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Nilpotent Elements in the Green Ring

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1. INTRODUCTION

In the modular and integral representation theory of finite groups one of the most often used operations is that of the tensor product of modules. In general, however, there has always existed the problem of finding the decomposition of a tensor product into a direct sum of indecomposable modules. Very few techniques exist which reveal or even hint at the nature of such decompositions. One method of addressing this problem is to consider the tensor operation as the product in the representation ring or Green ring of the group ring and to look for ring-theoretic properties of this object. This investigation began with J. A. Green's theorem [11] that the representation ring of a group algebra kG is semisimple in the case where G is a cyclic p-group and k is a field of characteristic p. Much subsequent work has centered on the semisimplicity question in the form: "Does the Green ring have (nonzero) nilpotent elements?" This question has been largely answered. If R is a rank one complete discrete valuation ring whose maximal ideal contains $p \neq 0$, then the Green ring of RG-lattices has nilpotent elements except in certain cases where the Sylow p-subgroup P of G is cyclic of order p or p^2 [14, 15, 17]. If k is a field of characteristic p, then the Green ring has nilpotent elements except when P is a cyclic group or an elementary abelian 2-group (the case of an elementary abelian 2-group of rank of least 3 is still open to the best of our knowledge) [7, 11, 13, 17, 18]. Nevertheless, the proofs given for these results were heavily computational, and neither explained the properties of nilpotent elements, nor indicated a general method for constructing them.

In this paper we give new approaches to these questions, as well as producing some results of independent interest concerning tensor products and endomorphism rings.

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Let G be a finite group, R a commutative ring with a unit, and k a field of characteristic p. By an RG-*lattice* (RG-*module* if R is a field) we shall mean a finitely generated RG-module which is projective as an R-module. If M is an RG-lattice let $[M]$ denote its isomorphism class. Let $a(RG)$ be the free abelian group generated by all symbols $[M]$, modulo the subgroup generated by all elements of the form $[M]-[L]-[N]$, where $0 \to L \to M \to N \to 0$ is a split exact sequence. If M and N are RG-lattices, $M \otimes_R N$ is an RG-lattice with G-action given by $g(m \otimes n) = gm \otimes gn$ for all $g \in G$, $m \in M$, $n \in N$. The *Green ring* or *representation ring* of RG is $a(RG)$ with product given by $[M] \cdot [N] = [M \otimes_R N]$.Let $A(RG) = a(RG) \otimes_{\mathbb{Z}} \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers.

In Section 2 we present a basic result which determines when the trivial kG-module k is a direct summand of a tensor product of two kG-modules; we also give a generalization of this to Scott modules. As an easy consequence of this result we exhibit an ideal $a(G;p) \subseteq a(kG)$ with the property that $a(kG)/a(G;p)$ has no nilpotent elements. Using some results from [2] we produce a new proof of the semisimplicity of the Green ring in the case where G has cyclic Sylow p-subgroups.

In the last two sections we consider the problem of existence of nilpotent elements. In Section 3 a general technique is given for producing nilpotent elements as well as idempotents. The results duplicate those of Zemanek [17, 18] except that instead of giving a few isolated examples, the method produces an infinity of nilpotent elements. The major feature of the technique is to replace Zemanek's difficult tensor product calculation with a comparatively easy cohomology calculation. Examples of the use of the method are given in Section 4.

Throughout this paper all lattices and modules will be assumed to be finitely generated. Maps will be written on the left, and all modules will be left modules. The symbol $\otimes$ will denote tensor product over the coefficient ring R unless otherwise indicated. Reecall that if M and N are RG-lattices then $\operatorname{Hom}_R(M, N)$ is an RG-lattice with G-action given by $(gf)(m) = g(f(g^{-1}m))$ for $g \in G$, $f \in \operatorname{Hom}_R(M, N)$ and $m \in M$. We have $\operatorname{Hom}_R(M, N) \cong M^* \otimes N$, where $M^* = \operatorname{Hom}_R(M, R)$ is the dual of M. Let $\operatorname{End}_R(M) = \operatorname{Hom}_R(M, M)$ and $\operatorname{End}_{RG}(M) = \operatorname{Hom}_{RG}(M, M)$. We say that M is a *component* of N if M is an indecomposable direct summand of N. In this case the *multiplicity* of M in N is the greatest integer s such that a direct sum of s copies of M is a direct summand of N.

2. Nonexistence of Nilpotent Elements

The basic theorem in this section is Theorem 2.1. We also deduce Proposition 2.4, which is a generalization to Scott modules of 2.1, although we shall not use this generalization in what follows. Theorem 2.1 was first proved in the absolutely irreducible case by Peter Landrock.

DEFINITION. We say a kG-module M is *absolutely indecomposable* if $\mathrm{End}_{kG}(M)/\mathrm{Rad}\,\mathrm{End}_{kG}(M) \cong k$.

Note that this implies but is not implied by the property that M is indecomposable under any extension of the field of scalars. If k is algebraically closed then all indecomposable modules are absolutely indecomposable.

THEOREM 2.1. *Suppose that M and N are absolutely indecomposable kG-modules. Then $M \otimes N$ has the trivial module k as a direct summand if and only if the following two conditions are satisfied.*

(i) $M \cong N^*$.

(ii) $p \nmid \dim(N)$.

Moreover if k is a component of $N^* \otimes N$ then it has multiplicity one.

Proof. The set of G-fixed points of $M \otimes N$ is isomorphic to $\mathrm{Hom}_{kG}(N^*, M)$, while the set of G-fixed points of $(M \otimes N)^*$ is isomorphic to $\mathrm{Hom}_{kG}(M, N^*)$. Thus the trivial kG-module k is a direct summand of $M \otimes N$ if and only if we can find homomorphisms

$$k \to M \otimes N \to k$$

whose composite is nonzero; namely, if and only if the composite map

$$\mathrm{Hom}_{kG}(N^*, M) \xhookrightarrow{i} M \otimes N \xrightarrow{p} (\mathrm{Hom}_{kG}(M, N^*))^*$$

is nonzero.

Associated to $p \circ i$, there is a map

$$\eta\colon\quad \mathrm{Hom}_{kG}(N^*, M) \otimes \mathrm{Hom}_{kG}(M, N^*) \to k$$

with the property that $p \circ i \neq 0$ if and only if $\eta \neq 0$. This map is given as follows. Choose a basis $n_1, \ldots, n_r$ for N and let $n'_1, \ldots, n'_r$ be the dual basis for N^*. Let $\alpha \in \mathrm{Hom}_{kG}(N^*, M)$ and $\beta \in \mathrm{Hom}_{kG}(M, N^*)$. Then

$$i(\alpha) = \sum_{j=1}^{r} \alpha(n'_j) \otimes n_j,$$

$$p(m \otimes n)(\beta) = \beta(m)(n),$$

and so by definition

$$\begin{aligned}
\eta(\alpha\otimes\beta) &= ((p\circ i)(\alpha))(\beta) \\
&= p\left(\sum_{j=1}^{r} \alpha(n_j')\otimes n_j\right)(\beta) \\
&= \sum_{j=1}^{r} p(\alpha(n_j')\otimes n_j)(\beta) \\
&= \sum_{j=1}^{r} \beta(\alpha(n_j'))(n_j) \\
&= \operatorname{tr}(\beta\circ\alpha).
\end{aligned}$$

Hence we may factor η as composition followed by trace.

