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DIAGRAMMATIC METHODS FOR MODULAR REPRESENTATIONS AND COHOMOLOGY

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To Maurice Auslander on his 60^{th} birthday

1. Introduction.

Diagrammatic methods have long been used to produce examples and generate intuition in group representation theory and in the representation theory of Artin algebras. However the techniques have seldom actually been used to prove anything, and, as a consequence, the literature contains very few articles describing the methods. Papers such as [1], [6] and [14] are examples of exceptions, but even these do little more than expostulate a diagrammatic scheme for modules. The problem with making calculations from such a scheme lies primarily in justifying the techniques.

In this paper G denotes a finite group and K a field of characteristic $p > 0$. Our aim is to develop, with complete

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justification, a system for constructing and using diagrams for KG-modules. The main application of the techniques is in the computation of cohomology groups and rings. For groups for which the projective modules have nice diagrams, the methods work amazingly well and yield results that would be exceptionally difficult to verify by other means. Nevertheless the reader should bear in mind that the methods have limited applications. Many modules simply do not have corresponding diagrams, as we have defined them. Moreover, because of an inability to visualize diagrams in more than two dimensions, we are constrained to considering groups of p-rank at most two.

In section 2, we begin with a variation on Alperin's definition of a module diagram and its representations. Basic properties of the diagrams are explored in this and the next three sections. In Section 6, 7 and 8 we investigate diagrams which are strings or rigid strings. Throughout the first half of the paper the emphasis is on the implications of a diagram's structure to that of the corresponding module. For example, under certain conditions, homomorphisms of modules must respect their diagrammatic structure and modules whose diagrams are strings must be indecomposable. The diagram for a module determines its socle and radical (Proposition 4.1). Section 9 is a digression into homological algebra. The principal result is that, with proper hypotheses, the cohomology ring for a module can be determined from only a few terms and relations.

The remainder of the paper is devoted to the consideration of some specific examples. Basic notational conventions are outlined in Section 10. The examples for $G = SL(3,2)$, M_{11} and A_6 are discussed in detail with the primary focus being on the calculation of their cohomology rings. Other examples are mentioned without detail. In the final section we give a variation on our methods for calculating the cohomology ring of a semi-dihedral group.

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2. Basic definitions.

In this section we define module diagrams, their representations and homomorphisms. The custom among experts in modular representations is to use different types of diagrams in different situations. We do not presume to claim that our definition is the only possible or even the best. Yet it is consistent, or nearly so, with current practice. We differ from Alperin [1] by adding the condition (2.1,iii). This requirement is a convenience that permits the proof of some of the theorems of the paper. More stringent (and also abstruse) conditions would yield better results but at a cost.

Definition 2.1. A KG-module diagram is a pair $D(X,f)$ consisting of the following data.

i) X is a finite directed graph with vertices $\{x_1, \ldots, x_n\}$. If there is an edge from x_i to x_j we denote it by $e(x_i, x_j)$. We write $x_i > x_j$ if there is a sequence $x_i = y_0, \ldots, y_t = x_j$

of vertices such that there exist edges $e(y_{\ell-1}, y_\ell)$ for $\ell = 1,\dots,t$. The graph X must satisfy the following conditions.

a) X has no loops or multiple edges. That is, there is no $x \in X$ such that $x < x$, and between any two points of X there is at most one edge.

b) If $x_1, x_2, x_3 \in X$ with $x_1 > x_2 > x_3$ then there is no edge $e(x_1, x_3)$.

ii) The function f assigns to each vertex $x \in X$ an irreducible KG-module $f(x)$ and to each edge $e(x,y)$ an extension class $f(e(x,y)) \in \mathrm{Ext}^1_{KG}(f(x),f(y))$. The modules $f(x)$ should be taken from a fixed set of representatives of the isomorphism classes of irreducible modules. The assignment must satisfy the condition given below.

iii) Suppose that $x, y_1,\dots,y_t$ are vertices with $f(y_1) \cong \dots \cong f(y_t) \cong N$. If there exist edges $e(x,y_i)$, $i = 1,\dots,t$, then the classes $f(e(x,y_i))$ are K-linearly independent in $\mathrm{Ext}^1_{KG}(f(x),N)$. Dually, if there exist edges $e(y_i,x)$ then the classes $f(e(y_i,x))$, $i = 1,\dots,t$, are linearly independent. In particular, for any edge $e(x,y)$, $f(e(x,y)) \neq 0$.

Suppose that $D(X,f)$ is a module diagram. The relation $>$ defines a topology on X. Namely, a subset $U \subseteq X$ is open provided that whenever $x \in U$ and $x > y$ then $y \in U$, and that an edge is in U if and only if it connects two vertices in U. Hence an open set in X is actually a subgraph of X, but it is determined <u>entirely</u> by its set of vertices. Consequently all set theoretic operations in X may be regarded as taking place on the level of vertices. The union of two open sets is the open set determined by the union of the sets of vertices. The complement of an open set U is the closed set U^c consisting of all vertices not in U and all edges that join two points neither of which is in U. So a closed set V must satisfy the condition that if $x \in V$ and $y > x$ then $y \in V$. Note that if U is an open (or closed) set in X, then there is a corresponding module diagram $D(U, f|_U)$ which we usually write as $D(U,f)$.

<u>Definition 2.2</u>. A <u>representation</u> for a module diagram $D(X,f)$ is a KG-module M and a function $U \to M_U$ from open sets in X to submodules of M which satisfy the following four conditions. Let U, V and W be open sets in X.

i) $M_X = M$, $M_\emptyset = \{0\}$ and $M_U \subseteq M_V$ whenever $U \subseteq V$.

ii) $M_{U\cap V} = M_U \cap M_V$ and $M_{U\cup V} = M_U + M_V$.

iii) If $V = U \cup \{x\}$ with $x \notin U$ then

we have an exact sequence $0 \to M_U \xrightarrow{i_{U,V}} M_V \xrightarrow{\lambda_{U,V}} f(x) \to 0$ with $i_{U,V}$ being the inclusion. If, in addition, $W = U \cup \{y\}$, $y \notin U$ then the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_U & \longrightarrow & M_V & \xrightarrow{\lambda_{U,V}} & f(x) & \longrightarrow & 0 \\
 & & {\scriptstyle i_{U,W}}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_W & \longrightarrow & M_{V\cup W} & \xrightarrow{\lambda_{W,V\cup W}} & f(x) & \longrightarrow & 0
\end{array}$$

commutes.

iv) Suppose that $V = U \cup \{x\}$, $W = V \cup \{y\}$, $x \notin U$, $y \notin V$ and there exist an edge $e(y,x)$. Then in the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_U & = & M_U & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M_V & \longrightarrow & M_W & \xrightarrow{\lambda_{V,W}} & f(y) & \longrightarrow & 0 \\
 & & {\scriptstyle \lambda_{U,V}}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & f(x) & \longrightarrow & M_W/M_U & \longrightarrow & f(y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

the bottom row represents the extension class $f(e(y,x))$ in $\mathrm{Ext}^1_{KG}(f(y),f(x))$.

If $X = \{x\}$ then any representation M of $D(X,f)$ has the form $M = M_X/M_\emptyset \cong f(x)$. If $X = \{x,y\}$ with no edges then by condition (iii) any representation of $D(x,f)$ is isomorphic to $f(x) \oplus f(y)$. Suppose that $X = \{x,y\}$ with $x < y$. By condition (iv) any representation of $D(x,f)$ is the (unique up to isomorphism) middle term of a short exact sequence

$$0 \to f(x) \to M \to f(y) \to 0$$

representing the extension class $f(e(y,x))$. For larger graphs X , diagrams may not have representations or the representations may not be unique up to isomorphism. By way of sample calculations we present the following.

Proposition 2.3. Suppose that $D(X,f)$ is a module diagram, $X = \{x_1,\dots,x_n\}$, $x_1 < x_2 < \dots < x_n$.

(a) If $n = 3$, then $D(x,f)$ has a representation if and only if the cup product $f(e(x_2,x_1)) \cdot f(e(x_3,x_2))$ is zero in $\mathrm{Ext}^2_{KG}(f(x_3),f(x_1))$. Moreover if the condition is satisfied and if $f(x_3) \not\cong f(x_1)$ with $f(x_1)$, $f(x_3)$ absolutely indecomposable, then the set of all isomorphism classes of representations of $D(x,f)$ has a natural structure as an affine space over the vector space $\mathrm{Ext}^1_{KG}(f(x_3),f(x_1))$.

(b) If $n = 4$, then $D(x,f)$ has a representation if and only if $f(e(x_2,x_1))\cdot f(e(x_3,x_2)) = 0$, $f(e(x_3,x_2))\cdot f(e(x_4,x_3)) = 0$ and the Massey triple product

$[f(e(x_2,x_1)),\ f(e(x_3,x_2)),\ f(e(x_4,x_3))] = 0$.

Proof. These facts are most easily checked by direct matrix calculation. For example if M is a representation of $D(x,f)$ as in (a) then there exists a K-basis for M relative to which the action of G is given by the matrices

$$g \longrightarrow \begin{bmatrix} f(x_3)_g & f(e(x_3,x_2))_g & A_g \\ 0 & f(x_2)_g & f(e(x_2,x_1))_g \\ 0 & 0 & f(x_1)_g \end{bmatrix}$$

for $g \in G$ with the obvious notation. An easy calculation shows that the coboundary of the map $g \to A_g$ represents the cup product $f(e(x_2,x_1)) \cdot f(e(x_3,x_2))$. Hence the product is zero. Also if we have another representation with A replaced by B then

$$g \longrightarrow \begin{pmatrix} f(x_3)_g & B_g - A_g \\ 0 & f(x_1)_g \end{pmatrix}$$

is an extension $f(x_3)$ by $f(x_1)$. This is a result of the cocycle condition

$$B_{gh} - A_{gh} = f(x_3)_g(B_h - A_h) + (B_g - A_g)f(x_1)_h \ .$$

The conditions on $f(x_1)$, $f(x_3)$ simply insure that the only endomorphisms of a representation are scalars. Otherwise the statement may be false as for example in the representations of a cyclic p-group.

Statement (b) of the proposition follows immediately from the definition of the Massey triple product (see [12] or [13]). For chains of five or more modules the higher Massey products play a rôle which is harder to describe directly. The reader is referred to [12] for a fuller discussion. We investigate this situation from a different angle in Section 5.

Proposition 2.4. Let M be a representation for a module diagram $D(X,f)$. If X has n vertices, then the composition length of M is n .

Proof. Index the vertices $x_1, \ldots, x_n$ of X so that $x_i < x_j$ implies $i < j$. Let $U_i = \{x_1, \ldots, x_i\}$. Then U_i is an open set and if $M_i = M_{U_i}$, then $\{0\} = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ is a composition series with $M_i/M_{i-1} \cong f(x_i)$.

Remarks (2.5): (1) We could equivalently have defined a representation of $D(X,f)$ in terms of closed subsets W of X . That is, to each closed W of X there is a corresponding quotient $\overline{M}_W$ and epimorphism $\psi_W : M \to \overline{M}_W$ such that the duals of conditions 2.2 (i),...,(iv) are satisfied. The equivalence of the definitions is proved by letting M_U be the kernel of ψ_W when U is the complement of W or, conversely, by letting $\bar{M}_W = M/M_U$ for U open and W its complement.

(2) Suppose that $D(X,f)$ is a module diagram. We define the dual diagram $D(X^*,f^*)$ as follows. The graph X^* has the same vertices as X but each edge $e(x,y)$ in X is replaced by $e(y,x)$ in X^*. For each $x \in X^*$ let $f^*(x)$ be the dual module $(f(x))^* = \mathrm{Hom}_K(f(x),K)$. If $e(x,y)$ is an edge in X^*, let $f^*(e(x,y)) = (f(e(y,x)))^* \in \mathrm{Ext}^1_{KG}(f^*(x),f^*(y))$. That is if $f(e(y,x))$ is represented by the extension

$$0 \longrightarrow f(x) \xrightarrow{\alpha} L \xrightarrow{\beta} f(y) \longrightarrow 0$$

then $f^*(e(x,y))$ is represented by

$$0 \longrightarrow (f(y))^* \xrightarrow{\beta^*} L^* \xrightarrow{\alpha^*} (f(x))^* \longrightarrow 0 .$$

It is easy to see using the previous remark that a KG-module M represents $D(X,f)$ if and only if M^* represents $D(X^*,f^*)$. Many of the theorems in the paper have dual statements which we will write out but not prove.

<u>Definition 2.6</u>. Suppose that $D(X,f)$ and $D(Y,g)$ are module diagrams. A <u>diagram isomorphism</u> $\phi\colon D(X,f) \to D(Y,g)$ is an isomorphism $\phi\colon X \to Y$ of directed graphs with the property that $g \circ \phi = f$ on both vertices and edges. A <u>diagram homomorphism</u> $\phi\colon D(X,f) \to D(Y,g)$ consists of a closed set $V \subseteq X$, an open set $U \subseteq Y$ and a diagram isomorphism $\phi_0\colon D(V,f) \to D(U,g)$.

The kernel of ϕ is the diagram $D(V^c,f)$ where $V^c = X - V$ is the open complement of V. The image of ϕ is the diagram $D(U,g)$. If $W \subseteq X$ is open we write $\phi(W)$ for $\phi_0(V \cap W)$. Also if $S \subseteq Y$ we write $\phi^{-1}(S)$ for $\phi_0^{-1}(U \cap S) \cup (X-V)$.

Suppose that M and N are representations of $D(X,f)$ and $D(Y,g)$ respectively. A homomorphism $\sigma: M \to N$ is a <u>diagrammatic homomorphism</u>, or <u>D-homomorphism</u>, if there exists a diagram homomorphism $\phi: D(X,f) \to D(Y,g)$ with $\sigma(M_U) = N_{\phi(U)}$ for all open sets $U \subseteq X$. In particular $\text{Ker}\ \sigma = M_{\text{Ker}\ \phi}$ and $\text{Im}\ \sigma = N_{\text{Im}\ \phi}$. Note that not all module homomorphisms are D-homomorphisms and conversely a diagram homomorphism may not be represented by a D-homomorphism of the modules.

