JOURNAL OF ALGEBRA 87, 290–331 (1984)

The Green Ring of a Finite Group

D. J. BENSON

Department of Mathematics, Yale University, New Haven, Connecticut 06520

AND

R. A. PARKER

D.P.M.M.S., 16 Mill Lane, Cambridge, England

Communicated by Walter Feit

Received September 26, 1981

Contents.

INTRODUCTION

The Brauer characters of elements of a finite group on modules do not separate them very well. Two modules have the same Brauer character if and only if they have the same composition factors. We investigate what happens when we try to enlarge the set of columns of the character table to achieve better separation. The most general candidate for a column is a linear character of the Green Ring, which we call a *species*, to distinguish it from a character of the group.

First we examine the general structure of the Green Ring $A(G)$. We impose two inner products,

$$(M, N) = \dim_k \mathrm{Hom}_{kG}(M, N)$$

and

$$\langle M, N \rangle = \text{rank of } \sum_{g \in G} g \text{ on } \mathrm{Hom}_k(M, N),$$

on the Green Ring, and show that there are elements u and v in $A(G)$ with the properties that $u \cdot v = 1$, $(M, N) = \langle v \cdot M, N \rangle$, and $\langle M, N \rangle = (u \cdot M, N)$ (Corollary 2.3). In Section 3, we show that these inner products are nonsingular on $A(G)$, by finding elements G_i for each indecomposable module V_i such that $\langle V_i, G_j \rangle = \delta_{ij}$ (Theorem 3.5). These elements G_i are

0021-8693/84 $3.00

called *atoms*, and they are the simple modules and the "irreducible glues," the latter being related to the Auslander–Reiten sequences for the group algebra. Any module may then be regarded as a formal sum of atoms, namely its composition factors and the glues holding it together.

In Section 4, as an application of this, we find the radical of the bilinear forms $\dim_k \operatorname{Ext}^n_{kG}(M, N)$ on $A(G)$ (Theorem 4.4).

Section 5 consists of some integrality theorems which are used in Section 6 and 7. In Section 6, we introduce the concept of a species. To each species, we associate two conjugacy classes of subgroups, called the origins and the vertices. In order to define the origins, we need a theorem of independent interest; namely, that if H is any subgroup of G, then $A(G)$ is the direct sum of the kernel of the restriction map and the image of the induction map, while $A(H)$ is the direct sum of the image of the restriction map and the kernel of the induction map (Theorem 6.7). Corollary 6.8 is also of independent interest.

In Section 7, we show that if H is an origin of a species s, then $O_p(H)$ is a vertex, and $H/O_p(H)$ is cyclic of order coprime to p (Proposition 7.4 and Theorem 7.8). Section 8 is devoted to the proof of the induction formula (8.3), which is a generalization of the usual formula for the character of an induced module. This formula rests on Theorem 8.2, which should be compared to Proposition 5.3.

In Section 9, we bring together the results of the previous sections to provide an extension of Brauer character theory. We project all the information we have onto a finite dimensional direct summand of $A(G)$ satisfying conditions (i)–(iv) at the beginning of Section 9. Under these conditions, there are only finitely many species. We define tables T_{ij} and U_{ij} called the Atom table and the Representation table, which satisfy the orthogonality relations (9.10) and (9.13). The usual Brauer tables correspond to the summand of $A(G)$ spanned by the projective modules. In this case, T_{ij} is the table of Brauer characters of irreducible modules, and U_{ij} is the table of Brauer characters of projective indecomposable modules. It turns out that the Brauer case is the unique minimal case (Remark 9.1(ii)). The analogues of the centralizer orders in the Brauer case do not always turn out to be positive or rational!

In Section 10, we examine another particular case of the theory developed in Section 9, namely the summand spanned by the cyclic vertex modules. In Appendix 2 we give examples of the tables resulting from this summand. The main tool for calculating these tables is the so-called "atom-copying theorem" (Theorem 11.2) which describes how the atoms of degree zero, namely the irreducible glues, are controlled by the local subgroups. This theorem corresponds to an important property of Auslander–Reiten sequences, namely their behaviour under induction. Finally, in Section 12, we provide a method for constructing species in characteristic two.

1. Notation

Let G be a finite group and k a field of characteristic p. Let $a(G) = a_k(G)$ be the Green Ring, or representation ring, formed from the finite dimensional right kG-modules. We refer the reader to [19] for the definition and standard results used here. Let $A(G) = A_k(G) = a_k(G) \otimes \mathbb{C}$. For each subgroup H of G we have a homomorphism $r_{G,H}: A(G) \to A(H)$ given by restriction of representations, and a linear map, which is not a ring homomorphism, $i_{H,G}: A(H) \to A(G)$ given by induction of representations. We shall also use the symbols $\downarrow_H$ and $\uparrow^G$ to denote these maps. Let $A(G, H)$ be the linear span in $A(G)$ of the direct summands of modules induced from H. Then $A(G, H)$ is an ideal in $A(G)$. If $\mathfrak{X}$ is a collection of subgroups of G, let $A(G, \mathfrak{X})$ be the ideal of $A(G)$ spanned by the set of ideals $A(G, H)$ for $H \in \mathfrak{X}$. Let $A'(G, H) = A(G, \mathfrak{X}(H))$, where $\mathfrak{X}(H)$ in the set of proper subgroups of H. Set $W(G, H) = A(G, H)/A'(G, H)$. Denote by Cyc the collection of cyclic subgroup of G. Thus $A(G, \mathrm{Cyc})$ is the ideal spanned by indecomposable modules whose vertex is cyclic. Moreover, $A(G, 1)$ is the ideal spanned by projective modules. Let $A_0(G, 1)$ be the ideal of $A(G)$ spanned by elements of the form $X - X' - X''$, where $0 \to X' \to X \to X'' \to 0$ is a short exact sequence. Then $A(G) = A(G, 1) \oplus A_0(G, 1)$. Finally, let $A(G, \mathrm{Triv})$ be the subring of $A(G)$ spanned by indecomposable modules whose source is the trivial module.

2. The Inner Products on $A(G)$

Let Ω denote the Heller operation, namely ΩX is the kernel of the projective cover of X (see [20]). Let Ω^{-1} denote the dual operation, namely $\Omega^{-1}X$ is the quotient of the injective hull of X by X. These operations are inverse modulo projective direct summands.

We define two different inner products on $A(G)$ as follows. If M and N are kG-modules, we let

$$(M, N) = \dim_k \mathrm{Hom}_{kG}(M, N).$$

Let $P_1 = (P_1)_{kG}$ be the projective cover of the trivial one-dimensional kG-module 1, and let

$$u_{kG} = u = P_1 - \Omega^{-1}(1),$$
$$v_{kG} = v = P_1 - \Omega(1),$$

as elements of $A(G)$. Then we define

$$\langle M, N \rangle = (u \cdot M, N) = (u, \mathrm{Hom}_k(M, N)).$$

This is the same as the multiplicity of P_1 in a direct sum decomposition of the kG-module $\mathrm{Hom}_k(M, N)$, and is also equal to the rank of $\sum_{g \in G} g$ in the matrix representation on $\mathrm{Hom}_k(M, N)$. In particular, $\langle M, N\rangle = \langle N, M\rangle$.

We extend $(\ ,\ )$ and $\langle\ ,\ \rangle$ bilinearly to give inner products on the whole of $A(G)$.

2.1. PROPOSITION. *Let M be an indecomposable kG-module with projective cover P_M and injective hull I_M. Then we have*

(i) $\Omega^{-1}(1) \otimes \Omega(M) \cong M \oplus$ *projectives.*

(ii) $u \cdot (P_M - \Omega(M)) = M = v \cdot (I_M - \Omega^{-1}(M))$ *and in particular* $u \cdot v = 1$.

(iii) $u \cdot M = I_M - \Omega^{-1}(M)$, $v \cdot M = P_M - \Omega(M)$.

Proof. We have short exact sequences

$$0 \to 1 \to P_1 \to \Omega^{-1}(1) \to 0 \qquad \text{and} \qquad 0 \to \Omega(M) \to P_M \to M \to 0.$$

Tensor the first of these with M, and the second with $\Omega^{-1}(1)$. Then applying Schanuel's lemma (see, e.g., Swan [25]) we get

$$\Omega^{-1}(1) \otimes \Omega(M) \oplus P_1 \otimes M \cong \Omega^{-1}(1) \otimes P_M \oplus M, \tag{2.2}$$

which proves (i).

Thus as elements of the Green Ring, we get

$$\begin{aligned} u \cdot (P_M &- \Omega(M)) \\ &= P_1 \cdot P_M - (P_1 \cdot P_M - P_1 \cdot M) - \Omega^{-1}(1) \cdot P_M + \Omega^{-1}(1) \cdot \Omega(M) \\ &= M \qquad \text{by the isomorphism (2.2).} \end{aligned}$$

This statement and its dual prove (ii), and then (iii) follows immediately. ∎

2.3. COROLLARY.

$$\begin{aligned} (M, N) &= \langle v, \mathrm{Hom}_k(M, N)\rangle = \langle v \cdot M, N\rangle \\ &= \langle \mathrm{Hom}_k(N, M), u\rangle = \langle M, u \cdot N\rangle, \\ \langle M, N\rangle &= (u, \mathrm{Hom}_k(M, N)) = (u \cdot M, N) \\ &= (\mathrm{Hom}_k(N, M), v) = (M, v \cdot N). \end{aligned}$$

Proof.

$$\begin{aligned}\langle v, \operatorname{Hom}_k(M, N)\rangle &= (u \cdot v, \operatorname{Hom}_k(M, N))\\ &= (1, \operatorname{Hom}_k(M, N)) \qquad \text{by Proposition 2.1(ii)}\\ &= (M, N).\end{aligned}$$

The rest are proved similarly. ∎

2.4. Corollary (Green). *If M is indecomposable and N is irreducible then*

$$\begin{aligned}\langle M, N\rangle &= d_N \qquad \text{if} \quad M = P_N,\\ &= 0 \qquad \textit{otherwise},\end{aligned}$$

where $d_N = \dim_k \operatorname{End}_{kG}(N)$.

Proof.

$$\begin{aligned}\langle M, N\rangle &= (M, v \cdot N) && \text{by Corollary 2.3}\\ &= (M, P_N - \Omega(N)) && \text{by Proposition 2.1(iii)}\\ &= d_N && \text{if } M = P_N,\\ &= 0 && \text{otherwise.}\end{aligned}$$ ∎

2.5. Lemma. *If $H \leqslant G$, then*

(i) $u_{kG}\downarrow_H = u_{kH}$,

(ii) $v_{kG}\downarrow_H = v_{kH}$.

Proof. We have two short exact sequences,

$$0 \to \Omega(1)_{kG}\downarrow_H \to (P_1)_{kG}\downarrow_H \to 1_{kH} \to 0,$$
$$0 \to \Omega(1)_{kH} \to (P_1)_{kH} \to 1_{kH} \to 0.$$

Lemma 2.5(ii) thus follows from Schanuel's lemma, and (i) is proved dually. ∎

2.6. Corollary (Knörr).

$$\langle V, W\downarrow_H\rangle = \langle V\uparrow^G, W\rangle$$

(This is the analogue of the usual Frobenius reciprocity statement.)

Proof.

$$\langle V, W{\downarrow_H}\rangle = (u_{kH}, \mathrm{Hom}_k(V, W{\downarrow_H}))$$
$$= (u_{kG}{\downarrow_H}, \mathrm{Hom}_k(V, W{\downarrow_H})) \qquad \text{by Lemma 2.5}$$
$$= (u_{kG}, \mathrm{Hom}_k(V, W{\downarrow_H}){\uparrow^G})$$

by the usual Frobenius reciprocity

$$= (u_{kG}, \mathrm{Hom}_k(V{\uparrow^G}, W))$$
$$= \langle V{\uparrow^G}, W\rangle. \quad \blacksquare$$

3. Almost Split Sequences

In this section we shall find elements of the Green Ring dual to the indecomposables with respect to the inner products $(\,,\,)$ and $\langle\,,\,\rangle$. We assume for this section that k is algebraically closed, although we could avoid this by dividing our dual elements by $\dim_k|\mathrm{End}_{kG}(V_i)/\mathrm{Rad}\,\mathrm{End}_{kG}(V_i)|$.

3.1. Definition. An *Almost split sequence*, or Auslander–Reiten sequence, is a short exact sequence,

$$0 \to A \to B \xrightarrow{\sigma} C \to 0,$$

satisfying the conditions,

(a) σ does not split;

(b) A and C are indecomposable;

(c) if $f: X \to C$ is not a split epimorphism then f factors through σ.

In our situation, namely that of finite dimensional modules over the group algebra of a finite group, the results of Auslander and Reiten can be stated as follows.

