For the proof of (2), observe that

$$\frac{d}{dx}\binom{n}{k}F(x)^k(1-F(x))^{n-k}$$
$$=\left(\binom{n-1}{k-1}F(x)^{k-1}(1-F(x))^{n-k}-\binom{n-1}{k}F(x)^k(1-F(x))^{n-1-k}\right)nf(x)$$
$$=-\nabla\binom{n-1}{k}F(x)^k(1-F(x))^{n-1-k}nf(x).$$

Thus

$$\binom{n}{k}F(c)^k(1-F(c))^{n-k}=-\nabla\int_a^c\binom{n-1}{k}F(x)^k(1-F(x))^{n-1-k}nf(x)\,dx,$$

and by (5),

$$\sum_{k=r}^{n}\binom{n}{k}F(c)^k(1-F(c))^{n-k}=-\left[\int_a^c\binom{n-1}{k}F(x)^k(1-F(x))^{n-1-k}nf(x)\,dx\right]_{r-1}^{n}$$
$$=\int_a^c\binom{n-1}{r-1}F(x)^{r-1}(1-F(x))^{n-r}nf(x)\,dx.$$

References

1. A. Chao, Another approach to incomplete integrals, The American Statistician, 36 (1982) 48.
2. W. Feller, An Introduction to Probability Theory and Its Applications, Vol 1, 3rd ed., Wiley, New York, 1968.

FINITE LINEAR GROUPS, THE COMMODORE 64, EULER AND SYLVESTER

DAVID BENSON, WALTER FEIT, AND ROGER HOWE
Department of Mathematics, Yale University, New Haven, CT 06520

For some examples in group theory, one of us was interested in knowing the number of conjugacy classes in $GL_n(2)$, where $GL_n(q)$ denotes the general linear group in dimension n over the finite field with q elements. A generating function was given in [3] for the number $p_n(q)$ of conjugacy classes in $GL_n(q)$:

$$1+\sum_{n=1}^{\infty}p_n(q)x^n=\prod_{i=1}^{\infty}\frac{(1-x^i)}{(1-qx^i)}. \tag{1}$$

Since he had available a Commodore 64 minicomputer, the aforementioned one of us let it compute not simply $p_n(2)$, but the full polynomials $p_n(q)$, for $n < 40$. Here is a sample of the results.

$$\begin{aligned}
p_1(q)&=q-1,\\
p_2(q)&=q^2-1,\\
p_4(q)&=q^4-q,\\
p_8(q)&=q^8-q^3-q^2+q,\\
p_{16}(q)&=q^{16}-q^7-q^6-q^5+2q^3+q^2-q,\\
p_{32}(q)&=q^{32}-q^{15}-q^{14}-q^{13}-q^{12}-q^{11}-q^{10}+q^9\\
&\quad+2q^8+4q^7+3q^6-4q^4-3q^3+q^2+q.
\end{aligned} \tag{2}$$

Evidently the Commodore 64 had uncovered something interesting. The polynomials $p_n(q)$ are very atypical, in that the coefficient of q^k vanishes for $n > k > (n-1)/2$, and equals -1 for $(n-1)/2 \geqslant k > (n-3)/3$. Particularly interesting from the group-theoretical viewpoint is the fact that q^n is an astoundingly good approximation to $p_n(q)$. Evidently a tremendous amount of cancellation occurs if $p_n(q)$ is computed using formula (1).

Intrigued by this behavior, another of us studied the Commodore's initial output, which was for $n < 20$, and conjectured the following formula:

$$(3)\quad \prod_{i=1}^{\infty} \frac{1-x^i}{1-qx^i} = 1 + \sum_{m\geqslant 0} q^m \left(\sum_{\substack{r\geqslant 0\\(m,r)\neq(0,0)}} (-1)^r x^{(r+1)m+r(3r-1)/2} \frac{(1-x^{m+2r})}{(1-x^{m+r})} \binom{m+r}{r}(x) \right),$$

where

$$(4)\quad \binom{m+r}{r}(x) = \frac{(1-x^{m+r})(1-x^{m+r-1})\cdots(1-x^{m+1})}{(1-x^r)(1-x^{r-1})\cdots(1-x)}.$$

The functions $\binom{m+r}{r}(x)$ are easily checked to satisfy the recursions

$$(5)\quad \binom{m+r}{r}(x) = \binom{m+r-1}{r}(x) + x^m \binom{m+r-1}{r-1}(x).$$

(See [6], p. 175.) Using (5) one sees by induction that $\binom{m+r}{r}(x)$ is a polynomial of degree mr with nonnegative coefficients. Hence there is relatively little cancellation of coefficients in the right-hand side of (3), so that it provides a much faster way of computing the polynomial $p_n(q)$ than does the deceptively compact right-hand side of (1). For example the next polynomial in the series (2) was computed by hand from (3) to be

$$(6)\quad p_{64}(q) = q^{64} - \sum_{i=21}^{31} q^i + 2q^{19} + 3q^{18} + 5q^{17} + 6q^{16} + 8q^{15} + 7q^{14} + 4q^{13} - 2q^{12} - 10q^{11}$$
$$-17q^{10} - 17q^9 - 6q^8 + 10q^7 + 20q^6 + 9q^5 - 6q^4 - 7q^3 + q.$$

Further computation by the Commodore 64 verified formula (3) up to $n = 39$, so we attempted to prove (3). We briefly sketch our derivation with comments. We thank George E. Andrews for providing us with the historical remarks below.