<u>Lemma 2.7</u>. Suppose that M and N are representations of $D(X,f)$ and $D(Y,g)$ respectively. Let $\sigma: M \to N$ be a D-homomorphism with underlying diagram homomorphism ϕ. If V is an open subset of Y then $\sigma^{-1}(N_V) = M_W$ where $W = \phi^{-1}(V)$

<u>Proof</u>. The lemma follows directly from the fact that $\text{Ker}\ \phi \subseteq W$.

3. Cutting and pasting.

The basic tool for cutting and pasting is the following.

<u>Proposition 3.1</u>. Let $D(X,f)$ be a module diagram. Let U, V be open sets in X with $X = U \cup V$. Suppose that M and N are representations of $D(U,f)$ and $D(V,f)$ respectively and that there exists a D-isomorphism $\sigma: M_{U \cap V} \to N_{U \cap V}$. Then the

pushout L defined by the commutative diagram

$$\begin{array}{ccccc} M_{U\cap V} & \xrightarrow{\sigma} & N_{U\cap V} & \longrightarrow & N \\ \downarrow & & & & \downarrow \\ M & & \longrightarrow & & L \end{array} \qquad (3.2)$$

is a representation of $D(X,f)$.

Dually, if U and V are closed sets with D-isomorphism $\sigma\colon \bar{M}_{U\cap V} \to \bar{N}_{U\cap V}$ then the pullback L' of the diagram

$$\begin{array}{ccccc} L' & & \longrightarrow & & N \\ \downarrow & & & & \downarrow \\ M & \longrightarrow & \bar{M}_{U\cap V} & \xrightarrow{\sigma} & \bar{N}_{U\cap V} \end{array}$$

is a representation of $D(X,f)$.

<u>Proof</u>. Assume the hypotheses with U and V open. If W is an open set in X , then L_W is defined to be the pushout

$$\begin{array}{ccccc} M_{W\cap U\cap V} & \xrightarrow{\sigma} & N_{W\cap U\cap V} & \longrightarrow & N_{W\cap V} \\ \downarrow & & & & \downarrow \\ M_{W\cap U} & & \longrightarrow & & L_W \end{array} \quad . \qquad (3.3)$$

L_W is identified with a submodule of L via the commutative cube of injections (3.3) $\to$ (3.2). From now on we identify M with L_U and N with L_V . We must check the four conditions of Definition (2.2). The first is obvious. If W, W' are open in X , then

$$\begin{aligned} L_{W\cup W'} &= M_{(W\cup W')\cap U} + N_{(W\cup W')\cap V} \\ &= M_{W\cap U} + M_{W'\cap U} + N_{W\cap V} + N_{W'\cap V} \\ &= L_W + L_{W'} \end{aligned}$$

It is clear that $L_{W\cap W'} \subseteq L_W \cap L_{W'}$ and we may show equality by counting composition lengths and using Prop. 2.4. Let $\ell(-)$ be the composition length function. Then by (3.3)

$$\begin{aligned} \ell(L_W) &= \ell(M_{W\cap U}) + \ell(N_{W\cap V}) - \ell(M_{W\cap U\cap V}) \\ &= |W| \end{aligned}$$

the number of vertices in W. Hence we have that

$$\begin{aligned} \ell(L_W \cap L_{W'}) &= \ell(L_W) - \ell(L_W/(L_W \cap L_{W'})) \\ &= |W| - \ell((L_W + L_{W'})/L_{W'}) \\ &= |W| - \ell(L_{W\cup W'}/L_{W'}) \\ &= |W| - (|W\cup W'| - |W'|) \\ &= |W\cap W'| = \ell(L_{W\cap W'}) . \end{aligned}$$

This proves condition (ii).

To check the last two conditions we need only note that any vertex or any edge of X must be either in U or in V. The dual statement is proved by the dual argument (see Remark 2.5 (2)).

Definition 3.4. Suppose that $D(X,f)$ and $D(Y,g)$ are module diagrams. Let $U \subseteq X$, $V \subseteq Y$ be open sets and assume that there exists a diagram isomorphism $\phi: D(U,f) \to D(V,g)$. The amalgamation $D(X,f)\times_\phi D(Y,G)$ is the diagram obtained by identifying $D(U,f)$ with $\phi(D(U,f))$. That is, it is the diagram $D(Z,h)$, with $Z = X \dot{\cup} Y/(x=\phi(x))$ and

$$h(z) = \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in Y \end{cases}$$

on both vertices and edges. Note that $D(Z,h)$ is a module diagram if and only if condition (iii) of Definition (2.1) is satisfied. Dually we can apply the same construction if U and V are closed.

<u>Proposition 3.5</u>. Let L, M, and N be representations of the module diagrams D(X,f), D(Y,g), and D(Z,h) respectively. Suppose that there exist D-homomorphisms σ_1: L → M, σ_2: L → N with underlying diagram homomorphisms ϕ_1: D(Z,h) → D(X,f), ϕ_2: D(Z,h) → D(Y,g). Let W, U, V be the closed sets

$$W = Z - (\mathrm{Ker}\ \phi_1 \cup \mathrm{Ker}\ \phi_2) \subseteq Z$$

$$U = \phi_1(Z) - \phi_1(\mathrm{Ker}\ \phi_2) \subseteq X$$

$$V = \phi_2(Z) - \phi_2(\mathrm{Ker}\ \phi_1) \subseteq Y .$$

Let ϕ denote the composite isomorphism

$$D(U,f) \xrightarrow{\phi_1^{-1}} D(W,h) \xrightarrow{\phi_2} D(V,G) .$$

Then the cokernel of (σ_1,σ_2): L → M ⊕ N is a representation of $D(X - \phi_1(\mathrm{Ker}\ \phi_2),f)\times_\phi D(Y - \phi_2(\mathrm{Ker}\ \phi_1),g)$ provided the latter is a module diagram.

Dually, let σ_1: M → L, σ_2: N → L be D-homomorphisms with underlying diagram homomorphisms ϕ_1, ϕ_2. Let $W = \mathrm{Im}(\phi_1) \cap \mathrm{Im}(\phi_2)$, $U = \phi_1^{-1}(W) - \mathrm{Ker}\ \phi_1$, $V = \phi_2^{-1}(W) - \mathrm{Ker}\ \phi_2$, and let ϕ be the composite $D(U,f) \xrightarrow{\phi_2^{-1}\phi_1} D(V,g)$. Then the kernel of (σ_1,σ_2): M ⊕ N → L is a representation of $D(\phi_1^{-1}(W),f) \times_\phi D(\phi_2^{-1}(W),g)$ provided this is a module diagram.

<u>Proof</u>. The proof follows by applying Proposition 3.1 to the pushout

$$\begin{array}{ccc} \bar{L}_W & \longrightarrow & \bar{N}_{Y-\phi_2(\mathrm{Ker}\ \phi_1)} \\ \downarrow & & \downarrow \\ \bar{M}_{X-\phi_1(\mathrm{Ker}\ \phi_2)} & \longrightarrow & \mathrm{Coker}\ (\sigma_1,\sigma_2) \end{array} ,$$

or the corresponding pullback in the dual case.

4. Socles and radicals.

For the purposes of this section we assume that all irreducible KG-modules are absolutely irreducible. Suppose that $D(X,f)$ is a module diagram. The radical and socle of X are defined by

$$\mathrm{Rad}\ X = \{x \in X \mid x < y \ \text{ for some } \ y \in X\}$$

$$\mathrm{Soc}\ X = \{x \in X \mid \text{there exists no } \ y \in X \ \text{ with } \ y < x\} .$$

That is, Soc X is the maximal open subset of X that has no edges and Rad X is the minimal open set whose complement has no edges.

<u>Proposition 4.1</u>. Let M be a representation of the module diagram $D(X,f)$. Then Rad $M = M_{\mathrm{Rad}\ X}$ and Soc $M = M_{\mathrm{Soc}\ X}$.

<u>Proof</u>. Let $U = \mathrm{Rad}\ X$ and let $X - U = \{y_1,\ldots,y_t\}$. For each i the set $V_i = X - \{y_i\}$ is open and there is a homomorphism $\psi_i\colon M \to f(y_i)$ with kernel M_{V_i} . So the sum

$$\psi = (\psi_1,\ldots,\psi_t)\colon M \to \bigoplus_{i=1}^{t} f(y_i)$$

is surjective and has kernel $\bigcap_{i=1}^{t} M_{V_i} = M_U$ by (2.2, ii). Since the image is semisimple we must have that $\text{Rad } M \subseteq M_U$.

Suppose now that $\text{Rad } M \neq M_U$ and that M is an example with minimal composition length with respect to this property. Then there exists a homomorphism $\theta: M \to N$ where N is irreducible and $\theta(M_U) \neq 0$. By minimality there is no non-empty open subset $W \subseteq X$ with $\theta(M_W) = 0$. Thus for each $x \in X$ with $\{x\}$ open, we must have that $f(x) \cong N$. On the other hand if $\{y\}$ is not open and V is any open set containing y then M_V has the desired property and, by minimality, $V = X$. Hence X has only one vertex y with $\{y\}$ not open. This all implies that $X = \{y, x_1, \ldots, x_r\}$ with edges $e(y, x_i)$, $i = 1, \ldots, r$ and $f(x_i) \cong N$. So $U = \text{Soc } X = \{x_1, \ldots, x_r\}$ and $M_U \cong \bigoplus_{i=1}^{r} f(x_i)$. Therefore we have exact sequences

$$0 \longrightarrow M_U \xrightarrow{i} M \longrightarrow f(y) \longrightarrow 0$$

and

$$\ldots \to \text{Hom}_{KG}(M,N) \xrightarrow{i^*} \text{Hom}_{KG}(M_U,N) \xrightarrow{\delta} \text{Ext}^1_{KG}(f(h),N) \to \ldots$$

Now because N is absolutely irreducible, $\text{Hom}_{KG}(M_U,N)$ has basis $\zeta_1, \ldots, \zeta_r$ where ζ_j is a homomorphism with kernel $\bigoplus_{i \neq j} f(x_i)$. Moreover $\delta(\zeta_j)$ is the extension class represented by the pushout of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_U & \longrightarrow & M & \longrightarrow & f(y) & \longrightarrow & 0 \\ & & \zeta_j \big\downarrow & & & & & & \\ & & f(x_j) & & & & & & \end{array}$$

That is, $\delta(\zeta_j) = f(e(y,x_j))$. By (2.1,iii), $\delta(\zeta_j), \ldots, \delta(\zeta_r)$ are linearly independent in $\text{Ext}^1_{KG}(f(y),N)$. Therefore δ is an injection and i^* is the zero map. This produces the desired contradiction since $\theta(M_U) = i^*(\theta)(M_U) = 0$. Hence $\mathrm{Rad}\ M = M_{\mathrm{Rad}\ X}$. The second statement is dual to the one we have proved.

5. Uniqueness.

Definition 5.1. A module diagram $D(X,f)$ is said to have a unique representation if it has a representation and any two representations are isomorphic as modules. If, in addition, any two representations are D-isomorphic, with underlying diagram isomorphism equal to the identity map on $D(X,f)$, then we say that the representation is D-unique. Finally, $D(X,f)$ has an absolutely D-unique representation if it has a D-unique representation whose only D-automorphisms, with underlying diagram automorphism equal the identity map, are scalar multiples of the identity.

Thus if $X = \{x\}$ then $D(X,f)$ has a D-unique representation which is absolutely D-unique if and only if $f(x)$ is absolutely irreducible. The following shows how D-uniqueness behaves under cutting and pasting.

Proposition 5.2. Let $D(X,f)$ be a module diagram, and suppose that U and V are open sets in X with $X = U \cup V$. If $D(U,f)$ and $D(V,f)$ have D-unique representations, and $D(U \cap V, f)$ has an absolutely D-unique representation then $D(X,f)$ has a D-unique representation.

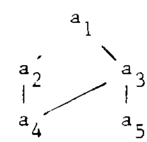
Proof. The existence of a representation is proved in Proposition 3.1. Suppose that M and N are two representations of $D(X,f)$. By hypothesis there exist D-isomorphisms $\sigma_1: M_U \to N_U$ and $\sigma_2: M_V \to N_V$. Since $D(U\cap V,f)$ has an absolutely D-unique representation σ_1 is a scalar multiple of σ_2 on $M_{U\cap V}$. Replacing σ_1 by a suitable scalar multiple we may assume that they coincide on $M_{U\cap V}$. However we know that $M = M_{U\cup V} = M_U + M_V \cong (M_U \oplus M_V)/\hat{M}_{U\cap V}$ where $\hat{M}_{U\cap V} = \{(m,-m) \mid m \in M_{U\cap V}\}$. The similar formulation for N and the isomorphism $(\sigma_1,\sigma_2): M_U \oplus M_V \to N_U \oplus N_V$ induce a D-isomorphism $\sigma: M \to N$.

Of course, Proposition 5.2 has a dual statement for closed sets U, V with $X = U \cup V$. If, in the proposition, $D(U\cap V,f)$ has only a D-unique representation, then it can be shown that $D(X,f)$ has a representation but it is not necessarily D-unique or even unique.

Proposition 5.3. Assume that all irreducible KG-modules are absolutely irreducible. If a module diagram $D(X,f)$ is represented by a projective module P then it has a unique representation (which is not necessarily D-unique).

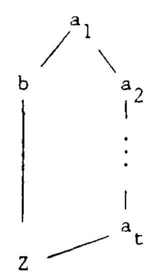
Proof. Suppose that M also represents $D(X,f)$. Let $U = \text{Rad } X$. Then by Proposition 4.1. $M_U = \text{Rad } M$, $P_U = \text{Rad } P$ and $M/M_U \cong \bigoplus_{y\in X-U} f(y) \cong P/P_U$. Since P is projective, the homomorphism $P \to P/P_U \cong M/M_U$ lifts to a homomorphism $\psi: P \to M$ which must be onto because $M_U = \text{Rad } M$. By Proposition 2.4, ψ is an isomorphism.