3.2. Theorem (Auslander and Reiten [2]). *Given an indecomposable kG-module C which is not projective, there exists an almost split sequence,*

$$0 \to A \to B \to C \to 0,$$

and any two such sequences are isomorphic. Moreover, $A \cong \Omega^2(C)$, where Ω^2 denotes the square of the Heller operation. As a module for $\mathrm{End}_{kG}(C)^{\mathrm{op}}$, *the group* $\mathrm{Ext}^1_{kG}(C, A)$ *has a simple socle of dimension* 1 *with the above extension as generator.*

3.3. LEMMA. *If C and D are indecomposable modules, and*

$$0 \to A \to B \to C \to 0$$

is the almost split sequence terminating in C, then the following hold:

(i) *If* $C \not\cong D$ *then* $0 \to \mathrm{Hom}_{kG}(D, A) \to \mathrm{Hom}_{kG}(D, B) \to \mathrm{Hom}_{kG}(D, C) \to 0$ *is exact.*

(ii) *The sequence* $0 \to \mathrm{Hom}_{kG}(C, A) \to \mathrm{Hom}_{kG}(C, B) \to \mathrm{Hom}_{kG}(C, C) \to \mathrm{Soc}(\mathrm{Ext}^1_{kG}(C, A)) \to 0$ *is exact, where this is the truncation of the long exact* Ext *sequence.*

Proof. This follows immediately from Theorem 3.2. ∎

3.4. THEOREM. *Let $\{V_i : i \in I\}$ be the set of indecomposable modules for kG. Let*

$$\begin{aligned} H_i &= V_i - \mathrm{Rad}(V_i) && \text{if } V_i \text{ is projective,} \\ &= V_i + \Omega^2(V_i) - X && \text{otherwise,} \end{aligned}$$

where

$$0 \to \Omega^2(V_i) \to X \to V_i \to 0$$

is the almost split sequence terminating in V_i. Then $(V_i, H_j) = \delta_{ij}$.

Proof. If V_j is not projective, then Lemma 3.3 shows that $(V_i, H_j) = \delta_{ij}$ since $\mathrm{Soc}(\mathrm{Ext}^1_{kG}(V_j, \Omega^2(V_j)))$ is one dimensional. If V_j is projective, then

$$\begin{aligned} (V_i, H_j) &= \langle V_i, \mathrm{Soc}(V_j) \rangle && \text{by Proposition 2.1(ii) and Corollary 2.3} \\ &= \delta_{ij} && \text{by Corollary 2.4. ∎} \end{aligned}$$

3.5. THEOREM. *Let $\{V_i : i \in I\}$ be the set of indecomposable modules for kG. Let*

$$\begin{aligned} G_i &= \mathrm{Soc}(V_i) && \text{if } V_i \text{ is projective,} \\ &= X - \Omega^{-1}(V_i) - \Omega(V_i) && \text{otherwise,} \end{aligned}$$

where

$$0 \to \Omega(V_i) \to X \to \Omega^{-1}(V_i) \to 0$$

is the almost split sequence terminating in $\Omega^{-1}(V_i)$. Then $\langle V_i, G_j \rangle = \delta_{ij}$.

Proof. If V_j is not projective, then

$$\begin{aligned}\langle V_i, G_j\rangle &= (u \cdot V_i, G_j) \\ &= (I_{V_i} - \Omega^{-1}(V_i), G_j) && \text{by Proposition 2.1(iii)} \\ &= (-\Omega^{-1}(V_i), G_j) && \text{since } G_j \in A_0(G, 1) \\ &= \delta_{ij} && \text{by Theorem 3.4.}\end{aligned}$$

If V_j is projective, then

$$\langle V_i, G_j\rangle = \delta_{ij} \qquad \text{by Corollary 2.4.} \quad \blacksquare$$

3.6. Definition. We define a semilinear map $\tau: A(G) \to A(G)$ via $\tau(\sum a_i V_i) = \sum \bar{a}_i G_i$, where the overbar denotes complex conjugation. Then

$$\langle x, \tau(x)\rangle = \sum |a_i|^2 \geqslant 0,$$

with equality if and only if $x = 0$.

3.7. Proposition. *The inner products* $\langle\ ,\ \rangle$ *and* $(\ ,\ )$ *are nonsingular on* $A(G)$ *in the sense that given* $x(\neq 0) \in A(G)$, *there is* $y \in A(G)$ *such that* $\langle x, y\rangle \neq 0$ *and* $z \in A(G)$ *such that* $(x, z) \neq 0$.

Proof. Take $y = \tau(x)$ and $z = v \cdot y$, and use Definition 3.6 and Corollary 2.3. $\blacksquare$

3.8. Definitions. The *atom* corresponding to V_i is G_i. The *glue* for a short exact sequence $0 \to X' \to X \to X'' \to 0$ is the element $X - X' - X''$ of the Green Ring. Thus if V_i is not projective, then G_i is a glue.

A glue is *irreducible* if its is nonzero and not the sum of two nonzero glues as an element of the Green Ring.

3.9. Lemma. *If* $X - X' - X''$ *is the glue for* $0 \to X' \to X \to X'' \to 0$, *then for any module* V, $\langle X - X' - X'', V\rangle \geqslant 0$.

Proof. The number of copies of P_1 in a direct sum decomposition of $X \otimes V$ is at least the sum of the number of copies in $X' \otimes V$ and the number of copies in $X'' \otimes V$, since P_1 is both injective and projective. $\blacksquare$

3.10. Theorem. (i) *Every nonzero glue can be written as the sum of an atom and a glue. Thus every irreducible glue is an atom.*

(ii) *Every atom is either a simple module or an irreducible glue.*

Proof. First we note that the sum of two glues is a glue, since we can add the exact sequences term by term as a direct sum.

(i) Suppose $0 \to Y' \to Y \xrightarrow{\pi} Y'' \to 0$ is an exact sequence with $Y - Y' - Y'' \neq 0$ its glue. If Y'' is decomposable, $Y'' = W'' \oplus Z''$, then $Y - Y' - Y''$ is the sum of the glues for

$$0 \to \pi^{-1}(W'') \to Y \to Z'' \to 0$$

and

$$0 \to Y' \to \pi^{-1}(W'') \to W'' \to 0.$$

At least one of these is nonzero, and so we may assume by induction that Y'' is indecomposable. Thus π is not a split epimorphism. Letting $0 \to \Omega^2(Y'') \to X \to Y'' \to 0$ be the almost split sequence terminating in Y'', we have the commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] \arrow[d, dashed] \arrow[dr, dashed] & Y \arrow[r] \arrow[d, dashed] \arrow[dr] & Y'' \arrow[r] \arrow[d, no head] & 0 \\ 0 \arrow[r] & \Omega^2(Y'') \arrow[r] & X \arrow[r] & Y'' \arrow[r] & 0. \end{tikzcd}$$

The left-hand square is a pushout diagram, and so we get an exact sequence

$$0 \to Y' \to Y \oplus \Omega^2(Y'') \to X \to 0.$$

The given glue is the sum of the glue for this sequence and the atom corresponding to the almost split sequence terminating in Y''.

(ii) If V_i is projective, G_i is a simple module. If V_i is not projective, then G_i is a glue. Suppose it is not irreducible. Then by (i) it is the sum of another atom, say G_j, and a glue. But $\langle G_i - G_j, V_j \rangle = -1$ by Theorem 3.5, contradicting Lemma 3.9. ∎

Informally, we think of every representation as consisting of (possibly infinitely many) atoms, namely, the simple composition factors and some irreducible glues.

$$x = \sum_i \langle x, V_i \rangle \cdot G_i. \tag{3.11}$$

This formal expression has the right inner product with any indecomposable module V_j, since

$$\left\langle \left(\sum_i \langle x, V_i\rangle \cdot G_i\right), V_j\right\rangle = \sum_i \langle\langle x, V_i\rangle \cdot G_i, V_j\rangle$$
$$= \sum_i \langle x, V_i\rangle\, \delta_{ij} \qquad \text{by Theorem 3.5}$$
$$= \langle x, V_j\rangle.$$

Then it has the right inner product with any element of $A(G)$, so that by Proposition 3.7 it is a valid formal sum.

We consider atoms to be in the same block as the corresponding indecomposable modules. Then in the formal sum (3.11), an indecomposable module can only involve atoms from the same block.

4. The Radical of $\dim_k \operatorname{Ext}^n_{kG}$

We define inner products $(\ ,\ )_n$ for $n \geqslant 1$ as follows. If M and N are kG-modules, we let

$$(M, N)_n = \dim_k \operatorname{Ext}^n_{kG}(M, N).$$

We extend this bilinearly to define inner products on the whole of $A(G)$.

4.1. Lemma.

$$(M, N)_n = (\Omega^n M, N) - \langle \Omega^n M, N\rangle$$
$$= ((1-u)\,\Omega^n M, N).$$

Proof. The short exact sequence

$$0 \to \Omega M \to P_M \to M \to 0$$

gives rise to a long exact sequence,

$$0 \to \operatorname{Hom}_{kG}(M, N) \to \operatorname{Hom}_{kG}(P_M, N) \to \operatorname{Hom}_{kG}(\Omega M, N) \to \operatorname{Ext}^1_{kG}(M, N)$$
$$\to \operatorname{Ext}^1_{kG}(P_M, N) \to \operatorname{Ext}^1_{kG}(\Omega M, N) \to \operatorname{Ext}^2_{kG}(M, N) \to \operatorname{Ext}^2_{kG}(P_M, N)$$
$$\to \cdots$$

Now $\operatorname{Ext}^i_{kG}(P_M, N) = 0$. Thus by Proposition 2.1 and Corollary 2.3,

$$(M, N)_1 = (M, N) + (\Omega M, N) - (P_M, N)$$
$$= (\Omega M, N) - \langle \Omega M, N\rangle$$
$$= ((1-u)\,\Omega M, N),$$

and for $n \geqslant 1$,

$$(M, N)_{n+1} = (\Omega M, N)_n. \quad \blacksquare$$

4.2. DEFINITION.

$$\mathrm{Rad}(\ ,\ )_n = \{x \in A(G) \colon (x, y)_n = 0 \text{ for all } y \in A(G)\}.$$

4.3 LEMMA. *Suppose M is a periodic kG-module with even period $2s$. Then as elements of $A(G)$,*

$$M = u^{2s} \cdot M.$$

Proof. By Proposition 2.1(iii), $M - u^{2s}M \in A(G, 1)$. Since 1 and u have the same Brauer character, $M - u^{2s}M$ has Brauer character zero, and is hence the zero element of $A(G, 1)$. $\blacksquare$

4.4. THEOREM. $\mathrm{Rad}(\ ,\ )_n$ *is the linear span in $A(G)$ of the projective modules and elements of the form*

$$\sum_{i=1}^{2s} (-1)^i \Omega^i(M),$$

for M a periodic module of even period $2s$.

Proof. Suppose $x = \sum a_i V_i \in \mathrm{Rad}(\ ,\ )_n$. Then for V_i nonprojective we have

$$\begin{aligned}
0 &= (x, \Omega^n G_i)_n \\
&= (\Omega^n x, \Omega^n G_i) - \langle \Omega^n x, \Omega^n G_i \rangle \qquad \text{by Lemma 4.1} \\
&= (x, G_i) - \langle x, G_i \rangle \\
&= -(\text{coefficient of } \Omega^{-1}(V_i)) - (\text{coefficient of } V_i),
\end{aligned}$$

since for V_i non projective, $G_i = -H_{\Omega^{-1}(i)}$, where $V_{\Omega^{-1}(i)} = \Omega^{-1}(V_i)$. Hence

$$(\text{coefficient of } V_i) = -(\text{coefficient of } \Omega V_i).$$

Thus if $a_i \neq 0$, V_i is projective or periodic of even period. Conversely, if M is periodic of even period $2s$, then by Lemma 4.3 we have

$$(1 - u)(1 + u + \cdots + u^{2s-1})M = 0,$$

and so $(1 + u + \cdots + u^{2s-1})M \in \mathrm{Rad}(\ ,\ )_n$. But $(1 + u + \cdots + u^{2s-1})\ M \equiv \sum_{i=1}^{2s} (-1)^i \Omega^i(M)$ modulo projectives by Proposition 2.1(iii). $\blacksquare$

5. Some Integrality Theorems

Let k_l be an extension of k. Then we have a homomorphism of Green Rings $e_{k,k_1}: A_k(G) \to A_{k_1}(G)$ given by $V \mapsto V \otimes_k k_1$ for V a representation in $A_k(G)$.

5.1. Proposition. (i) *e_{k,k_1} preserves the inner products $(\ ,\ )$ and $\langle\ ,\ \rangle$.*

(ii) *e_{k,k_1} is injective.*

Proof. (i) The identity

$$\mathrm{Hom}_{kG}(V, W) \otimes_k k_1 = \mathrm{Hom}_{k_1G}(V \otimes k_1, W \otimes k_1)$$

is proved in [14, (18.4)]. It is also clear that $(P_1)_{kG} \otimes k_1 = (P_1)_{k_1G}$. The result now follows from the definitions of $(\ ,\ )$ and $\langle\ ,\ \rangle$.

(ii) This is the Noether–Deuring theorem (see [12, (29.7)]), but also follows from (i) and Proposition 3.7. ∎

Because of Proposition 5.1, from now on we shall identify $A_k(G)$ with its image under e_{k,k_1} for any extension k_1 of k.

5.2. Proposition. *Suppose k_1 is a separable algebraic extension of k. Then $A_{k_1}(G)$ is integral as an extension of $A_k(G)$.*

Proof. Let k_2 be the normal closure of k_1 over k. Then k_2 is a normal separable algebraic extension of k, and it suffices to prove that $A_{k_2}(G)$ is integral over $A_k(G)$. Let $\alpha \in A_{k_2}(G)$. Then α has only finitely many images $\alpha_1,\dots,\alpha_r$ under the action of $\mathrm{Gal}(k_2/k)$. The equation $(x-\alpha_1)\dots(x-\alpha_r)=0$ is a monic polynomial satisfied by α, and its coefficients are in the fixed field of $\mathrm{Gal}(k_2/k)$, namely in $A_k(G)$. ∎

Finally, we wish to prove an integrality property over the image of the restriction map from a larger group.

