Journal of Pure and Applied Algebra 44 (1987) 13–34
North-Holland

VARIETIES FOR MODULES AND A PROBLEM OF STEENROD

D.J. BENSON*
Dept. of Mathematics, Northwestern University, Evanston, IL 60201, USA

N. HABEGGER**
Dept. of Mathematics, Yale University, New Haven, CT 06520, USA

Communicated by E.M. Friedlander and S. Priddy
Received 19 April 1985

1. Introduction

Let G be a finite group, R a commutative ring with a unit, and V an RG-module. An *RG-Moore space* of type (V, n) is a G-space K (i.e., a topological space equipped with an action of G as homeomorphisms) with

$$\tilde{H}_i(K; R) = \begin{cases} V, & i = n, \\ 0, & \text{otherwise} \end{cases}$$

as RG-modules. Here, $\tilde{H}_i$ denotes reduced singular homology with coefficients in R. We say V is *realizable* if there exists an RG-Moore space of type (V, n) for *some n*.

Steenrod's Problem. Which RG-modules are realizable?

We shall mainly be interested in the cases $R = \mathbb{Z}$ and $R = k$ a field of characteristic p, and in finitely generated RG-modules.

The original formulation of the problem by N. Steenrod in 1960 was the following (quoted from Swan [20]):

"Let A be a finitely generated abelian group and π a finite group of automorphisms of A. Let n be a positive integer. Is there a finite complex K with $\tilde{H}_i(K) = 0$ for $i \neq n$, $H_n(K) = A$ such that π acts on K, preserving the cellular structure, and induces the given action on $A = H_n(K)$?"

In this form, the problem was solved in 1969 by Swan [20], who gave a counterexample with $A = \mathbb{Z}/47\mathbb{Z}$ and π a subgroup of Aut(A) of order 23. However, his methods also showed that there is an infinite complex realizing this module, and in

* Partially supported by the NSF.
** Partially supported by NSF Grant no. DMS 84-01578.

1977 Arnold [1,2] showed that every finitely generated module for a cyclic p-group is realizable using an infinite complex. Vogel [23] has also proved that if $|G|$ has no square factor then every $\mathbb{Z}G$-module is realizable. Vogel (unpublished) has also extended these methods to show that if G has cyclic Sylow p-subgroups for all p, then every finitely generated $\mathbb{Z}G$-module is realizable.

If we delete the condition that the complex be finite from Steenrod's original formulation, the first counterexamples were produced by Carlsson [9] for the elementary abelian group of order p^2 (p any prime, although the details are only given for $p=2$).

In [13], Kahn gives an interpretation of Carlsson's results in terms of a construction of Waldhausen in algebraic K-theory, and uses this to produce some further examples of non-realizable modules. In this paper, we shall give another interpretation of G. Carlsson's results, this time in terms of J. Carlson's concept of associating algebraic varieties to modular representations [7]. We give necessary conditions for a modular representation to be realizable, in terms of the varieties associated to direct summands of the module (see Theorem 6.3 and the following remarks). Our methods may be used to construct a wide range of non-realizable modules, including those of Carlsson and Kahn. (Vogel has also obtained some of our examples independently.)

In Section 7 we investigate the homotopy theory relevant to realizability, and in particular we show that if two modules are cohomologically equivalent, one is realizable if and only if the other is. In the case where G is the Klein four group, Bašev [4] has classified the 2-modular representations (see also Conlon [11]), and we show using the results of Section 7, that in this case the condition given in Theorem 6.3 is necessary and sufficient. Calculations with the quaternion group of order eight indicate that this is not true in general (Vogel, private communication).

2. Preliminary remarks

In this section, we gather together some observations (mostly well known) which we will use in the course of the paper.

2.1. Since every topological space has the weak homotopy type of a CW-complex [26, V.3], it is no loss to replace topological spaces by CW-complexes in the original definition. By the same argument, we may also assume that the G-action is cellular.

2.2. If K is an RG-Moore space of type (V,n), then the unreduced suspension ΣK of K (i.e., $I\times K$ with $\{0\}\times K$ identified to a point and $\{1\}\times K$ identified to another point) is an RG-Moore space of type $(V,n+1)$. This shows that when talking about realizability, we may assume that G fixes some point of K, and work with pointed RG-Moore spaces. Also, this argument shows that it is interesting to ask for the *least* value of n for which an RG-Moore space of type (V,n) exists. We shall not investigate this question here.

2.3. If K is an RG-Moore space of type (V, n) and EG is a contractible CW-complex on which G acts freely, then $K \times EG$ (with diagonal G-action) is an RG-Moore space of type (V, n) on which G acts freely. Similarly if (K, x_0) is a pointed RG-Moore space of type (V, n) then $(K \times EG, x_0 \times EG)$ (with $x_0 \times EG$ identified to a point) is a pointed RG-Moore space of type (V, n) on which G acts freely outside the base point.

2.4. If V is an RG-module and $f: R \to R'$ is a morphism of commutative rings, then $V \otimes_R R'$ has a natural structure as an $R'G$-module. If V is R-flat, then by the universal coefficient spectral sequence, if K is an RG-Moore space of type (V, n), then K is also an $R'G$-Moore space of type $(V \otimes_R R', n)$. For example, if V is a $\mathbb{Z}$-torsion-free $\mathbb{Z}G$-module and K is a $\mathbb{Z}G$-Moore space of type (V, n), then K is also an $\mathbb{F}_p G$-Moore space of type $(V \otimes_{\mathbb{Z}} \mathbb{F}_p, n)$. Thus non-realizability of a liftable modular representation implies non-realizability of any lift of it.

3. Carlsson's method

The basic method of Carlsson [9] for producing non-realizable $\mathbb{Z}G$-modules is as follows. He shows that if K is a $\mathbb{Z}G$-Moore space of type (V, n), with V a $\mathbb{Z}$-torsion-free $\mathbb{Z}G$-module, then there is an action of the Steenrod algebra $\mathscr{A}(p)$ on $\mathrm{Ext}^*_{\mathbb{Z}G}(V, \mathbb{F}_p)$, which is compatible with the action of $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{F}_p)$ by cup-product. For the sake of convenience, we state a slightly generalized form of Carlsson's results here, with a more conceptual proof due to Vogel, avoiding the Serre spectral sequence. This has the advantage that not so many routine checks on transferring actions are necessary.

Definition 3.1 (Carlsson). Let A^* be a graded $\mathscr{A}(p)$-algebra (i.e., the Cartan formula $P^k(a_1 a_2) = \sum_{i+j=k} P^i(a_1) P^j(a_2)$ holds), and M^* a graded A^*-module equipped with an action of $\mathscr{A}(p)$. We say M^* is an *$\mathscr{A}(p)$-A^*-module* if the Cartan formula $P^k(\alpha m) = \sum_{i+j=k} P^i(\alpha) P^j(m)$ holds for all $\alpha \in A^*$ and $m \in M^*$.

Proposition 3.2. *Suppose $\mathfrak{p}$ is a (not necessarily non-zero) maximal ideal in R, and $R/\mathfrak{p} = k$ is a field of characteristic p. Suppose V is a realizable (not necessarily R-torsion-free) RG-module. Then $\mathrm{Ext}^*_{RG}(V, k)$ admits an $\mathscr{A}(p)$-$\mathrm{Ext}^*_{RG}(R, k)$-module structure.*

Proof. Suppose (K, x) is a pointed RG-Moore space of type (V, n) (see Remark 2.2). We form the singular chain complex $C_* = C_*(K \times EG, x \times EG; R)$ (cf. Remark 2.3) and the cochain complex

$$C^* = \mathrm{Hom}_{RG}(C_*, k) \cong \mathrm{Hom}_R\left(C_*\left(K \underset{G}{\times} EG, x \underset{G}{\times} EG; R\right), k\right).$$

Then by hypothesis, C_* is a 'shifted resolution' of V as a RG-module, in the sense that the C_i are free RG-modules, $C_i=0$ for $i<0$, and

$$H_i(C_*)=\begin{cases} V, & i=n, \\ 0, & \text{otherwise.} \end{cases}$$

Using this resolution to calculate $\mathrm{Ext}^i_{RG}(V,k)$ we get

$$\mathrm{Ext}^i_{RG}(V,k)\cong H^{i+n}(C^*)\cong H^{i+n}\left(K\underset{G}{\times}EG, x\underset{G}{\times}EG; k\right).$$

This gives the action of $\mathscr{A}(p)$ on $\mathrm{Ext}^*_{RG}(V,k)$, and the Cartan formula follows as in [9] from the fact that the $\mathrm{Ext}^*_{RG}(R,k)$-module structure on $\mathrm{Ext}^*_{RG}(V,k)$ is induced by the diagonal maps

$$K\underset{G}{\times}EG\to\left(\mathrm{pt}\underset{G}{\times}EG\right)\times\left(K\underset{G}{\times}EG\right)$$

and

$$x\underset{G}{\times}EG\to\left(\mathrm{pt}\underset{G}{\times}EG\right)\times\left(x\underset{G}{\times}EG\right). \qquad \square$$

Remarks. The proposition may be interpreted as providing a functor Φ making the following diagram commute.

$$\begin{array}{ccc} \{RG\text{-Moore spaces}\} & \xrightarrow{\ \Phi\ } & \{\mathscr{A}(p)\text{-}\mathrm{Ext}^*_{RG}(R,k)\text{-modules}\} \\ \downarrow{\scriptstyle \tilde{H}_*} & & \downarrow{\scriptstyle F} \\ \{RG\text{-modules}\} & \xrightarrow{\mathrm{Ext}^*_{RG}(-,k)} & \{\mathrm{Ext}^*_{RG}(R,k)\text{-modules}\} \end{array}$$

In this diagram, F denotes the forgetful functor.