For the purposes of computing examples we need the following combinatorial lemmas. Our convention throughout the rest of the paper will be to write diagrams with edges indicated by line segments with the greater vertex appearing above the lesser on the page. For example, if X is the graph in (5.4) then the edges in X are $e(a_1,a_2)$, $e(a_1,a_3)$, $e(a_2,a_4)$, $e(a_3,a_4)$ and $e(a_3,a_5)$.



(5.4)

Proposition 5.5. Suppose that all irreducible KG-modules are absolutely irreducible. Let $D(X,f)$ be a module diagram where X has the form of (5.6). Here Z is an open subdiagram with the property that the only edges connecting Z with vertices outside of Z are of the form $e(a_t,z)$ or $e(b,z)$ for some $z \in Z$. Assume that $f(b) \not\cong f(a_i)$ for $i = 2,\dots,t$ and that $D(X,f)$ is represented by an indecomposable projective module P . If Y is the closed subset of X with vertices $a_1,\dots,a_t$ then $D(Y,f)$ has a unique representation. (In 6.1 we show it is D-unique.)



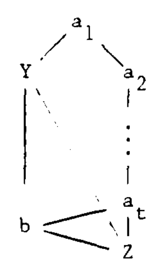
(5.6)

Proof. Note that $D(Y,f)$ has a representation, namely P/P_U where $U = X - Y = Z \cup \{b\}$. Let M be another representation. Then $M/\text{Rad } M \cong f(a_1)$ and since P is the projective cover of $f(a_1)$ there exists an epimorphism $\sigma\colon P \to M$. To prove the lemma we need only show that $P_U \subseteq \text{Ker } \sigma$.

Let V be the smallest open set containing b . By (4.1), $V \subseteq \text{Rad } X = X - \{a_1\}$. So $P_V \subseteq \text{Rad } P$ and $\sigma(P_V) \subseteq \text{Rad } M$.

Also Rad $V = V - \{b\}$ and $P_V/\text{Rad } P_V \cong f(b)$. By hypothesis no composition factor of Rad M is isomorphic to f(b) and $\sigma(P_V) = 0$.

Let W be the smallest open set containing a_t . By an easy induction $P_W \subseteq \text{Rad}^{t-1}(P)$ so $\sigma(P_W) \subseteq \text{Soc } M$ and $\sigma(P_{\text{Rad } W}) = 0$. The indecomposability assumption on P requires that for any $z \in Z$ either $z < b$ or $z < a_t$. That is either $z \in V$ or $z \in \text{Rad } W$. So $U = V \cup \text{Rad } W$, $P_U = P_V + P_{\text{Rad } W}$ and $\sigma(P_V) = 0$. This proves the lemma.



(5.8)

Lemma 5.7. Suppose that all irreducible KG-modules are absolutely irreducible. Let D(X,f) be a module diagram with X having the form of (5.8). Here Y, Z are subdiagrams with the property that every edge connecting a vertex of Y (resp. Z) to a vertex outside of Y (resp. Z) has the form $e(a_1,y)$, $e(y,b)$ or $e(y,z)$ (resp. $e(a_t,z)$, $e(b,z)$ or $e(y,z)$) for $y \in Y$, $z \in Z$. We are assuming that there exist edges $e(a_1,y)$ $e(y',b)$, $e(a_t,b)$ for some $y, y' \in Y$. Suppose the following conditions are satisfied.

i) D(X,f) is represented by an indecomposable projective module P .

ii) The subgraph $U = Y \cup \{a_1,b\}$ has a_1 as the unique maximal element and b as the unique minimal element.

iii) $f(y) \ncong f(a_i)$ for all $y \in Y$, $i = 2,\ldots,t$.

iv) If V is the smallest open subset containing a_2 , then $Z \subseteq V$.

Then $D(U,f)$ has no representation.

Proof. As in the proof of the last lemma, if M is a representation of $D(U,f)$ then there exists an epimorphism $\sigma: P \to M$. By condition (iii), $\sigma(P_V) \subseteq Soc(M) \cong f(b)$, and $P_{Rad\ V} \subseteq Ker\ \sigma$. By a dimension argument $P_{Rad\ V} = Ker\ \sigma$. Since $Soc(P/P_{Rad\ V})$ is not irreducible, $P/P_{Rad\ V}$ cannot be isomorphic to M .

6. String diagrams.

A module diagram $D(X,f)$ is called a uniserial diagram if the vertices $x_1,\ldots,x_n$ of X can be indexed in such a way that $x_1 > x_2 >\ldots> x_n$. It is a string diagram if the vertices can be indexed so that for each $i = 1,\ldots,n-1$ there is an edge $e(x_i,x_{i+1})$ or an edge $e(x_{i+1},x_i)$ and no other edges. In particular a string diagram is necessarily connected and, because of condition (2.1,iii), has no proper diagram automorphisms.

Proposition 6.1. Suppose that all irreducible KG-modules are absolutely irreducible.

a) A string diagram has a representation if and only if each uniserial subdiagram has a representation.

b) A string diagram has a D-unique representation if and only if every uniserial subdiagram has a unique representation.

Proof. Statement (a) is an immediate result of Proposition 3.1 and induction. For (b) we must show that uniserial diagrams with unique representations have D-unique representations. Suppose that M and N are representations of $D(X,f)$ where $X = \{x_1,\ldots,x_n\}$ and $x_1 > x_2 >\ldots> x_n$. The open sets in X are all of the form $U_i = \{x_{i+1},\ldots,x_n\}$, $i = 0,\ldots,n-1$, or $U_n = \emptyset$. But then $M_{U_i} = \mathrm{Rad}^i M$. Hence if $\sigma: M \to N$ is an isomorphism then $\sigma(M_{U_i}) = N_{U_i}$. The proof of (b) is completed by applying induction and Proposition 5.2. For example, if $D(X,f)$ is a string diagram with a minimal vertex x that is not at one of the ends of the string, then there are open sets U, V such that $U \cap V = \{x\}$, $U \cup V = X$ and $D(U,f)$, $D(V,f)$ are string diagrams. By induction and the fact that $D(\{x\},f)$ has an absolutely D-unique representation, we are done.

Corollary 6.2. Let $D(X,f)$ and $D(Y,g)$ be string diagrams with the property that every uniserial subdiagram has a unique representation. Suppose that M and N are representations of $D(X,f)$ and $D(Y,g)$ respectively. If $\phi: D(X,f) \to D(Y,g)$ is a diagram homomorphism then there exists a D-homomorphism $\sigma: M \to N$ corresponding to ϕ.

Proof. Let $U = \mathrm{Ker}\,\phi \subseteq X$ and $V = \mathrm{Im}\,\phi \subseteq Y$. Observe that $D(X - U,f)$ and $D(V,g)$ are each disjoint unions of string diagrams and they are isomorphic under ϕ. By hypothesis and the previous proposition they have D-unique representations, M/M_U and $\sigma(M)$ respectively, which are necessarily D-isomorphic under ϕ.

Lemma 6.3. Let M be a representation of the module diagram $D(X,f)$. Given an open set $W \subseteq X$ and $\alpha \in K$ we form a new module diagram $D(X,f')$ as follows. Let $f'(x) = f(x)$ for all vertices x. Let $f'(e(x,y)) = \alpha f(e(x,y))$ if $y \in W$, $x \notin W$ and let $f'(e(x,y)) = f(e(x,y))$ otherwise. Then M is also a representation of $D(X,f')$.

Proof. Let $U \to M_U$ be the function that defines M as a representation of $D(X,f)$. For $D(X,f')$ use the same assignment. However if U and V are open sets with $V = U \cup \{x\}$, $x \notin U$ then we choose the homomorphisms $\lambda'_{U,V}: M_V \to f'(x)$ as follows

i) If $x \in W$ then let $\lambda'_{U,V} = \lambda_{U,V}$.

ii) If $x \in W$, then we multiply by α as in the diagram

$$\begin{array}{ccc} M_V & \xrightarrow{\lambda_{U,V}} & f(x) \\ {\scriptstyle\alpha}\downarrow & & \| \\ M_V & \xrightarrow{\lambda'_{U,V}} & f'(x) \end{array}$$

It is straightforward to check that with these new identifications, M is a representation of $D(X,f')$.

Definitions 6.4. A diagram $D(X,f)$ is said to be *rigid* if whenever $e(x,y)$ is an edge in X, $\mathrm{Dim}_K \mathrm{Ext}^1_{KG}(f(x),f(y)) = 1$.

By condition (2.1,iii) any subdiagram of a rigid diagram that has the form

$$\begin{array}{ccc} & y & \\ / & & \backslash \\ x & & z \end{array} \qquad \text{or} \qquad \begin{array}{ccc} x & & z \\ \backslash & & / \\ & y & \end{array}$$

has the property that $f(x) \not\cong f(z)$. Moreover each $f(x)$ in a rigid string diagram is absolutely irreducible (see (8.4)).

Proposition 6.5. Suppose that $D(X,f)$ and $D(X,f')$ are rigid string diagrams such that $f(x) = f'(x)$ for all vertices $x \in X$. Then a KG-module represents $D(X,f)$ if and only if it represents $D(X,f')$.

Proof. This follows directly from Lemma 6.2.

The preceeding result implies that for studying the representations of rigid string diagrams there is no point in labeling the edges. This shall be our practice, when possible, in the examples.

7. Homomorphisms of rigid string diagrams.

In view of Proposition 6.4 we may assume that rigid string diagrams are normalized in the following sense. Given any two simple modules M and N with $\mathrm{Dim}\,\mathrm{Ext}^1_{KG}(M,N) = 1$ select a nonzero element $\xi(M,N) \in \mathrm{Ext}^1_{KG}(M,N)$. We say that the rigid string diagram $D(X,f)$ is normalized if for any edge $e(x,y) \in X$, $f(e(x,y)) = \xi(f(x), f(y))$. If $D(X,f)$ and $D(Y,g)$ are normalized and if there exist subsets $U \subseteq X$ $V \subseteq Y$, U closed, V open, and an isomorphism of directed graphs $\phi: U \to V$ with $g(\phi(u)) = f(u)$ for all $u \in U$, then ϕ defines a diagram homomorphism with kernel $X - U$ and image V. Moreover if M and N are representations $D(X,f)$ and $D(Y,g)$ and if uniserial subdiagrams have unique representations, then by Proposition 6.1 there exists a D-homomorphism $\theta: M \to N$ corresponding to ϕ.

Theorem 7.1. Suppose that $D(X,f)$ and $D(Y,g)$ are normalized rigid string diagrams and that any uniserial subdiagram of either has a unique representation. Let M and N be representations of $D(X,f)$ and $D(Y,g)$ respectively. Then every homomorphism $\sigma: M \to N$ is a K-linear combination of D-homomorphisms.

To prove the theorem we need the following.

Lemma 7.2. Let $D(X,f)$, $D(Y,g)$, M and N be as in the theorem. Assume that the conclusion of the theorem is true for any similar data $D(X',f')$, $D(Y',g')$, M', N' with $|X'| \leq |X|$ and $|Y'| \leq |Y|$. Let $\Phi = \Phi(X,f; Y,g) = \{\phi_1,\ldots,\phi_r\}$ be the set of all diagram homomorphisms $\phi: D(X,f) \to D(Y,g)$ such that $\mathrm{Im}\,\phi$ is a connected string in Y. (Note that $\mathrm{Im}\,\phi$ is connected if and only if $X-\ker\phi$ is connected). For each i choose a D-homomorphism $\theta_i: M \to N$ whose underlying diagram homomorphism is ϕ_i. Then $\{\theta_1,\ldots,\theta_r\}$ is a K-basis for $\mathrm{Hom}_{KG}(M,N)$.

Proof. The proof is by induction on $|X| + |Y|$. Suppose that $\alpha_1,\ldots,\alpha_r \in K$ and that $\Sigma\, \alpha_i\theta_i = 0$. Let $x \in X$ be a closed point and let $U \subseteq X$ be the minimal open set containing x. Let $\phi_1,\ldots,\phi_t$ be the elements of Φ with $x \notin \mathrm{Ker}\,\phi_i$. For $1 \leq i, j \leq t$, $\phi_i(x) = \phi_j(x)$ if and only if $i = j$. So if $m \in M_U$, $m \notin \mathrm{Rad}\, M_U$ then $\theta_1(m),\ldots,\theta_t(m)$ are K-linearly independent elements of N. Since $\sum_{i=1}^{r} \alpha_i\theta_i(m) = 0 = \sum_{i=1}^{t} \alpha_i\theta_i(m)$ we must have that $\alpha_1,\ldots,\alpha_t$ are zero. Now do the same for the other closed points of X . We conclude that $\theta_1, \ldots, \theta_r$ are linearly independent.

By hypothesis it is sufficient to prove that any D-homomorphism is in the span of $\theta_1, \ldots, \theta_n$. Let $\theta: M \to N$ be a D-homomorphism corresponding to diagram homomorphism ϕ . Suppose that there exists a closed point $y \in Y$ with $y \notin \operatorname{Im} \phi$. Then $\theta(M) \subseteq N_U$, $U = Y - \{y\}$. By induction we are finished because any $\phi' \in \Phi(X,f; U,g)$ coincides with some $\phi_i \in \Phi$ with $y \notin \operatorname{Im} \phi_i$, and the $\theta': M \to N_U$ corresponding to ϕ' may be taken to be θ_i . Similarly we are done if there exists an open point $x \in \operatorname{Ker} \phi$. Therefore we may assume that ϕ is a diagram isomorphism. In this case $\phi \in \Phi$, say $\phi = \phi_1$. Let y be a closed point in Y and let $\psi: N \to g(y)$ be a D-homomorphism with kernel N_U , $U = Y - \{y\}$. Because $g(y)$ is absolutely irreducible there exists $\alpha \in K$ with $\psi(\theta - \alpha\theta_1) = 0$. So $\theta - \alpha\theta_1: M \to N_U$ and by induction θ is a linear combination of $\theta_1, \ldots, \theta_r$.

Proof of Theorem 7.1. For convenience let $\tilde{1}, \ldots, \tilde{n}$ denote the vertices of X and let $\hat{1}, \ldots, \hat{m}$ be the vertices of Y . The proof is by induction on the sum of the lengths $n + m$. The theorem is obvious if either $n = 1$ or $m = 1$. The argument has three major steps which are reductions based on the assumption that the given data represent a minimal counterexample.

