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DIAGRAMS FOR MODULAR LATTICES

D.J. BENSON
Yale University, New Haven, CT 06520, USA

J.H. CONWAY
DPMMS, 16 Mill Lane, Cambridge, England

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1. Introduction

In the study of representation theory, many people find themselves drawing some kind of diagram describing the submodule structure of a module. The amount of information displayed varies: at one extreme we can show the complete lattice of submodules together with information about the isomorphism types of subquotients; at the other extreme we may only give the set of composition factors. Between these extremes we have for example the Loewy and socle series.

In the case where the module has no subquotient isomorphic to a direct sum of two isomorphic simple modules, there is an easy definition of a diagrammatic notation, in which the composition factors are given a partial order, and the lattice of submodules may be recovered from the partial order (see for example the discussion in Alperin [2]). The purpose of this paper is to describe a notation for arbitrary modular lattices satisfying an appropriate finiteness condition, in such a way that the lattice may always be recovered from the diagram, and if the lattice is finite then the diagram is also finite, and usually has significantly fewer vertices than the lattice has elements.

Finitely generated modules over an Artinian ring have the property that the lattice of submodules satisfies both the ascending and descending chain condition, and has a maximal and a minimal element. We say such a lattice has *finite length*.

Modular lattices of finite length in fact have a well defined length $l(\Gamma)$, and the number of modes in our diagram for Γ is at least $l(\Gamma)$. In the case of the lattice of submodules of a module, $l(\Gamma)$ corresponds to the number of composition factors, and equality occurs if and only if the module has no subquotient isomorphic to a direct sum of two isomorphic simple modules.

An unfortunate feature of our diagrams is that they do not behave well under duality, in the sense that it is difficult to relate the diagrams for a lattice and its

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opposite. Also, the diagram for an extension of two modules is difficult to relate to the diagrams for the two modules; but this appears to reflect the complexity of the extension problem.

For a basic reference on lattice theory, see Aigner [1].

2. Notation and definitions

Definitions. A *lattice* is a partially ordered set any two of whose elements have a greatest lower bound or 'meet' denoted $x \wedge y$, and a least upper bound or 'join' denoted $x \vee y$. A lattice is *complete* if every subset has a meet and a join.

A lattice has *finite length* if it has no infinite ascending or descending chains. It is clear that such a lattice is automatically complete, and has unique maximal and minimal elements, denoted 1 and 0.

A lattice is *modular* if $a \leq b \Rightarrow (a \vee c) \wedge b = a \vee (c \wedge b)$.

Let Γ be a lattice. For $x \in \Gamma$, let

$$\mu(x) = \{ y \in \Gamma : y < x \text{ and } \nexists z \in \Gamma, y < z < x \}.$$

If Γ has finite length, we have

$$\mu(x) = \emptyset \quad \Rightarrow \quad x = 0.$$

We define

$$\mathrm{Rad}(x) = \Big(\bigwedge_{y \in \mu(x)} y \Big) \wedge x,$$

$$n(x) = |\mu(x)| \qquad \text{(possibly } n(x) = \infty\text{)},$$

$$h_r(\Gamma) = \{ x \in \Gamma : n(x) = r \},$$

$$h(\Gamma) = h_1(\Gamma).$$

We denote by Γ^* the *opposite lattice* of Γ; namely the points of Γ^* are the same as the points of Γ, but $x \leq y$ in Γ^* if and only if $y \leq x$ in Γ. Thus the rôles of meet and join are reversed.

It is interesting to note that Dilworth [3] has proved that if Γ is finite, then for any r, $|h_r(\Gamma)| = |h_r(\Gamma^*)|$. This theorem with $r = 1$ will tell us that our diagrams for Γ and Γ^* have the same number of nodes, although there is in general no natural one–one correspondence.

3. Definition of the diagram

If Γ is a modular lattice of finite length, the diagram $D(\Gamma)$ for Γ consists of some points, some directed edges and some 'dotted lines' (subsets of the set of points) as follows.

The *points* of $D(\Gamma)$ are the elements of $h(\Gamma)$.

There is a *directed edge* from x to y, $x, y \in h(\gamma)$, if and only if $y<x$ and there is no $z \in h(\Gamma)$ with $y<z<x$.

A subset $X \subseteq h(\Gamma)$ forms a *dotted line* if and only if the following conditions are satisfied.

(a) $|X| \geq 3$ (possibly X is infinite).

(b) For all $x, y \in X$, $x \vee y = \bigvee(X)$.

(c) X is maximal with respect to properties (a) and (b).