Carlsson [9] then continues by providing a construction of a module V such that $\mathrm{Ext}^*_{\mathbb{Z}G}(V,\mathbb{F}_p)$ is not in the image of F. We shall instead investigate the consequences of the above proposition in terms of the cohomological varieties defined in the next section. We shall show that it forces the variety to have a very special form (Theorem 6.3), whereas without the realizability condition, almost anything can happen (Proposition 4.3).

4. Varieties for modules

In this section, we recall the basic definitions and properties of the cohomological varieties $X_G(V)$. For the remainder of this paper we shall only deal with *finitely generated* modules, except where otherwise stated.

4.1. Definitions. Let k be an algebraically closed field of characteristic p. If $p=2$, let $H^{\bullet}(G,k)$ denote $H^*(G,k)$, while if p is odd, let $H^{\bullet}(G,k)$ denote $H^{\mathrm{ev}}(G,k)$, the even part of the cohomology ring. Then $H^{\bullet}(G,k)$ is a commutative graded ring. Note that we must take $H^*(G,k)$ rather than $H^{\mathrm{ev}}(G,k)$ in case $p=2$ since we need an action of the total Steenrod operation in Section 5. We form the affine (homogeneous) variety $X_G = \operatorname{Spec} H^{\bullet}(G,k)$, and the projective variety $\bar{X}_G = \operatorname{Proj} H^{\bullet}(G,k)$ of one smaller dimension.

Denote by $I_G(V)$ the ideal of $H^{\bullet}(G,k)$ consisting of those elements x such that for all modules S, there exists a positive integer j with $H^*(G, V\otimes S)\cdot x^j = 0$ (cup-product action). Note that by [5, Lemma 2.25.2] it is sufficient to check this condition for S simple.

The variety $X_G(V)$ is defined to be $\operatorname{Spec}(H^{\bullet}(G,k)/I_G(V))$, as a homogeneous affine subvariety of X_G. Similarly we have a projective variety $\bar{X}_G(V) = \operatorname{Proj}(H^{\bullet}(G,k)/I_G(V)) \subseteq \bar{X}_G$, of one smaller dimension.

If H is a subgroup of G, denote by $t_{H,G}$ the map from X_H to X_G induced by $\operatorname{res}_{G,H}: H^{\bullet}(G,k) \to H^{\bullet}(H,k)$. The following theorem summarizes some of the main properties of these cohomological varieties, not all of which we shall need.

Theorem 4.2. *Let $H \leq G$, and V be a kG-module and W a kH-module.*

(i) $\dim X_G(V)$ *is equal to the* complexity *of V. Namely the rate of growth of the dimensions in a minimal projective resolution of V is bounded by a polynomial of degree* $\dim X_G(V)-1$, *but not by one of degree* $\dim X_G(V)-2$.

(ii) $X_G(V) = X_G(V^*) = X_G(V\otimes V^*) = X_G(\Omega V)$ *(here, Ω denotes the Heller operator, of taking the kernel of the projective cover).*

(iii) $X_H(V\downarrow_H) = t_{H,G}^{-1}(X_G(V))$.

(iv) $X_G(W\uparrow^G) = t_{H,G}(X_H(W))$.

(We use the symbols $\uparrow^G$ and $\downarrow_H$ to denote induction and restriction of representations between G and H.)

(v) *If $0\to V_1\to V_2\to V_3\to 0$ is a short exact sequence of kG-modules, then $X_G(V_i)\subseteq X_G(V_j)\cup X_G(V_k)$, $\{i,j,k\}=\{1,2,3\}$.*

(vi) $X_G(V\oplus V') = X_G(V)\cup X_G(V')$.

(vii) $X_G(V\otimes V') = X_G(V)\cap X_G(V')$.

(viii) $X_G(V)=\{0\}$ *if and only if V is projective.*

(ix) $X_G(V) = \bigcup_E t_{E,G}(X_E(V\downarrow_E))$ *as E ranges over the elementary abelian p-subgroups of G.*

(x) *Given a closed homogeneous subvariety $X\subseteq X_G$, there is a module V with $X_G(V)=X$.*

(xi) *If $X_G(V)\cap X_G(V')=\{0\}$, then* $\operatorname{Ext}^i_{kG}(V,V')=0$ *for all $i>0$.*

(xii) *If $X_G(V)\subseteq X_1\cup X_2$, where X_1 and X_2 are closed homogeneous subvarieties of X_G with $X_1\cap X_2=\{0\}$, then we may write $V=V_1\oplus V_2$ with $X_G(V_1)\subseteq X_1$ and $X_G(V_2)\subseteq X_2$. In particular, if V is indecomposable, then $\bar{X}_G(V)$ is connected in the Zariski topology.*

Proof. This is proved in [5, Theorems 2.26.9, 2.26.10, 2.27.3, 2.27.7 and 2.27.8], and are mostly due to Carlson [7,8] and Avrunin, Scott [3]. □

The following proposition, which is analogous to part (x) of the above theorem, will also be of interest to us.

Proposition 4.3. *Given a closed homogeneous* $\mathbb{F}_p$*-rational subvariety* ($\mathbb{F}_p$*-rational* means stable under the Frobenius morphism) $X \subseteq X_G$, *there is a* $\mathbb{Z}$*-free* $\mathbb{Z}G$*-module* U *with* $X_G(U \otimes_{\mathbb{Z}} k) = X$.

The proof of Proposition 4.3 depends on the following lemma.

Lemma 4.4. *Let* $x \in H^{2n}(G, \mathbb{Z}/p^k\mathbb{Z})$. *Then* x^p *is in the image of the natural map* $H^{2np}(G, \mathbb{Z}/p^{k+1}\mathbb{Z}) \to H^{2np}(G, \mathbb{Z}/p^k\mathbb{Z})$.

Proof. Corresponding to the short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^{k+1}\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z} \to 0$$

we have a long exact sequence

$$\cdots \to H^r(G, \mathbb{Z}/p^{k+1}\mathbb{Z}) \to H^r(G, \mathbb{Z}/p^k\mathbb{Z}) \xrightarrow{\delta} H^{r+1}(G, \mathbb{Z}/p\mathbb{Z}) \to \cdots$$

in which the Bockstein homomorphism δ is a derivation with respect to the cup product pairing $H^*(G, \mathbb{Z}/p^k\mathbb{Z}) \times H^*(G, \mathbb{Z}/p\mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z})$. Hence $\delta(x^p) = p(x^{p-1} \cup \delta(x)) = 0$ and so x^p is in the image of $H^{2np}(G, \mathbb{Z}/p^{k+1}\mathbb{Z}) \to H^{2np}(G, \mathbb{Z}/p^k\mathbb{Z})$. □

Lemma 4.5. *Let* $x \in H^{2n}(G, \mathbb{Z}/p\mathbb{Z})$. *If* $|G| = p^\alpha q$ *with* $(p, q) = 1$, *then* $x^{p^{\alpha+1}}$ *is in the image of the natural map*

$$H^*(G, \mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z}).$$

(cf. Maranda's theorem [12, Theorem 30.14]).

Proof. Applying Lemma 4.4 and induction, $x^{p^{\alpha+1}}$ is in the image of the natural map $H^*(G, \mathbb{Z}/p^{\alpha+1}\mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z})$, and hence also in the image of

$$H^*(G, \mathbb{Z}/p\,|G|\,\mathbb{Z}) = H^*(G, \mathbb{Z}/p^{\alpha+1}\mathbb{Z}) \oplus H^*(G, \mathbb{Z}/q\mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z}).$$

Denote by δ_1 the connecting homomorphism associated to the coefficient sequence $0 \to \mathbb{Z}/|G|\,\mathbb{Z} \to \mathbb{Z}/p\,|G|\,\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, by δ_2 the connecting homomorphism associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$, and by θ the coefficient homomorphism determined by $\mathbb{Z} \to \mathbb{Z}/|G|\,\mathbb{Z}$. Then $0 = \delta_1(x^{p^{\alpha+1}}) = \theta(\delta_2(x^{p^{\alpha+1}}))$, and since θ is injective, $\delta_2(x^{p^{\alpha+1}}) = 0$. Thus $x^{p^{\alpha+1}}$ is in the image of the natural map $H^*(G, \mathbb{Z}) \to H^*(G, \mathbb{Z}/p\mathbb{Z})$. □

4.6. Definitions. If $\zeta \in H^n(G, k) \cong \operatorname{Ext}^n_{kG}(k, k)$, we may represent ζ by a homomorphism $\tilde{\zeta}: \Omega^n(k) \to k$. If $\zeta \neq 0$, this homomorphism is surjective, and we denote by L_ζ its kernel. Thus we have a short exact sequence

$$0 \to L_\zeta \to \Omega^n(k) \xrightarrow{\tilde{\zeta}} k \to 0.$$

Similarly if $\xi \in H^n(G, \mathbb{Z}) \cong \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$, we may make an analogous construction. Let $\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$ be a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, where the P_i have minimal $\mathbb{Z}$-rank (cf. [19, 25]), and denote by $\Omega^i(\mathbb{Z})$ the ith kernel of this resolution. Then ξ is represented by a surjective map $\tilde{\xi}: \Omega^i(\mathbb{Z})(\oplus \mathbb{Z}G) \to \mathbb{Z}$, and we denote its kernel by L_ξ.