5.3. Proposition. *Let $H \leqslant G$. Then $A(H)$ is integral over* $\mathrm{Im}(r_{G,H})$.

Proof (Puig, private communication). Let $A(H)^{N(H)}$ be the set of fixed points of $A(H)$ under the action of $N_G(H)$ by conjugation. If $\alpha \in A(H)$, then α has only finitely many images $\alpha_1,\dots,\alpha_r$ under the action of $N_G(H)$. The equation $(x-\alpha_1)\dots(x-\alpha_r)=0$ is a monic polynomial satisfied by α, and its coefficients are in $A(H)^{N(H)}$. Thus $A(H)$ is integral over $A(H)^{N(H)}$, and so we must show that $A(H)^{N(H)}$ is integral over $\mathrm{Im}(r_{G,H})$.

Let $a \in A(H)^{N(H)}$. For any $K \lneqq H$ we set

$$X_K = \{r_{H^x,K}(a^x) \colon K < H^x\}.$$

We denote by $\langle \mathrm{Im}(r_{G,K}), X_K\rangle$ the subring of $A(K)$ generated by $\mathrm{Im}(r_{G,K})$ and X_K. We may assume inductively that $\langle \mathrm{Im}(r_{G,K}), X_K\rangle$ is finitely generated as a module for $\mathrm{Im}(r_{G,K})$. We claim that

$$\mathrm{Im}(r_{G,H}) + \sum_{K<H} i_{K,H}(\langle \mathrm{Im}(r_{G,K}), X_K\rangle)$$

is a subring of $A(H)$, finitely generated as a module for $\mathrm{Im}(r_{G,H})$ and containing the element a.

(i) We first prove that it is a ring. It is clear that

$$\mathrm{Im}(r_{G,H}) \cdot i_{K,H}(\langle \mathrm{Im}(r_{G,K}), X_K\rangle) \subseteq i_{K,H}(\langle \mathrm{Im}(r_{G,K}), X_K\rangle),$$

since

$$r_{G,H}(x) \cdot i_{K,H}(r_{G,K}(y)) = i_{K,H}(r_{G,K}(x \cdot y))$$

and

$$r_{G,H}(x) \cdot i_{K,H}(r_{H^y,K}(a^y)) = i_{K,H}(r_{G,K}(x) \cdot r_{H^y,K}(a^y)).$$

If $K < H$ and $L < H$, then

$$\begin{aligned} &i_{K,H}(\langle \mathrm{Im}(r_{G,K}), X_K\rangle) \cdot i_{L,H}(\langle \mathrm{Im}(r_{G,L}), X_L\rangle) \\ &\quad \subseteq \sum_{g\in H} i_{K\cap L^g,H}(r_{K,K\cap L^g}(\langle \mathrm{Im}(r_{G,K}), X_K\rangle) \cdot r_{L^g,K\cap L^g}(\langle \mathrm{Im}(r_{G,L^g}), X_L^g\rangle)),\end{aligned}$$

by the Mackey decomposition theorem. But

$$\begin{aligned} r_{K,K\cap L^g}(\langle \mathrm{Im}(r_{G,K}), X_K\rangle) &\subseteq \langle \mathrm{Im}(r_{G,K\cap L^g}), r_{K,K\cap L^g}(X_K)\rangle \\ &\subseteq \langle \mathrm{Im}(r_{G,K\cap L^g}), X_{K\cap L^g}\rangle.\end{aligned}$$

(ii) It is finitely generated as a module for $\mathrm{Im}(r_{G,H})$. By induction, for any $K < H$, we have

$$\langle \mathrm{Im}(r_{G,K}), X_K\rangle = \sum_{b\in Y_K} \mathrm{Im}(r_{G,K}) \cdot b,$$

with Y_K a finite set. So

$$i_{K,H}(\langle \mathrm{Im}(r_{K,G}), X_K\rangle) = \sum_{b\in Y_K} \mathrm{Im}(r_{G,H}) \cdot i_{K,H}(b).$$

(iii) It contains a. Since a is invariant under $N_G(H)$, we have

$$r_{G,H}(i_{H,G}(a)) = |N_G(H) \colon H|\, a + \sum_{\substack{HgH \\ \text{s.t. } H\cap H^g<H}} i_{H\cap H^g,H}(r_{H^g,H\cap H^g}(a^g)). \quad \blacksquare$$

6. Species, Vertices, and Origins

Let R be a subalgebra or ideal of $A(G)$.

6.1. Definition. A *species* of R is a nonzero linear algebra homomorphism $s: R \to \mathbb{C}$. If s is a species of R and $x \in R$, we write (s, x) for the value of s on x.

Since R is commutative we have

6.2. Lemma. (i) $\mathrm{Rad}(R) = \bigcap_s \mathrm{Ker}(s) = \{x \in R: x^n = 0$ *for some* $n\}$.
(ii) *If* $S \leqslant R$ *then* $\mathrm{Rad}(S) = S \cap \mathrm{Rad}(R)$.

6.3. *Comment.* It is shown in [18, 19, 21] that when G has a cyclic Sylow p-subgroup, $A(G)$ is semisimple. In [27] it is shown that if G has noncyclic Sylow p-subgroups, where p is odd, then $A(G)$ has nonzero nilpotent elements, while for $p = 2$, [7, 26] show that if the Sylow 2-subgroup of G is isomorphic to the Klein four-group then $A(G)$ is semisimple. In [28] it is shown that in certain cases for $p = 2$, $A(G)$ has nilpotent elements. However, there are still some unresolved cases for $p = 2$.

6.4. Lemma. *Let I be an ideal of R, and s a species of I. Then s extends uniquely to a species of R.*

Proof. Choose $x \in I$ with $(s, x) = 1$. Then if $y \in R$, for any extension t of s we must have

$$(t, y) = (t, y)(s, x) = (t, x \cdot y) = (s, x \cdot y),$$

and so t is uniquely determined by s. Moreover, the t so defined is indeed a species of R, since

$$\begin{aligned}(t, y)(t, z) &= (s, x \cdot y)(s, x \cdot z)\\ &= (s, x)(s, x \cdot y \cdot z)\\ &= (t, y \cdot z). \quad \blacksquare\end{aligned}$$

If s is a species of R, and R' is a subalgebra of R, then the restriction of s to R' is a species of R'. Thus species of R' correspond to equivalence classes of species for R, two such being equivalent if and only if their restriction to R' are the same.

6.5. Lemma. *Any set of distinct species of R is linearly independent.*

Proof. Suppose $\sum_{i=1}^r a_i s_i = 0$ is a linear relation of smallest size among the species for R. Choose $y \in R$ such that $(s_1, y) \neq (s_2, y)$. Then

$$0 = \sum_{i=1}^{r} a_i(s_i, x \cdot y) = \sum_{i=1}^{r} a_i(s_i, y)(s_i, x),$$

and so $\sum_{i=2}^r a_i((s_i, y) - (s_1, y))\, s_i = 0$. This is a smaller nonzero linear relation, contradicting the minimality of r. ∎

Thus if R' is a finite-dimensional semisimple subalgebra of R, with species $s_1, \ldots, s_r$, we can find idempotents $e_1, \ldots, e_r$ with $(s_i, e_j) = \delta_{ij}$. This gives a direct sum decomposition,

$$R = \bigoplus_{i=1}^{r} R e_i. \tag{6.6}$$

Thus every species of R is a species of some $R e_i$ and is zero on the $R e_j$, $j \neq i$.

6.7. THEOREM.[1] *Let H be a subgroup of G. Then*

(i) $A(G) = \mathrm{Im}(i_{H,G}) \oplus \mathrm{Ker}(r_{G,H})$ *as a direct sum of ideals,*

(ii) $A(H) = \mathrm{Im}(r_{G,H}) \oplus \mathrm{Ker}(i_{H,G})$ *as a direct sum of vector spaces.*

Proof. (i) We proceed by induction on $|H|$. If $|H| = 1$, then $\mathrm{codim}(\mathrm{Ker}(r_{G,1})) = \dim(\mathrm{Im}(i_{1,G})) = 1$. Since $i_{1,G}(1) \notin \mathrm{Ker}(r_{G,1})$ the result follows. So suppose $|H| > 1$, and that for any $K < H$, $A(G) = \mathrm{Im}(i_{K,G}) + \mathrm{Ker}(r_{G,K})$. Then $A(G) = \sum_{K<H} \mathrm{Im}(i_{K,G}) + \bigcap_{K<H} \mathrm{Ker}(r_{G,K})$, and so $\mathrm{Im}(r_{G,H}) = r_{G,H}(\sum_{K<H} \mathrm{Im}(i_{K,G})) + \bigcap_{K<H} \mathrm{Ker}_{\mathrm{Im}(r_{G,H})}(r_{H,K})$. Let $1 = a + b$ in this decomposition. Then since $b = 1 - a$ is invariant under $N_G(H)$, we have, by the Mackey decomposition theorem, $b \uparrow^G \downarrow_H = |N_G(H){:}\, H| \cdot b$, and so $b \in r_{G,H}(\mathrm{Im}(i_{H,G}))$. Hence $\mathrm{Im}(r_{G,H}) = r_{G,H}(\mathrm{Im}(i_{H,G}))$. Now if $x \in A(G)$, choose $y \in \mathrm{Im}(i_{H,G})$ with $x \downarrow_H = y \downarrow_H$. Then $x = y + (x - y) \in \mathrm{Im}(i_{H,G}) + \mathrm{Ker}(r_{G,H})$. Now write $1 = a' + b'$ in this decomposition. If $x \in \mathrm{Im}(i_{H,G}) \cap \mathrm{Ker}(r_{G,H})$, then $x = x.a' + x.b' = 0$.

(ii) Write $A_1 = \mathrm{Im}(r_{G,H})$, $A_2 = \mathrm{Ker}(i_{H,G})$. We show by induction on subgroups $K \leqslant H$, that if M is a module for K, then $M \uparrow^H \in A_1 + A_2$. By the Mackey theorem,

$$M \uparrow^G \downarrow_H = \sum M^g \downarrow_{K^g \cap H} \uparrow^H.$$

[1] We are grateful to L. Puig and P. Landrock for pointing out improvements on the original proof of theorem 6.7.

If $K^g \leqslant H$, then $M^g \downarrow_{K^g \cap H} \uparrow^H \equiv M \uparrow^H \bmod A_2$. If $K^g \nleqslant H$, then $M^g \downarrow_{K^g \cap H} \uparrow^H \in A_1 + A_2$ by the inductive hypothesis. Since $M \uparrow^G \downarrow_H \in A_1$, some positive multiple of $M \uparrow^H$ is in $A_1 + A_2$, and hence so is $M \uparrow^H$.

Now suppose $x \in A_1 \cap A_2$, with $x = u \downarrow_H$. By (i), we may assume $u \in \mathrm{Im}(i_{H,G})$. Let $e \uparrow^G$ be the idempotent generator for $\mathrm{Im}(i_{H,G})$. Write $e = v \downarrow_H + w$, with $v \in A(G)$ and $w \in A_2$. Then

$$\begin{aligned} u &= u \cdot e \uparrow^G = (u \downarrow_H \cdot e) \uparrow^G = (xe) \uparrow^G = (x.v \downarrow_H) \uparrow^G + (xw) \uparrow^G \\ &= x \uparrow^G \cdot v + u \cdot w \uparrow^G = 0. \quad \blacksquare \end{aligned}$$

6.8. Corollary. *Let $H \leqslant G$ and let V_1 and V_2 be kH-modules, and W_1 and W_2 be kG-modules.*

(i) *If $V_1 \uparrow^G \downarrow_H \cong V_2 \uparrow^G \downarrow_H$ then $V_1 \uparrow^G \cong V_2 \uparrow^G$.*

(ii) *If $W_1 \downarrow_H \uparrow^G \cong W_2 \downarrow_H \uparrow^G$ then $W_1 \downarrow_H \cong W_2 \downarrow_H$.*

Proof. (i) $V_1 \uparrow^G - V_2 \uparrow^G \in \mathrm{Im}(i_{H,G}) \cap \mathrm{Ker}(r_{G,H}) = 0$ by Theorem 6.7 (i).
(ii) $W_1 \downarrow_H - W_2 \downarrow_H \in \mathrm{Im}(r_{G,H}) \cap \mathrm{Ker}(i_{H,G}) = 0$ by Theorem 6.7 (ii). $\blacksquare$

6.9. Proposition. *Let s be a species for $A(G)$. The following conditions on a subgroup H are equivalent*:

(i) *Ker*$(s) \geqslant \mathrm{Ker}(r_{G,H})$.

(ii) $\mathrm{Ker}(s) \ngeqslant \mathrm{Im}(i_{H,G})$.

(iii) *There is a species t for $A(H)$ such that for all $x \in A(G)$, $(s, x) = (t, x \downarrow_H)$.*

Proof. The equivalence of (i) and (ii) follows from Theorem 6.7. The equivalence of (i) and (iii) follows from Proposition 5.3 and the going-up theorem (see, e.g., [1, p. 62]). $\blacksquare$

Note that in Proposition 6.9(iii) the species t need not be unique. We write $t \sim s$ and say *t fuses* to s if (iii) is satisfied. This is a generalization of the concept of fusion of p'-conjugacy classes of a subgroup, see Definition 6.12.

6.10. Definitions. A species s of $A(G)$ *factors through H* if and only if the equivalent conditions of Proposition 6.9 are satisfied. An *origin* of s is a subgroup minimal among those through which s factors. If s is a species for $A(H)$, $H \leqslant G$, then the *stabilizer* of s is defined by

$$\mathrm{Stab}_G(s) = \{ y \in N_G(H) \colon (s, x) = (s, x^y) \text{ for all } x \in A(H) \}.$$

Suppose R is spanned as a vector space by indecomposable modules. A *vertex* of a species s of R is a vertex of minimal size over indecomposable modules V for which $(s, V) \neq 0$.