$$\operatorname{Hom}_{kG}(N^*, M)\otimes \operatorname{Hom}_{kG}(M, N^*) \longrightarrow \operatorname{End}_{kG}(N^*) \xrightarrow{\operatorname{tr}} k.$$

Since N^* is absolutely indecomposable, $\operatorname{End}_{kG}(N^*)$ is a local ring, and every kG-endomorphism of N^* is of the form $\lambda I + n$ with n nilpotent. Now we have $\operatorname{tr}(n)=0$, and $\operatorname{tr}(I)=\dim N^* = \dim N$. Thus for k to be a direct summand of $M\otimes N$ (i.e., for η to be nonzero) we must have $p\nmid\dim N$. Moreover, we must have elements $\alpha\in\operatorname{Hom}_{kG}(N^*, M)$ and $\beta\in\operatorname{Hom}_{kG}(M, N^*)$ such that $\operatorname{tr}(\beta\circ\alpha)\neq 0$; namely, such that $\beta\circ\alpha$ is an isomorphism. Since M is indecomposable this means we must have $M\cong N^*$. Moreover, in the case where $p\nmid\dim N$ and $M\cong N^*$, it is clear that $\eta(\beta\circ\alpha)\neq 0$ for any isomorphisms α and β.

Finally, suppose k is a component of $N^*\otimes N$ with multiplicity greater than one. Then the image of $p\circ i$ has dimension greater than one. This means that there are subspaces of $\operatorname{Hom}_{kG}(N^*, M)$ and $\operatorname{Hom}_{kG}(M, N^*)$ of dimension greater than one, on which η is a nonsingular pairing. Thus there is a subspace of $\operatorname{Hom}_{kG}(N^*, M)$ of dimension greater than one all of whose nonzero elements are isomorphisms, and this we know to be impossible. ∎

PROPOSITION 2.2. *Suppose that M is absolutely indecomposable and $p\mid\dim M$. Then for any module N and any component U of $M\otimes N$ we have $p\mid\dim U$.*

Proof. Suppose that $p\nmid\dim U$. Then by Theorem 2.1, k is a component of $U\otimes U^*$, and hence of $(M\otimes N)\otimes U^*\cong M\otimes(N\otimes U^*)$. But by Theorem 2.1 again, this implies that $p\nmid\dim M$ and that M^* is isomorphic to a summand of $N\otimes U^*$, contradicting the hypothesis. ∎

In the statements of Lemma 2.3 and Proposition 2.4, the notation $\mathrm{Sc}_G(D)$ is used to denote the Scott module for G with vertex D. For the definition and basic results on Scott modules, see [3]. We would like to thank Peter Landrock for suggesting that proposition 2.4 might be true.

LEMMA 2.3. *Suppose that M is a D-projective kG-module. Then M has a direct summand $N \cong \mathrm{Sc}_G(G)$ if and only if there is a kG-module homomorphism from M to k which splits on restriction to D.*

Proof. Since the natural map from $\mathrm{Sc}_G(D)$ to k splits on restriction to D, the "only if" part is clear. Conversely, if there is a kG-module homomorphism from M to k which splits on restriction to D, choose a component N of M with the same property. Since N is D-projective, N must have vertex exactly D and trivial source. Hence $N \cong \mathrm{Sc}_G(D)$. ▮

PROPOSITION 2.4. *Suppose that M is an absolutely indecomposable kG-module with vertex D and source S. Then $\mathrm{Sc}_G(D)$ is a direct summand of $M \otimes M^*$ if and only if $p \nmid \dim S$.*

Proof. Without loss of generality, we may assume that k is algebraically closed. First, suppose that $p \mid \dim S$. Then by Green's indecomposability theorem [10, Theorem 8], and the Mackey decomposition theorem, $p \mid \dim U$ for every component U of $S\uparrow^G\downarrow_D$. Thus by Theorem 2.1, k is not a direct summand of $S\uparrow^G\downarrow_D \otimes (S^*)\uparrow^G\downarrow_D$ and hence it is not a direct summand of $(M \otimes M^*)\downarrow_D$. Since k is a direct summand of $\mathrm{Sc}_G(D)\downarrow_D$, this implies that $\mathrm{Sc}_G(D)$ is not a direct summand of $M \otimes M^*$.

Conversely, suppose that $p \nmid \dim S$. Denote by $\alpha\colon k \to (M \otimes M^*)\downarrow_D$ the kD-module homomorphism corresponding to the composite map $M\downarrow_D \twoheadrightarrow S \hookrightarrow M\downarrow_D$. Since $p \nmid \dim S$, α is a D-splitting for the kG-module homomorphism $\mathrm{tr}\colon M \otimes M^* \to k$. Hence by Lemma 2.3, $\mathrm{Sc}_G(D)$ is a direct summand of $M \otimes M^*$. ▮

Motivated by Proposition 2.2, we make the following definitions.

DEFINITIONS. A kG-module M is *absolutely p-divisible* if, for every extension field k_1 of k, and every direct summand M_1 of $k_1 \otimes_k M$ (as a k_1G-module), $p \mid \dim_{k_1}(M_1)$.

Let $a(G;p)$ denote the linear span in $a(G)$ of the absolutely p-divisible kG-modules, and let

$$A(G;p) = a(G;p) \otimes_{\mathbb{Z}} \mathbb{C} \subseteq A(G).$$

Note that

$$a(G;p) = A(G;p) \cap a(G).$$

LEMMA 2.5. (i) *$a(G;p)$ is an ideal in $a(G)$.*

(ii) *$A(G;p)$ is an ideal in $A(G)$.*

Proof. This follows immediately from Proposition 2.2. ▮

LEMMA 2.6. *Suppose that $x = \sum a_i[M_i] \in A(G)$. Define $x^* = \sum \bar{a}_i[M_i^*]$. If $xx^* \in A(G;p)$ then $x \in A(G;p)$.*

Proof. If $xx^* = \sum |a_i|^2[M_i \otimes M_i^*] + \sum_{i \neq j} a_i\bar{a}_j[M_i \otimes M_j^*]$ does not involve the trivial module $[k]$, then by Theorem 2.1, each $[M_i]$ lies in $A(G;p)$. ▮

THEOREM 2.7. *For an arbitrary finite group G, the ring $A(G)/A(G;p)$ has no nonzero nilpotent elements.*

Proof. If $A(G)/A(G;p)$ has a nonzero nilpotent element, then there is a nonzero element $x \in A(G)$, not in $A(G;p)$, but with $x^2 \in A(G;p)$. Let $y = xx^*$. Then $yy^* = (xx^*)^2 \in A(G;p)$. Applying Lemma 2.6 twice, we deduce first that $y \in A(G;p)$, and then that $x \in A(G;p)$. ▮

Now, we wish to use the above results to give a new proof that $A(G)$ has no nilpotent elements in the case where G has cyclic Sylow subgroups; and more generally that $A(G, \mathrm{Cyc})$, the linear span in $A(G)$ of the modules with cyclic vertex,has no nilpotent elements.

Recall from [2] that a subgroup H of G is called *p-hypoelementary* if $H/O_p(H)$ is cyclic. We also use the notations $r_{G,H}$ and $i_{H,G}$ to denote the linear maps between representation rings given by restriction and induction of representations.

First, we must study the case of a p-hypoelementary group H with $O_p(H)$ cyclic.