Lemma. *Any subset $X \subseteq h(\Gamma)$ satisfying* (a) *and* (b) *is contained in a maximal such subset.*

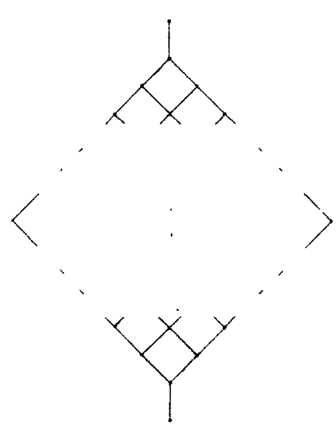
Proof. If X_1 and X_2 satisfy (a) and (b) with $X_1 \subseteq X_2$, then $\bigvee(X_1) = \bigvee(X_2)$. Thus the union of any ascending chain of such subsets is again such a subset. The result now follows from Zorn's lemma. □

Note that there may be more than one dotted line containing a given set of points. Note also that since there are no directed loops, the direction of a directed edge may be (and will be in our examples) indicated by placing the tail of the arrow higher on the page than the head, and drawing an undirected edge.

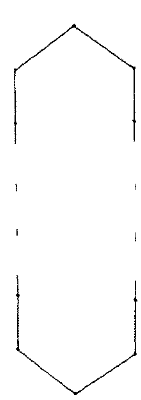
If Γ is the lattice of submodules of a module, then we may mark each point $x \in D(\Gamma)$ with the isomorphism type of $x/\mathrm{Rad}(x)$, which is a simple module.

4. Examples

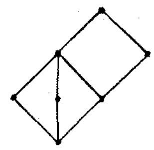
(1) Let B be a block of a group algebra kG, with cyclic defect. The principal indecomposable modules in B are either uniserial, or have the following lattice of submodules and diagram:



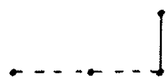
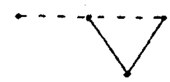
lattice



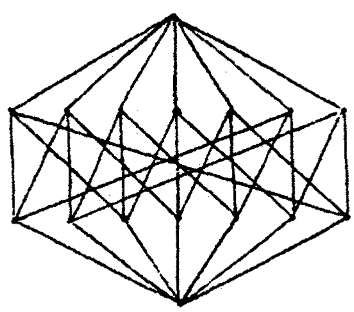
diagram

(2) Let Γ_1 denote the lattice 

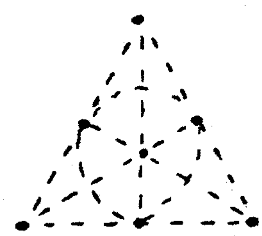
Then

$D(\Gamma_1) =$  and $D(\Gamma_1^*) =$ 

(3) The direct sum of three copies of an absolutely irreducible representation of a finite group over GF(2) has its lattice of submodules and diagram as follows.

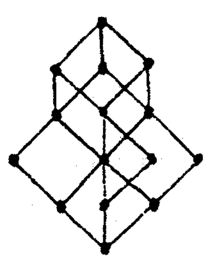


lattice

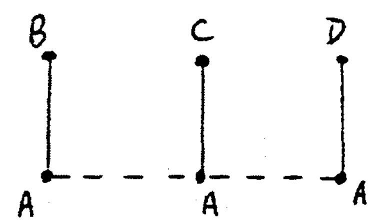


diagram

(4) Again working over GF(2), we may glue three different simple modules B, C and D on top of $A \oplus A$ to obtain a module with the following lattice of submodules and diagram.



lattice



diagram

5. Recovering the lattice

Let D be a diagram arising from a modular lattice as in Section 3. We define a lattice $\Gamma(D)$ as follows.

The points of $\Gamma(D)$ are the subsets M of the points of D satisfying the following conditions.

(α) If $x \in M$ and there is a directed edge $x \to y$, then $y \in M$.

(β) If X is a dotted line, then either $|X \cap M| \leq 1$ or $X \subseteq M$.

We give $\Gamma(D)$ a poset structure by defining $M_1 \leq M_2$ if and only if $M_1 \subseteq M_2$ as subsets of D. Our main result is the following.

Main Theorem. *Suppose Γ is a modular lattice of finite length. Then $\Gamma(D(\Gamma))$ has meets and joins, giving it the structure of a modular lattice naturally isomorphic to Γ.*

6. Proof of the Main Theorem

Before proving this theorem, we need some preliminary lemmas.