6.11. PROPOSITION. (i) *If W is an indecomposable module with $(s, W) \neq 0$ then every vertex of s is contained in a vertex of W.*

(ii) *The vertices of s form a single conjugacy class of p-subgroups of G.*

Proof. (i) Suppose D is a vertex of s, and of V with $(s, V) \neq 0$. If $(s, W) \neq 0$ then $(s, V \otimes W) \neq 0$ and so $(s, X) \neq 0$ for some indecomposable direct summand X of $V \otimes W$. But every vertex of X is contained in both a vertex of V and a vertex of W. By minimality, D is a vertex of X and is contained in a vertex of W.

(ii) Follows immediately from (i). ∎

6.12. DEFINITION. A *Brauer species* is a species whose vertex is the trivial subgroup. Thus choosing an isomorphism between the $|G|$th roots of unity in $\bar{k}$ and in $\mathbb{C}$, the Brauer species are put in one–one correspondence with conjugacy classes of elements of order coprime to p. The value of a species on a module is the value of the Brauer character of the module on the corresponding element. Clearly the origin of a Brauer species is the cyclic group generated by the corresponding element.

6.13. PROPOSITION. *Let s be a species of $A(G)$. Then the origins of s form a single conjugacy class of subgroups.*

Proof. Let H_1 and H_2 be two origins of s. Then since $\mathrm{Ker}(s)$ is a prime ideal,

$$\mathrm{Ker}(s) \not\geqslant \mathrm{Im}(i_{H_1,G}) \cdot \mathrm{Im}(i_{H_2,G}) \leqslant \sum_{x \in G} \mathrm{Im}(i_{H_1 \cap H_2^x, G}).$$

So for some $x \in G$, $\mathrm{Ker}(s) \not\geqslant \mathrm{Im}(i_{H_1 \cap H_2^x})$. Hence by minimality $H_1 = H_2^x$. ∎

We now investigate what happens when we extend the field k.

6.14. PROPOSITION. *Let k_1 be a separable algebraic extension of k, and let s be a species of $A_k(G)$. Then s extends to a species of $A_{k_1}(G)$. For any such extension t of s we have*

(i) *A vertex of s is a vertex of t,*

(ii) *an origin of s is an origin of t.*

Proof. The fact that s extends to a species of $A_{k_1}(G)$ follows from Proposition 5.2 and the going-up theorem [1, p. 62].

(i) In [13, Lemma 4.6], it is shown that if M is an indecomposable kG-module and N is an indecomposable component of $M \otimes_k k_1$, then a vertex of M is also a vertex of N. Thus a vertex of s contains a vertex of t.

To prove the converse, we let k_2 be the normal closure of k_1. It suffices to prove the proposition with k_2 in place of k_1. Let N be an indecomposable k_2G-module with vertex D and $(t, N) \neq 0$. Let $N_1,..., N_r$ be the images of N under $\mathrm{Gal}(k_2/k)$. Then the minimal equation of N over $A_k(G)$ is

$$(X - N_1) \cdot \cdots \cdot (X - N_r) = X^r - A_{r-1}X^{r-1} + \cdots \pm A_0,$$

with $A_i \in A_k(G)$. Hence we have

$$(t, N)^r - (s, A_{r-1})(t, N)^{r-1} + \cdots \pm (s, A_0) = 0.$$

Since $(t, N) \neq 0$, at least one of the (s, A_i) is nonzero, and hence for some indecomposable direct summand A_i' of A_i, $(s, A_i') \neq 0$. But D contains a vertex of every such A_i', and hence contains a vertex of s.

(ii) This follows from the fact that the decomposition of Theorem 6.7 is preserved under field extensions, in the sense that $i_{H,G}(A_k(H)) \leqslant i_{H,G}(A_{k_1}(H))$ and $\mathrm{Ker}_{A_k(G)}(r_{G,H}) \leqslant \mathrm{Ker}_{A_{k_1}(G)}(r_{G,H})$. ▮

7. The Trivial Source Subring

7.1. Definition. A group H is said to be *p-hypoelementary* if and only if $H/O_p(H)$ is cyclic. Note that this is not the same as the p-hyperelementary subgroups used in, for example, the theory of Schur indices, see [12, p. 302]. Let $\mathrm{Hyp}_p(G)$ be the collection of p-hypoelementary subgroups of G.

The following is a construction for species of $A(G, \mathrm{Triv})$. Let $H \in \mathrm{Hyp}_p(G)$, and let V be an indecomposable module for kG with trivial source. Let $V\downarrow_H = W_1 \oplus W_2$, where W_1 is a direct sum of modules with vertex $O_p(H)$ and $W_2 \in A'(G, H)$. Then $O_p(H)$ acts trivially on W_1, and so W_1 is a module for $H/O_p(H)$. Let b be a Brauer species of $H/O_p(H)$, and define $(s_{H,b}, V) = (b, W_1)$. Then it is easy to check that $s_{H,b}$ is a species for $A(G, \mathrm{Triv})$, and in [9, 10] Conlon proves

7.2. Theorem. *$A(G, Triv)$ is semisimple, and the species for $A(G$,* Triv) *are precisely the $s_{H,b}$ defined above.*

Following (6.6) we have a direct sum decomposition,

$$A(G) = \bigoplus_{H,b} A(G) \cdot e_{H,b}. \tag{7.3}$$

In this decomposition, H runs through a subset of $\mathrm{Hyp}_p(G)$ containing one representative from each conjugacy class. For a given H, b runs through a set containing one representative of each conjugacy class under the action of

$N_G(H)$ of Brauer species of $H/O_p(H)$ whose origin is the whole of $H/O_p(H)$; $e_{H,b}$ is the corresponding idempotent in $A(G, \mathrm{Triv})$. In [10] Conlon also shows that $r_{G,N_G(H)}\colon A(G) \cdot e_{H,b} \to A(N_G(H)) \cdot e_{H,b}$ is an isomorphism, and that $r_{N_G(H),H}\colon A(N_G(H)) \cdot e_{H,b} \to A(H) \cdot e_{H,b}$ is an injection. If $K < H$, then $e_{H,b} \downarrow_K = 0$. Thus if s is a species of $A(G)$, then it is a species of a unique $A(G) \cdot e_{H,b}$, and H is an origin of s. This, together with Proposition 6.13, proves

7.4. Proposition. *Let s be a species of $A(G)$. Then the origins of s form a single conjugacy class of hypoelementary subgroups of G.*

In order to analyse the relationship between vertices and origins, we need another result of Conlon, and some preliminary lemmas.

7.5. Lemma (Conlon [8]). *For any $H \leqslant G$, $A(G, H)$ is a direct summand of $A(G)$. Contained in $A(G, H)$ there is a canonical direct summand $A''(G, H)$ of $A(G)$, with the properties that*

$$A''(G, H) \cong W(G, H) \cong W(N_G(H), H),$$

and $A(G, H) = A'(G, H) \oplus A''(G, H)$. This gives a direct sum decomposition,

$$A(G, H) = \bigoplus_D A''(G, D).$$

In this decomposition, D runs through a set containing exactly one representative from each conjugacy class in G of p-subgroups of H. The map

$$r_{G,H_G(H)}\colon A''(G, H) \to A''(N_G(H), H)$$

is an isomorphism.

7.6. Lemma. *If D is a vertex for s then s factors through $N_G(D)$.*

Proof. Here s is nonzero on $A(G, D)$ and zero on $A'(G, D)$. Hence it is a species of $A''(G, D)$ and factors through $r_{G,N_G(D)}$ by Lemma 7.5. ∎

7.7. Lemma. *Suppose k is a separably closed field and V is an indecomposable kG-module. Then V is absolutely indecomposable.*

Proof. Let $E = \mathrm{End}_{kG}(V)$ and k_1 be the algebraic closure of k. By [14, (18.4)] we have $E \otimes_k k_1 = \mathrm{End}_{k_1G}(V \otimes_k k_1)$. Since $J(E) \otimes_k k_1 \subseteq J(E \otimes_k k_1)$, we need only show that $(E/J(E)) \otimes_k k_1$ is a local ring. Now by Noether's theorem [29, 3.2.1, p. 78], $E/J(E)$ is a field, since k is separably closed. Let $e = \sum a_i \otimes \lambda_i$ be an idempotent in $(E/J(E)) \otimes_k k_1$. Since k_1 is purely inseparable over k, there is an integer n such that for all i, $\lambda_i^{p^n} \in k$. Then

$$e = e^{p^n} = \left(\sum a_i \otimes \lambda_i\right)^{p^n} = \sum a_i^{p^n} \otimes \lambda_i^{p^n} \in E.$$

Hence $e = 1$, and so $(E/J(E)) \otimes_k k_1$ is a local ring. ∎

7.8. THEOREM. *Let s be a species of $A(G)$. If H is an origin of s then $O_p(H)$ is a vertex of s.*

Proof. By Proposition 6.14 we may replace the field k by its separable closure. Then by Lemma 7.7, any indecomposable kG-module is absolutely indecomposable.

By Proposition 6.9, s is nonzero on some module induced from H. Thus H contains a vertex of s, and hence so does $O_p(H)$. Suppose $D < O_p(H)$ is a vertex of s. Let W be a kG-module with vertex D and $(s, W) \neq 0$. Let t be a species of $A(H)$ which fuses to s, and let V be an indecomposable summand of $W\downarrow_H$ with $(t, V) \neq 0$. Then a vertex of V is contained in some G-conjugate D^x of D, and hence so is a vertex D_1 of t. Since H is an origin for t, Lemma 7.6 shows that $D_1 \lhd H$.

Let V_1 be a module for kH with vertex D_1, such that $(t, V_1) \neq 0$. Let U be a source of V_1, and let I be the inertial group for U in H. Let E be the inverse image in H of a Hall p'-subgroup of H/D_1 containing a Hall p'-subgroup of I/D_1. Let $E_1 = E \cap I$. Then by Clifford theory we have $U\uparrow^E = X_1 \oplus \cdots \oplus X_r$, where $r = |E_1 : D_1|$, $\dim(X_i) = q \cdot \dim(U)$, $q = |E : E_1|$, $(qr, p) = 1$, and the X_i are indecomposable. Since V_1 is a direct summand of $U\uparrow^H$, this means that $q \cdot \dim(U)$ divides $\dim(V_1)$, by the Mackey decomposition theorem and the fact that E_1 contains the inertial groups in E for all H-conjugates of U. Now U is absolutely indecomposable, and so Green's theorem [17, Theorem 8] implies that $U\uparrow^{O_p(H)}$ is indecomposable. Thus $U\uparrow^H\downarrow_{O_p(H)}$ is a sum of indecomposable modules each having dimension $|O_p(H) : D_1| \cdot \dim(U)$. Thus $|O_p(H) : D_1| \cdot \dim(U)$ divides $\dim(V_1)$. But so does $q \cdot \dim(U)$, and so $\dim(V_1)$ is divisible by $q \cdot |O_p(H) : D_1| \cdot \dim(U)$. Thus V_1 is induced from one of the X_i, and so by Proposition 6.9, $(t, V_1) = 0$. This contradiction shows that $D = O_p(H)$, and proves the theorem. ∎

8. THE INDUCTION FORMULA

Let s be a species of $A(G)$ with origin $H \leqslant G$, and let V be a module for $K \leqslant G$. We want a formula for $(s, V\uparrow^G)$ in terms of species of K. Let t be a species of H fusing to s. Then

$$\begin{aligned}(s, V\uparrow^G) &= (t, V\uparrow^G\downarrow_H)\\ &= \sum_{HgK} (t, V^g\downarrow_{H\cap K^g}\uparrow^H),\end{aligned}$$

by the Mackey decomposition theorem. Now if $H \not\leqslant K^g$, $(t, V^g\downarrow_{H\cap K^g}\uparrow^H)$ is zero by Proposition 6.9, since H is an origin of t. Thus we have the formula

$$(s, V\uparrow^G) = \sum_{H^g \leqslant K} |N_G(H^g): N_K(H^g)|\,(t^g, V\downarrow_{H^g}). \tag{8.1}$$

The sum runs over K-conjugacy classes of G-conjugates of H contained in K. In order to convert this into a formula involving species of K, we must examine the number of species of K fusing to s.

8.2. THEOREM. *Let s be a species of $A(G)$ with origin H. Regard s as a species of $\mathrm{Im}(r_{G,H})$. Then s extends uniquely to a species t of $A(H)^{N_G(H)}$, and $N_G(H)$ is transitive on the extensions $t_1,\dots,t_r$ of t to $A(H)$. The number of extensions is $r = |N_G(H): \mathrm{Stab}_G(t_1)|$.*

Proof. By Proposition 6.9(iii), s certainly extends to species of $A(H)^{N(H)}$ and $A(H)$. Let t be an extension of s to $A(H)^{N(H)}$. Then for $x \in A(H)^{N(H)}$, the Mackey decomposition and the fact that H is an origin for s imply that $(t, x\uparrow^G\downarrow_H) = |N_G(H): H|(t, x)$. Thus $(t, x) = (1/|N_G(H): H|)(s, x\uparrow^G)$ is uniquely determined by s.