PROPOSITION 2.8. *Assume that k is algebraically closed. Suppose that H is p-hypoelementary and $1 \neq D = O_p(H)$ is cyclic. Let H_1 be a subgroup of H of index p. Then*

$$A(H;p) = \mathrm{Im}(i_{H_1,H}).$$

Proof. Suppose that M is an indecomposable kH-module. Then M is uniserial and $M\downarrow_D$ is indecomposable. Let D_1 be the subgroup of D of index p. Since indecomposable kD-modules are simply Jordan blocks with eigenvalue one and length at most $|D|$, it is clear by Green's Indecomposability Theorem [10, Theorem 8] that such a module has dimension divisible by p if and only if it is induced from a kD_1-module. Thus M has dimension divisible by p if and only if it is H_1-projective. Hence it only remains to show that if N is an indecomposable kH_1-module then $N\uparrow^H$ is indecomposable. But this is clear since $N\uparrow^H\downarrow_D \cong N\downarrow_{D_1}\uparrow^D$ by the Mackey Decomposition Theorem. ▮

PROPOSITION 2.9. *If H is p-hypoelementary with $D = O_p(H)$ cyclic, then $A(H)$ has no nilpotent elements.*

Proof. Since extension of scalars gives an injection of Green rings [2, Proposition 5.1], we may assume without loss of generality that k is algebraically closed.

If $|D| = 1$ then H has order coprime to p, and the result follows from ordinary character theory. If $|D| > 1$, then by Proposition 2.8 and [2, Theorem 6.7]

$$A(H) = \operatorname{Im}(i_{H_1,H}) \oplus \operatorname{Ker}(r_{H,H_1})$$
$$= A(H;p) \oplus \operatorname{Ker}(r_{H,H_1})$$

as a direct sum of ideals. Assume inductively that the result is true for H_1. Then r_{H,H_1} maps $A(H;p)$ injectively into $A(H_1)$ and so $A(H;p)$ has no nilpotent elements. Since $A(H)/A(H;p)$ has no nilpotent elements by Theorem 2.7, the result follows. ∎

We pass from p-hypoelementary groups to arbitrary groups using a method of Conlon. If H is a subgroup of G, we denote by $A(G, H)$ the linear span in $A(G)$ of the H-projective kG-modules, and by $A'(G, H)$ the linear span in $A(G)$ of the modules which are K-projective for some proper subgroup K of H.

LEMMA 2.10 (Conlon [6]). *For any $H \leqslant G$, $A(G, H)$ is an ideal direct summand of $A(G)$. Contained in $A(G, H)$ there is a canonical ideal direct summand $A''(G, H)$ of $A(G)$, with the properties that*

$$A''(G, H) \cong A(G, H)/A'(G, H) \cong A(N_G(H), H)/A'(N_G(H), H)$$

and

$$A(G, H) = A'(G, H) \oplus A''(G, H).$$

This gives a direct sum decomposition

$$A(G, H) = \bigoplus_D A''(G, D).$$

In this decomposition, D runs through a set containing exactly one representative from each conjugacy class in G of p-subgroups of H. The map

$$r_{G,N_G(H)}\colon A''(G, H) \to A''(N_G(H), H)$$

is an isomorphism. ∎

PROPOSITION 2.11. *Let $\mathscr{H}$ be a set consisting of one representative of each conjugacy class of p-hypoelementary subgroups $H \leqslant G$ with $O_p(H)$ cyclic. Then the map*

$$A(G, \mathrm{Cyc}) \to \bigoplus_{H \in \mathscr{H}} A(H)$$

given by the sum of the restriction maps, is injective.

Proof. From Lemma 2.10, we have

$$A(G) = \bigoplus_{D} A''(G, D),$$

where D runs through a complete set of nonconjugate p-subgroups of G. Moreover $r_{G,N_G(D)}$: $A''(G, D) \to A''(N_G(D), D)$ is an isomorphism. If D is not cyclic, then $A''(G, D) \cap A(G, \mathrm{Cyc}) = \{0\}$, and so we may restrict our attention to D cyclic. The result now follows from the fact [8, Sect. 5] that the map

$$A''(N_G(D), D) \to \bigoplus_{\substack{H \in \mathscr{H} \\ O_p(H) = D}} A''(H, D)$$

given by the sum of the restriction maps, is injective. ∎

THEOREM 2.12. *For an arbitrary finite group G, the ideal $A(G, \mathrm{Cyc})$ has no nilpotent elements.*

Proof. This follows from Propositions 2.9 and 2.11. ∎

COROLLARY 2.13. *If G has cyclic Sylow p-subgroups then $A(G)$ has no nilpotent elements.*

Proof. In this case $A(G) = A(G, \mathrm{Cyc})$. ∎

Finally, we would like to point out that we can express the basic Theorem (2.1) in terms of almost split sequences. Since Proposition 2.15 is false in greater generality, we shall assume for the rest of section two that k is algebraically closed. We start with a preliminary discussion of short exact sequences.

Suppose $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ are short exact sequences with $A \cong A'$, $B \cong B'$, and $C \cong C'$. Then it is not necessarily true that there is an isomorphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\sigma} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| \wr & & \| \wr & & \| \wr & & \\
0 & \longrightarrow & A' & \xrightarrow{\tau} & B' & \longrightarrow & C' & \longrightarrow & 0.
\end{array}$$

For example, we could let $G = V_4 = \langle x, y \mid x^2 = y^2 = (xy)^2 = 1 \rangle$, $B = B' = \Omega^2(k)$, and $A \cong A' \cong k$. The socle of $\Omega^2(k)$ has dimension two. Choose any basis v_1, v_2 of the socle and let $\sigma(A) = kv_1$, $\tau(A') = kv_2$. Then $C = B/\sigma(A) \cong k \oplus \Omega(k) \cong B'/\tau(A') = C'$. But because $\Omega^2(k)$ is absolutely indecomposable, any automorphism of $\Omega^2(k)$ fixes the submodules $\sigma(A)$ and $\tau(A')$.

For split and almost split sequences, however, the situation is different. It follows by counting dimensions of spaces of homomorphisms to C, that if $0 \to A \to B \to C \to 0$ is split (resp. almost split), then $0 \to A' \to B' \to C' \to 0$ is also split (resp. almost split).

Suppose $0 \to \Omega(k) \to X \to \Omega^{-1}(k) \to 0$ is the almost split sequence with $X \cong P_1 \oplus \mathrm{Rad}(P_1)/\mathrm{Soc}(P_1)$ (here, P_1 is the projective cover of the trivial one-dimensional kG-module k, Ω denotes the Heller operation of taking the kernel of the projective cover, and Ω^{-1} denotes the dual operation of taking the cokernel of the injective hull). Tensoring with an indecomposable module M, we obtain a short exact sequence

$$0 \to \Omega(k) \otimes M \to X \otimes M \to \Omega^{-1}(k) \otimes M \to 0.$$

Since $\Omega(k) \otimes M \cong \Omega(M) \oplus$ projectives and $\Omega^{-1}(k) \otimes M \cong \Omega^{-1}(M) \oplus$ projectives, and since projective modules are injective, we may remove some projective summands from the sequence if necessary, to obtain a short exact sequence

$$0 \to \Omega(M) \to Y \to \Omega^{-1}(M) \to 0. \tag{2.14}$$

PROPOSITION 2.15. *The sequence* (2.14) *is always either split or almost split. It is split if and only if* $p \mid \dim M$.

Proof. Let $\alpha = [X] - [\Omega(k)] - [\Omega^{-1}(k)]$ as an element of $A(G)$. Then by [2, Theorem 3.5], for N indecomposable

$$\langle \alpha, [N] \rangle = \begin{cases} 1 & \text{if } N \cong k \\ 0 & \text{otherwise} \end{cases}$$

(the bilinear form $\langle\ ,\ \rangle$ is the one introduced in [2, Sect. 2]).