Step I. Let r be the least index such that $\hat{r} \in Y$ is open, and let t be the least integer with $t > r$ and $\hat{t}$ closed. Then $t = m$. That is, Y has only one open point.

Suppose that $t < m$. Let $U = \{\hat{i} \mid i < t\}$, $V = \{\hat{i} \mid i > t\}$, $\bar{U} = Y - U$, and $\bar{V} = Y - V$. Then we have the following commutative diagrams of diagram homomorphisms and corresponding D-homomorphisms:

$$\begin{array}{ccc} D(Y,g) & \xrightarrow{\phi_U} & D(\bar{U},g) \\ \phi_V \downarrow & \searrow^{\phi_{U\cup V}} & \downarrow \phi'_V \\ D(\bar{V},g) & \xrightarrow[\phi'_U]{} & D(\{\hat{t}\},g) \end{array} , \qquad \begin{array}{ccc} N & \xrightarrow{\psi_U} & \bar{N}_{\bar{U}} \\ \psi_V \downarrow & \searrow^{\psi_{U\cup V}} & \downarrow \psi'_V \\ \bar{N}_{\bar{V}} & \xrightarrow[\psi'_U]{} & \bar{N}_{\{\hat{t}\}} \end{array} \cong g(t)$$

As in Lemma 7.2, let $\Phi(X,f;\bar{U},g) = \{\phi_1,\ldots,\phi_q\}$ and $\Phi(X,f;\bar{V},g) = \{\phi'_1,\ldots,\phi'_s\}$, and choose for each i and each j corresponding D-homomorphisms $\theta_i\colon M \to \bar{N}_{\bar{U}}$ and $\theta'_j\colon M \to \bar{N}_{\bar{V}}$. By induction, and Lemma 7.2, there exist α_i, $\beta_j \in K$ such that $\psi_U\sigma = \Sigma\, \alpha_i\theta_i$ and $\psi_V\sigma = \Sigma\, \beta_j\theta'_j$. Also $\psi_{U\cup V}\sigma = \Sigma\, \alpha_i\psi'_V\theta_i = \Sigma\, \beta_j\psi'_U\theta'_j$.

Let $\tilde{k}$ be a closed point in X. Note that there is at most one index i and at most one j such that $\phi_i(\tilde{k}) = \hat{t}$ and $\phi'_j(\tilde{k}) = \hat{t}$. This is a consequence of rigidity and the fact that $\operatorname{Im}\phi_i$ is an open substring (connected) in U and X-$\ker\phi_i$ is an isomorphic closed substring. Suppose that $\alpha_i\psi'_V\theta_i \neq 0$ and $\phi_i(\tilde{k}) = \hat{t}$. Then $\phi'_U\phi_i(\tilde{k}) = \hat{t}$ and $f(k) = g(t)$. Let W be the smallest open set in X that contains $\tilde{k}$. Then $\psi'_V\psi_U\sigma(M_U) = \psi'_U\psi_V\sigma(M_U) \neq \{0\}$ and there must be an index j such that $\phi'_j(\tilde{k}) = \hat{t}$ and $\beta_j\psi'_U\theta'_j \neq 0$. Now $\phi_i^{-1}(U) = U_1$ is a closed substring in X. By reversing the ordering on the vertices of X, if necessary, we get that $U_1 = \{\tilde{\ell} \mid c \le \ell \le k\}$ for some $c \ge 1$ and that $f(k-d) = g(t-d)$ for

$d=0,\ldots,k-c$. Because $g(t+1) \not\cong g(t-1)$, the closed set $(\phi_j')^{-1}(V) = V_1$ must have the form $V_1 = \{\tilde{\ell} \mid k \le \ell \le d\}$ for some d . Therefore $U_1 \cup V_1$ is a closed connected subset of X and there is a diagram isomorphism from $D(U_1 \cup V_1, f)$ to $D(\mathrm{Im}\ \phi_i \cup \mathrm{Im}\ \phi_j', g)$. Let $\phi_i'': D(X,f) \to D(Y,g)$ be the corresponding diagram homomorphism and $\theta_i'': M \to N$ a corresponding D-homomorphism. Note that $\phi_U \phi_i'' = \phi_i$ and $\phi_V \phi_i'' = \phi_j'$. Because $g(t)$ is absolutely irreducible there exist η , η' in K with $\alpha_i \psi_V' \theta_i = \eta \psi_{U\cup V} \theta_i''$ and $\beta_j \psi_U' \theta_j' = \eta' \psi_{U\cup V} \theta_i''$. Since $\alpha_i \psi_V' \theta_i$ coincides with $\beta_j \psi_U' \theta_j'$ on M_W we must have that $\eta = \eta'$ and

$$\psi_{U\cup V}(\sigma - \eta\theta_i'')\ (M_W) = \{0\} .$$

Continuing in this fashion we can find D-homomorphisms $\theta_1'',\ldots,\theta_q''$ from M to N (where $\theta_i'' = 0$ if $\alpha_i \psi_V' \theta_i = 0$) and elements $\eta_1,\ldots,\eta_q \in K$ so that

$$\psi_{U\cup V}(\sigma - \Sigma\ \eta_i \theta_i'') = 0 .$$

That is $(\sigma - \Sigma\ \eta_i \theta_i'')(M) \subseteq N_{U\cup V}$. By induction we are finished unless $t = m$.

<u>Step II</u>. Let s and t be the least indices such that $\tilde{s} \in X$ is closed, $t > s$ and $\tilde{t} \in X$ is open. Then $t = n$. That is, X has exactly one closed point.

This step is exactly dual to step I. That is, if $n > t$ then $X = U \cup V$ where U and V are open sets with $U \cap V = \{\tilde{t}\}$. Now show that there exist D-homomorphisms $\theta_i'': M \to N$ and $\eta_i \in K$ such that $(\sigma - \Sigma\ \eta_i \theta_i'')(M_{\{\tilde{t}\}}) = 0$. Again apply the induction.

<u>Step III</u>. By I and II we have that $X = \{\tilde{1},\ldots\tilde{n}\}$ with unique closed point $\tilde{s}$ and $Y = \{\hat{1},\ldots,\hat{m}\}$ with unique open point r. By reversing the ordering on the indices of necessary, we may assume that $s - 1 \geq n - s$ and $r - 1 \geq m - r$. Observe first that $s = r$. For if $s > r$ then $\sigma(M_{\{\tilde{1}\}}) \subseteq \sigma(\mathrm{Rad}^{s-1}M) \subseteq \mathrm{Rad}^{s-1}N = \{0\}$. Then by induction the induced homomorphism $\sigma'\colon M/M_{\{\tilde{1}\}} \to N$ is a linear combination of D-homomorphisms. On the other hand if $s < r$ then $\sigma(M) \subseteq \mathrm{Soc}^{s-1}(N) \subseteq N_V$ for $V = Y - \{\hat{1}\}$, and again we are done by induction.

Since $M_{\{\hat{1}\}}$ is not in the kernel of σ we must have that $f(\tilde{1}) \cong g(\tilde{r})$ ($\cong f(\tilde{n})$ if $n \neq s$). Likewise since $\sigma(M) \not\subseteq N_V$ for any proper $V \subseteq Y$, $f(\tilde{s}) = g(\hat{1})$ ($= g(\hat{m})$ if $r \neq m$).

Suppose that $s > 2$. Let $U = \{\tilde{i} \mid 1 \leq i < s\} \subseteq X$. Then $\sigma(M_U) \subseteq \mathrm{Soc}^{s-2}(N) \subseteq N_V$, $V = \{\hat{j} \mid 2 \leq j \leq m\}$. By induction, σ restricted to M_U is a sum of D-homomorphisms, at least one of which is not zero on $M_{\{\tilde{1}\}}$. Hence there exists a diagram homomorphism $\phi\colon D(U,f) \to D(Y,g)$ such that $\phi(\tilde{1}) = \hat{s} = \hat{r}$. Since $\mathrm{Im}\,\phi$ is an open connected uniserial subset of Y we have exactly three possibilities.

1. $\mathrm{Im}\,\phi = \{\hat{i} \mid 2 \leq i \leq r\}$ and $f(i) = g(r-i+1)$, $i = 1,\ldots, s-1$. However we know that $f(s) = g(1)$. Therefore there is a diagram homomorphism $\phi'\colon D(X,f) \to D(Y,g)$ with kernel $\{\tilde{i} \mid i > s\}$. If θ is a corresponding D-homomorphism then there exists some $\alpha \in K$ such that $(\sigma-\alpha\theta)(M_{\{\tilde{1}\}}) = 0$. By induction we are finished.

2. $\mathrm{Im}\,\phi = \{\hat{i} \mid r \leq i < m\}$. This requires that $m - r = s - 1 = r - 1$. So reverse the ordering on the vertices of Y and apply case 1.

3. $\operatorname{Im} \phi = \{\hat{i} \mid 1 \le i \le m\}$. So $m - r = s - 2$, and $f(\tilde{i}) = g(r+i-1)$ for $i = 1,\dots,s-1$. Let $A = \{\tilde{1},\tilde{2}\}$. Since $g(r-1) \not\cong g(r+1) \cong f(2)$, $\sigma(M_A) \subseteq N_B$ where $B = \{r,r+1\}$. Also $\sigma(M_A) \not\subseteq N_{\{r\}}$ since otherwise $\sigma(M_{\{1\}}) = \sigma(\operatorname{Rad} M_A) = \{0\}$. There exists $\alpha \in \operatorname{Rad}^{s-2}KG$, $m \in M_A$ such that $\alpha m \in M_A$ and $\alpha m \notin M_{\{1\}}$. Then $\sigma(\alpha m) \in \operatorname{Rad}^{s-2}(N) = N_W$ where $W = \{r-1,r\}$. This is impossible.

Finally suppose that $s = 2$. By our assumptions $m = 2$ or 3 . However it is not possible to have $m = 3$ because $g(1) \cong g(3)$. Similarly if $m = 2$, $D(X,f)$ is isomorphic to $D(Y,g)$ and σ is a D-homomorphism.

Remark 7.3. Without the assumption that uniserial string diagrams have unique representations, Theorem 7.1 fails except in some very special cases. For example if $D(X,f)$ and $D(Y,g)$ are normalized rigid uniserial diagrams with representations M and N , then every KG-homomorphism from M to N is itself a D-homomorphism. This can be easily proved by looking at powers of the radical and using Proposition 4.1.

8. The indecomposability theorem.

The purpose of this section is to show that, under the same assumptions as in Section 7, rigid string diagrams have only indecomposable representations. Specifically we prove the following.

Theorem 8.1. Suppose that $D(X,f)$ is a normalized rigid string diagram such that any uniserial subdiagram has a unique representation. If M is a representation of $D(X,f)$ then M is indecomposable.

To prove the theorem it is sufficient to show that $\mathrm{Hom}_{KG}(M,M)$ is a local ring or that $\mathrm{Hom}_{KG}(M,M)/\mathrm{Rad}\ \mathrm{Hom}_{KG}(M,M) \cong K$. Hence the theorem is an immediate consequence of the following.

Proposition 8.2. Let $D(X,f)$ and M be as in Theorem 8.1. Let $\Phi = \Phi(x,f;\ x,f) = \{\phi_1,\ldots,\phi_r\}$ be the set of all diagram homomorphisms from $D(X,f)$ to itself, whose images are connected strings. Assume that ϕ_1 is the identity. Then if $i \geq 2$, ϕ_i has nonempty kernel. For each i choose a D-homomorphism $\theta_i\colon M \to M$ whose underlying diagram homomorphism is ϕ_i . Then $\{\theta_1,\ldots,\theta_r\}$ is a basis for $\mathrm{Hom}_{KG}(M,M)$, and $\{\theta_2,\ldots,\theta_r\}$ is a basis for $\mathrm{Rad}\ \mathrm{Hom}_{KG}(M,M)$.

Proof. As noted before, there is only one diagram homomorphism from $D(X,f)$ to itself that is bijective, because $D(X,f)$ is rigid. By Corollary 6.2, the homomorphisms $\theta_1,\ldots,\theta_r$ exist and in Theorem 7.1 and Lemma 7.2 we showed that they form a basis for $\mathrm{Hom}_{KG}(M,M)$. Therefore to prove the proposition it is only necessary to demonstrate that the subspace J spanned by $\theta_2,\ldots,\theta_r$ is a nilpotent ideal.

The key idea in the proof is the observation that if $\phi\colon D(x,f) \to D(x,f)$ is a diagram homomorphism which is not the identity then there exists no element $x_i \in X$ such that $\phi(x_i) = x_i$. For suppose otherwise. Because $\ker \phi$ and $\mathrm{Im}\ \phi$ must be open sets and ϕ must give an isomorphism from $D(X\text{-}\mathrm{Ker}\ \phi, f)$ to $D(\mathrm{Im}\ \phi,\ f)$, neither x_{i-1} nor x_{i+1} can be in the kernel of ϕ . So $\phi(x_{i-1}) = x_{i-1}$ and $\phi(x_{i+1}) = x_{i+1}$ since $D(X,f)$ is rigid. Continuing in this fashion we get that ϕ is the identity.

Suppose that $\gamma_1, \ldots, \gamma_t$ are elements in the set $\{\phi_2, \ldots, \phi_r\}$. We can see that for any i, the sequence $y_0 = x_i$, $y_1 = \gamma_1(y_0)$, $y_2 = \gamma_2(y_1), \ldots, y_t = \gamma_t(y_{t-1})$ is a nonrepeating sequence of element in X. That is $y_i \neq y_j$ for $i \neq j$. This assumes that $y_{i-1} \notin \mathrm{Ker}\,\gamma_i$, $i=1,\ldots,t$. The justification for this step is the preceding paragraph, noting that if $1 \leq r \leq s \leq t$ then $\gamma_s \circ \ldots \circ \gamma_r$ is a diagram homomorphism with nonempty kernel. Consequently the composition of any $n = |X|$ elements in $\{\phi_2, \ldots, \phi_r\}$ has all of X in its kernel. From the definition of D-homomorphisms it follows that $\theta_i \circ \theta_j$ is a D-homomorphism with underlying diagram homomorphism $\phi_i \circ \phi_j$. Therefore the subspace J has the property that $J^n = \{0\}$.