Now suppose t_1 and t_2 are two extensions of t to $A(H)$, and that $t_1 \neq t_2^g$ for all $g \in N_G(H)$. Then by Lemma 6.5 there is an element $x \in A(H)$ such that $(t_1, x) = 0$ and $(t_2, x^g) = (t_2^{g^{-1}}, x) = 1$ for all $g \in N_G(H)$. Let $y = \prod_{g\in N_G(H)} x^g$. Then $0 = (t_1, y) = (t, y) = \prod_{g \in N_G(H)} (t_2, x^g) = 1$. This contradiction proves the theorem. The formula for the number of extensions is clear. ∎

By Theorem 8.2, the contribution in (8.1) from a particular conjugate H^g is

$$\sum_{t^g \sim s} |\mathrm{Stab}_G(t^g): N_K(H^g)|\,(t^g, V\downarrow_{H^g}).$$

In this expression, t^g runs over the species of H^g fusing to s. If s_0 is a species for K fusing to s, and with origin H^g, then by Theorem 8.2, the number of t^g fusing to s_0 is

$$|N_K(H^g): \mathrm{Stab}_K(t^g)| = |N_G(H^g) \cap \mathrm{Stab}_G(s_0): \mathrm{Stab}_G(t^g)|.$$

Thus

$$\sum_{t^g \sim s_0} |\mathrm{Stab}_G(t^g): N_K(H^g)|\,(t^g, V\downarrow_{H^g})$$
$$= |N_G(H^g) \cap \mathrm{Stab}_G(s_0): N_K(H^g)|\,(s_0, V).$$

Thus we can rewrite (8.1) as

$$(s, V\uparrow^G) = \sum_{s_0 \sim s} |N_G(\mathrm{Orig}(s_0)) \cap \mathrm{Stab}_G(s_0): N_K(\mathrm{Orig}(s_0))|\,(s_0, V). \quad (8.3)$$

In this expression, s_0 runs over the species of K fusing to s, and $\mathrm{Orig}(s_0)$ is any origin of s_0.

The expression (8.3) is called the *induction formula*, since it is a generalization of the usual formula for the value of an induced character (see also (9.17)).

9. Finite Dimensional Summands of $A(G)$

We now investigate what happens when we project everything onto a finite dimensional summand. Suppose $A(G) = A \oplus B$ is an ideal direct sum decomposition, with projections π_1: $A(G) \to A$ and $\pi_2: A(G) \to B$. Suppose the following conditions are satisfied:

(i) A is finite dimensional.

(ii) A is semisimple as a ring.

(iii) A is freely spanned as a vector space by indecomposable modules.

(iv) A is closed under taking dual modules.

Under these conditions we will define tables T_{ij} and U_{ij} resembling the tables of modular irreducible characters and projective indecomposable characters.

9.1. *Remarks.* (i) Any finite dimensional semisimple ideal I is a direct summand since

$$A(G) = I \oplus \bigcap_s \ker(s).$$

(ii) Any direct summand satisfying the above conditions automatically contains $A(G, 1)$. This follows from Lemma 9.8, since by tensoring P_1 by itself and taking direct summands, we obtain all projective modules for $G/O_{p'}(G)$, and hence we obtain the idempotent generator for $A(G, 1)$.

9.2. EXAMPLES. (i) Since the Brauer species separate elements of $A(G, 1)$, and $A(G) = A(G, 1) \oplus A_0(G, 1)$, this means that $A(G, 1)$ satisfies the above conditions.

(ii) In [21], it is shown that for H cyclic, $A(G, H)$ is a finite dimensional semisimple ideal. Thus $A(G, \mathrm{Cyc})$ satisfies the above conditions. We write $A(G) = A(G, \mathrm{Cyc}) \oplus A_0(G, \mathrm{Cyc})$. This case will be studied in more detail in Section 10.

(iii) It can be seen from the tables in Appendix 1 that the Green Ring of the Klein four-group has infinitely many such summands.

9.3. LEMMA. *The inner products* $\langle\ ,\ \rangle$ *and* $(\ ,\ )$ *are nonsingular on* A.

Proof. Given $x \in A$, choose y and z in $A(G)$ as in Proposition 3.7. Then $\langle x, \pi_2(y)\rangle = \langle 1, \bar{x} \cdot \pi_2(y)\rangle = \langle 1, 0\rangle = 0$ using property (iv) and the fact that $A \cdot B = 0$. Hence $0 \neq \langle x, y\rangle = \langle x, \pi_1(y)\rangle$. Similarly $(x, \pi_1(z)) \neq 0$. ∎

9.4. DEFINITIONS. Let $s_1, \ldots, s_n$ be the species of A, and $V_1, \ldots, V_n$ the indecomposable modules freely spanning A. Let $G_1, \ldots, G_n$ be the corresponding atoms. The *atom table* of A is the matrix

$$T_{ij} = (s_j, G_i) = (s_j, \pi_1(G_i)).$$

The *representation table* of A is the matrix

$$U_{ij} = (s_j, V_i).$$

Let $\Gamma = \pi_1(a(G))$.

9.5. LEMMA. Γ *is a lattice in* A.

Proof. Let $x \in a(G)$. Then for each i, $\langle \pi_1(x), V_i\rangle \in \mathbb{Z}$. ∎

9.6. LEMMA. *Let V be a kG-module in $a(G)$, and let s be a species of A. Then (s, V) is an algebraic integer.*

Proof. The $\mathbb{Z}$-span in A of the tensor powers of $\pi_1(V)$ form a sublattice of Γ. Since this lattice satisfies the ascending chain condition, this implies that for some n,

$$\otimes^n(\pi_1(V)) \in \langle 1, \pi_1(V), \ldots, \otimes^{n-1}(\pi_1(V))\rangle.$$

This gives a monic equation with integer coefficients satisfied by the value of every species of A on V. ∎

9.7. *Question.* (i) Under the conditions of Lemma 9.6, is it true that (s, V) is always a cyclotomic integer?

(ii) $\Gamma/(a(G) \cap A)$ is a finite abelian group of order $\det(\langle V_i, V_k \rangle)$. Is it a p-torsion group?

9.8. LEMMA. $P_1 \in A$.

Proof. Since $\pi_1(1)$ is the identity element of A, it is nonzero, and hence for some V_j, $\langle V_j, 1 \rangle = \langle V_j, \pi_1(1) \rangle \neq 0$ by Lemma 9.3. Thus by Corollary 2.4 some V_j is equal to P_1. ∎

We choose our notation so that $P_1 = V_1$. By Lemma 6.5, the matrix U_{ij} is invertible. We define

$$m_i = (U^{-1})_{i1} = \sum_j (U^{-1})_{ij} \langle 1, V_j \rangle.$$

Then $\langle 1, V_i \rangle = \sum_j U_{ij} m_j$, and so for any $x \in A(G)$ and $y \in A$ we have the equations,

$$\langle 1, y \rangle = \sum_j (s_j, y)\, m_j,$$

$$\langle x, y \rangle = \langle 1, \bar{x} \cdot y \rangle = \sum_j (s_j, \bar{x})(s_j, y)\, m_j.$$

Now by Lemma 9.3 this means the m_j are nonzero, and so we can define

$$c_j = c(s_j) = c_G(s_j) = 1/m_j. \tag{9.9}$$

Thus we have, for $x \in A(G)$ and $y \in A$,

$$\langle x, y \rangle = \sum_j \frac{(s_j, \bar{x})(s_j, y)}{c_j}. \tag{9.10}$$

9.11. EXAMPLE. If $A = A(G, 1)$, then the c_j are the orders of the centralizers of the origins of the s_j, as is well known from the orthogonality relations of Brauer character theory. For a more general A, the c_j need not, however, be positive or rational. Indeed, they are not, in general, even for $A = A(G, \mathrm{Cyc})$.

Now let $p_i = (s_i, u) = (s_i, \pi_1(u))$. By Corollary 2.3, for $x \in A(G)$ and $y \in A$,

$$(x, y) = \langle x, u \cdot y \rangle,$$

and so by (9.10) we have

$$(x, y) = \sum_j \frac{p_j(s_j, \bar{x})(s_j, y)}{c_j}. \tag{9.12}$$

Now let $U^\#$ be the matrix obtained by transposing U and replacing each representation by its dual. Let C be the diagonal matrix of c_is. The orthogonality relations (9.10) can be expressed as

$$TC^{-1}U^\# = 1. \tag{9.13}$$

9.14. *Question.* Is it true in general that $U^\# = U^+$, the Hermitian adjoint of U? In other words, is it true that $(s, \bar{x}) = \overline{(s, x)}$?

9.15. *Remark.* If A_1 and A_2 are two direct summands of $A(G)$ satisfying (i)–(iv) and s is a species which is not identically zero on A_1 or A_2, then the two definitions of $c(s)$ obtained by viewing s as a species of A_1 and A_2 coincide. This follows from the ensuing more general discussion.

Now suppose s_i is a species of A which factors through a subgroup $H \leqslant G$. Let $r_{G,H}(A) \leqslant A'$ with A' also satisfying (i)–(iv) (e.g., in Examples 9.2(i), (ii) let $A' = A(H, 1)$ and $A(H, \mathrm{Cyc})$, resp.). Choose a species t_j of A' fusing to s_i. We wish to compare $c_G(s_i)$ with $c_H(t_j)$.

Choose $x_i \in A$ with $(s_k, x_i) = \delta_{ik}$ and $y_j \in A'$ with $(t_m, \bar{y}_j) = \delta_{jm}$. Then $\langle y_j \uparrow^G, x_i \rangle = \langle y_j, x_i \downarrow_H \rangle$ by Corollary 2.6. Hence

$$\sum_k \frac{(s_k, \bar{y}_j \uparrow^G)(s_k, x_i)}{c_G(s_k)} = \sum_m \frac{(t_m, \bar{y}_j)(t_m, x_i \downarrow_H)}{c_H(t_m)},$$

by (9.10). Thus by the choice of x_i and y_j,

$$\frac{(s_i, \bar{y}_j \uparrow^G)}{c_G(s_i)} = \frac{(t_j, x_i \downarrow_H)}{c_H(t_j)} = \frac{(s_i, x_i)}{c_H(t_j)} = \frac{1}{c_H(t_j)}.$$

Hence $c_G(s_i) = (s_i, \bar{y}_j \uparrow^G)\, c_H(t_j)$. Now using the induction formula (8.3), we obtain the desired equation,

$$c_G(s_i) = |N_G(\mathrm{Orig}(t_j)) \cap \mathrm{Stab}_G(t_j) : N_H(\mathrm{Orig}(t_j))| \cdot c_H(t_j). \tag{9.16}$$

Conversely, we may rewrite the induction formula in the form

$$(s, V \uparrow^G) = \sum_{s_0 \sim s} \frac{c_G(s)}{c_K(s_0)} (s_0, V). \tag{9.17}$$

In this formula, s_0 runs over the species of A' fusing to s.

10. The Cyclic Vertex Ideal $A(G, \mathrm{Cyc})$

In this section we shall assume that k is algebraically closed. As in Section 9, we let $\pi_1 : A(G) \to A(G, \mathrm{Cyc})$ and $\pi_2 : A(G) \to A_0(G, \mathrm{Cyc})$ be the projections. By Example 9.2(ii), we may apply all the results of Section 9 to the ideal $A(G, \mathrm{Cyc})$.

By definition of vertex, the species of $A(G, \mathrm{Cyc})$ are precisely the species with cyclic vertex. Hence by Proposition 7.4 and Theorem 7.8 these are the species whose origins are the semidirect product of a normal cyclic p-group by a cyclic group of order coprime to p. We shall call such a group $p \cdot p'$-metacyclic. By Proposition 6.9, our first task is to examine the species for $p \cdot p'$-metacyclic groups.

Let H be a $p \cdot p'$-metacyclic group of order $p^r \cdot m$ with $(p, m) = 1$. Let

$$H = \langle x, y : x^{p^r} = y^m = 1, x^y = x^a \rangle,$$

where a is a primitive dth root of 1 modulo p^r, d divides $p-1$ and d divides m. Let θ be a primitive mth root of 1 in k with $a \equiv \theta^{m/d}$ as elements of the prime field of k. There are m irreducible modules $X_1(\theta^q)$, $1 \leqslant q \leqslant m$, for H, which are one dimensional and are given by $x \mapsto (1)$, $y \mapsto (\theta^q)$ as matrices. If $1 \leqslant n \leqslant p^r$, there are m indecomposable modules of dimension n. These are denoted $X_n(\theta^q)$, $1 \leqslant q \leqslant m$. These account for all the irreducible modules. $X_n(\theta^q)$ is uniserial, with Loewy layers $L_i(X_n(\theta^q)) \cong X_1(a^{n-i}\theta^q)$. We write X_n for $X_n(1)$.