The following lemma is clear from the definitions.

Lemma 4.7. *If $\xi \mapsto \zeta$ under the natural map $H^n(G, \mathbb{Z}) \to H^n(G, \mathbb{Z}/p\mathbb{Z}) \subseteq H^n(G, k)$, then*

$$L_\xi \otimes_{\mathbb{Z}} k \cong L_\zeta \oplus \text{projective}.$$

Lemma 4.8 (Carlson [8]). *If $\zeta \in H^n(G, k)$, then $X_G(L_\zeta)$ is the hypersurface $X_G(\zeta)$ given by considering ζ as an element of the coordinate ring of X_G.*

We are now ready to prove Proposition 4.3.

Proof of 4.3. If X is an $\mathbb{F}_p$-rational homogeneous subvariety of X_G, then let $I = \langle \zeta_1, \dots, \zeta_r \rangle$ be the corresponding ideal in $H^\bullet(G, k)$, with ζ_i homogeneous and in the image of $H^\bullet(G, \mathbb{Z}/p\mathbb{Z}) \to H^\bullet(G, k)$. By Lemma 4.5, if we replace ζ_i by some power of ζ_i, we may assume that it is the image of some $\xi_i \in H^\bullet(G, \mathbb{Z})$. Then by 4.7, 4.2(vii) and 4.8,

$$\begin{aligned} X_G((L_{\xi_1} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L_{\xi_n}) \otimes_{\mathbb{Z}} k) &= X_G(L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_n}) \\ &= X_G(L_{\zeta_1}) \cap \cdots \cap X_G(L_{\zeta_n}) \\ &= X_G(\zeta_1) \cap \cdots \cap X_G(\zeta_n) \\ &= X_G(\langle \zeta_1, \dots, \zeta_n \rangle) = X. \end{aligned}$$

Thus $L_{\xi_1} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L_{\xi_n}$ is a $\mathbb{Z}$-free $\mathbb{Z}G$-module with the desired properties. □

5. The total Steenrod operation

Denote by T the total Steenrod operation, namely $\sum_{i=0}^\infty P^i$ if p is odd, and $\sum_{i=0}^\infty \mathrm{Sq}^i$ if $p = 2$. Then T is an algebra endomorphism of $H^\bullet(G, k)$, since P^i has even degree if $p \neq 2$. We also denote by T the corresponding map $X_G \to X_G$.

If X is a homogeneous subvariety of X_G, we denote by $X^{(T)}$ the homogeneous span of the images of X under powers of T. Since T is not homogeneous, $X^{(T)}$ will usually have greater dimension than X.

Theorem 5.1. *Suppose X is a homogeneous subvariety of X_G. If X is not of the form $\bigcup_E t_{E,G}(X_E)$ for some collection of elementary abelian p-subgroups E of G, then $X^{(T)} > X$. In particular, $X^{(T)}$ is always of the above form.*

Proof. If G is elementary abelian, this is proved in [17, Proposition 1]. For G arbitrary, the result now follows from the fact that $X_G = \bigcup_E t_{E,G}(X_E)$ as E ranges over the collection of all elementary abelian p-subgroups of G (this is 4.2(ix) in the case where V is the trivial module. See also [14, 15]). □

Definition 5.2. Let M^* be a graded A^*-module, as in Definition 3.1. Somewhat weaker than having an $\mathscr{A}(p)$-A^*-module structure is just having an action of the total Steenrod operation T, without requiring the Adem relations to hold. We say M^* is a *T-A^*-module* if there is a linear map $\mathscr{T} = \sum_{i=0}^{\infty} \mathscr{T}_i$ on M^*, satisfying the following conditions.

(i) $\mathscr{T}_i$ is homogeneous of the same degree as P^i (or Sq^i if $p=2$).
(ii) $\mathscr{T}_0$ is the identity map.
(iii) $\mathscr{T}(xy) = T(x)\mathscr{T}(y)$ for $x \in A^*$, $y \in M^*$.

Any $\mathscr{A}(p)$-A^*-module is a T-A^*-module by letting $\mathscr{T} = T$.

Proposition 5.3. *Let M^* be a T-A^*-module. Then any direct summand (as A^*-module) N^* of M^* is also a T-A^*-module.*

Proof. Let i denote the inclusion of N^* in M^*, and π the projection of M^* onto N^*. Then $\mathscr{T}' = \pi \circ \mathscr{T} \circ i$ is the desired linear operation on N^*. □

Thus by Proposition 3.2, we have the following.

Proposition 5.4. *Suppose $\mathfrak{p}$ is a (not necessarily non-zero) maximal ideal in a commutative ring R, and $R/\mathfrak{p} = k$ is a field of characteristic p. Suppose V is a realizable RG-module. Then for every summand W of V, $\mathrm{Ext}^*_{RG}(W,k)$ admits a T-$\mathrm{Ext}^*_{RG}(R,k)$-module structure.*

Corollary 5.5. *If H is a subgroup of G and V is an RH-module such that $\mathrm{Ext}^*_{RH}(V,k)$ does not admit a T-$\mathrm{Ext}^*_{RH}(R,k)$-module structure, then $\mathrm{Ext}^*_{RG}(V\uparrow^G, k)$ does not admit a T-$\mathrm{Ext}^*_{RG}(R,k)$-module structure. In particular, $V\uparrow^G$ is not realizable.*

Proof. This follows from Proposition 5.3 since $V\uparrow^G\downarrow_H \cong V \oplus W$ for some module W. The last remark follows from Proposition 5.4. □

Corollary 5.6. *Suppose k' is a finite separable extension of the field k. Let V be a $k'G$-module and let W be the kG-module obtained from V by restriction of scalars.*

Then $\mathrm{Ext}^*_{k'G}(V,k')$ *is a* T-$\mathrm{Ext}^*_{k'G}(k',k')$*-module if and only if* $\mathrm{Ext}^*_{kG}(W,k)$ *is a* T-$\mathrm{Ext}^*_{kG}(k,k)$*-module.*

Proof. One way follows from the observation that

$$\begin{aligned}\mathrm{Ext}^*_{kG}(W,k)\otimes_k k' &\cong \mathrm{Ext}^*_{k'G}(W\otimes_k k',k') \cong \mathrm{Ext}^*_{k'G}(V\otimes_{k'}(k'\otimes_k k'),k')\\ &\cong \mathrm{Ext}^*_{k'G}(V\oplus V',k') \cong \mathrm{Ext}^*_{k'G}(V,k')\oplus \mathrm{Ext}^*_{k'G}(V',k')\end{aligned}$$

for some $k'G$-module V', combined with Proposition 5.3. The other way follows from the fact that the restriction to k of $\mathrm{Ext}^*_{k'G}(V,k')$ is $\mathrm{Ext}^*_{kG}(W,k)$. □

6. The main theorem

We begin with some lemmas.

Lemma 6.1. *If H is a subgroup of G, and V is a kG-module, then*

$$\mathrm{res}_{G,H}(I_G(V)) \subseteq I_H(V{\downarrow_H}).$$

Proof. By Shapiro's lemma [18, p. 116–7], if S is a kH-module, we have an isomorphism

$$H^*(H, V{\downarrow_H}\otimes S) \cong H^*(G,(V{\downarrow_H}\otimes S){\uparrow^G}) = H^*(G, V\otimes(S{\uparrow^G})),$$

with the property that the cup product action of $x\in H^\bullet(G,k)$ on $H^*(G,V\otimes(S{\uparrow^G}))$ is the same as that of $\mathrm{res}_{G,H}(x)$ on $H^*(H,V{\downarrow_H}\otimes S)$. □

Lemma 6.2. *Let P be a Sylow p-subgroup of G and V a kG-module. Then*

$$I_G(V) = \mathrm{res}^{-1}_{G,P}(I_P(V{\downarrow_P})).$$

Proof. This follows from Lemma 6.1 and the fact that

$$\mathrm{res}_{G,P} : H^*(G,k) \to H^*(P,k)$$

is injective. □

Theorem 6.3. *Suppose R is a commutative ring with a unit, $\mathfrak{p}$ is a maximal ideal in R with $R/\mathfrak{p} = k_0 \subseteq k$ an algebraically closed field of characteristic p, and U is a (finitely generated) realizable RG-module. Then for every direct summand V of $U\otimes_R k$ (as a kG-module), $X_G(V)$ is T-stable, and hence of the form $\bigcup_{E\leq G} t_{E,G}(X_E)$ for some collection of elementary abelian subgroups E of G.*

Proof. In order to show that $X_G(V)$ is T-stable, we must show that $T(I_G(V)) \subseteq I_G(V)$. Let P be a Sylow p-subgroup of G. Since the action of T commutes with $\mathrm{res}_{G,P}: H^{\bullet}(G,k) \to H^{\bullet}(P,k)$, by Lemma 6.2 it suffices to show that $T(I_P(V\downarrow_P)) \subseteq I_P(V\downarrow_P)$. Thus we may assume that $G=P$ is a p-group. In this case, there is only one simple module, namely the trivial one-dimensional module k, and so by Theorem 4.2(ii) and the definitions, $I_G(V)=I_G(V^*)$ is the set of $x \in H^{\bullet}(G,k)$ such that for some $j>0$, $\mathrm{Ext}^*_{kG}(V,k) \cdot x^j = H^*(G,V^*) \cdot x^j = 0$.