Thus by Theorem 2.1,

$$\langle \alpha, [M^* \otimes N] \rangle = \begin{cases} 1 & \text{if } M \cong N \text{ and } p \nmid \dim M \\ 0 & \text{otherwise.} \end{cases}$$

But $\langle \alpha, [M^* \otimes N] \rangle = \langle \alpha \cdot [M], [N] \rangle$, and so by [2, Proposition 3.7],

$\alpha \cdot [M]$ is equal to the difference of the almost split sequence terminating in $\Omega^{-1}(M)$ if $p \nmid \dim M$, and zero if $p \mid \dim M$. The result now follows from the above discussion of short exact sequences. ∎

Remark. One may also prove Theorem 2.1 by first proving Proposition 2.15 and then using the above argument backwards. For further details, see [1].

EXAMPLE 2.16. The following example indicates why M needs to be *absolutely* indecomposable in Theorem 2.1, and why we take k algebraically closed for Proposition 2.15.

Let $G = A_4$, the alternating group on four letters, $k = \mathbb{F}_2$, the field of two elements, and M be the irreducible two-dimensional module (which splits upon extension to $\mathbb{F}_4$ as a sum of two one-dimensional modules). Then

$$M^* \otimes M \cong k \oplus k \oplus M,$$

and in the notation of Proposition 2.15,

$$\langle \alpha, [M^* \otimes M] \rangle = 2.$$

EXAMPLE 2.17. The following example shows that the ring $A(G)/A(G;p)$ discussed in Theorem 2.17 can be quite complicated in structure.

Let $G = \langle x, y \mid x^p = y^p = [x, y] = 1 \rangle \cong C_p \times C_p$, p odd, and let k be an infinite field of characteristic p. Let

$$M = M_\alpha = kG/\langle (x-1)(y-1), (y-1)^2 - \alpha(x-1)^{p-1} \rangle \qquad (\alpha \in k),$$

so that $\dim M = p+1$. By direct computation, it can be shown that $M \cong M^*$, and $M\downarrow_{\langle x \rangle} \cong k \oplus F$, where F is a free $k\langle x \rangle$-module. We apply the following lemmas to this situation.

LEMMA 2.18. *Suppose M is an indecomposable kG-module such that for some element x of order p in G, $M\downarrow_{\langle x \rangle} \cong k \oplus F$, where F is a free $k\langle x \rangle$-module. Then $[M] + a(G;p)$ is a unit in $a(G)/a(G;p)$, with inverse $[M^*] + a(G;p)$.*

Proof. $(M \otimes M^*)\downarrow_{\langle x \rangle} \cong k \oplus F \oplus F^* \oplus F \otimes F^*$. Also $M \otimes M^* \cong k \oplus L$ for some kG-module L. Hence L is free as a $k\langle x \rangle$-module and so every component of L is in $a(G;p)$. ∎

Applying this lemma to our situation, we find that

$$([M_\alpha] + a(G;p))^2 = [k] + a(G;p).$$

Also $M_\alpha \not\cong M_\beta$ if $\alpha \neq \beta$, and so in this case $a(G)/a(G;p)$ has an infinite number of units of order 2. Note that we may also use this to manufacture idempotents in $A(G)/A(G;p)$ since if $u^2=1$ in $A(G)/A(G;p)$ then $\frac{1}{2}(1-u)$ and $\frac{1}{2}(1+u)$ are orthogonal idempotents.

3. Existence of Nilpotent Elements

In this section we describe a general method for producing nilpotent elements in the Green ring $a(RG)$. The method appears to have some validity for any commutative ring R, but for convenience we assume that R is an integral domain. In Section 4 we give specific examples of nilpotent elements using this method. We begin with a general result, which is a generalization of Schanuel's lemma.

Proposition 3.1. *Let A be a ring with a unit element. Suppose that*

$$0 \longrightarrow W_1 \longrightarrow U_1 \xrightarrow{\sigma} V \longrightarrow 0$$

and

$$0 \longrightarrow W_2 \longrightarrow P_2 \xrightarrow{\mu} V \longrightarrow 0$$

are exact sequences of A-modules such that P_2 is a projective A-module and the homomorphism σ factors through a projective A-module. Then

$$W_1 \oplus P_2 \cong U_1 \oplus W_2.$$

Proof. By hypothesis there exists a projective A-module P_1 and homomorphisms $\alpha\colon U_1 \to P_1$, $\tau\colon P_1 \to V$ such that $\tau \circ \alpha = \sigma$. Since μ is onto and P_1 is projective, there exists a map $\beta\colon P_1 \to P_2$ with $\mu \circ \beta = \tau$. Now construct the pull-back diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & W_2 \arrow[rr, equal] \arrow[d] & & W_2 \arrow[d] & \\
0 \arrow[r] & W_1 \arrow[r, "\zeta"] \arrow[dd, equal] & X \arrow[rr] \arrow[dd, "\delta"'] & & P_2 \arrow[r] \arrow[dd, "\mu"] \arrow[ll, dashed, bend right, "\gamma"'] & 0 \\
 & & & P_1 \arrow[ur, "\beta"] \arrow[dr, "\tau"] & & \\
0 \arrow[r] & W_1 \arrow[r, "\varepsilon"] & U_1 \arrow[rr, "\sigma"] \arrow[ur, "\alpha"] \arrow[d] & & V \arrow[r] \arrow[d] & 0 \\
 & & 0 & & 0 &
\end{tikzcd}$$

Because P_2 is projective there exists a splitting homomorphism $\gamma: P_2 \to X$. Note that $\sigma = \sigma\delta\gamma\beta\alpha$. So let $\theta: U_1 \to X$ be defined by $\theta = \gamma\beta\alpha + (\zeta\varepsilon^{-1})(1 - \delta\gamma\beta\alpha)$. Then $\delta\theta$ is the identity on U_1, and so θ is a splitting for δ. Hence

$$U_1 \oplus W_2 \cong X \cong W_1 \oplus P_2. \quad \blacksquare$$

Suppose that R is an integral domain such that the prime p is not a unit in R. Let G be a finite group with $p \mid |G|$. For any RG-lattice M, let $\tilde{\Omega}^n(M)$ be an RG-lattice of minimal rank such that there exists an exact sequence

$$0 \to \tilde{\Omega}^n(M) \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where $F_0, \ldots, F_{n-1}$ are free RG-lattices. We insist that $F_0, \ldots, F_{n-1}$ be free lattices rather than just projective for the sake of convenience. That is, it makes the statements of the results easier. In case R is a field or a complete discrete valuation ring, the RG-lattices satisfy the Krull–Schmidt Theorem, and we may substitute $\Omega^n(M)$ for $\tilde{\Omega}^n(M)$. In any case $\tilde{\Omega}^n(M)$ has a sort of stable uniqueness in the following sense:

LEMMA 3.2. *Suppose that*

$$0 \to N \to E_{n-1} \to \cdots \to E_0 \to M \to 0$$

is a long exact sequence of RG-lattices with $E_0, \ldots, E_{n-1}$ free. Then there exist free RG-lattices P_1 and P_2 such that $N \oplus P_1 \cong \tilde{\Omega}^n(M) \oplus P_2$. Moreover, if $\operatorname{Rank}(N) = \operatorname{Rank}(\tilde{\Omega}^n(M))$ *then* $[N] = [\tilde{\Omega}^n(M)]$ *as elements of $a(RG)$.*