10.1. *The Case $r = 1$*

In this case H has order $p \cdot m$, $(p, m) = 1$. The following relations are sufficient to determine the structure of the Green Ring:

$$\begin{aligned}
X_1(\theta^q) \otimes X_n &\cong X_n(\theta^q), && \\
X_2 \otimes X_n &\cong X_{n-1}(a) \oplus X_{n+1} && \text{if} \quad 1 \leqslant n < p, \\
&\cong X_p(a) \oplus X_p && \text{if} \quad n = p.
\end{aligned} \tag{10.2}$$

Thus the species are as follows. Let

$$\begin{aligned}
f_n(x) &= x^{n-1} + x^{n-3} + \cdots + x^{-n+1} \\
&= \frac{x^n - x^{-n}}{x - x^{-1}} \qquad \text{when } x \neq \pm 1.
\end{aligned}$$

Let ε be a primitive $2p$th root of unity and λ a primitive $2m$th root of unity in $\mathbb{C}$. Define

$$
\begin{aligned}
(s(1), X_n(\theta^q)) &= n, \\
(s(\lambda^{2t}), X_n(\theta^q)) &= \lambda^{2qt}\left(\frac{1-\lambda^{2mnt/d}}{1-\lambda^{2mt/d}}\right), \\
(s(\varepsilon^{t_1}, \lambda^t), X_n(\theta^q)) &= \lambda^{2qt+t(n-1)m/d} f_n(\varepsilon^{t_1}),
\end{aligned} \tag{10.3}
$$

for $t_1 \not\equiv 0 \bmod p$. Then $s(\varepsilon^{t_1}, \lambda^t) = s(\varepsilon^{-t_1}, \lambda^t)$, and the functions defined by (10.3) satisfy the relations given by (10.2). Thus $s(1)$, $s(\lambda^{2t})$, and $s(\varepsilon^{t_1}, \lambda^t)$ are the $p \cdot m$ species of $A(H)$. This means the representation depicted in Table I.

10.4. Example. Let $p = 2$ and $m = 1$. Then $H = \langle x : x^2 = 1 \rangle$. The two species are $s(1)$ and $s(-1, -1)$. Their values are $(s(1), V) = \dim(V)$ and $(s(-1, -1), V) = \dim(\mathrm{Ker}(x+1)/\mathrm{Im}(x+1))$. See also Section 12.

10.5. *The Case* $m = 1$

In this case H is cyclic of order p^r. It is shown in [18, Sect. 2.3], that the following relations define $A(H)$ as an extension of the Green Ring of the quotient H/H_0 of order p^{r-1}.

$$
\begin{aligned}
&(X_{p^{r-1}+1} - X_{p^{r-1}-1}) \cdot X_n \\
&\quad = X_{p^{r-1}+n} - X_{p^{r-1}-n}, && 1 \leqslant n \leqslant p^{r-1}, \\
&\quad = X_{p^{r-1}+n} + X_{n-p^{r-1}}, && p^{r-1} < n < (p-1)p^{r-1}, \\
&\quad = X_{n-p^{r-1}} + 2X_{p^r} - X_{2p^r-(n+p^{r-1})}, && (p-1)p^{r-1} \leqslant n \leqslant p^r.
\end{aligned} \tag{10.6}
$$

The species are defined inductively as follows. Let s_t, $1 \leqslant t \leqslant p$ be the species for C_p given in Table I, and let $s_{t_1, \ldots, t_{r-1}}$, $1 \leqslant t_i \leqslant p$, be the species of H/H_0. Define

$$
p_{t_1, \ldots, t_{r-1}} = (s_{t_1, \ldots, t_{r-1}}, X_{p^{r-1}} - X_{p^{r-1}-1}).
$$

TABLE I

		Brauer Species	Non-Brauer Species
Species / Rep^n	$s(1)$	$s(\lambda^{2t})\quad (t \neq 0)$	$s(\varepsilon^{\pm t_1}, \lambda^t)\quad (t_1 \neq 0)$
$X_n(\theta^q)$	n	$\lambda^{2qt}\left(\frac{1-\lambda^{2mnt/d}}{1-\lambda^{2mt/d}}\right)$	$\lambda^{2qt+t(n-1)m/d} f_n(\varepsilon^{t_1})$

(These are the p_i defined in Sect. 9, and are all ± 1. In fact p_i is the sign of $c(s_i)$ in this case). Let $n = n_0 p^{r-1} + n_1$, with $1 \leqslant n_0 \leqslant p$ and $1 \leqslant n_1 < p^{r-1}$. Then the species $s_{t_1,\ldots,t_r}$ of H are defined by

$$(s_{t_1,\ldots,t_r}, X_{n+1} - X_n) = (s_{t_1,\ldots,t_{r-1}}, X_{n_1+1} - X_{n_1})(s_{t_r}, X_{n_0+1} - X_{n_0})\, p^{n_0}_{t_1,\ldots,t_r}. \tag{10.7}$$

Thus to obtain the representation table for C_{p^r} we take the successive differences of the rows of the table for C_p, form the tensor rth power of the resulting table, multiply by the appropriate signs and sum back up again.

10.8. EXAMPLE. Let $p = 3$. The representation table of C_3 and its successive differences are as follows, where the top row of the first table gives $c(s_i)$, (see (9.9)).

3	6	−2
1	1	1
2	−1	1
3	0	0

1	1	1
1	−2	0
1	1	−1

Tensor squaring and multiplying by the appropriate signs, we get the first of the following tables. Summing back up again gives the second, which is the representation table of C_9.

1	1	1	1	1	1	1	1	1
1	−2	0	1	−2	0	1	−2	0
1	1	−1	1	1	−1	1	1	−1
1	1	−1	−2	−2	2	0	0	0
1	−2	0	−2	4	0	0	0	0
1	1	1	−2	−2	−2	0	0	0
1	1	1	1	1	1	−1	−1	−1
1	−2	0	1	−2	0	−1	2	0
1	1	−1	1	1	−1	−1	−1	1

9	18	−6	18	36	−12	−6	−12	4
1	1	1	1	1	1	1	1	1
2	−1	1	2	−1	1	2	−1	1
3	0	0	3	0	0	3	0	0
4	1	−1	1	−2	−2	3	0	0
5	−1	−1	−1	2	2	3	0	0
6	0	0	−3	0	0	3	0	0
7	1	1	−2	1	1	2	−1	−1
8	−1	1	−1	−1	1	1	1	−1
9	0	0	0	0	0	0	0	0

10.9. *The General $p \cdot p'$-Metacyclic Group*

We conjecture that the following formula is the appropriate generalization of (10.3) and (10.7). Let

$$\begin{aligned} s_{t_r,t} &= s(\varepsilon^{\pm t_r}, \lambda^t) && \text{if} \quad t_r \neq 0, \\ &= s(\lambda^{2t}) && \text{if} \quad t_r = 0, \end{aligned}$$

be the species given in (10.3). The remaining definitions are as in case 10.5. Then the species of H are defined inductively by

$$\begin{aligned}(s_{t_1,\dots,t_r,t}, X_{n+1}(\theta^q) - X_n(\theta^q))\\ = (s_{t_1,\dots,t_{r-1},t}, X_{n_1+1}(\theta^q) - X_{n_1}(\theta^q))(s_{t_r,t}, X_{n_0+1}(\theta^q)\\ - X_{n_0}(\theta^q)) \cdot p^{n_0}_{t_1,\dots,t_{r-1},t}.\end{aligned} \tag{10.10}$$

10.11. *Comments.* If a representation is held on a computer as matrices, then calculation of values of cyclic vertex species is very easy. They can be calculated from the ranks of elements of the form $(X - \lambda I)^i$ of the group algebra.

We now wish to prove a copying theorem for species with cyclic origin on trivial source representations. Let R be a complete discrete valuation ring in characteristic 0 with maximal ideal $\mathfrak{p}$ and $R/\mathfrak{p} = k$.

10.12. LEMMA. *Every indecomposable kG-module with trivial source lifts uniquely to an R-free RG-module with trivial source.*

Proof. See [23, Proposition 1, p. 102]. ∎

10.13. LEMMA. *Let V be an R-free indecomposable RG-module with trivial source, and let $g \in G$. Write $g = xy = yx$, with x a p-element and y a p'-element. Then $\chi_V(g) = \chi_W(y)$, with $W = C_V(x)$, the space of fixed points of x on V.*

Proof. $V\downarrow_{\langle g\rangle}$ is a sum of R-free trivial source indecomposable $R\langle g\rangle$-modules. These are the modules of the form $X = X_1 \otimes X_2$, where X_1 is a permutation module for $R\langle x\rangle$ and X_2 is a one-dimensional module for $R\langle y\rangle$. Then

$$\chi_X(g) = \chi_{X_1}(x)\,\chi_{X_2}(y) = \chi_{C_X(x)}(y).$$

Hence $\chi_V(g) = \chi_W(y)$. ∎

10.14. THEOREM. *Let V be a trivial source kG-module and s a species of $A(G)$ with cyclic origin $\langle g\rangle$. Let $\hat{V}$ be the lift of V to an R-free RG-module with trivial source (see Lemma 10.12). Then $(s, V) = \chi_{\hat{V}}(g^r)$ for some generator g^r of $\langle g\rangle$.*

Proof. Let $g = xy = yx$, with x a p-element and y a p'-element. By Theorem 7.2, the species s is evaluated on V by finding the fixed space of x and evaluating a faithful Brauer character of $\langle y\rangle$. Now by Lemma 10.13, $C_{\hat{V}}(x) \otimes k = C_V(x)$. Hence there is a generator y^r of $\langle y\rangle$ such that $(s, V) = \chi_{C_{\hat{V}}(x)}(y^r)$, by definition of Brauer characters. Hence by Lemma 10.13, $(s, V) = \chi_{\hat{V}}(x \cdot y^r)$. ∎

10.15. *Remark.* If $\mathrm{char}(k) = 2$, then every species with cyclic vertex has cyclic origin.

10.16. PROBLEM. Let $D = \langle x \rangle$ be a cyclic p-subgroup of G. How many representations does G have with vertex D?

Solution. Let $|D| = p^r$ and let $D' = \langle x^p \rangle$. We must calculate $\dim_{\mathbb{C}}(A(G, D)) - \dim_{\mathbb{C}}(A(G, D'))$. Since these rings are semisimple (see Example 9.2(ii)) this is equal to the number of species of G with vertex D.

Let H be a hypoelementary subgroup with $O_p(H) = D$, and let $|H| = p^r \cdot s$, $(p, s) = 1$. Then H has $p^{r-1}\ (p-1)\, s$ species with vertex D. By Theorem 8.2, two such fuse to the same species of $A(G)$ if and only if they are conjugate under $N_G(H) \leqslant N_G(D)$.

Thus if h is the number of $N_G(D)$-conjugacy classes of p'-elements of $N_G(D)$, then the total number of distinct species of $A(G)$ with vertex D is $h \cdot p^{r-1}(p-1)$. Hence this is also the number of indecomposable representations of G with vertex D.

11. ATOM COPYING

In this section, we prove a theorem about local control of the atoms. The tools for this section are the Green correspondence and an extension due to Burry and Carlson.

11.1. PROPOSITION (Burry and Carlson [4, Theorem 5]). *Let D be a p-subgroup of G, and let H be a subgroup of G with $N_G(D) \leqslant H$. Let V be an indecomposable kG-module such that $V\downarrow_H$ has a direct summand U with vertex D. Then V has vertex D and V is the Green correspondent of U.*

11.2. THEOREM (Atom copying by induction). *Let D be a p-subgroup of G, and let H be a subgroup of G with $N_G(D) \leqslant H$. Let V_i be an indecomposable kG-module with vertex D and atom G_i. Let the kH-module V_j' be the Green correspondent of V_i, and let G_j' be the corresponding atom. Then $G_j'\uparrow^G = G_i$.*

Proof. By Proposition 11.1, if V_k is an indecomposdable kG-module, then $V_k\downarrow_H$ has V_j' as a direct summand if and only if $i = k$, and then only once. Thus by Corollary 2.6 and Theorem 3.5 we have

$$\langle V_k, G_j'\uparrow^G \rangle = \langle V_k\downarrow_H, G_j' \rangle = \delta_{ik}.$$

Hence $\langle V_k, G_j'\uparrow^G - G_i \rangle = 0$, and so by Proposition 3.7, $G_j'\uparrow^G - G_i = 0$. ∎

11.3. *Remark.* For V_i a module whose vertex is a full Sylow subgroup, we also have $G_i \downarrow_H = G_j'$. The proof is trivial.

11.4. EXAMPLE. Let $G = L_3(2)$ and $p = 3$. Since a Sylow p-subgroup is cyclic of order 3, we may determine the whole atom table and representation table. Since the Sylow 3-normalizer is S_3, we must first determine the tables for S_3. From Table I we see that these are as given in Appendix 2. Thus from the atom copying theorem 11.2 and the 3-modular Brauer character table of $L_3(2)$ we have the following portion of the atom table for $L_3(2)$ (see Appendix 2 for an explanation of the notation).

168	8	4	7	7	12	−4	−4	−4
1*A*	2*A*	4*A*	7*A*1	7*A*2	3*A*1	3*A*2	S_3A1	S_3A2
0	0	0	0	0	3	−1	−1	−1
0	0	0	0	0	3	−1	1	1
0	0	0	0	0	−3	−1	i	$-i$
0	0	0	0	0	−3	−1	$-i$	i
1	1	1	1	1				
3	−1	1	*b*7	**				
3	−1	1	**	*b*7				
6	2	0	−1	−1				
7	−1	−1	0	0				

The value of species on the simple modules apart from the one of dimension 7 are easy enough to see, since they are all trivial or projective. How can we find the values on the one of dimension 7?