Now by Proposition 3.2, $\mathrm{Ext}^*_{RG}(U,k)$ admits an $\mathscr{A}(p)$-$\mathrm{Ext}^*_{RG}(R,k)$-module structure, i.e., $\mathrm{Ext}^*_{kG}(U \otimes_R k, k)$ admits an $\mathscr{A}(p)$-$H^*(G,k)$-module structure. Hence by Proposition 5.3, $\mathrm{Ext}^*_{kG}(V,k)$ admits a T-$H^*(G,k)$-module structure. It is now easy to see from the above description of $I_G(V)$, that $T(I_G(V)) \subseteq I_G(V)$ since the action of T commutes with cup products. $\square$

Remarks. We may combine the results of Proposition 4.3 and Theorem 6.3 as follows. If U is a finitely generated $\mathbb{Z}$-free realizable $\mathbb{Z}G$-module, then by 2.4, $U \otimes_{\mathbb{Z}} k$ is a realizable kG-module. By Theorem 6.3, for every summand V of $U \otimes_{\mathbb{Z}} k$, $X_G(V)$ is T-stable, and hence of the form $\bigcup_E t_{E,G}(X_E)$ for some collection of elementary abelian p-subgroups E of G. On the other hand, Proposition 4.3 shows that for every $\mathbb{F}_p$-rational homogeneous subvariety X of X_G, there is a $\mathbb{Z}$-free $\mathbb{Z}G$-module U with $X_G(U \otimes_{\mathbb{Z}} k) = X$. As soon as the p-rank of G is at least 2, almost all $\mathbb{F}_p$-rational homogeneous subvarieties are not of the form $\bigcup_E t_{E,G}(X_E)$, and so we have a plentiful supply of non-realizable $\mathbb{Z}$-free $\mathbb{Z}G$-modules, expressed as tensor products of the modules L_ξ appearing in the proof of Proposition 4.3. Note that these modules are not even direct summands of realizable modules.

These methods say nothing in the case where G has p-rank one. In this case either the Sylow p-subgroups of G are cyclic, and all modules should be realizable (see the introduction), or $p=2$ and the Sylow 2-subgroups of G are generalized quaternion. We shall deal with the latter case in Section 9.

If $p=2$ and $G \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, the Klein four group, let

$$H^*(G,\mathbb{Z}) = \mathbb{Z}[x,y,z]/(2x, 2y, 2z, z^2 - x^2y - xy^2)$$

with $\deg(x)=\deg(y)=2$ and $\deg(z)=3$. Then the counterexample of Carlsson [9] is the module L_ξ, where $\xi = x^2+xy+y^2$. Since $L_\xi \otimes k$ is a direct sum of two four-dimensional modules $L_{\bar{x}+\omega\bar{y}}$ and $L_{\bar{x}+(\omega+1)\bar{y}}$ where ω and $\omega+1$ are the primitive cube roots of unity in k, $X_G(L_\xi \otimes k)$ is a pair of lines through the origin in $\mathbb{A}^2(k)$ of slopes ω and $\omega+1$, swapped by the Frobenius morphism. We shall say more about this case in Section 8.

Amir Assadi has pointed out to us the following corollary to Theorem 6.3.

Corollary 6.4. *Suppose R is a commutative ring with a unit, and suppose U is a f.g. realizable RG-module. Then U is projective if and only if $U\downarrow_{\langle x\rangle}$ is projective for each cyclic subgroup $\langle x\rangle$ of G of prime order.*

7. Cohomology equivalence and realizability of modules

In this section, we develop the technical machinery necessary for demonstrating that a given module is realizable. We use some of this machinery in the next section to investigate modules for the Klein four group.

Because of Remark 2.1, all spaces in this section will be assumed to be *CW*-complexes with cellular *G*-action. For the purpose of this section, however, we shall *not* assume that the *RG*-modules which we are dealing with are finitely generated, since with this restriction Lemma 7.2 is no longer true.

7.1. Definitions. A map $V \xrightarrow{h} V'$ of *RG*-modules is *realizable* if there is a *G*-equivariant map $K \xrightarrow{f} K'$ of *RG*-Moore spaces of type $(V, n), (V', n)$ for some n, with $H_n(f) = h$.

Warning. It is not necessarily true that the composite of two realizable maps is realizable.

An *RG*-module V is *cohomologically trivial* if for all subgroups $H \leq G$, $\hat{H}^*(H, V) = 0$ (Tate cohomology). A homomorphism $V \xrightarrow{h} V'$ is a *cohomology equivalence* if for all subgroups $H \leq G$, $\hat{H}^*(H, h) : \hat{H}^*(H, V) \to \hat{H}^*(H, V')$ is an isomorphism.

An *RG*-module V is *weakly injective* if given a monomorphism $0 \to W \to W'$ and a homomorphism $W \to V$ which extends to an R-linear map $W' \to V$, then it extends to an *RG*-homomorphism $W' \to V$.

Standard properties of cohomologically trivial modules are given in [10, 16, 18]. In particular, we shall need the following.

Lemma 7.2. *Let V be an RG-module, with $R = \mathbb{Z}$ or $\mathbb{F}_p$. Then the following two properties are equivalent.*

(i) *V is cohomologically trivial.*

(ii) *There is a short exact sequence $0 \to F_1 \to F_2 \to V \to 0$ with F_1 and F_2 free.*

Properties (i) *and* (ii) *are implied by:*

(iii) *V is weakly injective.*

Proof. This follows from [16] and [27, XII.1.1]. □

Lemma 7.3. *Let $R = \mathbb{Z}$ or $\mathbb{F}_p$. Suppose a homomorphism $V \xrightarrow{h} V'$ factors through a weakly injective module. Then h factors through the natural inclusion $V \to V \otimes_R RG$ given by $v \mapsto \sum_{g \in G} v \otimes g$.*

Proof. Suppose h factors through $h' : V \to W$ with W weakly injective. Since the natural map $\alpha : V \to V \otimes_R RG$ is an R-split monomorphism h' factors through α as R-modules, and hence as *RG*-modules since W is weakly injective. Thus h factors through α. □

Lemma 7.4. *Let* $0 \to V_1 \xrightarrow{h_1} V_2 \xrightarrow{h_2} V_3 \to 0$ *be a short exact sequence of RG-modules. If either* h_1 *or* h_2 *is realizable, so is the other.*

Proof. Consider the mapping sequence $K_1 \xrightarrow{f} K_2 \xrightarrow{j} C_f \xrightarrow{\varrho} \Sigma K_1$. If f realizes h_1, then j realizes h_2. If f realizes h_2, then ϱ realizes h_1. $\square$

Lemma 7.5. *Let* K_1, K_2 *be simply connected* $\mathbb{Z}$*-Moore spaces (i.e.,* $G=1$*) of type* (V_1, n) *and* (V_2, n) *respectively. Then the map* $H_n : [K_1, K_2] \to \mathrm{Hom}_{\mathbb{Z}}(V_1, V_2)$ *is surjective, i.e., all homomorphisms of* $\mathbb{Z}$*-modules (abelian groups) are realizable.*

Proof. Let F be a free $\mathbb{Z}$-module with basis B and surjecting onto V_1. Let F' be the kernel, again free by [28, Theorem 14.5] with basis B'. The maps $F \to V_1$ and $F' \to F$ are represented by maps $S^n \wedge B_+ \xrightarrow{f} K_1$ and $S^n \wedge B'_+ \xrightarrow{g} S^n \wedge B_+$, where B_+, B'_+ denote the discrete spaces B, B' with base point added. Moreover, since the composite map $F' \to F \to V_1$ is trivial, the map $f \circ g$ extends to a map $D^{n+1} \wedge B'_+ \to K_1$, i.e., we get a map $C_g \to K_1$ inducing an isomorphism on homology and hence a homotopy equivalence. Thus if $V_1 \to V_2$ is any homomorphism, we can construct a map $C_g \to K_2$ realizing it. Thus the map $H_n : [K_1, K_2] = [C_g, K_2] \to \mathrm{Hom}_{\mathbb{Z}}(V_1, V_2)$ is surjective. $\square$

Warning. If V_1 and V_2 are $\mathbb{Z}G$-modules with trivial G-action (but G not necessarily the trivial group), and K_1, K_2 are simply connected $\mathbb{Z}G$-Moore spaces of type (V_1, n) and (V_2, n) respectively, then it is not necessarily true that $H_n : [K_1, K_2]_G \to \mathrm{Hom}_{\mathbb{Z}G}(V_1, V_2) = \mathrm{Hom}_{\mathbb{Z}}(V_1, V_2)$ is surjective.

