:	++	7	1	-1	3	1	1	-1	1	0	0
o	45_1	-3	0	1	1	0	0	b7	**	1	1	1	:	oo	3	5	3	1	0	1	0	0	b7	**
o	$\overline{45}_1$	-3	0	1	1	0	0	**	b7	1	1	1	:	oo	3	5	3	1	0	1	0	0	**	b7
+	55	7	1	3	1	0	1	1	-1	1	0	0	:	++	13	5	1	1	1	-1	0	1	-1	1
+	99_1	3	0	3	1	-1	0	1	1	1	0	0	:	++	15	1	3	1	0	-1	1	0	1	1
+	154_1	10	1	2	2	-1	1	0	0	0	0	0	:	++	14	6	2	2	-1	0	1	1	0	0
+	210_1	2	3	-2	2	0	1	0	0	0	1	1	:	++	14	-10	2	2	-1	0	0	1	0	0
+	231_1	7	3	-1	1	1	1	0	0	1	0	0	:	++	7	-9	9	9	1	-1	1	-1	0	0
o	280_1	8	1	0	0	0	1	0	0	0	b11	**	⌈	+	0	0	0	0	0	0	0	0	0	0
o	280_2	8	1	0	0	0	1	0	0	0	**	b11	⌊											
+	385	1	-2	1	1	0	2	0	0	1	0	0	:	++	21	5	3	3	0	1	0	0	0	0
ord	1	2	3	4	4	5	6	7	7	8	11	11			2	2	4	4	6	8	10	12	14	14
	2	2	6	4		10	6	14	14	8	22	22			2		4		6		10	12	14	14
o	10	2	1	2	0	0	-1	b7	**	0	1	-1	:	oo	4	0	2	0	1	0	0	1	b7	**
o	$\overline{10}$	2	1	2	0	0	-1	**	b7	0	1	-1	:	oo	4	0	2	0	1	0	0	1	**	-b7
+	56_1	-8	2	0	0	1	-2	0	0	0	1	1	:	++	0	0	0	0	0	0	r5	0	0	0
+	120_1	-8	3	0	0	0	1	1	1	0	1	-1	:	++	8	0	4	0	1	0	0	1	1	1
o	126_1	6	0	-2	0	1	0	0	0	0	b11	**	⌈	+	0	0	0	0	0	0	0	0	0	0
o	126_2	6	0	-2	0	1	0	0	0	0	**	b11	⌊											
o	154_2	2	1	-2	0	1	1	0	0	2i	0	0	⌈	+	0	0	0	0	0	0	0	0	0	0
o	$\overline{154}_2$	2	1	-2	0	1	-1	0	0	2i	0	0	⌊											
+	210_2	10	3	2	0	0	1	0	0	0	1	1	:	++	28	0	2	0	1	0	0	1	0	0
+	330_1	2	3	2	0	0	-1	1	1	0	0	0	:	++	20	0	-2	0	1	0	0	1	-1	-1
+	440	8	1	0	0	0	1	1	1	0	0	0	:	++	8	0	-4	0	1	0	0	-1	1	1

Appendix continued

APPENDIX: ORDINARY CHARACTERS OF $12M_{22}$ IN "ATLAS" FORMAT[1] (*continued*)

ord	1	2	3	4	8	5	6	7	7	8	11	11			2	4	8	4	6	8	20	24	14	14
	4	4	12	4		20	12	28	28	8	44	44			2		8		6		20	24	14	14
	2		6			10		14	14	8	22	22												
	4		12			20		28	28	8	44	44												
o2	56_2	0	2	0	0	1	0	0	0	2*z*8	1	1	↓	o2										
o2	56_3	0	2	0	0	1	0	0	0	−2*z*8	1	1	*											
o2	144_1	0	0	0	0	−1	0	−*b*7	**	0	1	1	*	o										
o2	144_2	0	0	0	0	−1	0	**	−*b*7	0	1	1	*	o										
o2	160_1	0	−2	0	0	0	0	−1	−1	0	−*b*11	**	↓	+2										
o2	160_2	0	−2	0	0	0	0	−1	−1	0	**	−*b*11	*											
02	176	0	−4	0	0	1	0	1	1	0	0	0	*	+										
02	560	0	2	0	0	0	0	0	0	0	−1	−1	*	+										
ord	1	2	3	4	4	5	6	7	7	8	11	11			2	2	4	4	6	8	10	12	14	14
	3	6		12	12	15	6	21	21	24	33	33												
	3	6		12	12	15	6	21	21	24	33	33												
o2	21_2	5	0	1	1	1	2	0	0	−1	−1	−1	*	+										
o2	45_3	−3	0	1	1	0	0	*b*7	**	−1	1	1	*	o										
o2	45_4	3	0	1	1	0	0	**	*b*7	−1	1	1	*	o										
o2	99_2	3	0	3	−1	−1	0	1	1	−1	0	0	*	+										
o2	105_1	9	0	1	1	0	0	0	0	1	−*b*11	**	↓	+2										
o2	105_2	9	0	1	1	0	0	0	0	1	**	−*b*11	*											
o2	210_3	2	0	−2	2	0	2	0	0	0	1	1	*	+										
02	231_2	7	0	−1	−1	1	−2	0	0	−1	0	0	*	+										
02	231_3	9	0	3	−1	1	0	0	0	1	0	0	*	+										
o2	330_2	−6	0	−2	2	0	0	1	1	0	0	0	*	+										
o2	384_1	0	0	0	0	−1	0	−1	−1	0	−1	−1	*	+										
ord	1	2	3	4	4	5	6	7	7	8	11	11			2	2	4	4	6	8	10	12	14	14
	6	6	6	12	12	30	6	42	42	24	66	66			2		4		6		10	12	14	14
	3	6		12	12	15	6	21	21	24	33	33												
	2	2		4		10	6	14	8	22	22													
	3	6		12		15	6	21	21	24	33	33												
	6	6		12		30	6	42	42	24	66	66												