Dualizing the portion of the table we have so far, using (9.13), we get the following portion of the representation table

168	8	4	7	7	12	−4	−4	−4
1*A*	2*A*	4*A*	7*A*1	7*A*2	3*A*1	3*A*2	S_3A1	S_3A2
					1	1	1	1
					1	1	−1	−1
					−1	1	$-i$	i
					−1	1	i	$-i$
9	1	1	2	2	0	0	0	0
3	−1	1	*b*7	**	0	0	0	0
3	−1	1	**	*b*7	0	0	0	0
6	2	0	−1	−1	0	0	0	0
15	−1	−1	1	1	0	0	0	0

Which of the first four rows corresponds to the 7-dimensional irreducible? The first must be the trivial module, and the third and fourth are not self-dual. Thus the only possibility is the second row, and so the completed atom table is

168	8	4	7	7	12	-4	-4	-4
$1A$	$2A$	$4A$	$7A1$	$7A2$	$3A1$	$3A2$	S_3A1	S_3A2
0	0	0	0	0	3	-1	-1	-1
0	0	0	0	0	3	-1	1	1
0	0	0	0	0	-3	-1	i	$-i$
0	0	0	0	0	-3	-1	$-i$	i
1	1	1	1	1	1	1	1	1
3	-1	1	$b7$	**	0	0	0	0
3	-1	1	**	$b7$	0	0	0	0
6	2	0	-1	-1	0	0	0	0
7	-1	-1	0	0	1	1	-1	-1

Dualizing this, the complete representation table is

168	8	4	7	7	12	-4	-4	-4
$1A$	$2A$	$4A$	$7A1$	$7A2$	$3A1$	$3A2$	S_3A1	S_3A2
1	1	1	1	1	1	1	1	1
7	-1	-1	0	0	1	1	-1	-1
8	0	0	1	1	-1	1	$-i$	i
8	0	0	1	1	-1	1	i	$-i$
9	1	1	2	2	0	0	0	0
3	-1	1	$b7$	**	0	0	0	0
3	-1	1	**	$b7$	0	0	0	0
6	2	0	-1	-1	0	0	0	0
15	-1	-1	1	1	0	0	0	0

From this table, we can work out, for example, the decomposition of tensor products, and dimensions of spaces of homomorphisms.

11.5. *Question.* Let G be a finite group and k a field. Let M be an indecomposable kG-module with cyclic vertex, and S a simple kG-module. Can it ever happen that $\dim_k \operatorname{Hom}_{kG}(S, M) > 1$?

Answer. Yes. Let $G = M_{11}$ and k an algebraically closed field of characteristic 2. By the techniques of Sections 9–11, we can show that the cyclic vertex tables for kG are as given in Appendix 2. Using (9.12) and the fact that p_i is just the sign of the corresponding c_i, we can see that the 200-dimensional cyclic vertex indecomposable module has two copies of the 44-dimensional irreducible module in its socle.

12. The Middle of a Module

Let G be a finite group, and k a field of characteristic 2. Let t be an involution in G, and $H = C_G(t)$. Let $d = 1 + t$ as an element of the group algebra of H. If V is a module for H, we also use the symbol d to denote the map given by right multiplication by d.

12.1. *Definition.* $\lambda_t(V) = \lambda_{t,H}(V) = \mathrm{Ker}_V(d)/\mathrm{Im}_V(d)$ is called the *middle* of V with respect to t. (Note that $d^2 = 0$.) If W is a kG-module, the *middle* of W with respect to t is defined to be $\lambda_t(r_{G,H}(W))$.

If $f\colon V \to W$ is a kH-module homomorphism, then $f(\mathrm{Ker}_V(d)) \subseteq \mathrm{Ker}_W(d)$ and $f(\mathrm{Im}_V(d)) \subseteq \mathrm{Im}_W(d)$. Thus f induces a homomorphism $\lambda_t(f)\colon \lambda_t(V) \to \lambda_t(W)$. We shall use the symbol λ_t to denote both this functor and the corresponding map of Green Rings $\lambda_t\colon A(H) \to A(H/\langle t\rangle)$.

12.2. Lemma. *The map λ_t is a ring homomorphism.*

Proof. We must show that for V and W kH-modules,

$$\frac{\mathrm{Ker}_V(d)}{\mathrm{Im}_V(d)} \otimes \frac{\mathrm{Ker}_W(d)}{\mathrm{Im}_W(d)} \cong \frac{\mathrm{Ker}_{V\otimes W}(d)}{\mathrm{Im}_{V\otimes W}(d)}.$$

The action of d on $V \otimes W$ is

$$(v \otimes w)\, d = v \otimes wd + vd \otimes w + vd \otimes wd. \tag{12.3}$$

Let $i\colon \mathrm{Ker}_V(d) \otimes \mathrm{Ker}_W(d) \hookrightarrow V \otimes W$ be the inclusion map. Then by (12.3) we have

(i) $\mathrm{Im}(i) \subseteq \mathrm{Ker}_{V\otimes W}(d)$,

(ii) $(\mathrm{Ker}_V(d) \otimes \mathrm{Im}_W(d))\, i \subseteq \mathrm{Im}_{V\otimes W}(d)$,

(iii) $(\mathrm{Im}_V(d) \otimes \mathrm{Ker}_W(d))\, i \subseteq \mathrm{Im}_{V\otimes W}(d)$,

(iv) $\mathrm{Im}_{V\otimes W}(d) \subseteq (\mathrm{Ker}_V(d) \otimes \mathrm{Im}_W(d))\, i + (\mathrm{Im}_V(d) \otimes \mathrm{Ker}_W(d))\, i$.

Thus i induces an injection,

$$j\colon \frac{\mathrm{Ker}_V(d)}{\mathrm{Im}_V(d)} \otimes \frac{\mathrm{Ker}_W(d)}{\mathrm{Im}_W(d)} = \frac{\mathrm{Ker}_V(d) \otimes \mathrm{Ker}_W(d)}{\mathrm{Im}_V(d) \otimes \mathrm{Ker}_W(d) + \mathrm{Ker}_V(d) \otimes \mathrm{Im}_W(d)} \hookrightarrow \frac{\mathrm{Ker}_{V\otimes W}(d)}{\mathrm{Im}_{V\otimes W}(d)}.$$

Comparing dimensions (cf. Example 10.4) we see that j is an isomorphism. ∎

We can use λ_t to obtain new species for G. If s is a species for H/t, we define $\phi_t s$ as a species for G by

$$(\phi_t s, V) = (s, \lambda_t(r_{G,H}(V))). \tag{12.4}$$

12.5. LEMMA. *Let $\langle t\rangle \leqslant K \leqslant H$, $t \in Z(H)$. If V is a module for K, then*

$$(\lambda_{t,K}(V))\uparrow^{H/\langle t\rangle} \cong \lambda_{t,H}(V\uparrow^{H}).$$

Proof. Consider $V\uparrow^H$ as $V \otimes_{kK} kH$. Then since $t \in Z(H)$, we have $(v \otimes g)\, d = vd \otimes g$. Hence $\mathrm{Ker}_{V\uparrow^H}(d) = (\mathrm{Ker}_V(d))\uparrow^H$ and $\mathrm{Im}_{V\uparrow^H}(d) = (\mathrm{Im}_V(d))\uparrow^H$, as submodules of $V\uparrow^H$. Hence the isomorphism. ∎

12.6. PROPOSITION. *Let s be a species for $H/\langle t\rangle$ with origin $K/\langle t\rangle$ and vertex $D/\langle t\rangle$. Then $\phi_t s$ is a species for G with origin K and vertex D.*

Proof. It is clear that $\phi_t s$ factors through K. Suppose it factors through $K_1 < K$. Then by Proposition 6.9 there is a module V for K_1 with $0 \neq (\phi_t s, V\uparrow^G) = (s, \lambda_t(V\uparrow^G\downarrow_H))$. By the Mackey decomposition and Lemma 12.5, s is nonzero on some $(\lambda_{t,\langle K_1,t\rangle}(W))\uparrow^{H/\langle t\rangle}$, where W is a module for $\langle K_1, t\rangle$ induced from K_1. Since $K/\langle t\rangle$ is an origin for s, this means that $K = \langle K_1, t\rangle = K_1 \times \langle t\rangle$. But then W is projective as a module for $\langle t\rangle$, and so $\lambda_{t,\langle K_1,t\rangle}(W) = 0$. This contradiction shows that K is an origin for $\phi_t s$. Now by Theorem 7.8 it follows that D is a vertex for $\phi_t s$. ∎

12.7. PROPOSITION. *Let V be an indecomposable kG-module, and suppose $\lambda_t(r_{G,H}(V))$ has an indecomposable direct summand X with vertex $D/\langle t\rangle$. Then D is contained in a vertex of V.*

Proof. Since the vertices of every direct summand of $r_{G,H}(V)$ are contained in vertices of V, we may take $G = H$. Let D_1 be a vertex of V. Let W be a $k\langle D_1, t\rangle$-module, induced from a kD_1-module, with $W\uparrow^H = V \oplus V_0$. Then by Lemma 12.5, we have

$$\lambda_{t,\langle D_1,t\rangle}(W)\uparrow^{H/\langle t\rangle} = \lambda_{t,H}(W\uparrow^H) = \lambda_{t,H}(V) \oplus \lambda_{t,H}(V_0).$$

Thus $\lambda_{t,H}(V)$ is $\langle D_1, t\rangle$-projective. If $t \notin D_1$, then W is projective as a $\langle t\rangle$-module, and so $\lambda_{t,\langle D_1,t\rangle}(W) = 0$. Thus $t \in D$, and the proposition is proved. ∎

12.8. *Remarks.* (i) By Proposition 12.6, we now have a plentiful supply of species with a given vertex. For example, we may take a central series for the vertex

$$1 < \langle t_1\rangle < \langle t_1, t_2\rangle < \cdots < \langle t_1, ..., t_r\rangle = D,$$

with each t_i an involution modulo $\langle t_1, ..., t_{i-1}\rangle$. If b is a Brauer species of $C_G(D)$, then $\phi_{t_r} \cdots \phi_{t_1} \cdot b$ is a species of $A(G)$ with vertex D.

(ii) Proposition 2.7 gives us a method for finding a lower bound for the vertex of an indecomposable kG-module V.

APPENDIX 1: REPRESENTATIONS OF THE KLEIN FOUR-GROUP

In this Appendix, we give the complete representation table and atom table of the Klein four-group V_4, over an algebraically closed field of characteristic 2.

The set of species for $A(V_4)$ falls naturally into three subsets:

(i) The dimension.

(ii) A set of species parametrized by the nonzero complex numbers $z \in \mathbb{C}\{0\}$.

(iii) A set of species parametrized by the set of ordered pairs (N, λ) with $N \in \mathbb{N}\backslash\{0\}$ and $\lambda \in \mathbb{P}^1(k)$.

The set of indecomposable representations also falls naturally into three subsets:

(i) The projective indecomposable representation of dimension four.

(ii) A set of representations parametrized by the integers $m \in \mathbb{Z}$, and of dimension $2|m| + 1$. These are the syzygies of the trivial module.

(iii) A set of representations parametrized by the set of ordered pairs (n, λ) with $n \in \mathbb{N}\backslash\{0\}$ and $\lambda \in \mathbb{P}^1(k)$, and having dimension $2n$.

Define infinite matrices A, B, C and D as

$A\diagdown\xrightarrow{N}$		1	2	3	4	5	$\cdots$
$n\downarrow$	1	2	0	0	0	0	
	2	2	2	0	0	0	
	3	2	2	2	0	0	
	4	2	2	2	2	0	
	5	2	2	2	2	2	
	$\vdots$						

$B\diagdown\xrightarrow{N}$		1	2	3	4	5	$\cdots$
$n\downarrow$	1	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	
	2	2	2	0	0	0	
	3	2	2	2	0	0	
	4	2	2	2	2	0	
	5	2	2	2	2	2	
	$\vdots$						

$C \diagdown \xrightarrow{N}$		1	2	3	4	5	$\cdots$
$n \downarrow$	1	-2	2	0	0	0	
	2	0	-2	2	0	0	
	3	0	0	-2	2	0	
	4	0	0	0	-2	2	
	5	0	0	0	0	-2	
	$\vdots$						

$D \diagdown \xrightarrow{N}$		1	2	3	4	5	$\cdots$
$n \downarrow$	1	$2-2\sqrt{2}$	$2+2\sqrt{2}$	0	0	0	
	2	$\sqrt{2}-2$	$-\sqrt{2}-2$	2	0	0	
	3	0	0	-2	2	0	
	4	0	0	0	-2	2	
	5	0	0	0	0	-2	
	$\vdots$						

Let 0 represent an infinite matrix of zeros. Then the representation table and atom table for V_4 are:

Representation Table for V_4

Parameters	dim	z	(N, ∞)	$(N, 0)$	$(N, 1)$	(N, λ_1)	(N, λ_2)	(N, λ_3)	$\cdots$
(Projective)	4	0	0	0	0	0	0	0	
m	$2\|m\|+1$	z^m	1	1	1	1	1	1	
(n, ∞)	$2n$	0	A	0	0	0	0	0	
$(n, 0)$	$2n$	0	0	A	0	0	0	0	
$(n, 1)$	$2n$	0	0	0	A	0	0	0	
(n, λ_1)	$2n$	0	0	0	0	B	0	0	
(n, λ_2)	$2n$	0	0	0	0	0	B	0	
(n, λ_3)	$2n$	0	0	0	0	0	0	B	
$\vdots$									

Atom Table for V_4

Parameters	dim	z	(N, ∞)	$(N, 0)$	$(N, 1)$	(N, λ_1)	(N, λ_1)	(N, λ_3)	$\cdots$
(Simple)	1	1	1	1	1	1	1	1	
m	0	$-z^{m-1}(1-z)^2$	0	0	0	0	0	0	
(n, ∞)	0	0	C	0	0	0	0	0	
$(n, 0)$	0	0	0	C	0	0	0	0	
$(n, 1)$	0	0	0	0	C	0	0	0	
(n, λ_1)	0	0	0	0	0	D	0	0	
(n, λ_2)	0	0	0	0	0	0	D	0	
(n, λ_3)	0	0	0	0	0	0	0	D	
$\vdots$									

APPENDIX 2: SOME CYCLIC VERTEX SPECIES TABLES

In this Appendix, we give some examples of the tables defined in Section 9, for the summand $A(G, \text{Cyc})$. These were calculated using the results of Sections 9–11.