Proposition 7.6. *Let* K *be a simply connected pointed* $\mathbb{F}_pG$*-Moore space of type* (V, n) *with* $n \geq 3$*. Then there is a G-equivariant map* $K \to K'$ *inducing an* $\mathbb{F}_p$*-equivalence (i.e.,* $H_*(K; \mathbb{F}_p) \xrightarrow{\cong} H_*(K'; \mathbb{F}_p)$*) with* K' $(n-1)$*-connected.*

Proof. Suppose K is $(m-1)$-connected with $1 < m < n$. We shall show that there is a G-equivariant $\mathbb{F}_p$-equivalence $K \to K'$ with K' m-connected, and the result follows by composing these maps. Let $V = H_m(K) \cong \pi_m(K)$. If $m < n-1$, we have $V \otimes_{\mathbb{Z}} \mathbb{F}_p = 0$ and $\mathrm{Tor}(V, \mathbb{F}_p) = 0$ by the universal coefficient theorem. Let L be a $\mathbb{Z}$-Moore space of type (V, m). Then L is an $\mathbb{F}_p$-Moore space of type 0. Moreover, by Lemma 7.5, there is a map $L \to K$ inducing an isomorphism $H_n(L) \to H_n(K)$. Denote by G_+ the discrete space G with base point added and left G-action. Then there is a G-equivariant map $L \wedge G_+ \xrightarrow{f} K$ inducing an epimorphism $H_m(L \wedge G_+) \to H_m(K)$. Then $K' = C_f$, the mapping cone of f, is m-connected and $K \to K'$ is an $\mathbb{F}_p$-equivalence, by the long exact sequence of homology.

If $m = n-1$ and $H_n(K) \to H_n(K; \mathbb{F}_p)$ is surjective, we may proceed as above, finishing the proof. If not, we reduce to this case by showing that there is an $\mathbb{F}_p$-equivalence $K \to K'$, with K' $(n-2)$-connected and $H_n(K') \to H_n(K'; \mathbb{F}_p)$ surjective.

We have an exact sequence

$$0 \to H_n(K) \otimes_{\mathbb{Z}} \mathbb{F}_p \to H_n(K; \mathbb{F}_p) \to \mathrm{Tor}(H_{n-1}(K); \mathbb{F}_p) \to 0$$

(universal coefficients). Let $V = \mathrm{Tor}(H_{n-1}(K); \mathbb{F}_p) = \{x \in H_{n-1}(K) : px = 0\}$, an $\mathbb{F}_p$ vector space, say with basis B. Since $H_{n-1}(K)$ is p-divisible, there is a map $\bigoplus_B \mathbb{Z}/p^\infty\mathbb{Z} \xrightarrow{\phi} H_{n-1}(K)$ inducing an isomorphism on $\mathrm{Tor}(-; \mathbb{F}_p)$, where $\mathbb{Z}/p^\infty\mathbb{Z}$ denotes $\varinjlim \mathbb{Z}/p^n\mathbb{Z}$. Let L be a $\mathbb{Z}$-Moore space of type $(\mathbb{Z}/p^\infty\mathbb{Z}, n-1)$. Note that since $n \geq 3$, we can suppose $L = \Sigma L_0$ (see 2.2), and hence that a coproduct $L \to L \vee L$ exists. Since K is $(n-2)$-connected, by Lemma 7.5 there is a map $L \wedge B_+ \xrightarrow{\psi} K$ inducing the map ϕ on homology, where B_+ is the discrete space B with base point added.

Similarly let $W = H_n(K) \otimes_{\mathbb{Z}} \mathbb{F}_p$, with basis B'. There is a map $\bigoplus_{B'} \mathbb{Z} \xrightarrow{\phi'} H_n(K)$ inducing an isomorphism when tensored with $\mathbb{F}_p$. Since K is $(n-2)$-connected, the Hurewicz homomorphism $\pi_n(K) \to H_n(K)$ is surjective. Hence there is a map $S^n \wedge B'_+ \xrightarrow{\psi'} K$ inducing the map ϕ' on homology. Then $\psi \vee \psi'$ is a map from $L_1 = (L \wedge B_+) \vee (S^n \wedge B'_+)$ to K inducing an $\mathbb{F}_p$-equivalence.

Now $H^i(L; \pi_i(S^n)) = 0$ for $i \neq n$, and so $H^i(L; \mathbb{Z}) = [L, S^n]$. Since $H^n(L; \mathbb{Z}) \to H^n(L; \mathbb{F}_p) = \mathbb{F}_p$ is surjective, there is a map $L \to S^n$ inducing an $\mathbb{F}_p$-equivalence. Thus there is a map η from L_1 to $L_2 = (S^n \wedge B_+) \vee (S^n \wedge B'_+) = S^n \wedge (B_+ \vee B'_+)$, inducing an $\mathbb{F}_p$-equivalence.

The map $L_1 \to K$ extends to a G-equivariant map $L_1 \wedge G_+ \xrightarrow{\eta'} K$. Since L and S^n have a coproduct, so does L_1, and hence so does $L_1 \wedge G_+$. Denote by f the composite G-equivariant map

$$L_1 \wedge G_+ \to (L_1 \wedge G_+) \vee (L_1 \wedge G_+) \xrightarrow{(\eta \wedge G_+, \eta')} (L_2 \wedge G_+) \vee K.$$

Since η is an $\mathbb{F}_p$-equivalence, the map $K \to C_f$ is also an $\mathbb{F}_p$-equivalence. Moreover C_f is $(n-2)$-connected, and the diagram

$$\begin{array}{ccc} H_n(L_2 \wedge G_+; \mathbb{Z}) & \longrightarrow & H_n(C_f; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_n(L_2 \wedge G_+; \mathbb{F}_p) & \twoheadrightarrow & H_n(C_f; \mathbb{F}_p) \end{array}$$

shows that $H_n(C_f; \mathbb{Z}) \to H_n(C_f; \mathbb{F}_p)$ is surjective. $\square$

Lemma 7.7. *Let V be a realizable RG-module with $R = \mathbb{Z}$ or $\mathbb{F}_p$. Let F be a free RG-module with basis B. Then every RG-module homomorphism $F \to V$ is realized by a G-equivariant map $S^n \wedge B_+ \wedge G_+ \to K$, for some n, and some RG-Moore space K of type (V, n).*

Proof. Let K be an RG-Moore space of type (V, n). By suspending if necessary, we may suppose K is pointed and simply connected (Remark 2.2). If $R = \mathbb{Z}$, then K is $(n-1)$-connected, and $\pi_n(K) \cong H_n(K) \cong V$. Choosing representative maps $S^n \to K$ for elements of the basis B and extending G-equivariantly, we obtain a G-equivariant map realizing $F \to V$. If $R = \mathbb{F}_p$, by Proposition 7.6 we may replace K by an $(n-1)$-connected RG-Moore space of type (V, n), after suspending if necessary to ensure $n \geq 3$. Then $\pi_n(K) \cong H_n(K) \to H_n(K; \mathbb{F}_p) \cong V$ is surjective and we may proceed as before. □

Lemma 7.8. *Denote by J the cokernel of the map $\mathbb{Z} \to \mathbb{Z}G$ given by $1 \mapsto \sum_{g \in G} g$. Then for large n, the map $\mathbb{Z} \to \mathbb{Z}G$ is realizable by a map $S^n \to S^n \wedge G_+$, and J is realizable.*

Proof. Let V be a finite-dimensional faithful $\mathbb{R}G$-module with a fixed point (e.g., $V = \mathbb{R}G$). Then the unit sphere in V is a $\mathbb{Z}G$-Moore space of type $(\mathbb{Z}, n)$ for some n, and with a free G-orbit. Contracting the complement of a small neighborhood of this free orbit to a point gives us a realization of the map $\mathbb{Z} \to \mathbb{Z}G$. By Lemma 7.4, this implies that J is realizable. □

Lemma 7.9. *Let V be any $\mathbb{Z}G$-module. Then there is a $\mathbb{Z}$-free $\mathbb{Z}G$-module V' and a surjection $V' + V$ which is a cohomology equivalence. Moreover, if V is realizable, then so is V'. If V is finitely generated, we may take V' also to be finitely generated.*

Proof. Let F be a free module surjecting onto $V \otimes_{\mathbb{Z}} \mathbb{Z}G$, and let V' be the kernel of the composite map $F \to V \otimes_{\mathbb{Z}} \mathbb{Z}G \to V \otimes_{\mathbb{Z}} J$, where J is the cokernel of the map $\mathbb{Z} \to \mathbb{Z}G$ given by $1 \mapsto \sum_{g \in G} g$. Let V'' be the kernel of the composite map $V' \otimes_{\mathbb{Z}} \mathbb{Z}G \to V \otimes_{\mathbb{Z}} \mathbb{Z}G \to V \otimes_{\mathbb{Z}} J$, and W denote the kernel of the map $V' \to V \to 0$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & W \arrow[r, equal] \arrow[d] & W \arrow[d] & & \\
0 \arrow[r] & V' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & V \otimes_{\mathbb{Z}} \mathbb{Z}G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[d] & V \otimes_{\mathbb{Z}} \mathbb{Z}G \arrow[r] \arrow[d] & V \otimes_{\mathbb{Z}} \mathbb{Z}G \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Now applying the snake lemma to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & V' \arrow[d] & V'' \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & V'\otimes_{\mathbb{Z}} \mathbb{Z}G \arrow[r] \arrow[d] & V'\otimes_{\mathbb{Z}} \mathbb{Z}G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W\otimes_{\mathbb{Z}} J \arrow[r] \arrow[d] & V'\otimes_{\mathbb{Z}} J \arrow[r] \arrow[d] & V\otimes_{\mathbb{Z}} J \arrow[r] \arrow[d] & 0 \\
 & W\otimes_{\mathbb{Z}} J & 0 & 0 &
\end{tikzcd}$$

we obtain a short exact sequence $0\to V'\to V''\to W\otimes_{\mathbb{Z}} J\to 0$.