o2	66_1	−6	0	2	0	1	0	$b7$	**	0	0	0	*	o												
o2	66_2	−6	0	2	0	1	0	**	$b7$	0	0	0	*	o												
o2	120_2	−8	0	0	0	0	2	1	1	0	1	−1	*	+												
o2	126_3	6	0	−2	0	1	0	0	0	0	$b11$	**	┐	+2												
o2	126_4	6	0	−2	0	1	0	0	0	0	**	$b11$	┘*													
o2	210_4	10	0	2	0	0	2	0	0	0	1	1	*	+												
o2	210_5	6	0	2	0	0	0	0	0	$2i$	1	1	┐	+2												
o2	210_6	6	0	2	0	0	0	0	0	$2i$	1	1	┘*													
o2	330_3	2	0	2	0	0	2	1	1	0	0	0	*	+												
o2	384_2	0	0	0	0	1	0	−1	−1	0	−1	−1	*	+												
ord	1	2	3	4	8	5	6	7	7	8	11	11			4	2	8	4	6	8	20	24	14	14		
	12	12	12	12	24	60	12	84	84	24	132	132				2	8		6		20	24	14	14		
	6	6	6	12	24	30	6	42	42	24	66	66														
	4	4	12	4		20	12	28	28	8	44	44														
	3	6		12		15	6	21	21	24	33	33														
	12	12		12		60	12	84	84	24	132	132														
	2					10		14	14	8	22	22														
	12					60		84	84	24	132	132														
	3					15		21	21	24	33	33														
	4					20		28	28	8	44	44														
	6					30		42	42	24	66	66														
	12					60		84	84	24	132	132														
o4	$120_{3,5}$	0	0	0	0	0	0	1	1	$\overline{2z8}$	−1	−1	┐	o4												
o4	$120_{4,6}$	0	0	0	0	0	0	1	1	$-\overline{2z8}$	−1	−1	┘**													
o4	$144_{3,5}$	0	0	0	0	−1	0	$-b7$	**	0	1	1	**	o2												
o4	$144_{4,6}$	0	0	0	0	−1	0	**	$b7$	0	1	1	**	o2												
o4	$336_{1,3}$	0	0	0	0	1	0	0	0	0	$b11$	**	┐	+4												
o4	$336_{2,4}$	0	0	0	0	1	0	0	0	0	**	$b11$	┘**													
o4	$384_{3,4}$	0	0	0	0	1	0	1	−1	0	−1	−1	**	+2												

[1] See [3].

Computing characters, we see that $V \otimes 21_2$ has Brauer character $\underline{\overline{160}}_{1,2} + \overline{144}_1 + \overline{144}_2 + \overline{56}_2$. Since the corresponding tree for $\hat{G}$ has to have an automorphism corresponding to complex conjugation, we see that the given planar embedding is correct except that possibly the 144's may be the other way around.

Finally, let W we the 36-dimensional simple module in the block corresponding to α^{10}, so that $f(W) = X_3(\theta)$. We have

$$\begin{aligned} f(V \otimes W) \oplus \text{projective} &= X_2(\theta^4) \otimes X_3(\theta) \\ &= X_2(\theta^4) \oplus X_4(1) \end{aligned}$$

and so

$$V \otimes W = \begin{array}{cc} & 144_{2/1} \\ 144_{1/2} \oplus & 144_{1/2} \\ 56_3 & 56_3 \\ & \underline{160}_{1,2} \\ & \underline{160}_{1,2} \end{array}$$

Comparing characters, we see that the 144's are as shown.

References

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