Proof. The extended Schanuel lemma [16, (1.4)] states that

$$N \oplus F_{n-1} \oplus E_{n-2} \oplus \cdots \cong \tilde{\Omega}^n(M) \oplus E_{n-1} \oplus F_{n-2} \oplus \cdots.$$

The last statement follows because the rank condition implies that $P_1 \cong P_2$. Therefore

$$[N] = [N \oplus P_1] - [P_1] = [\tilde{\Omega}^n(M) \oplus P_2] - [P_2] = [\tilde{\Omega}^n(M)]. \quad \blacksquare$$

THEOREM 3.3 (see Carlson [5]). *Let $\zeta: \tilde{\Omega}^n(R) \to R$ be an epimorphism with kernel L_ζ. Let $\hat{\zeta}$ be the cohomology class in* $\operatorname{Ext}^n_{RG}(R, R)$ *represented by ζ. Suppose that M is an RG-lattice such that $\hat{\zeta}$ annihilates* $\operatorname{Ext}^*_{RG}(M, M)$ *(with the cup-product action of* $\operatorname{Ext}^*_{RG}(R, R)$ *on* $\operatorname{Ext}^*_{RG}(M, M)$*). Then there exist free RG-lattices P_1 and P_2 such that*

$$(L_\zeta \otimes M) \oplus P_1 \cong \tilde{\Omega}^n(M) \oplus \tilde{\Omega}(M) \oplus P_2.$$

Proof. Suppose that

$$0 \to \tilde{\Omega}^n(R) \to F_{n-1} \to \cdots \to F_0 \to R \to 0$$

is a long exact sequence of RG-lattices with $F_0, \ldots, F_{n-1}$ free. Let $I \in \operatorname{Ext}^0_{RG}(M, M) = \operatorname{Hom}_{RG}(M, M)$ be the identity homomorphism. Then the product $\hat{\zeta} I$ is represented by the exact sequence which is the pushout of the diagram

$$\begin{array}{l} 0 \to \tilde{\Omega}^n(R) \otimes M \to F_{n-1} \otimes M \to \cdots \to F_0 \otimes M \to M \to 0 \\ \qquad\qquad \Big\downarrow {\scriptstyle \zeta \otimes I} \\ \qquad\qquad M \end{array}$$

(see [12]). But $\hat{\zeta} I = 0$ and hence $\zeta \otimes I$ is a coboundary. This means that it factors through the projective module $F_{n-1} \otimes M$. Hence we have exact sequences

$$0 \to \tilde{\Omega}(M) \to E \to M \to 0$$

and

$$0 \to L_\zeta \otimes M \to \tilde{\Omega}^n(R) \otimes M \xrightarrow{\zeta \otimes I} M \to 0,$$

where E is a free RG-module and $\zeta \otimes 1$ factors through a projective. Applying Proposition 3.1 we obtain

$$(L_\zeta \otimes M) \oplus E \cong \tilde{\Omega}(M) \oplus (\tilde{\Omega}^n(R) \otimes M).$$

By Lemma 3.2 there exist free RG-lattices Q_1 and Q_2 with

$$(\tilde{\Omega}^n(R) \otimes M) \oplus Q_1 \cong \tilde{\Omega}^n(M) \oplus Q_2.$$

This proves the theorem. ▌

We are now ready to set up the method for producing nilpotent elements in the Green ring.

THEOREM 3.4. *Let $\zeta: \tilde{\Omega}^{2n}(R) \to R$ $(n > 0)$ be an epimorphism with kernel L_ζ. Suppose that L_ζ satisfies the following three conditions.*

(i) *For some $m > 0$, there exist free RG-lattices $F_0, \ldots, F_{2m-1}$ and a long exact sequence*

$$0 \to L_\zeta \to F_{2m-1} \to \cdots \to F_0 \to L_\zeta \to 0.$$

That is, L_ζ is periodic with even period $2m$.

(ii) $\sum_{i=1}^{2m}(-1)^i \operatorname{rank}(\tilde{\Omega}^i(L_\zeta))=0$.

(iii) *The cohomology class* ζ *annihilates* $\operatorname{Ext}^*_{RG}(L_\zeta, L_\zeta)$.

Then in the Green ring $\sum_{i=1}^{2m}(-1)^i[\tilde{\Omega}^i(L_\zeta)]$ *is a nilpotent element whose square is zero, while* $(1/4m)\sum_{i=1}^{2m}[\tilde{\Omega}^i(L_\zeta)]-F$ *is idempotent, provided* F *is a suitable multiple of* $[RG]$ *to ensure that the total rank is zero.*

Proof. By Theorem 3.3 and condition (iii) there exist free RG-lattices P_1 and P_2 such that

$$(L_\zeta \otimes L_\zeta) \oplus P_1 \cong \tilde{\Omega}^{2n}(L_\zeta) \oplus \tilde{\Omega}(L_\zeta) \oplus P_2.$$

Now suppose that

$$0 \to \tilde{\Omega}(L_\zeta) \to E \to L_\zeta \to 0$$

is a short exact sequence of RG-lattices with E free. Then

$$0 \to \tilde{\Omega}(L_\zeta) \otimes L_\zeta \to E \otimes L_\zeta \to L_\zeta \otimes L_\zeta \to 0$$

is exact and $E \otimes L_\zeta$ is free. Thus there exist free RG-lattices P_3 and P_4 such that

$$(\tilde{\Omega}(L_\zeta) \otimes L_\zeta) \oplus P_3 \cong \tilde{\Omega}^{2n+1}(L_\zeta) \oplus \tilde{\Omega}^2(L_\zeta) \oplus P_4.$$

Continuing in this way, we find that there are free RG-lattices P_{ij} and Q_{ij} such that

$$(\tilde{\Omega}^i(L_\zeta) \otimes \tilde{\Omega}^j(L_\zeta)) \oplus P_{ij} \cong \tilde{\Omega}^{2n+i+j}(L_\zeta) \oplus \tilde{\Omega}^{1+i+j}(L_\zeta) \oplus Q_{ij}.$$

Hence working modulo free modules in $a(RG)$ we have

$$\begin{aligned}
&\left(\sum_{i=1}^{2m}(-1)^i[\tilde{\Omega}^i(L_\zeta)]\right)^2 \\
&\quad = \sum_{i=1}^{2m}\sum_{j=1}^{2m}(-1)^{i+j}[\tilde{\Omega}^i(L_\zeta) \otimes \tilde{\Omega}^j(L_\zeta)] \\
&\quad = \sum_{i=1}^{2m}\sum_{j=1}^{2m}(-1)^{i+j}([\tilde{\Omega}^{2n+i+j}(L_\zeta)] + [\tilde{\Omega}^{1+i+j}(L_\zeta)]).
\end{aligned}$$

By the periodicity statement (i), this is equal to

$$2m\left(\sum_{k=1}^{2m}(-1)^k[\tilde{\Omega}^{2n+k}(L_\zeta)] + \sum_{k=1}^{2m}(-1)^k[\tilde{\Omega}^{1+k}(L_\zeta)]\right).$$

Again by periodicity, and the fact that $2n-1$ is odd, this expression is equal to zero.