Now suppose V is realizable. Since J is also realizable by Lemma 7.8, this shows that $V\otimes_{\mathbb{Z}} J$ is realizable (take the smash-product of the complexes). Applying Lemmas 7.4 and 7.7 to the short exact sequence

$$0\to V''\to V'\otimes_{\mathbb{Z}} \mathbb{Z}G\to V\otimes_{\mathbb{Z}} J\to 0$$

shows that V'' is realizable, since $V'\otimes_{\mathbb{Z}} \mathbb{Z}G$ is free.

Since $V\otimes_{\mathbb{Z}} \mathbb{Z}G$ is free, the sequence $0\to W\to F\to V\otimes_{\mathbb{Z}} \mathbb{Z}G\to 0$ splits and so W is free. This shows that $W\otimes_{\mathbb{Z}} J$ is free, and so $V''\cong V'\oplus W\otimes_{\mathbb{Z}} J$. We may thus apply Lemmas 7.4 and 7.7 again to the injection $W\otimes_{\mathbb{Z}} J\to V''$ to show that V' is realizable. □

Proposition 7.10 (compare [23]). *Let $R=\mathbb{Z}$ or $\mathbb{F}_p$.*

(i) *Every cohomologically trivial RG-module is realizable.*

(ii) *Let V_1 and V_2 be realizable RG-modules. Then any homomorphism $\phi: V_1\to V_2$ which factors through a cohomologically trivial module is realizable.*

(iii) *Let $0\to V_1\to V_2\to V_3\to 0$ be a short exact sequence of RG-modules. If V_i is realizable and V_j is cohomologically trivial then V_k is realizable, where $\{i,j,k\}=\{1,2,3\}$.*

(iv) *If $\phi: V_1\to V_2$ is a cohomology equivalence, and either V_1 or V_2 is realizable, then V_1, V_2 and ϕ are all realizable.*

Proof. (i) If P is a cohomologically trivial RG-module, then by Lemma 7.2, there is a short exact sequence $0\to F_1\to F_2\to V\to 0$ with F_1 and F_2 free. The result now follows from Lemmas 7.4 and 7.7.

(ii) Suppose first that the map $\phi: V_1\to V_2$ factors through a weakly injective module. Then by Lemma 7.3, ϕ factors through the natural inclusion $V_1\to$

$V_1 \otimes_R RG$. We shall realize the maps $V_1 \to V_1 \otimes_R RG$ and $V_1 \otimes_R RG \to V_2$ by maps $K_1 \to L$ and $L \to K_2$.

The map $\mathbb{Z} \to \mathbb{Z}G$ is realizable by Lemma 7.8, and so by smashing this map with a realization of V_1, we obtain a realization of $V_1 \to V_1 \otimes_R RG$ by some map $K_1 \to L = K_1 \wedge G_+$. Using Proposition 7.6 in case $R = \mathbb{F}_p$, we may assume that K_1 and hence L are realizations in degree n, say, which are $(n-1)$-connected. Suspending K_1 and L if necessary to increase n, and again using Proposition 7.6, we may also choose an $(n-1)$-connected realization K_2 for V_2 in degree n.

If $R = \mathbb{Z}$, we may realize the abelian group homomorphism $V_1 \otimes_R 1 \to V_2$ by a map $K_1 \to K_2$, by Lemma 7.5. We now extend this to a G-equivariant map $K_1 \wedge G_+ \to K_2$ realizing $V_1 \otimes_R RG \to V_2$.

If $R = \mathbb{F}_p$, then $V_1 \otimes_R RG$ is RG-free with basis B, say. By Lemma 7.7, there are maps $S^n \wedge B_+ \wedge G_+ \to L$ and $S^n \wedge B_+ \wedge G_+ \to K_2$ realizing the identity map on $V_1 \otimes_R RG$ and the map $V_1 \otimes_R RG \to V_2$. Finally, we replace $K_1 \to L$ by $K_1' \to L'$ defined by the homotopy pullback diagram

$$\begin{array}{ccc} K_1' & \longrightarrow & L' = S^n \wedge B_+ \wedge G_+ \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & L \end{array}$$

We now deal with the case where $R = \mathbb{Z}$, and $\phi : V_1 \to V_2$ only factors through some cohomologically trivial module P, say, which is not necessarily weakly injective. Using Lemma 7.9, we may find a realizable $\mathbb{Z}$-free $\mathbb{Z}G$-module V_1' and a surjective cohomology equivalence $V_1' \to V_1$. The kernel F of this map is $\mathbb{Z}$-free and cohomologically trivial, and hence $\mathbb{Z}G$-free. The composite map $V_1' \to V_1 \to V_2$ factors through a weakly injective module, and so by the first part of the proof it is realizable by some map $K_1' \to K_2$. The proof of Lemma 7.7 in case $R = \mathbb{Z}$ allows us to choose our $\mathbb{Z}G$-Moore space, and so the map $F \to V_1'$ is realizable by some map $S^n \wedge G_+ \wedge B_+ \xrightarrow{f} K_1'$. Since the composite $F \to V_1' \to V_1$ is trivial, the composite $S^n \wedge G_+ \wedge B_+ \to K_1' \to K_2$ extends to a map $D^{n+1} \wedge G_+ \wedge B_+ \to Y$. Thus taking $K_1 = C_f$ we get a map $K_1 \to K_2$ realizing $V_1 \to V_2$.

(iii) By (i) and (ii), if V_2 is either realizable or cohomologically trivial then one map, and hence by Lemma 7.4 both, are realizable.

If V_1 is realizable and V_3 is cohomologically trivial, let F be a free RG-module which surjects onto V_2. Then there is a short exact sequence $0 \to V_4 \to V_1 \oplus F \to V_2 \to 0$ with V_4 cohomologically trivial and $V_1 \oplus F$ realizable. Hence V_2 is realizable.

If V_3 is realizable and V_1 is cohomologically trivial, let I be the injective hull of V_2. Then there is a short exact sequence $0 \to V_2 \to V_3 \oplus I \to V_4 \to 0$ with V_4 cohomologically trivial and $V_3 \oplus I$ realizable. Hence V_2 is realizable.

(iv) Consider the short exact sequence

$$0 \to V_1 \to V_2 \oplus V_1 \otimes_R RG \to V_3 \to 0.$$

The modules V_3 and $V_1 \otimes_R RG$ are cohomologically trivial. Thus if V_1 or V_2 is realizable, so is the map $V_1 \to V_2 \oplus V_1 \otimes_R RG$ by (iii). Let $K_1 \to K_2$ be a realization of this map. We realize the inclusion $V_1 \otimes_R RG \to V_2 \oplus V_1 \otimes_R RG$ by a map $K_3 \xrightarrow{f} K_2$ as follows. If $R = \mathbb{Z}$ we take $K_3 = K_1 \wedge G_+$, while if $R = \mathbb{F}_p$, we take $K_3 = L \wedge G_+$, where L is a bouquet of spheres. The composite $K_1 \to K_2 \to C_f$ then represents $\phi : V_1 \to V_2$. $\square$

Finally, we shall need the following lemma.

Lemma 7.11. *Let k be a field, and also denote by k the one-dimensional kG-module with trivial action.*

(a) *If $\phi_1 : k \to V_1$ and $\phi_2 : k \to V_2$ are realizable, then so are $(\phi_1, \phi_2) : k \to V_1 \oplus V_2$, the cokernel map $V_1 \oplus V_2 \to (V_1 \oplus V_2)/k = \mathrm{coker}(\phi_1, \phi_2)$, and the composite maps $V_1 \to (V_1 \oplus V_2)/k$ and $V_2 \to (V_1 \oplus V_2)/k$.*

(b) *If $\phi_1' : V_1 \to k$ and $\phi_2' : V_2 \to k$ are realizable, then so are $(\phi_1', \phi_2') : V_1 \oplus V_2 \to k$, the kernel map $\ker(\phi_1', \phi_2') \to V_1 \oplus V_2$, and the composite maps $\ker(\phi_1', \phi_2') \to V_1$ and $\ker(\phi_1', \phi_2') \to V_2$.*

Proof. (a) If $\psi_1 : X \to Y$, $\psi_2 : X' \to Z$ realize ϕ_1 and ϕ_2, then so do $(\psi_1 \wedge X') : X \wedge X' \to Y \wedge X'$ and $(X \wedge \psi_2) : X \wedge X' \to X \wedge Z$. Thus we may assume $X = X'$. Suspending if necessary, we have a coproduct map $X \to X \vee X$. The composite $f : X \to X \vee X \to Y \vee Z$ represents $(\phi_1, \phi_2) : k \to V_1 \oplus V_2$. By Lemma 7.4, $Y \vee Z \to C_f$ represents $V_1 \oplus V_2 \to (V_1 \oplus V_2)/k$, and the composite maps $Y \to Y \vee Z \to C_f$ and $Z \to Y \vee Z \to C_f$ represent $V_1 \to (V_1 \oplus V_2)/k$ and $V_2 \to (V_1 \oplus V_2)/k$.