We begin with two famous identities of Euler.

$$(7)\quad \prod_{i=1}^{\infty}(1-x^i) = 1 + \sum_{r=1}^{\infty}(-1)^r\left(x^{r(3r+1)/2} + x^{r(3r-1)/2}\right) = \sum_{r=-\infty}^{\infty}(-1)^r x^{r(3r-1)/2}$$

$$(8)\quad \sum_{i=1}^{\infty} \frac{1}{1-qx^i} = \sum_{m=0}^{\infty} q^m \frac{x^m}{(1-x^m)(1-x^{m-1})\cdots(1-x)}.$$

See [5] p. 284 and [1] p. 19. Note that formula (7) results from (3) by specializing q to be zero. From (8) we can write

$$(9)\quad \prod_{i=1}^{\infty} \frac{1-x^i}{1-qx^i} = \sum_{m=0}^{\infty} q^m x^m \prod_{j=m+1}^{\infty}(1-x^j).$$

If the product in the right-hand side of (9) is expanded one sees the coefficient of q^m is

$$x^m - x^{2m+1} - x^{2m+2} \cdots - x^{3m} + \text{higher terms}.$$

This explains the most obvious features of the coefficients of $p_n(q)$. However the right-hand side of (9) still involves much more cancellation than the right-hand side of (3). We can get (3) from (9) if the identity

$$(10)\quad \prod_{j=m+1}^{\infty}(1-x^j) = \sum_{r=0}^{\infty}(-1)^r x^{mr+r(3r-1)/2} \frac{(1-x^{m+2r})}{(1-x^{m+r})} \binom{m+r}{r}(x)$$

holds. (In (10) if $m = r = 0$, we agree that $(1 - x^0)/(1 - x^0) = 1$.) Or multiplying both sides of equation (10) by $\prod_{j=1}^{m}(1 - x^j)$ we get the equivalent relation

$$\prod_{j=1}^{\infty}(1 - x^j) = \sum_{r=0}^{\infty}(-1)^r A_m(r), \tag{11}$$

where the $A_m(r)$ are polynomials given by

$$A_m(r) = x^{mr + r(3r-1)/2}\frac{(1 - x^{m+2r})}{(1 - x^{m+r})}\prod_{l=1}^{m}(1 - x^{r+l}).$$

(Note that for $m = 0$, the product $(1 - x^{m+r}) \cdots (1 - x^{r+1})$ equals 1, while for $m \geqslant 1$, the terms $(1 - x^{m+r})$ cancel.)

As we have remarked, equation (11) for $m = 0$ is just Euler's identity (7). It turns out that the generalization (11) of (7) is a special case of a result of Sylvester [7] and was put into the more elegant form

$$\prod_{j=1}^{\infty}(1 - x^j) = \sum_{r=-\infty}^{\infty}(-1)^r x^{mr + r(3r-1)/2}(1 - x^{r+m-1}) \cdots (1 - x^{r+1}) \tag{12}$$

(corresponding to the third expression in (7)) relatively recently by I. J. Good [4]. Modulo notation, Sylvester's generalization of (11) can be found in [**1**]. p. 140.

For the benefit of the reader who, with us, is not familiar with Sylvester's identity, we indicate the proof we devised of (11). By induction on m, beginning with $m = 0$ (Euler's identity (7)), it is enough to show that the right-hand sides of (11) for m and $m + 1$ are equal. In other words, we want to show

$$\sum_{r=0}^{\infty}(-1)^r(A_{m+1}(r) - A_m(r)) = 0.$$

This will follow by formal manipulation if we can find polynomials $B_m(r)$ such that $B_m(0) = 0$ and

$$A_{m+1}(r) - A_m(r) = B_m(r + 1) + B_m(r). \tag{13}$$

But, indeed, if we set

$$B_m(r) = -x^{mr + r(3r-1)/2}\prod_{j=r}^{m+r-1}(1 - x^j)$$

for $m \geqslant 0$, $r \geqslant 0$, $mr \neq 0$, and $B_0(0) = 0$, then the verification of (13) is straightforward. The reader should note the similarity between the $B_m(r)$ and the terms of the sum in (12). This argument for identity (11) turns out to be essentially like one given by Sylvester's friend, Arthur Cayley [**2**], Vol. 12, pp. 217–219.

Thus these classical identities, long cultivated for their own sake, find an application in group theory. Although formulas analogous to (1) are known for the other classical groups over finite fields, none of these other formulas seem to give rise to intricate cancellations of the sort yielding expressions as in (2).

References

1. G. E. Andrews, The Theory of Partitions, Addison-Wesley, Reading, MA, 1976.

2. A. Cayley, Collected Papers.

3. W. Feit and N. J. Fine, Pairs of commuting matrices over a finite field, Duke Math. J., 27 (1960) 91–94.

4. I. J. Good, A generalization of the Bernoulli-Euler partition formula, Scripta Math., 28 (1968) 319–320.

5. G. M. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th Ed., Oxford University Press, Oxford, England, 1959.

6. T. Nagel, Introduction to Number Theory, Wiley, New York, 1951.

7. J. J. Sylvester, Collected Papers, 4 (1883) 91, 93–94.