If we put back in the free modules, then by condition (ii) we still get zero, and the first statement is proved. The second statement follows from an entirely analogous calculation. ∎

Remarks. (i) In the applications (see Sect. 4) of this theorem, we shall only be using the case $n=1$. In this case the result takes on the form

$$([L_\zeta]-[\tilde{\Omega}(L_\zeta)])^2=[L_\zeta\otimes L_\zeta]-2[L_\zeta\otimes\tilde{\Omega}(L_\zeta)]+[\tilde{\Omega}(L_\zeta)\otimes\tilde{\Omega}(L_\zeta)]=0.$$

(ii) In fact it follows from the conditions of Theorem 3.4 that in the case where R is a field of characteristic p, G must be a group of p-rank at most two (i.e., G has no elementary abelian subgroup of order p^3). The method may be used to construct nilpotent elements in the case where G is an abelian group of larger rank, for example, by putting some of the group into the kernel of the representations used.

(iii) It is interesting to note that for $R=k$, the nilpotent elements constructed in Theorem 3.4 all lie in the subspace Rad dim Ext^n_{kG} of $A(kG)$ investigated in Sect. 4 of [2]. It is not true in general that all nilpotent elements lie in this subspace, as may be observed by using the device mentioned in the second remark, that is, the inclusion $A(G/N)\hookrightarrow A(G)$.

4. Examples

In Section 3 we outlined a general method for finding nilpotent elements in $a(G)$. It remains to show that there exist modules L_ζ which satisfy the conditions of Theorem 3.4. In this section we produce a large class of examples satisfying the conditions. We work only with the group $G=\mathbb{Z}/(p^r)\times\mathbb{Z}/(p^s)$, for r and s positive integers. A similar thing can be done for the dihedral 2-groups. In the case of a quaternion 2-group, not many examples can be constructed by our method since $\Omega^{4n}(R)\cong R$ for all $n>0$. The details of some of the calculations are straightforward but rather lengthy and hence in some cases we shall give only sketches of the proofs.

Let $G=\langle x,y\mid x^{p^r}=y^{p^s}=[x,y]=1\rangle$ and let R be an integral domain in which p is not a unit. Define projective resolutions of R with $R\langle x\rangle$-lattices and $R\langle y\rangle$-lattices.

$$\cdots\longrightarrow U_2\xrightarrow{\partial_2'}U_1\xrightarrow{\partial_1'}U_0\xrightarrow{\varepsilon'}R\longrightarrow 0 \qquad (U,\varepsilon')$$

$$\cdots\longrightarrow V_2\xrightarrow{\partial_2''}V_1\xrightarrow{\partial_1''}V_0\xrightarrow{\varepsilon''}R\longrightarrow 0 \qquad (V,\varepsilon'')$$

as follows. Let $U_i=R\langle x\rangle\cdot u_i\cong R\langle x\rangle$, $V_j=R\langle y\rangle\cdot v_j\cong R\langle y\rangle$. Let ε' and

ε'' be the augmentation maps $\varepsilon'(u_0)=1$, $\varepsilon''(v_0)=1$. The boundary homomorphisms are given by

$$\partial_i'(u_i)=\begin{cases}(x-1)\,u_{i-1} & \text{for } i \text{ odd}\\ N_x\,u_{i-1} & \text{for } i \text{ even,}\end{cases}$$

$$\partial_j''(v_j)=\begin{cases}(y-1)\,v_{j-1} & \text{for } j \text{ odd}\\ N_y\,v_{j-1} & \text{for } j \text{ even}\end{cases}$$

where $N_x=\sum_{i=1}^{p^r} x^i$ and $N_y=\sum_{j=1}^{p^s} y^j$. Now form the double complex $(W,\varepsilon)=(U,\varepsilon')\otimes(V,\varepsilon'')$ where $\varepsilon=\varepsilon'\otimes\varepsilon'', W_n=\bigoplus_{i+j=n} U_i\otimes V_j$ and $\partial_n\colon W_n\to W_{n-1}$ is given by

$$\partial_n(u_i\otimes v_j)=\partial_i'(u_i)\otimes v_j+(-1)^i\,u_i\otimes\partial_j''(v_j)$$

for $u_i\in U_i$, $v_j\in V_j$, $i+j=n$. Then by the Künneth tensor formula, (W,ε) is a projective resolution of R with RG-lattices. It is in fact the tensor product of R with the usual $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Moreover, if $\mathscr{P}$ is a maximal ideal of R containing (p), and $k=R/\mathscr{P}$, then $(k\otimes_R W, 1\otimes\varepsilon)$ is a minimal projective resolution of k with kG-modules. Hence the following result is clear.

LEMMA 4.1. *W_n is a free RG-module with RG-basis $\{w_{ij}\mid i+j=n\}$ where $w_{ij}=u_i\otimes v_j$. Let $\Omega^n(R)$ denote the kernel of $\partial_{n-1}\colon W_{n-1}\to W_{n-2}$. Then $\Omega^{2n}(R)$ is generated as an RG-module by the elements $a_0,\ldots,a_{2n}$ where*

$$a_{2i}=\partial_{2n}(w_{2i,2n-2i})=N_x w_{2i-1,2n-2i}+N_y w_{2i,2n-2i-1}\qquad (0\leqslant i\leqslant n),$$

$$a_{2i+1}=\partial_{2n}(w_{2i+1,2n-2i-1})=(x-1)\,w_{2i,2n-2i-1}-(y-1)\,w_{2i+1,2n-2i-2}$$
$$(0\leqslant i\leqslant n-1).$$

Moreover these elements also satisfy the relations

$$(x-1)\,a_{2i}=N_y\,a_{2i+1},\qquad (y-1)\,a_{2i}=-N_x a_{2i-1}.$$

(For notational convenience we are assuming that $w_{i,n-i}$ and a_i are zero whenever $i<0$ or $i>n$.)

If i and j are nonnegative integers with $i+j=n$, then let $\phi_{i,j}\colon W_{2n}\to R$ be given by

$$\phi_{i,j}(w_{k,l})=\begin{cases}1 & \text{if } (k,l)=(2i,2j)\\ 0 & \text{otherwise.}\end{cases}$$

That is, $\phi_{i,j}$ is the augmentation map on $RG\cdot w_{i,j}$. Let ζ_1 be the cohomology class of $\phi_{1,0}$ in $\mathrm{Ext}^2_{RG}(R,R)$ and let ζ_2 be the class of $\phi_{0,1}$. It is easy to check that the cohomology element $\zeta_1^i\cdot\zeta_2^j$ is represented by the cocycle $\phi_{i,j}\colon W_{2n}\to R$ for $i+j=2n$.

Let α be any element of R. Let $\zeta = \alpha\zeta_1 + \zeta_2 \in \mathrm{Ext}^2_{RG}(R, R)$ and $\gamma = \zeta^n = \sum_{k=0}^n \binom{n}{k} \alpha^k \zeta_1^k \zeta_2^{n-k} = \mathrm{cls}(\phi)$, where

$$\phi = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \phi_{2k, 2(n-k)}: \quad W_{2n} \to R.$$