(b) Dually, if $\psi_1' : Y \to X$ and $\psi_2' : Z \to X'$ represent ϕ_1' and ϕ_2', then so do $(\psi_1' \wedge X') : Y \wedge X' \to X \wedge X'$ and $(X \wedge \psi_2') : X \wedge Z \to X \wedge X'$. Thus we may assume $X = X'$. The map $f' = \psi_1' \vee \psi_2' : Y \vee Z \to X$ then represents $(\phi_1', \phi_2') : V_1 \oplus V_2 \to k$. By Lemma 7.4, the map $C_{f'} \to \Sigma(Y \vee Z) = \Sigma Y \vee \Sigma Z$ represents the kernel map $\ker(\phi_1', \phi_2') \to V_1 \oplus V_2$, and the composite maps $C_{f'} \to \Sigma Y \vee \Sigma Z \to \Sigma Y$ and $C_{f'} \to \Sigma Y \vee \Sigma Z \to \Sigma Z$ (given by sending one wedge summand to the base point) represent $\ker(\phi_1', \phi_2') \to V_1$ and $\ker(\phi_1', \phi_2') \to V_2$. $\square$

8. Modules for the Klein Four group in characteristic 2

Throughout this section, let G denote the Klein four group (i.e., the elementary abelian group of order four) and k an algebraically closed field of characteristic two. We discuss here which finitely generated kG-modules are realizable. We shall see that the condition given in Theorem 6.3 is necessary and sufficient in this case.

Denote by Ω the Heller operator of taking the kernel of the projective cover of a module, and by Ω^{-1} the dual operator of taking the cokernel of the injective hull. Note that

$$V \cong \Omega^{-1}(\Omega(V)) \oplus \text{projective} \cong \Omega(\Omega^{-1}(V)) \oplus \text{projective}.$$

Let $H^*(G,k)=k[x,y]$, with $\deg(x)=\deg(y)=1$. It follows from the classification given in Bašev [4] (see also Conlon [11]) that every indecomposable kG-module is isomorphic either to kG, of dimension 4, to $\Omega^{\pm n}(k)$, of dimension $2n+1$ (where k denotes the trivial one-dimensional module), or to $L_{(\alpha x+\beta y)^m}$, of dimension $2m$, where the definition of this module is the same as in Definitions 4.6. The only isomorphisms between these modules are $L_{(\alpha x+\beta y)^m}\cong L_{(\alpha' x+\beta' y)^m}$ if $\alpha\beta'=\alpha'\beta$. Thus the modules of dimension $2m$ are parametrized by $\mathbb{P}^1(k)$, and by Lemma 4.8, $\bar{X}_G(L_{(\alpha x+\beta y)^m})$ is the point $(\alpha:\beta)\in\mathbb{P}^1(k)$.

By Theorem 6.3, $L_{(\alpha x+\beta y)^m}$ is not a direct summand of a realizable kG-module if $(\alpha:\beta)\notin\{(1:0),(0:1),(1:1)\}$. We shall show that all the other indecomposable kG-modules are realizable.

The free module kG, the trivial module k, and the modules L_x, L_y and $L_{(x+y)}$, are all permutation modules, and are hence realizable by bouquets of spheres permuted by G in the same fashion. By Lemmas 7.5 and 7.6, if V is realizable, then so are $\Omega(V)$ and $\Omega^{-1}(V)$. Thus the modules $\Omega^{\pm n}(k)$ are realizable. We shall realize the remaining modules by a process of gluing together permutation modules as we now describe.

The remaining modules are the modules L_{x^m}, L_{y^m} and $L_{(x+y)^m}$ with $m>1$. By a change of notation (i.e., a change of basis of G) we need only consider L_{x^m}.

Theorem 8.1. *The modules L_{x^m} are all realizable.*

Proof. *Step 1.* We realize the map $\tilde{x}:\Omega(k)\to k$ (see 4.6) as follows. Denote by $\bigvee^k(S^n)$ a wedge of k copies of the n-sphere. Then there are obvious maps $S^n\xrightarrow{\alpha}\bigvee^2(S^n)$ and $\bigvee^4(S^n)\xrightarrow{\beta}\bigvee^2(S^n)$ realizing the maps $k\rightarrowtail L_x$ and $kG\twoheadrightarrow L_x$. By Lemma 7.5, the mapping cone $C_{\alpha\vee\beta}$ of

$$S^n\vee\bigvee^4(S^n)\xrightarrow{\alpha\vee\beta}\bigvee^2(S^n)$$

realizes the kernel of $k\oplus kG\to L_x$, namely $\Omega(k)$, and the inclusion $C_{\alpha\vee\beta}\hookrightarrow\Sigma(S^n\vee\bigvee^4(S^n))$ realizes the inclusion $\Omega(k)\hookrightarrow k\oplus kG$. It is easy to check that projecting onto $\Sigma S^n\cong S^{n+1}$ gives us a map $\eta:C_{\alpha\vee\beta}\to S^{n+1}$ realizing $\tilde{x}:\Omega(k)\to k$.

Step 2. We realize $(\tilde{x}^m,0):\Omega^m(k)\oplus(\text{free})\to k$ as follows. Let $\eta:K_1\to K_2$ be the map constructed above realizing $\tilde{x}$. Then

$$(\eta\wedge 1)\circ(1\wedge\eta):K_1\wedge K_1\to K_1\wedge K_2\to K_2\wedge K_2$$

realizes the map

$$\tilde{x}\circ(1\otimes\tilde{x}):\Omega(k)\otimes\Omega(k)\to\Omega(k)\to k.$$

But $\Omega(k)\otimes\Omega(k)\cong\Omega^2(k)\oplus kG$, and $\tilde{x}\circ(1\otimes\tilde{x})$ is the map $(\tilde{x}^2,0):\Omega^2(k)\oplus kG\to k$. We repeat this procedure inductively as follows.

$$(\eta\wedge 1\wedge\cdots\wedge 1)\circ(1\wedge\eta\wedge\cdots\wedge 1)\circ\cdots\circ(1\wedge 1\wedge\cdots\wedge\eta)$$

is a map from $K_1\wedge\cdots\wedge K_1$ to $K_2\wedge\cdots\wedge K_2$ realizing the map

$$\tilde{x} \circ (1 \otimes \tilde{x}) \circ \cdots \circ (1 \otimes \cdots \otimes \tilde{x}) : \bigotimes^m (\Omega(k)) \to k.$$

But $\bigotimes^m (\Omega(k)) \cong \Omega^m(k) \oplus F$ with F free, and the above map is $(\tilde{x}^m, 0) : \Omega^m(k) \oplus F \to k$.

Step 3. By Lemma 7.4, the cone on the map constructed in Step 2 realizes $L_{x^m} \oplus F$. Since $L_{x^m} \oplus F$ is cohomologically equivalent to L_{x^m}, it follows from Proposition 7.10(iv) that L_{x^m} is realizable.

Alternative proof avoiding use of 7.10. We give the above proof because it shows how a construction follows closely from the definition of L_{x^m}. The following construction avoids some of the heavy work in Section 7, but is slightly less transparent.

We proceed inductively. Assume by induction that we have realized the short exact sequence

$$0 \to L_{x^{m-1}} \to \Omega^{m-1}(k) \to k \to 0.$$

We treat the case $m = 1$ exactly as in Step 1 above.

Since we also have a realizable map $L_x \to k$, we may use Lemma 7.11 to obtain a realizable map $\Omega^{m-1}(k) \oplus L_x \to k$ whose kernel is L_{x^m}. This gives us a realizable map $L_{x^m} \to \Omega^{m-1}(k)$ whose kernel is k. As before, we also have a realizable map from k to L_y, so we may add these, again using Lemma 7.11, to obtain a realizable short exact sequence

$$0 \to k \to L_{x^m} \oplus L_y \to \Omega^m(k) \to 0,$$

and hence a realizable short exact sequence

$$0 \to L_{x^m} \to \Omega^m(k) \to k \to 0$$

as required. $\square$

Combining Theorem 8.1 with the preceding discussion proves the following theorem.

Theorem 8.2. *Let G be the Klein four group. A finitely generated kG-module U is realizable if and only if for all summands V of U, $X_G(V)$ is of the form $\bigcup_{E \le G} t_{E,G}(X_E)$ for some collection of subgroups E of G.*

Comment. Using the techniques of Section 7, together with the description in [6] of the indecomposable $\hat{\mathbb{Z}}_2 G$-lattices, it should be possible to determine precisely which $\hat{\mathbb{Z}}_2 G$-lattices are realizable.

9. Modules for the quaternion group Q_8

Throughout this section, let G denote the quaternion group of order eight, and let k be an algebraically closed field of characteristic two. In this case, the variety X_G is a single line through the origin, and so the only homogeneous subvarieties are X_G and $\{0\}$. Thus Theorem 6.3 gives us no information in this case about realizability of finitely generated kG-modules. In this section, we produce an example of a non-realizable kG-module. This module is the extension of an $\mathbb{F}_4G$-module V to a kG-module $V\otimes_{\mathbb{F}_4}k$, and V is the reduction modulo (2) of a non-realizable $\mathbb{Z}[\omega]$-free $\mathbb{Z}[\omega]G$-module U. Here, and for the rest of this section, $\{1,\omega,\bar{\omega}\}$ are the cube roots of unity either in $\mathbb{C}$ or in k, so that for example $\mathbb{F}_4=\{0,1,\omega,\bar{\omega}\}$. We shall use a restriction of scalars argument to give a non-realizable $\mathbb{F}_2G$-module V_0 and a lift to a $\mathbb{Z}$-free non-realizable $\mathbb{Z}G$-module. This example is a modification of an example of Vogel, who shows that V_0 is a non-realizable $\mathbb{Z}$-torsion $\mathbb{Z}G$-module.