Let $I \in \mathrm{Hom}_{KG}(M,M)$ be the identity element. Then $\{I, \theta_2, \ldots, \theta_t\}$ is a basis for $\mathrm{Hom}_{KG}(M,M)$. Hence if $\theta \in \mathrm{Hom}_{KG}(M,M)$ then $\theta = aI + \gamma$ for $a \in K$ and $\gamma \in J$. So $\theta^{p^n} = a^{(p^n)}I + \gamma^{p^n} = a^{(p^n)}I$. Thus the map $\psi\colon \mathrm{Hom}_{KG}(M,M) \to KI \cong K$ given by $\psi(\theta) = \theta^{(p^n)}$ is a ring homomorphism (though not a K-algebra homomorphism). Since the kernel of ψ is J, J is a nilpotent ideal. This completes the proof.

<u>Corollary 8.3</u>. Suppose that $D(x,f)$ and M are as in Theorem 8.1. Then M is absolutely indecomposable.

9. Rank two groups.

One of the uses that we make of the diagrammatic methods is that of calculating the modules $\Omega^j(K)$ and subsequently unveiling the structure of the cohomology ring $H^*(G,K)$. One question arises. Once we have determined the structure of $\Omega^r(K)$ for several values of r , how do we know that the patterns we see persist indefinitely? In this section we show how the use of the varieties of modules enables us to reduce the problem to a finite calculation for groups G of p-rank 2. In fact the method works more generally for calculating $\Omega^r(M)$ for any module M of complexity two.

We refer the reader to [4] for a comprehensive treatment of varieties for modules. Briefly, $V_G(K)$ is the maximal ideal spectrum of $H^*(G,K)$ if $p = 2$ and of $H^{ev}(G,K) = \sum_{n=0}^{\infty} H^{2n}(G,K)$ if $p > 2$. For a KG-module M , $V_G(M)$ is the subvariety of $V_G(K)$ corresponding to the ideal of all elements in the cohomology ring that annihilates $\mathrm{Ext}^*_{KG}(M,M)$. Hence $V_G(M)$ is a homogeneous affine variety and its dimension is the complexity, $c_G(M)$, of M . The dimension of $V_G(K)$ is the p-rank of G .

For a KG-module M , let $\cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$ be a minimal projective resolution of M . The module $\Omega^n(M)$ is defined to be the kernel of $\partial_{n-1}: P_{n-1} \to P_{n-2}$. An element $\zeta \in \mathrm{Ext}^n_{KG}(M,M)$ is represented by a cocycle $\hat{\zeta}: \Omega^n(M) \to M$ and if $\hat{\zeta}$ is onto then we have an exact sequence

$$0 \longrightarrow M_\zeta \longrightarrow \Omega^n(M) \xrightarrow{\hat{\zeta}} M \longrightarrow 0 \qquad (9.1)$$

The isomorphism class of the kernel, M_ζ , depends only on ζ . Similarly if $\gamma \in \mathrm{Ext}^r_{KG}(K,K)$, let L_γ denote the kernel of a representing cocycle $\hat{\gamma}: \Omega^r(K) \to K$. By [8] ,

$V_G(L_\gamma) = V_G(\gamma)$, the hypersurface determined by γ as an element of the coordinate ring of $V_G(K)$.

Suppose that M is a periodic KG-module. Then $\dim V_G(M) = 1$ [4], and there exists some $\gamma \in \mathrm{Ext}^r_{KG}(K,K) = H^r(G,K)$ with $V_G(M) \cap V_G(\gamma) = V_G(M \otimes L_\gamma) = \{0\}$. So $M \otimes L_\gamma$ is projective and the exact sequence

$$0 \longrightarrow M \otimes L_\gamma \longrightarrow M \otimes \Omega^r(K) \xrightarrow{1\otimes\hat{\gamma}} M \longrightarrow 0$$

splits. That is $\Omega^r(M) \cong M$ and we say that γ generates the periodicity of M .

Suppose that $\zeta \in \mathrm{Ext}^n_{KG}(M,M)$ is an element represented by an epimorphism $\hat{\zeta}\colon \Omega^n(M) \to M$. For each $j \geq 0$, define a projective module $Q_j = Q_j(M,\zeta)$ as follows. Tensor the sequence (8.1) by $\Omega^j(M)$. Since KG is a self-injective ring any projective submodule in either of the end terms can be factored out. Then Q_j is the projective factor in the middle term of the resulting sequence:

$$0 \longrightarrow \Omega^j(M_\zeta) \longrightarrow \Omega^{n+j}(M) \oplus Q_j \xrightarrow{\Omega^j(\hat{\zeta})} \Omega^j(M) \longrightarrow 0 \ . \quad (9.2)$$

<u>Theorem 9.3</u>. Suppose M is KG-module with $c_G(M) = \dim V_G(M) = 2$. Let $\zeta \in \mathrm{Ext}^n_{KG}(M,M)$ have an epimorphic representative $\hat{\zeta}$ as in (9.1). Suppose that M_ζ is periodic and that its periodicity is generated by $\gamma \in \mathrm{Ext}^r_{KG}(K,K)$. If $Q_j = Q_j(M,\zeta) = \{0\}$ for $j = 0,\ldots, r-1$, then $Q_j = \{0\}$ for all $j \geq 0$.

<u>Proof</u>. Assume that $Q_j = \{0\}$. We wish to show that $Q_{j+r} = \{0\}$ and prove the theorem by induction. The assumption implies that

$$0 \longrightarrow \Omega^j(M_\zeta) \longrightarrow \Omega^{n+j}(M) \longrightarrow \Omega^j(M) \longrightarrow 0$$

is exact. Tensor this with the sequence

$$0 \longrightarrow L_\gamma \longrightarrow \Omega^r(K) \xrightarrow{\hat{\gamma}} K \longrightarrow 0$$

to obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^j(M_\zeta) \otimes L_\gamma & \longrightarrow & \Omega^{n+j}(M) \otimes L_\gamma & \longrightarrow & \Omega^j(M) \otimes L_\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^j(M_\zeta) \otimes \Omega^r(K) & \longrightarrow & \Omega^{n+j}(M) \otimes \Omega^r(K) & \longrightarrow & \Omega^j(M) \otimes \Omega^r(K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^j(M_\zeta) & \longrightarrow & \Omega^{n+j}(M) & \longrightarrow & \Omega^j(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Because $\Omega^j(M_\zeta) \otimes L_\gamma$ is projective we may factor this and other projectives from the corners to get the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega^{n+j}(M \otimes L_\gamma) & \xrightarrow{\mu} & \Omega^j(M \otimes L_\gamma) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^{j+r}(M_\zeta) & \xrightarrow{\alpha} & \Omega^{n+j+r}(M) \oplus P & \xrightarrow{\beta} & \Omega^{j+r}(M) \oplus P' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\
0 & \longrightarrow & \Omega^j(M_\zeta) & \longrightarrow & \Omega^{n+j}(M) & \longrightarrow & \Omega^j(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where μ, ν are isomorphisms, P and P' are projective. Because

β is surjective, it must map P onto P'. Suppose that S is a simple submodule of P with $\beta(S) = \{0\}$. Then $\theta(S) = \{0\}$ because $\Omega^{n+j}(M)$ has no projective submodules, i.e. $\mathrm{Soc}(P) \subseteq \mathrm{Ker}\ \theta$. However this is impossible because $S \subseteq \mathrm{Im}\ \alpha$ and $\theta\alpha$ is injective. Therefore $P \cap \mathrm{Ker}\ \beta = \{0\}$ and $P \cong P'$. This proves the theorem.

<u>Proposition 9.4</u>. Suppose that $\zeta \in \mathrm{Ext}^n_{KG}(M,M)$ is represented by an epimorphic $\hat{\zeta}\colon \Omega^n(M) \to M$. For any $j \geq 0$ the following are equivalent.

a) $Q_j(M,\zeta) = \{0\}$

b) For any irreducible module S, the map

$$\zeta\colon \mathrm{Ext}^j_{KG}(M,S) \to \mathrm{Ext}^{n+j}_{KG}(M,S),$$

given by cup product with ζ, is an injection.

<u>Proof</u>. Because S is irreducible $\mathrm{Ext}^j_{KG}(M,S) \cong \mathrm{Hom}_{KG}(\Omega^j(M),S)$. That is, no homomorphism from $\Omega^j(M)$ to S can factor through this inclusion of $\Omega^j(M)$ into P_{j-1}. Moreover cup product with ζ is the same as composition with $\Omega^j(\hat{\zeta})$ as in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^j(M_\zeta) & \longrightarrow & \Omega^{n+j}(M) \oplus Q_j & \xrightarrow{\Omega^j(\hat{\zeta})} & \Omega^j(M) & \longrightarrow & 0 \\
 & & & & \uparrow & & \downarrow{\scriptstyle \theta} & & \\
 & & & & \Omega^{n+j}(M) & \xrightarrow[\zeta\theta]{\qquad} & S & &
\end{array}$$

Clearly if $Q_j = 0$ then $\zeta\cdot$ is injective. On the other hand if $Q_j \neq \{0\}$, then the map $\Omega^{n+j}(M)/\mathrm{Rad}\ \Omega^{n+j}(M)$ to $\Omega^j(M)/\mathrm{Rad}\ \Omega^j(M)$ induced by $\Omega^j(\hat{\zeta})$ can not be onto since $\Omega^j(M_\zeta)$ has no projective submodules. So in this case $\zeta\cdot$ is not injective.

Theorem 9.5. Suppose that M and $\zeta \in \mathrm{Ext}^n_{KG}(M,M)$ satisfy the hypotheses of Theorem 9.3. If $Q_j(M,\zeta) = \{0\}$ for $0 \le j \le r-1$, then for any irreducible module S, and any $j \ge 0$

$$\mathrm{Dim}_K \mathrm{Ext}^{n+j}_{KG}(M,S) = \mathrm{Dim}_K \mathrm{Ext}^j_{KG}(M,S) + \mathrm{Dim}_K \mathrm{Ext}^j_{KG}(M_\zeta,S) .$$

Also the Poincaré series $P_{M,S}(t) = \sum_{j=0}^{\infty} t^j \,\mathrm{Dim}_K\, \mathrm{Ext}^j_{KG}(M,S)$ satisfies

$$P_{M,S}(t) = \frac{1}{(1-t^n)} \left\{ \sum_{j=0}^{n-1} t^j \mathrm{Dim}_K\, \mathrm{Ext}^j_{KG}(M,S) + t^n P_{M_\zeta,S}(t) \right\} .$$

Proof. By Theorem 9.3, $Q_j = \{0\}$ for all j. From sequence (9.2) we have the long cohomology sequence

$$\cdots \longrightarrow \mathrm{Ext}^j_{KG}(M,S) \xrightarrow{\zeta\cdot} \mathrm{Ext}^j_{KG}(\Omega^n(M),S) \longrightarrow \mathrm{Ext}^j_{KG}(M_\zeta,S) \longrightarrow \cdots$$

By Proposition 9.4 the connecting homomorphisms are zero. Since $\mathrm{Ext}^j_{KG}(\Omega^n(M),S) \cong \mathrm{Ext}^{n+j}_{KG}(M,S)$ the first statement is proved. The second is an easy consequence of the first.

In the applications of these results the following lemma will be needed. The proof is found in [8] (Lemma 4.1).

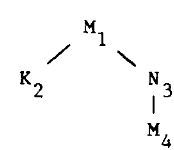
Lemma 9.6. Suppose that $\gamma \in \mathrm{Ext}^n_{KG}(K,K)$ and that $V_G(\gamma) = V_1 \cup V_2 \cup \ldots \cup V_t$ where each V_i is a closed projectively connected subvariety of $V_G(K)$ and $V_i \cap V_j = \{0\}$ for $i \neq j$. Then $L_\gamma = L_1 \oplus \ldots \oplus L_t$ where $V_G(L_i) = V_i$ and L_i is indecomposable for $i = 1, \ldots, t$.

10. Introduction and notation for the examples.

The specialized nature of the examples in the next three sections allows us to employ conventions which are outlined in this section.

(10.1). Notation. Use capital letters to denote the irreducible KG-modules. Let $D(X,f)$ be a module diagram. In place of a vertex x_i, we write a symbol consisting of a letter, denoting the isomorphism class of $f(x_i)$ and a subscript i indicating the index of the vertex. Every edge is denoted by a line between the symbols corresponding to the vertices of the edge. If there is an edge $e(x_i,x_j)$, then the relation $x_i > x_j$ is indicated by placing A_i above B_j $(f(x_i) \cong A,\ f(x_j) \cong B)$ on the page. Edges are labelled with extension classes only when necessary. For example, in a rigid string diagram no labeling on the edges is necessary (Proposition 6.4); it being assumed that the diagram is normalized (see Section 7).

As an example consider the diagram $D(X,f)$ in (10.2). It is a string diagram with vertices $x_1,\ldots,x_4$. Here $x_2 < x_1 > x_3 > x_4$, and $f(x_1) \cong f(x_4) \cong M$, $f(x_2) \cong K$ and $f(x_3) \cong N$.