Note that ϕ is a cocycle, and hence we have an induced homomorphism $\hat{\phi}: \Omega^{2n}(R) \to R$ given by $\hat{\phi}(a_j) = \binom{n}{j} \alpha^j$. The kernel L_γ of $\hat{\phi}$ is generated as an RG-module by the elements

$$b_i = a_{2i} - \binom{n}{i} \alpha^i a_0$$

$$c_i = a_{2i-1} \qquad (1 \leqslant i \leqslant n).$$

LEMMA 4.2. *The RG-lattice L_γ is indecomposable.*

Proof. The easiest proof of the lemma requires several results from the theory of varieties for modules (see [4] or [5]). Consider $M = L_\gamma \otimes_R \hat{k}$, where $\hat{k}$ is the algebraic closure of $R/\mathscr{P}$ and $\mathscr{P}$ is a maximal ideal in R containing (p). The point is that M is the kernel of $\gamma' = \gamma \otimes 1$: $\tilde{\Omega}^{2n}(R) \otimes_R \hat{k} \to \hat{k}$, and γ' can be regarded as an element of $\mathrm{Ext}^{2n}_{\hat{k}G}(\hat{k}, \hat{k})$. If $p > 2$ then $\gamma' = \theta^n$ for some irreducible element $\theta \in \mathrm{Ext}^2_{\hat{k}G}(\hat{k}, \hat{k})$. If $p = 2$ then $\gamma' = \theta^{2n}$ for some $\theta \in \mathrm{Ext}^1_{\hat{k}G}(\hat{k}, \hat{k})$. In either case the variety of M is the same as the variety of the ideal generated by θ, and this is a connected subvariety of the variety of G. Therefore by Lemma 4.1 of [5], M is indecomposable. ∎

LEMMA 4.3. *There exist free RG-lattices F_0, F_1, each on $2n$ generators, and an exact sequence*

$$0 \longrightarrow L_\gamma \xrightarrow{\theta} F_1 \xrightarrow{\tau} F_0 \xrightarrow{\sigma} L_\gamma \longrightarrow 0.$$

Proof. Let

$$F_0 = \sum_{i=1}^{2n} RG \cdot \beta_i, \qquad F_1 = \sum_{i=1}^{2n} RG \cdot \gamma_i$$

be free RG-lattices with RG-bases $\beta_1, ..., \beta_{2n}$ and $\gamma_1, ..., \gamma_{2n}$, respectively. Then we define θ, τ and σ as follows.

$$\theta(b_i) = N_x \gamma_{2i} + N_y \gamma_{2i+1} - \binom{n}{i} \alpha^i N_y \gamma_1 \qquad (1 \leqslant i \leqslant n-1),$$

$$\theta(b_n) = N_x \gamma_{2n} - \alpha^n N_y \gamma_1,$$

$$\theta(c_i) = (x-1)\gamma_{2i-1} - (y-1)\gamma_{2i} \qquad (1 \leqslant i \leqslant n),$$

$$\tau(\gamma_{2i-1}) = d_i = (y-1)\,\beta_{2i} + N_x\beta_{2i-1} \qquad (1 \leqslant i \leqslant n),$$

$$\tau(\gamma_{2i}) = e_i = (x-1)\,\beta_{2i} + N_y\left(-\beta_{2i+1} + \binom{n}{i}\alpha^i\beta_1\right) \qquad (1 \leqslant i \leqslant n-1),$$

$$\tau(\gamma_{2n}) = e_n = (x-1)\,\beta_{2n} - \alpha^n N_y\beta_1,$$

$$\sigma(\beta_{2i-1}) = c_i \qquad (1 \leqslant i \leqslant n),$$

$$\sigma(\beta_{2i}) = b_i \qquad (1 \leqslant i \leqslant n). \quad \blacksquare$$

It is not always true that $L_\gamma \not\cong \Omega(L_\gamma)$ (see [9]). In the following lemma, we list some cases where it is true.

LEMMA 4.4. *In each of the following cases* $[L_\gamma] \neq [\Omega(L_\gamma)]$ *as elements of* $a(RG)$.

(i) $p > 2$.

(ii) $p = 2$, $r > 1$ *and* $s > 1$.

(iii) $p = 2$, $r > 1$ *and* $\alpha = 0$.

(iv) $p = 2$, $r = s = 1$, $\alpha \in \{0, 1\}$ *and R has characteristic zero.*

Proof. The first three cases may be proved by reduction modulo a maximal ideal $\mathscr{P}$ containing (p). Let $\hat{k}$ be the algebraic closure of $R/\mathscr{P}$ and let $M = L_\gamma \otimes_R \hat{k}$. Then we have an exact sequence

$$0 \to M \to \Omega^{2n}(\hat{k}) \xrightarrow{\gamma'} \hat{k} \to 0,$$

where $\gamma' = \gamma \otimes 1$: $\Omega^{2n}(\hat{k}) \cong \tilde{\Omega}^{2n}(R) \otimes \hat{k} \to \hat{k}$. For convenience we identify M with its image in $\Omega^{2n}(\hat{k})$. Clearly it is sufficient to show that $M \not\cong \Omega(M)$.

In case (i) let $x_0 = x^{p^{r-1}}$, $y_0 = y^{p^{s-1}}$ and $u = 1 + (x_0 - 1) + (\bar{\alpha})^{1/p}(y_0 - 1) \in \hat{k}G$, where $\bar{\alpha}$ is the image of α under the homomorphism $R \to \hat{k}$. Then u is a unit of order p in $\hat{k}G$ and $\mathrm{res}_{G,\langle u\rangle}(\gamma') = 0$ (see Proposition 2.20 of [4]). Since $\Omega^{2n}(\hat{k})\downarrow_{\langle u\rangle} \cong \hat{k}_{\langle u\rangle} \oplus (\mathrm{proj})$, we have that $M\downarrow_{\langle u\rangle} \cong \hat{k}_{\langle u\rangle} \oplus \Omega(\hat{k}_{\langle u\rangle}) \oplus (\mathrm{proj})$ where $\Omega(\hat{k}_{\langle u\rangle}) \not\cong \hat{k}_{\langle u\rangle}$ is the uniserial $\hat{k}\langle u\rangle$-module of dimension $p-1$ (>1). Let $a \in \Omega^{2n}(\hat{k})$ be a generator for the component of M isomorphic to $\hat{k}_{\langle u\rangle}$. Then $\gamma'(a) = 0$. On the other hand, there does exist an element $\theta \in \mathrm{Ext}^{2n}_{\hat{k}G}(\hat{k}, \hat{k})$ such that $\mathrm{res}_{G,\langle u\rangle}(\theta) \neq 0$. Consequently $\theta(a) \neq 0$ and $a \notin \mathrm{Rad}_{\hat{k}G}(\Omega^{2n}(\hat{k})) \supseteq \mathrm{Rad}_{\hat{k}G}(M)$. So there exists a $\hat{k}G$-homomorphism σ: $M \to \hat{k}$ which is $\hat{k}\langle u\rangle$-split. We claim further that a is not in the socle of M. For otherwise σ would be $\hat{k}G$-split and $\hat{k}$ would be a direct summand of M. This is impossible because M is periodic. We see then that there exists no $\hat{k}G$-homomorphism $\tau : \hat{k} \to M$ which is $\hat{k}\langle u \rangle$-split.

Now consider $\Omega(M) \cong \Omega^{-1}(M)$. Let P be the projective cover of $\Omega^{2n-1}(\hat{k})$. The following diagram is commutative with exact rows and columns.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \hat{k} \arrow[d, "\phi"'] & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & \Omega^{-1}(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{2n}(\hat{k}) \arrow[r] \arrow[d, "\gamma'"] & P \arrow[r] & \Omega^{2n-1}(\hat{k}) \arrow[r] \arrow[d] & 0 \\
 & \hat{k} \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Note that $\Omega^{2n-1}(\hat{k}) \downarrow_{\langle u \rangle} \cong \Omega(\hat{k}_{\langle u \rangle}) \oplus (\text{proj})$ and $\Omega^{-1}(M) \downarrow_{\langle u \rangle} \cong \hat{k}_{\langle u \rangle} \oplus \Omega(\hat{k}_{\langle u \rangle}) \oplus (\text{proj})$. So ϕ must be $\hat{k}\langle u \rangle$-split, because the only non-split extension of $\hat{k}_{\langle u \rangle}$ by $\Omega(\hat{k}_{\langle u \rangle})$ is the free module. Hence by the previous paragraph $M \not\cong \Omega^{-1}(M)$. This proves case (i).