Lemma 1. *Suppose Γ is a lattice satisfying the descending chain condition. Then for any $x \in \Gamma$,*

$$x = \bigvee \{y \in h(\Gamma) : y \leq x\}$$

Proof. Suppose false, and choose a minimal $u \in \Gamma$ which does not have the given property. Let $u' = \bigvee \{y \in h(\Gamma) : y \leq u\}$. Then if $v < u$, $v \vee u' = u'$ and hence $v \leq u'$. Thus $u \in h(\Gamma)$, and we have a contradiction. □

Lemma 2 (Replacement Lemma). *Suppose Γ is a modular lattice and*

$$x = x_1 \vee \cdots \vee x_m = y_1 \vee \cdots \vee y_n$$

with x_i and y_j elements of $h(\Gamma)$. Then for some i, $1 \leq i \leq n$,

$$x = x_1 \vee \cdots \vee x_{m-1} \vee y_i.$$

In particular any two minimal such expressions have equal length.

Proof. See Aigner [1, 2.23]. □

Lemma 3. *Suppose Γ is a modular lattice of finite length. Then $\Gamma(D(\Gamma))$ has arbitrary meets and joins.*

Proof. Suppose $\{M_i : i \in I\} \subseteq \Gamma(D(\Gamma))$. Then $\bigcap_{i \in I} M_i \in \Gamma(D(\Gamma))$, and so we have $\bigwedge_{i \in I} M_i = \bigcap_{i \in I} M_i$.

Now $\bigcup_{i \in I} M_i$ is not necessarily an element of $\Gamma(D(\Gamma))$. However, $h(\Gamma)$ is the maximal element of $\Gamma(D(\Gamma))$, and so $\{M \in \Gamma(D(\Gamma)) : M \supseteq \bigcup_{i \in I} M_i\} \neq \emptyset$. Thus we have

$$\bigvee_{i \in I} M_i = \bigwedge \left\{ M \in \Gamma(D(\Gamma)) : M \supseteq \bigcup_{i \in I} M_i \right\}. \quad \square$$

Proof of Main Theorem. By Lemma 3, $\Gamma(D(\Gamma))$ has meets and joins making it a lattice. We define a map

$$\varrho : \Gamma \to \Gamma(D(\Gamma))$$

via

$$x \mapsto M_x = \{y \in h(\Gamma) : y \leq x\}.$$

We easily check from the definitions that $M_x \in \Gamma(D(\Gamma))$. For example, property (b) of the dotted lines shows that (β) is satisfied.

By Lemma 1, $x \leq x'$ if and only if $M_x \leq M_{x'}$, and $x < x'$ if and only if $M_x < M_{x'}$. Thus ϱ is injective and order-preserving.

To show that ϱ is surjective we must show that every element of $\Gamma(D(\Gamma))$ is of the form M_x for some x. Given $M \in \Gamma(D(\Gamma))$, let $x = \bigvee(M)$. We must show that if $y \in h(\Gamma)$ and $y \leq x$, then $y \in M$.

Suppose false. Then there exists M, $x = \bigvee(M)$ and $y \in h(\Gamma)$ with $y \leq x$ and $y \notin M$. Let

$$n_y = \inf\{\text{Card}(U) : U \subseteq M, y \leq \bigvee(U)\}$$

(n_y is finite since Γ has finite length), and choose the above data so as to minimize n_y. Note that $n_y \geq 2$ by property (α) of M. Choose $U = \{u_1, \ldots, u_n\}$ as above with $n = n_y$ so as to minimize $y \vee u_1$. If $u_1 \leq y$, then

$$y = y \wedge \bigvee(u) = u_1 \vee (y \wedge (u_2 \vee \cdots \vee u_n)) = y \wedge (u_2 \vee \cdots \vee u_n)$$

since $y \in h(\Gamma)$. This contradicts the minimality of Card(U), and so $u_1 \not\leq y$. Let $a = (u_1 \vee y) \wedge (u_2 \vee \cdots \vee u_n)$, and write $a = a_1 \vee \cdots \vee a_t$ with $a_i \in h(\Gamma)$. Since $a_i \leq u_2 \vee \cdots \vee u_n$, minimality of Card($U$) implies $a_i \in M$. Now

$$u_1 \vee y = (u_1 \vee y) \wedge (u_1 \vee \cdots \vee u_n) = u_1 \vee a = u_1 \vee a_1 \vee \cdots \vee a_t$$

and so by Lemma 3, $u_1 \vee y = b_1 \vee b_2 = u_1 \vee b_1$ for some $\{b_1, b_2\} \subseteq \{u_1, a_1, \ldots, a_t\}$, so that $n_y = 2$. By minimality of $u_1 \vee y$, we have $u_1 \vee y = b_1 \vee y$, and so $\{y, u_1, b_1\}$ is contained in a dotted line. Hence $y \in M$. □

Acknowledgement

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