Let $G=\langle x,y\,|\,x^2=y^2,x^4=1,x^{-1}yx=y^{-1}\rangle$. The representation U is given as 2×2 matrices over $\mathbb{Z}[\omega]$ as follows.

$$x\mapsto\begin{pmatrix}-1 & 1\\ -2 & 1\end{pmatrix},\qquad y\mapsto\begin{pmatrix}\bar{\omega}-\omega & \omega\\ 2\bar{\omega} & \omega-\bar{\omega}\end{pmatrix}.$$

It is easily verified that these matrices satisfy the given relations, and hence give a representation of Q_8. Modulo (2), we obtain the $\mathbb{F}_4G$-module V given by matrices as follows.

$$x\mapsto\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\qquad y\mapsto\begin{pmatrix}1 & \omega\\ 0 & 1\end{pmatrix}.$$

Note that $Z(G)$ is in the kernel of this $\mathbb{F}_4G$-module. However, it should be pointed out that it does not follow from the fact that the corresponding $\mathbb{F}_4[G/Z(G)]$-module is not realizable (see Section 8) that this $\mathbb{F}_4G$-module is not realizable. We need some further argument.

Recall that $H^*(G,\mathbb{F}_2)=\mathbb{F}_2[x,y,z]/(x^2+xy+y^2,x^2y+xy^2)$ where $\deg x=\deg y=1$, $\deg z=4$, and the action of the Steenrod algebra $\mathscr{A}(2)$ is given by $\mathrm{Sq}^1(x)=x^2$, $\mathrm{Sq}^1(y)=y^2$, $\mathrm{Sq}^1(z)=\mathrm{Sq}^2(z)=0$, and $\mathrm{Sq}^4(z)=z^2$.

Lemma 9.1. *As a module over $H^*(G,\mathbb{F}_4)$, $\mathrm{Ext}^*_{\mathbb{F}_4G}(V,\mathbb{F}_4)\cong H^*(G,V^*)$ is generated by elements a_0 in degree 0 and b_1 in degree 1, subject to the relations $(x+\omega y)b_1=(x^2+\omega y^2)a_0$ and $(x+\omega y)a_0=0$.*

Proof. This follows from an examination of the spectral sequence

$$\begin{array}{ccc} H^p(G/Z(G),H^q(Z(G),V^*)) & \Longrightarrow & H^{p+q}(G,V^*)\\ \wr\big\| & & \\ H^p(G/Z(G),V^*)\otimes_{\mathbb{F}_4}H^q(Z(G),\mathbb{F}_4). & & \end{array}\qquad\square$$

Theorem 9.2. $\mathrm{Ext}^*_{\mathbb{F}_4G}(V,\mathbb{F}_4)$ *has no structure as a T-$H^*(G,\mathbb{F}_4)$-module extending the action of $H^*(G,\mathbb{F}_4)$ given in Lemma 9.1.*

Proof. Suppose $\mathscr{T}$ is a linear map satisfying the conditions of Definition 5.2. Then

$$\begin{aligned}0&=\mathscr{T}((x+\omega y)a_0)=\mathscr{T}(x+\omega y)\mathscr{T}(a_0)\\&=(x+\omega y)\mathscr{T}(a_0)+(x^2+\omega y^2)\mathscr{T}(a_0),\end{aligned}$$

and

$$\begin{aligned}(x^2+\omega y^2)\mathscr{T}(a_0)&=\mathscr{T}((x^2+\omega y^2)a_0)=\mathscr{T}((x+\omega y)b_1)\\&=(x+\omega y)\mathscr{T}(b_1)+(x^2+\omega y^2)\mathscr{T}(b_1).\end{aligned}$$

Comparing terms in degree 3, we see that

$$\begin{aligned}(x^2+\omega y^2)b_1&=(x^2+\omega y^2)\mathscr{T}_1(a_0)+(x+\omega y)\mathscr{T}_1(b_1)\\&=(x+\omega y)(\mathscr{T}_2(a_0)+\mathscr{T}_1(b_1))\in H^*(G,\mathbb{F}_4)\cdot a_0.\end{aligned}$$

This contradicts the fact that $(x^2+\omega y^2)b_1$ is non-zero in

$$\mathrm{Ext}^*_{\mathbb{F}_4G}(V,\mathbb{F}_4)/H^*(G,\mathbb{F}_4)\cdot a_0,$$

which is an $H^*(G,\mathbb{F}_4)$-module with a single generator b_1 subject to the relation $(x+\omega y)b_1=0$. □

We now apply Proposition 5.4, to deduce that V is not realizable.

Corollary 9.3. (i) *V is a non-realizable $\mathbb{F}_4G$-module.*

(ii) *U is a non-realizable $\mathbb{Z}[\omega]G$-module.*

(iii) *Let V_0 be the module V considered as an $\mathbb{F}_2G$-module by restriction of scalars. Then V_0 is not realizable.*

(iv) *Let U_0 be the module U considered as a $\mathbb{Z}$-free $\mathbb{Z}G$-module by restriction of scalars. Then U_0 is not realizable.*

Proof. (i) This follows from Theorem 9.2 and Proposition 5.4.

(ii) This follows from (i) and 2.4.

(iii) This follows from (i) and Corollary 5.6.

(iv) This follows from (iii) and 2.4. □

Remark. Vogel has used the fact that V_0 is a $\mathbb{Z}$-torsion non-realizable $\mathbb{Z}Q_8$-module to show that the following conditions on a finite group G are equivalent.

(i) Every $\mathbb{Z}G$-module is realizable.

(ii) All Sylow subgroups of G are cyclic.

References

[1] J. Arnold, A solution of a problem of Steenrod for cyclic groups of prime order, Proc. Amer. Math. Soc. 62 (1977).

[2] J. Arnold, On Steenrod's problem for cyclic p-groups, Canad. J. Math. 29 (1977) 421–428.

[3] G.S. Avrunin and L.L. Scott, Quillen stratification for modules, Invent. Math. 66 (1982) 277-286.

[4] V.A. Bašev, Representations of the group $Z_2 \times Z_2$ in a field of characteristic 2, (Russian), Dokl. Akad. Nauk. SSSR 141 (1961) 1015-1018.

[5] D.J. Benson, Modular Representation Theory: New trends and Methods, Lecture Notes in Math. 1081 (Springer, Berlin, 1984).

[6] M.C.R. Butler, The 2-adic representations of Klein's four group, Proc. Second Internat. Conf. Theory of Groups, Canberra 1973, Lecture Notes in Math. 372 (Springer, Berlin, 1974).

[7] J.F. Carlson, Varieties and the cohomology ring of a module, J. Algebra 85 (1983) 104-143.

[8] J.F. Carlson, The variety of an indecomposable module is connected, Invent. Math. 77 (1984) 291-299.

[9] G. Carlsson, A counterexample to a conjecture of Steenrod, Invent. Math. 64 (1981) 171-174.

[10] J.W.S. Cassels and A. Fröhlich, Algebraic Number Theory, Proc. Instr. Conf. LMS (Academic Press, New York, 1967).

[11] S.B. Conlon, Certain representation algebras, J. Austral. Math. Soc. 5 (1965) 83-99.

[12] C.W. Curtis and I. Reiner, Methods of Representation Theory, with Applications to Finite Groups and Orders, Vol. I (Wiley-Interscience, New York, 1981).

[13] P. Kahn, Steenrod's problem and k-invariants of certain classifying spaces, Algebraic K-Theory, Proc. Oberwolfach 1980, Part 2, Lecture Notes in Math. 967 (Springer, Berlin, 1982).

[14] D. Quillen, The spectrum of an equivariant cohomology ring, I, Ann. of Math. 94 (1971) 549-572.

[15] D. Quillen, The spectrum of an equivariant cohomology ring, II, Ann. of Math. 94 (1971) 573-602.

[16] D.S. Rim, Modules over finite groups, Ann. of Math. 69 (1959) 700-712.

[17] J.-P. Serre, Sur la dimension cohomologique des groupes profinis, Topology 3 (1965) 412-420.

[18] J.-P. Serre, Local Fields, Graduate Texts in Math. 67 (Springer, Berlin, 1979).

[19] R.G. Swan, Minimal resolutions for finite groups, Topology 4 (1965) 193-208.

[20] R.G. Swan, Invariant rational functions and a problem of Steenrod, Invent. Math. 7 (1969) 148-158.

[21] R.G. Swan and E.G. Evans, K-Theory of Finite Groups and Orders, Lecture Notes in Math. 149 (Springer, Berlin, 1970).

[22] R.M. Switzer, Algebraic Topology – Homotopy and Homology, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, B.212 (Springer, Berlin, 1975).

[23] P. Vogel, On Steenrod's problem for non-abelian finite groups, Proc. Algebraic Topology Conf., Aarhus 1982, Lecture Notes in Math. 1051 (Springer, Berlin, 1984).

[24] C.T.C. Wall, Finiteness conditions for CW-complexes, Ann. of Math. 81 (1965) 56-69.

[25] P.J. Webb, Restricting $\mathbb{Z}G$-lattices to elementary abelian subgroups, Proc. Conf. on Integral Representations and Applications, Oberwolfach 1980, Lecture Notes in Math. 882 (Springer, Berlin, 1981).

[26] G.W. Whitehead, Elements of Homotopy Theory, Graduate texts in Math. 61 (Springer, Berlin, 1978).

[27] H. Cartan and S. Eilenberg, Homological Algebra (Princeton Univ. Press, Princeton, 1956).

[28] L. Fuchs, Infinite Abelian Groups (Academic Press, New York, 1970).