(10.2)

(10.3). The diagram of $\Omega(M)$. Assume that M and all of the indecomposable projective summands in its projective cover, P_M, are representations of diagrams. Proposition 4.1 says we may write $P_M = \Sigma\ P_i$ where $x_1,\ldots,x_t$ are the closed points in the diagram $D(X,f)$ for M and $P_i = P_{f(x_i)}$. Let $V_i \subseteq X$ be the smallest open set containing x_i. Suppose that for each i there is a D-homomorphism $\theta_i : P_i \to M$ whose underlying diagram homomorphism ϕ_i has image V_i. Suppose further that $x_1, \ldots, x_t$ can be ordered in such a way that $V_i \cap V_j = \emptyset$ unless $i-1 \le j \le i+1$ and if $U_i = V_i \cap V_{i+1}$, then $U_i \cap U_j = \emptyset$ if $i \neq j$. Visually X must have the form

$$\begin{array}{ccccccccccc} W_1 & & & & W_2 & & & & W_{t-1} & & W_t \\ & \diagdown & & \diagup & & \diagdown & \diagup \;\cdots\; \diagup & & & \diagdown\;\diagup & \\ & & U_1 & & & & U_2 \quad \cdots & & & U_{t-1} & \end{array}$$

where $V_i = W_i \cup U_i \cup U_{i-1}$ $(U_0 = \emptyset = U_{t+1})$. By our assumptions in the diagram $D(X_i, f_i)$ X_i must have the form

$$\begin{array}{ccc} & W_i & \\ \diagup & & \diagdown \\ U_{i-1} & & U_i \\ \diagdown & & \diagup \\ & T_i & \end{array}$$

Then by repeated use of Proposition 3.5 the diagram $D(Y,g)$ for $\Omega(M)$ is the amalgamation.

$$D(Y,g) = D(\phi_1^{-1}(U_1), f_1) \times_{\phi_2^{-1}\phi_1} D(\phi_2^{-1}(U_1 \cup U_2), f_2) \times \cdots \times_{\phi_t^{-1}\phi_{t-1}} D(\phi_t^{-1}(U_{t-1}), f_t)$$

Thus Y has the form

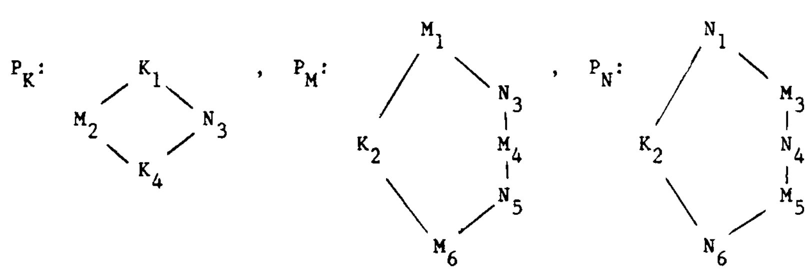
$$\begin{array}{ccccccccc} & U_1 & & U_2 & & U_{t-1} & & U_t & \\ \diagup & & \diagdown\;\diagup & & \diagdown \quad \cdots \quad \diagup & & \diagdown\;\diagup & & \diagdown \\ T_1 & & T_2 & & \cdots & & T_{t-1} & & T_t \end{array} .$$

(<u>10.4</u>). <u>Computation of $Ext^n_{KG}(M,N)$</u>. Here it is assumed that we have diagrams for $\Omega^n(M)$ and for N. If N is irreducible or if $\Omega^n(M)$ and N satisfy the hypothesis of Theorem 9.3, representatives for a basis for $Ext^n_{KG}(M,N)$ can be chosen as D-homomorphisms from $\Omega^n(M)$ to N. It may be necessary to establish that such a homomorphism does not factor through a projective module, thus being cohomologous to zero. This is clear if the underlying diagram homomorphism takes a closed point to a closed point, since then the image of $\Omega^n(M)$ would not be in the radical of N.

To compute cup products we use the following procedure. Suppose that $\zeta \in \mathrm{Ext}^m_{KG}(M,M)$, $\gamma \in \mathrm{Ext}^n_{KG}(M,N)$ are represented by D-homomorphisms $\hat{\zeta}\colon \Omega^m(M) \to M$ and $\hat{\gamma}\colon \Omega^n(M) \to N$. Find a D-homomorphism $\Omega^n(\hat{\zeta})\colon \Omega^{m+n}(M) \to \Omega^n(M)$ as in (9.2). The cup product $\gamma\zeta$ is represented by the composition $\hat{\gamma} \circ \Omega^n(\hat{\zeta})$. If $\mathrm{Ext}^n_{KG}(M,N)$ has dimension one, then any D-homorphism from $\Omega^{n+m}(M)$ to $\Omega^n(M)$ that does not factor through a projective will serve as $\Omega^n(\hat{\zeta})$ (at least up to scalar multiple). Otherwise we may compute $\Omega^n(\hat{\zeta})$ inductively by lifting $\Omega^{n-1}(\hat{\zeta})$ to a homomorphism of projective covers and observing the action on the kernels.

11. Example: $G = SL(3,2)$, characteristic of $K = 2$.

The principal block for KG has three irreducible modules which we denote K,M and $N \cong M^*$. They have dimension 1, 3 and 3 respectively. The only other irreducible KG-module is the Steinberg module which is projective. The projective covers of K,M and N have the following diagrams (see (9.1)) .



<u>Lemma 11.1</u>. If D(X,f) is a representable string diagram of modules in the principal block of KG , then it has a D-unique representation.

Proof. By Proposition 6.1 it is sufficient to show that all representable uniserial diagrams have unique representations. Note that every string diagram in the block is rigid, and hence we may assume it is normalized. By examining the projective modules and applying Lemma 5.7 it can be seen that the following is a complete list of uniserial diagrams of maximal length.

(11.2)

(i)	(ii)	(iii)	(iv)	(v)	(vi)
N	M	K	K	M	N
|	|	|	|	|	|
M	N	M	N	K	K
|	|				
N	M				
|	|				
M	N				

Clearly the four diagrams of length 2 have unique representations. Diagrams (i) and (ii) are the Ω translates of (vi) and (v) respectively. Likewise, by examining composition factors, we see that a uniserial module of length 3 must be either $\mathrm{Rad}^2(P_M)$ or $\mathrm{Rad}^2(P_N)$.

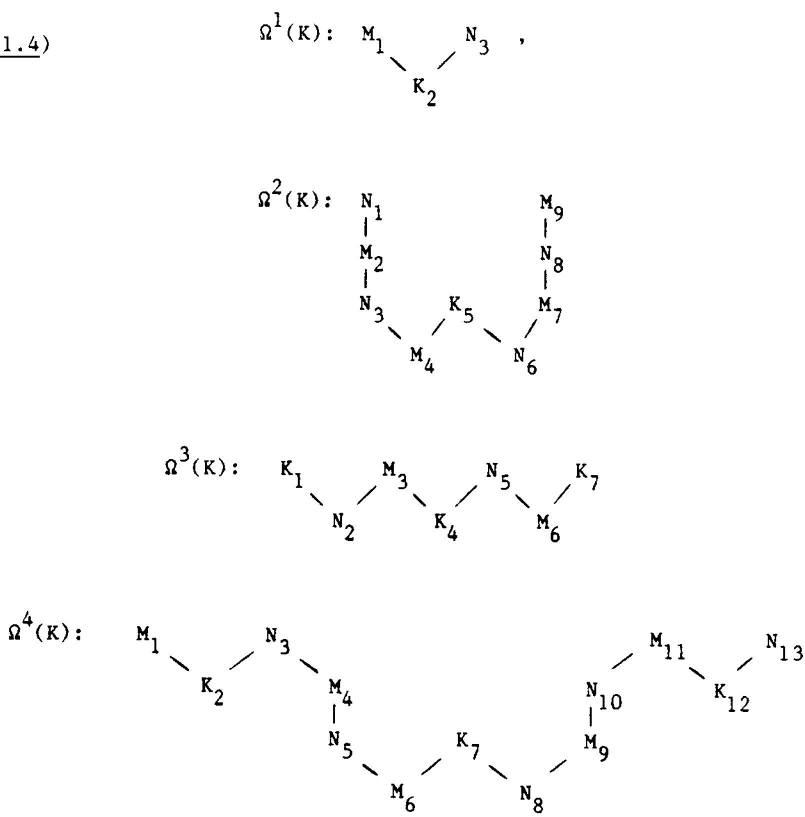
Theorem 11.3. The modules $\Omega^i(K)$, $1 \le i \le 6$, have diagrams as given in (11.4).

Proof. This follows by repeated use of Proposition 6.5 as outlined in (10.3). For the sake of clarity we show one calculation in detail. The diagram for $\Omega(K)$ is clear from that of P_K . By Proposition 4.1, the projective cover of $\Omega(K)$ is isomorphic to $P_M \oplus P_N$. Let $\sigma_1\colon P_M \to \Omega(K)$ and $\sigma_2\colon P_N \to \Omega(K)$ be D-homomorphisms corresponding respectively to diagram homomorphisms ϕ_1 and ϕ_2 which are as follows

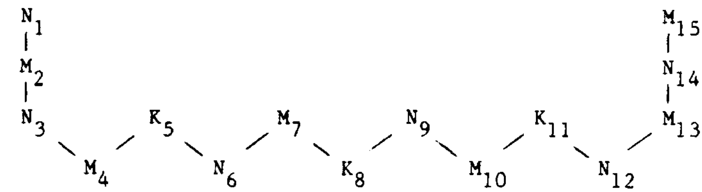
vertex of P_M	$\xrightarrow{\phi_1}$	vertex of $\Omega(K)$	vertex of P_N	$\xrightarrow{\phi_2}$	vertex of $\Omega(K)$
1		1	1		3
2		2 ,	2		2 .

By D-uniqueness there is no problem finding D-homomorphisms corresponding to ϕ_1 and ϕ_2. Both ϕ_1 and ϕ_2 take the vertex K_2 (in P_M and P_N respectively) to K_2 in the diagram for $\Omega(K)$. So $\phi_1^{-1}(\{K_2\})$ has vertices $\{K_2, N_3, N_4, M_5, M_6\}$ while $\phi_2^{-1}(\{K_2\})$ has vertices $\{K_2, M_3, N_4, M_5, N_6\}$ The diagram for $\Omega^2(K)$ is obtained by identifying the K_2 vertices as in (11.5).

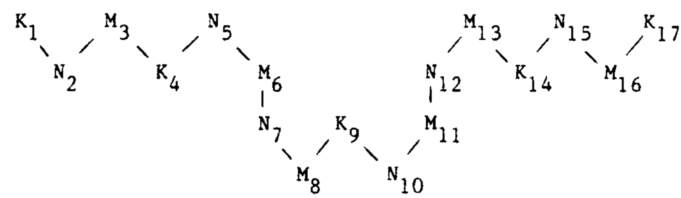
(11.4)



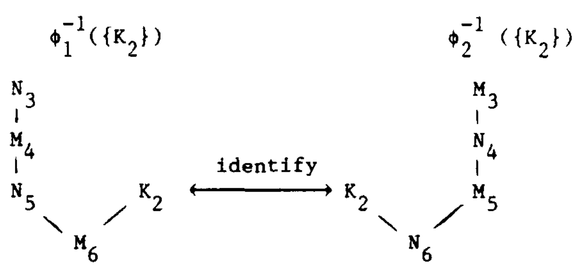
$\Omega^5(K)$:



$\Omega^6(K)$:



(<u>11.5</u>)



The diagrams for $\Omega^3(K)$,..., $\Omega^6(K)$ are worked out by repeated applications of this technique.

It should be noticed that in (11.4) the diagram for $\Omega^{r+2}(K)$ is obtained from that of $\Omega^r(K)$ by adjoining a uniserial (open) diagram at the end of each string. The diagrams that are added appear to depend on the residue class of r modulo 3. Using the results of section 9 it can be shown that the pattern persists for all r.

Clearly $\mathrm{Ext}^2_{KG}(K,K)$ has dimension 1. Let ζ be a generator. Then $\hat{\zeta}: \Omega^2(K) \to K$ is the obvious D-homomorphism, with kernel $L_\zeta \cong A \oplus B$ where A and B represent the diagrams (11.2) (i) and (ii) respectively. By examining the diagrams it can be seen that $\Omega^3(A) \cong A$, $\Omega^3(B) \cong B$ and $\Omega(A)$, $\Omega(B)$, $\Omega^2(A)$, $\Omega^2(B)$ are representations of (11.2) (iii), (iv), (v) and (vi) respectively. In particular $\Omega^3(L_\zeta) \cong L_\zeta$. By Lemma 8.6, $V_G(L_\zeta) = V_G(\zeta)$ is a union of two lines a and b with $V_G(A) = a$, $V_G(B) = b$.

Let $\hat{\gamma}_1, \hat{\gamma}_2 \colon \Omega^3(K) \to K$ be the D-homomorphisms corresponding to the closed points $\{K_1\}$ and $\{K_7\}$ in the diagram for $\Omega^3(K)$. We compute $\Omega(\hat{\gamma}_1)\colon \ \Omega^4(K) \to \Omega(K)$ by diagrammatically lifting $\hat{\gamma}_1$ to a homomorphism of the projective covers and seeing what happens in the kernels. Similarly we compute $\Omega^2(\hat{\gamma}_1)$, etc. We get that $\Omega^i(\hat{\gamma}_j)$ are D-homomorphisms with underlying diagram homomorphism given in (11.6).

(11.6)	vertex in domain	$\longrightarrow$ vertex in range	for values of r
$\Omega^1(\hat{\gamma}_1)$	r	r	r=1,2,3
$\Omega^2(\hat{\gamma}_1)$	r	r	r=1,...,7
$\Omega^3(\hat{\gamma}_1)$	r	r	r=1,...,6
$\Omega^1(\hat{\gamma}_2)$	r	r-10	r=11,12,13
$\Omega^2(\hat{\gamma}_2)$	r	r-6	r=9,...,15
$\Omega^3(\hat{\gamma}_2)$	r	r-10	r=12,...,17
$\Omega(\hat{\zeta})$	r	r-2	r=3,4,5
$\Omega^2(\hat{\zeta})$	r	r-2	r=3,...,11
$\Omega^3(\hat{\zeta})$	r	r-4	r=5,...,11

Note that $\hat{\gamma}_2 \circ \Omega^3(\hat{\gamma}_1) = 0 = \hat{\gamma}_1 \circ \Omega^3(\hat{\gamma}_2)$. So if γ_1 , $\gamma_2 \in \mathrm{Ext}^3_{KG}(K,K)$ are elements represented by $\hat{\gamma}_1$, $\hat{\gamma}_2$ then $\gamma_1\gamma_2 = \gamma_2\gamma_1 = 0$. Also $\hat{\zeta}\ \Omega^3(\hat{\gamma}_1)$ and $\hat{\zeta}\ \Omega^3(\hat{\gamma}_2)$ are

D-homomorphisms from $\Omega^5(K)$ to K corresponding to the closed points $\{K_5\}$ and $\{K_{11}\}$ respectively in $\Omega^5(K)$. Hence they represent a basis for $\mathrm{Ext}^5_{KG}(K,K)$, namely the elements $\zeta\gamma_1$ and $\zeta\gamma_2$. Similarly $\mathrm{Ext}^6_{KG}(K,K)$ has basis $\gamma_1^2, \zeta^3, \gamma_2^2$.