The notation used is an adaption of the "Atlas" conventions (see [11]), and is as follows.

The top row gives the value of $c(s)$, defined in (9.9) and calculated using the formula (9.16). The second row gives the isomorphism type of an origin of s, followed by a letter distinguishing the conjugacy class of the origin, and a number distinguishing the species with that origin, if there is more than one. The last column gives the conjugacy class of the vertex of the *representation*, in both the representation table and the atom table. If there is more than one possible source with a given vertex, the dimension of this source is given in brackets. For each group, the atom table is given first, followed by the representation table.

Irrationalities

The irrationalities we find in these tables are as follows

$$\begin{aligned} bn &= \tfrac{1}{2}(-1+\sqrt{n}) \qquad \text{if} \quad n \equiv 1 \pmod 4, \\ &= \tfrac{1}{2}(-1+i\sqrt{n}) \qquad \text{if} \quad n \equiv 3 \pmod 4, \end{aligned}$$

i.e., the Gauss sum of half the primitive nth roots of unity.

$zn = e^{2\pi i/n}$ is a primitive nth root of unity,

$rn = \sqrt{n}$,

$in = i\sqrt{n}$;

$*m$ denotes the image of the adjacent irrationality under the Galois automorphism $zn \mapsto (zn)^m$; $*$ denotes $*2$; $**$ denotes $*(-1)$.

$G = C_2$ $p = 2$

2	−2			2	−2	
1*A*	2*A*	*vtx*		1*A*	2*A*	*vtx*
0	−2	2*A*		1	1	2*A*
1	1	1*A*		2	0	1*A*

$G = C_4$ $p = 2$

4	−4	−4	4			4	−4	−4	4	
1*A*	2*A*	4*A*1	4*A*2	*vtx*		1*A*	2*A*	4*A*1	4*A*3	*vtx*
0	0	−2	2	4*A*(1)		1	1	1	1	4*A*(1)
0	−2	2	0	2*A*		2	2	0	0	2*A*
0	0	−2	−2	4*A*(3)		3	1	1	−1	4*A*(3)
1	1	1	1	1*A*		4	0	0	0	1*A*

$G = V_4$ $p = 2$

4	−4	−4	−4			4	−4	−4	−4	
1*A*	2*A*	2*B*	2*C*	*vtx*		1*A*	2*A*	2*B*	2*C*	*vtx*
0	−2	0	0	2*A*		2	2	0	0	2*A*
0	0	−2	0	2*B*		2	0	2	0	2*B*
0	0	0	−2	2*C*		2	0	0	2	2*C*
1	1	1	1	1*A*		4	0	0	0	1*A*

$G = S_3$ $p = 2$

6	3	−2			6	3	−2	
1*A*	3*A*	2*A*	*vtx*		1*A*	3*A*	2*A*	*vtx*
1	1	1	1*A*		2	2	0	1*A*
2	−1	0	1*A*		2	−1	0	1*A*
0	0	−2	2*A*		1	1	1	2*A*

$G = A_5 \qquad p = 2$

60	3	5	5	−4			60	3	5	5	−4		
1A	3A	5A1	5A2	2A	vtx		1A	3A	5A1	5A2	2A	vtx	
1	1	1	1	1	1A		12	0	2	2	0	1A	
2	−1	b5	*	0	1A		8	−1	*	−b5	0	1A	
2	−1	*	b5	0	1A		8	−1	−b5	*	0	1A	
4	1	−1	−1	0	1A		4	1	−1	−1	0	1A	
0	0	0	0	−2	2A		6	0	1	1	2	2A	

$G = L_3(2) \qquad p = 2$

168	3	7	7	−8	−8	8			168	3	7	7	−8	−8	8	
1A	3A	7A1	7A2	2A	4A1	4A2	vtx		1A	3A	7A1	7A2	2A	4A1	4A2	vtx
1	1	1	1	1	1	1	1A		8	2	1	1	0	0	0	1A
3	0	b7	**	1	1	−1	1A		16	1	b7 − 1	**	0	0	0	1A
3	0	**	b7	1	1	−1	1A		16	1	**	b7 − 1	0	0	0	1A
8	−1	1	1	0	0	0	1A		8	−1	1	1	0	0	0	1A
0	0	0	0	−2	2	0	2A		20	2	−1	−1	4	0	0	2A
0	0	0	0	0	−2	2	4A(1)		26	2	−2	−2	2	2	2	4A(1)
0	0	0	0	0	−2	−2	4A(3)		14	2	0	0	2	2	−2	4A(3)

$G = M_{11} \qquad p = 2$

7920	18	5	11	11	−48	−6	−16	16	−16	16	−16	−16	
1A	3A	5A	11A1	11A2	2A	6A	4A1	4A2	8A1	8A2	8A3	8A4	vtx
1	1	1	1	1	1	1	1	1	1	1	1	1	1A
10	1	0	−1	−1	2	−1	2	2	0	0	0	0	1A
16	−2	1	b11	**	0	0	0	0	0	0	0	0	1A
16	−2	1	**	b11	0	0	0	0	0	0	0	0	1A
44	−1	−1	0	0	4	1	0	0	0	0	0	0	1A
0	0	0	0	0	−2	−2	2	0	2	0	0	−2	2A
0	0	0	0	0	−4	2	4	0	−2	−2	2	2	2A
0	0	0	0	0	0	0	−2	2	2	0	−2	0	4A(1)
0	0	0	0	0	0	0	−2	−2	2	0	2	0	4A(3)
0	0	0	0	0	0	0	0	0	−2	2	2	−2	8A(1)
0	0	0	0	0	0	0	0	0	−2	−2	−2	−2	8A(3)
0	0	0	0	0	0	0	0	0	−2	2	−2	2	8A(5)
0	0	0	0	0	0	0	0	0	−2	−2	−2	−2	8A(7)

$G = M_{11}$ $p = 2$ (*continued*)

7920	18	5	11	11	−48	−6	−16	16	−16	16	16	−16	
1A	3A	5A	11A1	11A2	2A	6A	4A1	4A2	8A1	8A2	8A3	8A4	vtx
112	4	2	2	2	0	0	0	0	0	0	0	0	1A
96	6	1	−3	−3	0	0	0	0	0	0	0	0	1A
16	−2	1	b11	**	0	0	0	0	0	0	0	0	1A
16	−2	1	**	b11	0	0	0	0	0	0	0	0	1A
144	0	1	1	1	0	0	0	0	0	0	0	0	1A
200	2	0	2	2	8	2	0	0	0	0	0	0	2A
120	3	0	−1	−1	8	−1	0	0	0	0	0	0	2A
220	4	0	0	0	12	0	4	4	0	0	0	0	4A(1)
372	12	2	−2	−2	12	0	4	−4	0	0	0	0	4A(3)
110	2	0	0	0	6	0	2	2	2	2	2	2	8A(1)
90	0	0	2	2	2	2	2	2	2	−2	2	−2	8A(3)
286	7	1	0	0	10	1	2	−2	2	2	−2	−2	8A(5)
242	8	2	0	0	6	0	2	−2	2	−2	−2	2	8A(7)

$G = S_3$ $p = 3$

6	2	12	−4	−4	−4	6	2	12	−4	−4	−4
1A	2A	3A1	3A2	S_3A1	S_3A2	1A	2A	3A1	3A2	S_3A1	S_3A2
0	0	3	−1	−1	−1	1	1	1	1	1	1
0	0	3	−1	1	1	1	−1	1	1	−1	−1
0	0	−3	−1	i	−i	2	0	−1	1	−i	i
0	0	−3	−1	−i	i	2	0	−1	1	i	−i
1	1	1	1	1	1	3	1	0	0	0	0
1	−1	1	1	−1	−1	3	−1	0	0	0	0

$G = A_5$ $p = 3$

60	4	5	5	12	−4	−4	−4	60	4	5	5	12	−4	−4	−4
1A	2A	5A1	5A2	3A1	3A2	S_3A1	S_3A2	1A	2A	5A1	5A2	3A1	3A2	S_3A1	S_3A2
1	1	1	1	1	1	1	1	6	2	1	1	0	0	0	0
3	−1	−b5	*	0	0	0	0	3	−1	−b5	*	0	0	0	0
3	−1	*	−b5	0	0	0	0	3	−1	*	−b5	0	0	0	0
4	0	−1	−1	1	1	−1	−1	9	1	−1	−1	0	0	0	0
0	0	0	0	3	−1	−1	−1	1	1	1	1	1	1	1	1
0	0	0	0	3	−1	1	1	4	0	−1	−1	1	1	−1	−1
0	0	0	0	−3	−1	i	−i	5	1	0	0	−1	1	−i	i
0	0	0	0	−3	−1	−i	i	5	1	0	0	−1	1	i	−i

$G = C_5 \quad p = 5$

5	$10(1 - b5)$	*	$-2(3 + b5)$	*
$1A$	$5A1$	$5A2$	$5A3$	$5A4$
0	$2 - b5$	*	$-2 - b5$	*
0	$-1 + 3b5$	*	$1 + b5$	*
0	$1 - 3b5$	*	$1 + b5$	*
0	$-2b + 5$	*	$-2 - b5$	*
1	1	1	1	1
5	$10(1 - b5)$	*	$-2(3 + b5)$	*
$1A$	$5A1$	$5A2$	$5A3$	$5A4$
1	1	1	1	1
2	$b5$	*	$-b5$	*
3	$-b5$	*	$-b5$	*
4	-1	-1	1	1
5	0	0	0	0

Acknowledgments

We are much indebted to J. H. Conway, P. Landrock, L. Puig, J. G. Thompson, and P. Webb for helpful conversations and for their interest in this work.

References

1. M. F. Atiyah and I. G. Macdonald, "Introduction to Commutative Algebra," Addison–Wesley, Reading, Mass., 1969.
2. M. Auslander and I. Reiten, Representation theory of Artin algebras, III: almost split sequences, *Comm. Algebra* **3**(3) (1975), 239–294.
3. M. Auslander and I. Reiten, Representation theory of Artin algebras, IV: invariants given by almost split sequences, *Comm. Algebra* **5** (5) (1977), 443–518.
4. D. W. Burry and J. F. Carlson, Restrictions of modules to local subgroups, *Proc. Amer. Math. Soc.* **84** (1982), 181–184.
5. S. B. Conlon, Twisted group algebras and their representations, *J. Austral. Math. Soc.* **4** (1964), 152–173.
6. S. B. Conlon, Certain representation algebras, *J. Austral. Math. Soc.* **5** (1965), 83–99.
7. S. B. Conlon, The modular representation algebra of groups with Sylow 2-subgroups $Z_2 \times Z_2$, *J. Austral. Math. Soc.* **6** (1966), 76–88.
8. S. B. Conlon, Structure in representation algebras, *J. Algebra* **5** (1967), 274–279.
9. S. B. Conlon, Relative components of representations, *J. Algebra* **8** (1968), 478–501.
10. S. B. Conlon, Decompositions induced from the Burnside algebra, *J. Algebra* **10** (1968), 102–122.
11. J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, An atlas of finite groups, to appear, OUP 1984.
12. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Wiley–Interscience, New York/London.

13. K. Erdmann, Principal blocks of groups with dihedral Sylow 2-subgroups, *Comm. Algebra* **5** (7) (1977), 665–694.
14. W. Feit, "The Representation Theory of Finite Groups," North-Holland, Amsterdam, 1982.
15. P. Gabriel, Auslander–Reiten sequences and representation-finite algebras, "Representation theory I, Proceedings of the Ottawa Conference," Springer Lecture Notes in math., No. 831, 1979.
16. P. Gabriel and Ch. Riedtmann, Group representations without groups, *Comment. Math. Helv.* **54** (1967), 240–287.
17. J. A. Green, On the indecomposable representations of a finite group, *Math. Z.* **70** (1959), 430–445.
18. J. A. Green, The modular representation algebra of a finite group, *Illinois J. Math.* **6** (4) (1962), 607–619.
19. J. A. Green, A transfer theorem for modular representations, *J. Algebra* **1** (1964), 73–84.
20. A. Heller, The loop-space functor in homological algebra, *Trans. Amer. Math. Soc.* **17** (1974), 197–213.
21. M. F. O'Reilley, On the semisimplicity of the modular representation algebra of a finite group, *Illinois J. Math.* **9** (1965), 261–276.
22. J.-C. Renaud, The decomposition of products in the modular representation ring of a cyclic group of prime power order, *J. Algebra* **58** (1979), 1–11.
23. L. L. Scott, Modular permutation representations, *Trans. Amer. Math. Soc.* **175**, (1973), 101–121.
24. B. Srinivasan, The modular representation ring of a cyclic p-group, *Proc. London Math. Soc.* **14** (3) (1974), 677–688.
25. R. G. Swan, Induced representations and projective modules, *Ann. of Math.* (2) **71** (1960), 552–578.
26. W. D. Wallis, Factor ideals of some representation algebras, *J. Austral. Math. Soc.* **9** (1969), 109–123.
27. J. R. Zemanek, Nilpotent elements in representation rings, *J. Algebra* **19** (1971), 453–469.
28. J. R. Zemanek, Nilpotent elements in representation rings over fields of characteristic 2, *J. Algebra* **25** (1973), 534–553.
29. I. N. Herstein, "Noncommutative Rings," Wiley, New York, 1968.