In case (ii) let $u = 1 + (x_1 - 1) + (\bar{\alpha})^{1/4}(y_1 - 1)$ where $x_1 = x^{2^{r-2}}$, $y_1 = y^{2^{s-2}}$. In case (iii) let $u = x$. In both cases, $\operatorname{res}_{G,\langle u \rangle}(\gamma') = 0$ and, since $u^2 \neq 1$, $\Omega(\hat{k}_{\langle u \rangle}) \not\cong \hat{k}_{\langle u \rangle}$. Hence the same argument works.

In case (iv) we proceed as follows. Let $\hat{R}$ denote the $\mathscr{P}$-adic completion of R for some maximal ideal $\mathscr{P}$ lying above (p) in R. Then we have a Krull–Schmidt theorem for $\hat{R}G$-modules, and so it is sufficient to prove that $L_\gamma \otimes_R \hat{R} \not\cong \Omega(L_\gamma) \otimes_R \hat{R}$. But if $M = (x+1)(L_\gamma \otimes_R \hat{R})$ then $(M/\mathscr{P}M)\downarrow_{\langle y \rangle}$ is a direct sum of trivial $k\langle y \rangle$-modules. However, the same is not true of $M' = (x+1)(\Omega(L_\gamma) \otimes_R \hat{R})$. ▮

LEMMA 4.5. *For $\gamma = \zeta^n$ as above, γ annihilates* $\operatorname{Ext}^*_{RG}(L_\gamma, L_\gamma)$.

Proof. Let $I \in \operatorname{Ext}^0_{RG}(L_\gamma, L_\gamma) = \operatorname{Hom}_{RG}(L_\gamma, L_\gamma)$ be the identity homomorphism. As noted previously it is sufficient to show that $\gamma \cdot I = 0$. We construct a lifting of projective resolutions as follows.

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$$\begin{array}{ccccccccc}
0 \longrightarrow & L_\gamma & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & L_\gamma & \longrightarrow 0 \\
& \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \| & \\
0 \longrightarrow & \Omega^2(R)\otimes L_\gamma & \longrightarrow & W_1\otimes L_\gamma & \longrightarrow & W_0\otimes L_\gamma & \xrightarrow{\varepsilon\otimes I} & L_\gamma & \longrightarrow 0 \\
& \downarrow{\scriptstyle \zeta\otimes 1} & & & & & & & \\
& L_\gamma & & & & & & &
\end{array}$$

F_0 and F_1 are as given in the proof of Lemma 4.3, and for $1 \leqslant i \leqslant n$ we set

$$\mu_0(\beta_{2i-1}) = w_{0,0} \otimes c_i,$$

$$\mu_0(\beta_{2i}) = w_{0,0} \otimes b_i,$$

$$\mu_1(\gamma_{2i-1}) = w_{0,1} \otimes yb_i + \sum_{l=1}^{p^r-1} \eta_{l,x}\, w_{1,0} \otimes x^l c_i,$$

$$\mu_1(\gamma_{2i}) = w_{1,0} \otimes xb_i - \sum_{k=1}^{p^s-1} \eta_{k,y} w_{0,1} \otimes y^k \left(c_{i+1} - \binom{n}{i} \alpha^i c_1 \right),$$

$$\begin{aligned}
\mu'(b_i) = {} & a_2 \otimes xb_i + a_0 \otimes yb_{i+1} - \binom{n}{i} \alpha^i a_0 \otimes yb_1 \\
& - \sum_{k=1}^{p^s-1} \sum_{l=1}^{p^r-1} \eta_{k,y}\eta_{l,x} a_1 \otimes x^l y^k c_{i+1} \\
& + \binom{n}{i} \alpha^i \sum_{k=1}^{p^s-1} \sum_{l=1}^{p^r-1} \eta_{k,y}\eta_{l,x}\, a_1 \otimes x^l y^k c_1 \\
& - a_2 \otimes N_y c_{i+1} + \binom{n}{i} \alpha^i a_2 \otimes N_y c_1 + a_0 \otimes N_x c_{i+1} \\
& - \binom{n}{i} \alpha^i a_0 \otimes N_y c_1,
\end{aligned}$$

$$\mu'(c_i) = a_1 \otimes xyb_1 + a_0 \otimes \left(c_{i+1} - \binom{n}{i} \alpha^i c_1 \right) + a_2 \otimes c_i,$$

where $\eta_{l,x} = 1 + x + \cdots + x^{l-1}$ and a_0, a_1, a_2 are generators for $\Omega^2(R)$ as in Lemma 4.1, and $c_{i+1} = 0$ for $i = n$.

Now ζI is represented by the cocycle $(\zeta \otimes I) \circ \mu' = f$, and $\gamma \cdot I = \zeta^n \cdot I$ is represented by f^n. Consequently it is sufficient to show that $f^n = 0$. From the above expression for μ' we may conclude that for $1 \leqslant i \leqslant n$,

$$f(b_i) = \alpha b_i + b_{i+1} - \binom{n}{i} \alpha^i b_1$$

and

$$f(c_i) = \alpha c_i + c_{i+1} - \binom{n}{i} \alpha^i c_1.$$

Hence the matrix of f on the space generated by $b_1, ..., b_n$ is

$$A = \begin{pmatrix} \alpha - n\alpha & -\binom{n}{2}\alpha^2 & \cdots & -\binom{n}{n}\alpha^n \\ 1 & \alpha & & \bigcirc \\ & 1 & \ddots & \\ \bigcirc & & 1 & \alpha \end{pmatrix}.$$

Writing I_n for the $n \times n$ identity matrix, we have $A = \alpha I_n + C$, where C is the companion matrix for the polynomial $(X + \alpha)^n$. By the Cayley–Hamilton Theorem, C satisfies its characteristic polynomial and so $A^n = (C + \alpha I)^n = 0$. The matrix of f on the space generated by $c_1, ..., c_n$ is also A, and so we have $f^n = 0$, and we are done. ∎

PROPOSITION 4.6. *Suppose that α, γ, L_γ are as above, and we are in one of the cases listed in Lemma* 4.4. *Then the element* $[L_\gamma] - [\Omega(L_\gamma)]$ *is a non-zero nilpotent element of* $a(RG)$ *of class* 2.

Proof. By Lemma 4.3, L_γ satisfies conditions (i) and (ii) of Theorem 3.4, with $m = 1$. By Lemma 4.5, condition (iii) is also satisfied, and so it follows from Theorem 3.4 that $([L_\gamma] - [\Omega(L_\gamma)])^2 = 0$. Moreover, it follows from Lemma 4.4 that in the cases listed, $[L_\gamma] - [\Omega(L_\gamma)] \neq 0$. ∎

Remarks. (i) The question of the existence of nilpotent elements in $A(kG)$ when G is an elementary abelian 2-group, which is still open for $n \geqslant 3$, may not be resolved using these techniques. The problem is that every periodic kG-module has period one [4, Lemma 8.1].

(ii) It is not always true that γ annihilates $\mathrm{Ext}^*_{RG}(L_\gamma, L_\gamma)$. For example, it follows from [4, 11.3] that this does not hold for certain modules for an elementary abelian group of order four in characteristic two.

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