Now $L_{\zeta\gamma_1} = \ker(\hat{\zeta}\ \Omega^2(\hat{\gamma}_1)) \cong A \oplus C$ where C is an extension of the form $0 \to B \to C \to L_{\gamma_1} \to 0$. Since C is indecomposable by Theorem 8.1, it follows that $b \subseteq V_G(\gamma_1) = V_G(C)$. By Lemma 9.6, $a \not\subseteq V_G(\gamma_1)$. Now $V_G(\gamma_1) \cup V_G(\gamma_2) = V_G(K)$ since $\gamma_1\gamma_2 = 0$. Hence $V_G(\gamma_1 + \gamma_2) = V_G(\gamma_1) \cap V_G(\gamma_2)$ contains neither a nor b. We have established that $\gamma = \gamma_1 + \gamma_2$ generates the periodicity of L_ζ. Consequently, by Theorem 9.3, for every $j \geq 0$ there is an exact sequence

$$0 \longrightarrow \Omega^j(L_\zeta) \longrightarrow \Omega^{j+2}(K) \xrightarrow{\Omega^j(\zeta)} \Omega^j(K) \longrightarrow 0$$

Proposition 9.4 implies that ζ is not a zero divisor in $\mathrm{Ext}^*_{KG}(K,K)$. Theorem 9.5 provides the Poincaré series formula

$$P_{K,K}(t) = \frac{1}{1 - t^2}\,(1 + t^2\, P_{L_\zeta,K}(t))$$

From the periodicity of L_ζ and (11.2) we have that

$$P_{L_\zeta,K}(t) = 2t + 2t^4 + 2t^7 + \ldots \quad = \frac{2t}{1 - t^3}\,.$$

Hence $P_{K,K}(t) = \dfrac{1 + t^3}{(1-t^2)(1-t^3)} = \dfrac{1 - t^6}{(1-t^2)(1-t^3)^2}$. The final consequence of this analysis is the following.

<u>Theorem 11.7</u>. $H^*(G,K) \cong \mathrm{Ext}^*_{KG}(K,K) \cong K[\zeta,\gamma_1,\gamma_2]/(\gamma_1\gamma_2)$ where $\deg \zeta = 2$, $\deg \gamma_1 = \deg \gamma_2 = 3$.

Proof. By the above we have a homomorphism $\theta\colon K[x,y_1,y_2]/(y_1y_2) \to H^*(G,K)$ given by $\theta(x) = \zeta$, $\theta(y_1) = \gamma_1$, $\theta(y_2) = \gamma_2$. We need only show that θ is an isomorphism. Because the Poincaré series for the two rings are identical it suffices to show that θ is injective. By induction and the fact that multiplication by ζ is injective we have the following.

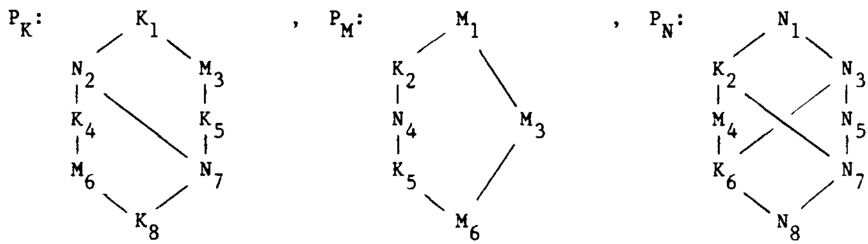
(1) If 3 does not divide r then $\mathrm{Dim}_K \mathrm{Ext}^r_{KG}(K,K) = \mathrm{Dim}_K \mathrm{Ext}^{r-2}_{KG}(K,K)$ and we are done.

(2) If $r = 3t$, then $\mathrm{Dim}_K \mathrm{Ext}^r_{KG}(K,K) = \mathrm{Dim}_K \mathrm{Ext}^{r-2}_{KG}(K,K) + 2$. Hence it is sufficient to exhibit two linearly independent elements in the image of θ in $\mathrm{Ext}^r_{KG}(K,K)/(\zeta \cdot \mathrm{Ext}^r_{KG}(K,K))$. We claim that γ_1^t, γ_2^t are such elements. This is because $V_G(\zeta) = a \cup b$ but $a \not\subseteq V_G(\gamma_2)$ and $b \not\subseteq V_G(\gamma_1)$.

12. Example: $G = M_{11}$, characteristic of $K = 2$.

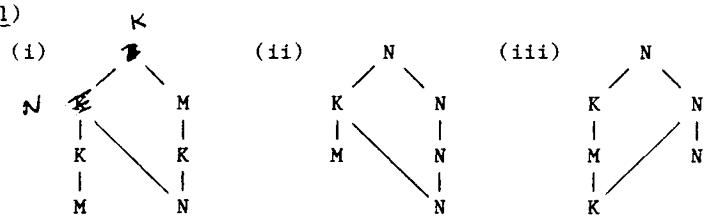
Having giving one example in the last section we abbreviate all arguments which are similar in nature. The cohomology ring of M_{11}, the Mathieu group on 11 letters has not been previously calculated. We shall also compute the action of the Steenrod algebra on the cohomology ring.

In the principal block of KG there are three irreducible modules K, M and N having dimensions 1, 44 and 10 respectively. The projective covers of these modules are representations of the following diagrams. As before we are not giving the assignments on the edges.

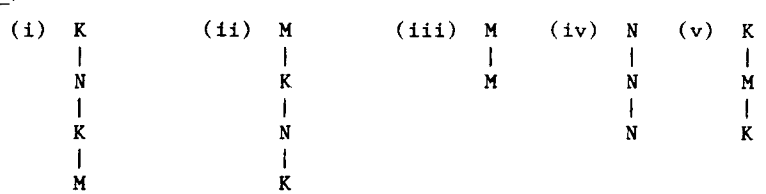


The structures of these modules were calculated independently by Alperin (unpublished) and Schneider [16]. They are not difficult to compute using the permutation modules for M_{11} , and the modules for the subgroup $M_{10} = A_6 \cdot 2$.

(12.1)



(12.2)



Lemma 12.3. For each of the diagrams in (12.1) there exists an assignment of extension classes so that the diagram has a representation. Moreover any two representations of one of the diagrams are isomorphic regardless of the (representable) assisgnments and if the assignments are the same they are D-isomorphic. The same result holds for all closed subdiagrams

of those in (12.1) and for their duals. In (12.2) is a complete list of representable uniserial diagrams of maximal length. All representable uniserial diagrams have unique representations.

Proof. Observe first that all string diagrams are rigid and hence can be assumed to be normalized. The proof uses Propositions 4.1 and 5.2 and a simple count of composition factors. For example, any representation of (12.2,i) must be the Ω translate of the unique representation of (12.2,iii). Similarly Ω takes (12.2,i) to (12.2,ii). A representation of (12.2,iv) must be isomorphic to the unique submodule of $\mathrm{Rad}^2\, P_N$ having those composition factors. The remainder of the proof concerning uniserial modules is similar.

By Proposition 4.1 any module representing one of the diagrams in (12.1) must be a factor of P_K or of P_N. The uniqueness of the kernels implies the uniqueness of the representations of (12.1). The D-uniqueness is likewise implied by the fact that the kernel is the unique submodule of the given projective with the prescribed composition factors.

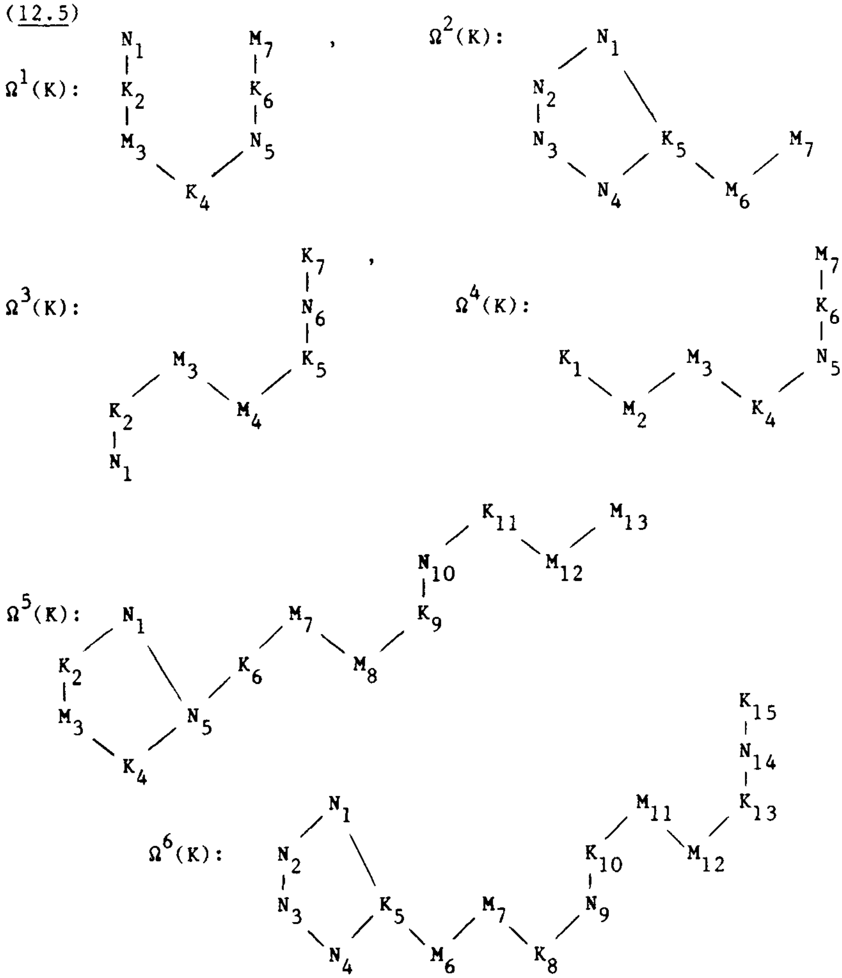
Theorem 12.4. The modules $\Omega^i(K)$, $i=1, \dots, 7$ are the D-unique representations of the diagrams in (12.5).

Proof. The diagrams are constructed using (10.3) as in the proof of Theorem 11.3. The D-uniqueness follows from Lemma 12.3 and Proposition 5.2 and its dual.

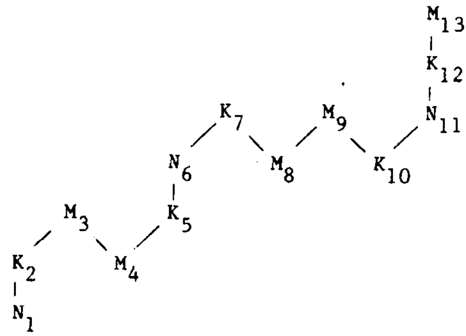
We proceed now to calculate the cohomology ring $H^*(G,K)$. Notice that $\mathrm{Dim}\, H^i(G,K)$ is zero for $i = 1$ and 2 and one for

$i = 3,4,5,6$ and 7 . Let α, β and γ be nonzero cohomology elements in degrees 3, 4 and 5 respectively. Then α is represented by a D-homomorphism $\hat{\alpha}: \Omega^3(K) \to K$. As in the last section we apply the methods of (10.4) to compute the underlying diagram homomorphisms for the maps $\Omega^j(\hat{\alpha}): \Omega^{3+j}(K) \to \Omega^j(K)$. The results are given in table (12.6). It is immediate that

(<u>12.5</u>)



$\Omega^7(K)$:



(12.6)	vertex in domain		→ vertex in range		for values of r
$\hat{\alpha}$	7	$(\Omega^3(K))$	1	(K)	
$\Omega(\hat{\alpha})$	r	$(\Omega^4(K))$	r	$(\Omega(K))$	$3 \le r \le 7$
$\Omega^2(\hat{\alpha})$	r	$(\Omega^5(K))$	$r-6$	$(\Omega^2(K))$	$10 \le r \le 13$
$\Omega^3(\hat{\alpha})$	r	$(\Omega^6(K))$	$r-8$	$(\Omega^3(K))$	$9 \le r \le 15$
$\Omega^4(\hat{\alpha})$	r	$(\Omega^7(K))$	$r-6$	$(\Omega^4(K))$	$7 \le r \le 13$
$\hat{\beta}$	1	$(\Omega^4(K))$	1	(K)	
$\Omega(\hat{\beta})$	r	$(\Omega^5(K))$	r	$(\Omega(K))$	$1 \le r \le 7$
$\Omega^2(\hat{\beta})$	r	$(\Omega^6(K))$	r	$(\Omega^2(K))$	$1 \le r \le 7$
$\Omega^3(\hat{\beta})$	r	$(\Omega^7(K))$	r	$(\Omega^3(K))$	$1 \le r \le 7$
$\hat{\gamma}$	11	$(\Omega^5(K))$	1	(K)	
$\Omega(\hat{\gamma})$	r	$(\Omega^6(K))$	$r-4$	$(\Omega(K))$	$7 \le r \le 11$
$\Omega^2(\hat{\gamma})$	r	$(\Omega^7(K))$	$r-2$	$(\Omega^2(K))$	$6 \le r \le 9$

$\Omega^6(\hat{\beta})$	r	$(\Omega^{10}(K))$	r	$(\Omega^6(K))$	$1 \le r \le 15$
$\Omega^5(\hat{\gamma})$	r	$(\Omega^{10}(K))$	$r-4$	$(\Omega^5(K))$	$7 \le r \le 17$

$\hat{\alpha} \circ \Omega^3(\hat{\alpha})$ and $\hat{\alpha} \circ \Omega^3(\hat{\beta})$ represent nonzero cohomology elements. Hence $H^6(G,K)$ and $H^7(G,K)$ are generated by α^2 and $\alpha\beta$ respectively. By direct calculation of the projective resolutions it can be seen that the kernels L_α , L_β of $\hat{\alpha}$ and $\hat{\beta}$ are periodic of periods 4 and 3 respectively. By Theorem 8.1

and Lemma 9.6 the varieties $V_G(L_\alpha) = V_G(\alpha)$ and $V_G(L_\beta) = V_G(\beta)$ are lines in $V_G(K)$. Also by Lemma 9.6, $V_G(\alpha\beta) = V_G(L_{\alpha\beta}) = V_G(\alpha) \cup V_G(\beta)$ has two components since $L_{\alpha\beta}$ is not indecomposable. Therefore $V_G(\alpha) \cap V_G(\beta) = \{0\}$, and α generates the periodicity of L_β . Since $\Omega^i(\hat{\beta})$ is onto for $i = 1,2,3$, we have an exact sequence

$$0 \longrightarrow \Omega^j(L_\beta) \longrightarrow \Omega^{j+4}(K) \xrightarrow{\Omega^j(\hat{\beta})} \Omega^j(K) \longrightarrow 0 \qquad (12.7)$$

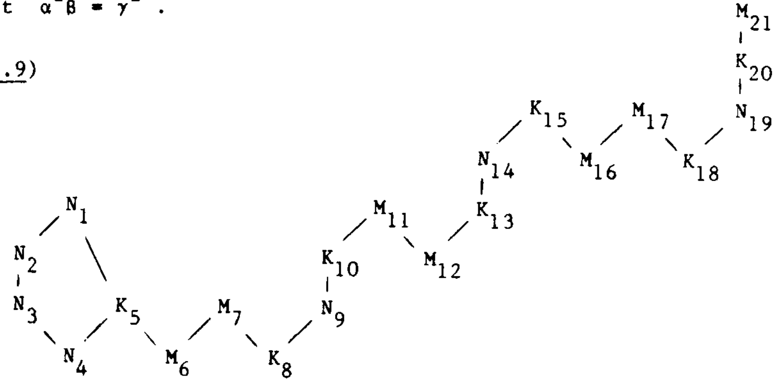
for all $j \geq 0$ by Theorem 9.3. Using Theorem 9.5 and the series $P_{L_\beta,K}(t) = (t + t^2)/(1 - t^3)$ we obtain the Poincaré series.

$$P_{K,K}(t) = \frac{1 + t^5}{(1-t^4)(1-t^3)} = \frac{1 - t^{10}}{(1-t^3)(1-t^4)(1-t^5)} .$$

<u>Lemma 12.8</u>. For proper choice of γ , $\alpha^2\beta = \gamma^2$ in $\mathrm{Ext}^{10}_{KG}(K,K)$.

<u>Proof</u>. By continuing the calculation in Theorem 12.4 or by using (12.7) we may establish that $\Omega^{10}(K)$ is a representation of the diagram in (12.9). The underlying diagram homomorphisms for $\Omega^6(\hat{\beta})$, $\Omega^5(\hat{\gamma})$ are given in (12.6). So $\hat{\alpha} \circ \Omega^3(\hat{\alpha}) \circ \Omega^6(\hat{\beta})$ has the same underlying diagram homomorphism as $\hat{\gamma} \circ \Omega^5(\hat{\gamma})$ and replacing γ by a suitable K-multiple, if necessary, we get that $\alpha^2\beta = \gamma^2$.

(<u>12.9</u>)



Theorem 12.10. $H^*(G,K) \cong \mathrm{Ext}^*_{KG}(K,K) \cong K[\alpha,\beta,\gamma]/(\alpha^2\beta+\gamma^2)$ where $\deg \alpha = 3$, $\deg \beta = 4$ and $\deg \gamma = 5$.

Proof. There exists a homomorphism

$$\theta\colon K[X_3,X_4,X_5]/(X_3^2X_4 - X_5) \longrightarrow H^*(G,K)$$

defined by $\theta(X_3) = \alpha$, $\theta(X_4) = \beta$, $\theta(X_5) = \gamma$. Since the Poincaré series for the two rings are identical, θ is an isomorphism if it is injective. Injectivity is proved by induction on degree and the fact that cup product with β is injective. There are three cases to consider.

1) If 3 divides $r-1$, then $\mathrm{Dim}_K H^r(G,K) = \mathrm{Dim}_K H^{r-4}(G,K)$ and we are done.

2) If 3 divides r, then $\mathrm{Dim}_K H^r(G,K) = \mathrm{Dim}_K H^{r-4}(G,K) + 1$. In this case $\alpha^{r/3}$ is an element in $H^r(G,K)$ that is not in the image of β, since $V_G(\alpha) \cap V_G(\beta) = \{0\}$.

3) If 3 divides $r - 2$, then $\mathrm{Dim}_K H^r(G,K) = \mathrm{Dim}_K H^{r-4}(G,K) + 1$. We may assume $r \geq 5$. Let $n = (r-5)/3$. We need only show that $\alpha^n\gamma$ is not a multiple of β. Suppose that $\alpha^n\gamma = \mu\beta$ for $\mu \in H^{r-4}(G,K)$. Then $\mu\beta\gamma = \alpha^n\gamma^2 = \alpha^{n+2}\beta$. As multiplication by β is injective, $\mu\gamma = \alpha^{n+2}$. This implies that $V_G(\alpha) = V_G(\mu) \cup V_G(\gamma)$. But since $\gamma^2 = \alpha^2\beta$, $V_G(\gamma) = V_G(\alpha) \cup V_G(\beta)$. However $V_G(\beta) \not\subseteq V_G(\alpha)$ and we have a contradiction. This completes the proof.

Now consider the action of the Steenrod algebra. Evens and Priddy [11] have computed the cohomology ring of the semi-dihedral

2-group, $Q_n = SD(2^n)$, of order 2^n . They showed that

$$H^*(Q_n,K) = K[x,y,z,w]/(xy,\ x^3,\ xz,\ z^2 + wy^2)$$

where the degrees of x, y, z and w are 1,1,3 and 4 respectively. Also the Steenrod algebra **A**(2) acts as follows.

	Sq^1	Sq^2	Sq^4
x	x^2	0	0
y	y^2	0	0
z	0	$y^2z + wy$	0
w	0	z^2	w^2

The Sylow 2-subgroup Q of $G = M_{11}$ is semi-dihedral of order 16. To compute the action of **A**(2) on the cohomology of M_{11} , it is only necessary to know the embedding given by the restriction map $\phi = \mathrm{res}_{G,Q}: H^*(G,K) \to H^*(Q,K)$.

Lemma 12.11. The restriction map is given as follows: $\phi(\alpha) = z$, $\phi(\beta) = w + y^4$, $\phi(\gamma) = y^2z + wy$.

Proof. Because these calculations may be performed over the prime field $\mathbf{F}^2$ there are three possibilities for $\phi(\alpha)$, namely y^3, z, and $y^3 + z$. Suppose first that $\phi(\alpha) = y^3$. Since $Sq^1(y^3) = y^4 \neq 0$ we must have that $\phi(Sq^1(\alpha)) = Sq^1(\phi(\alpha)) = \phi(\beta) = y^4$. But then $\phi(\gamma^2) = \phi(\alpha^2\beta) = y^{10}$ and the restriction map ϕ is not injective. This is not possible. So suppose that $\phi(\alpha) = y^3 + z$. Again $Sq^1(\phi(\alpha)) = y^4 = \phi(\beta)$, and $\phi(\gamma^2) = \phi(\alpha^2\beta) = (y^3+z)^2\, y^4 = (y^5 + y^2z)^2$. It is easily seen that the squaring map from $H^5(Q,K)$ to $H^{10}(Q,K)$ is an injection. Hence $\phi(\gamma) = y^5 + y^2z$. But then $Sq^1(\phi(\gamma)) = y^6$ is not in the image of ϕ .

We can conclude that $\phi(\alpha) = z$. Because $Sq^2(z) = y^2z + wy \neq 0$, it must be that $\phi(\gamma) = y^2z + wy$. Hence $\phi(\gamma^2) = \phi(\alpha^2\beta) = y^4z^2 + w^2y^2 = (y^4 + w)z^2$. Since $H^4(Q,K)$ has basis $\{y^4, yz, w\}$ it is necessary that $\phi(\beta) = y^4 + w$. This proves the lemma.

Theorem 12.12. The action of the Steenrod algebra $\mathbf{A}(2)$ on $H^*(M_{11},K)$ is given by the table

	degree	Sq^1	Sq^2	Sq^4
α	3	0	γ	0
β	4	0	α^2	β^2
γ	5	α^2	0	$\alpha^3 + \beta\gamma$

Remark 12.13. It is easy to see from the diagrams for $\Omega^r(K)$ that the trivial module K has a minimal projective resolution that is the total complex of the almost periodic double complex in Figure 1 of [5]. Moreover the double grading on $H^*(G,K)$, coming from the double complex, is compatible with the cup-product structure. In particular the double degrees of α , β and γ are (3,0), (2,2) and (4,1) respectively. Unfortunately the double grading does not fit in well with the action of the Steenrod algebra. Hence there seems to be no good diagrammatic interpretation of this action.

Using methods similar to those of Theorem 12.10 it is also possible to calculate the cohomology rings for the other two irreducible modules.

<u>Theorem 12.14</u>. $\mathrm{Ext}^*_{KG}(M,M) \cong K\langle \gamma_1, \gamma_2 \rangle / (\gamma_2^2,\ \gamma_1^2\gamma_2 - \gamma_2\gamma_1^2)$ (non-commutative) where $\deg(\gamma_i) = i$.
$\mathrm{Ext}^*_{KG}(N,N) \cong K[\mu_1, \mu_2, \mu_4]/(\mu_1^2, \mu_2^2)$, where $\deg(\mu_i) = i$.

We leave the details of the proof to the reader. The module N is periodic of period 4 and $\mathrm{Ext}^*_{KG}(N,N)$ is commutative. However, $\mathrm{Ext}^*_{KG}(M,M)$ is definitely not commutative, even modulo its radical (see [9]).

The modules $\Omega^r(M)$ are all representations of string diagrams, and it can be seen that the minimal resolution of M is the total complex of the double complex given in (12.15).

(<u>12.15</u>)

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
P_M & \longleftarrow & P_M & \longleftarrow & P_K & \longleftarrow & P_M & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
P_M & \longleftarrow & P_K & \longleftarrow & P_M & \longleftarrow & P_M & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
P_K & \longleftarrow & P_M & \longleftarrow & P_M & \longleftarrow & P_K & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
P_M & \longleftarrow & P_M & \longleftarrow & P_K & \longleftarrow & P_M & \longleftarrow & \cdots
\end{array}$$

It appears that the cohomology has a basis, any element of which, when viewed as a map of double complexes is either orientation preserving or reversing. That is, $\zeta(C_{p+a,q+b}) \subseteq C_{p,q}$ or

$\zeta(C_{p+a,q+b}) \subseteq C_{q,p}$ for $\deg \zeta = a + b$. We say that such a map has bidegree $(a,b)^+$ in the first case or $(a,b)^-$ in the second. In these terms $2\text{-deg}(\gamma_1) = (1,0)^-$, $2\text{-deg}(\gamma_2) = (0,2)^-$, so that $2\text{-deg}(\gamma_1\gamma_2) = (3,0)^+$ while $2\text{-deg}(\gamma_2\gamma_1) = (0,3)^+$. This seems related to the fact that the underlying diagram homomorphisms for $\Omega^r(\hat{\gamma}_1)$ and $\Omega^r(\hat{\gamma}_2)$ reverse the seemingly natural orientation on the strings.

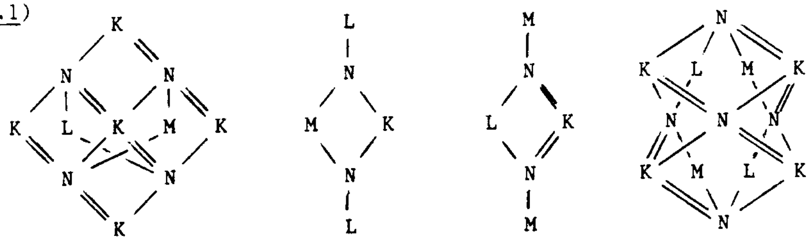
13. Example: $G = A_6$, characteristic of $K = 3$.

This is the most complicated example that we have tried to tackle by these methods. Since $G = A_6 \cong PSL(2,9)$ the cohomology ring $H^*(G,K)$ is known [7] and we will not repeat the calculation. Rather the interest of this example lies in the explicit diagrams for $\Omega^r(K)$ and in the double complex whose total complex is the minimal projective resolution for K. The details of the calculation are left to the reader.

In the principal block of KG there are four irreducible modules K, L, M and N of dimensions 1, 3, 3 and 4 respectively. Each of these modules is self dual. Since $\operatorname{Dim} \operatorname{Ext}^1_{KG}(K,N) = 2$ we must be careful about the labeling of the edges in diagrams. We have chosen bases for $\operatorname{Ext}^1_{KG}(K,M)$ and $\operatorname{Ext}^1_{KG}(M,K)$ and in each case we represent one basis element by a single line and the other by a double line. The indecomposable projective modules represent the diagrams (13.1) [2].

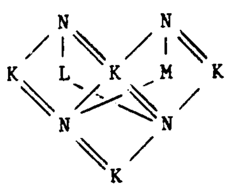
<u>Theorem 13.2</u>. The modules $\Omega^i(K)$, $1 \leq i \leq 9$ are representations of the diagrams in (13.3).

(<u>13.1</u>)

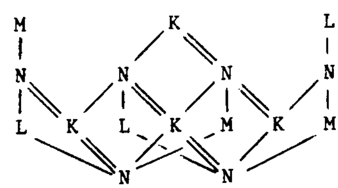


(<u>13.3</u>)

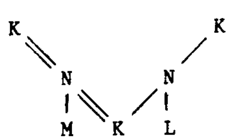
$\Omega^1(K)$:



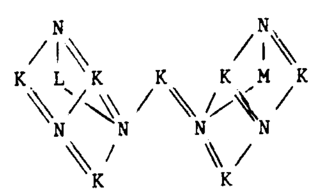
$\Omega^2(K)$:



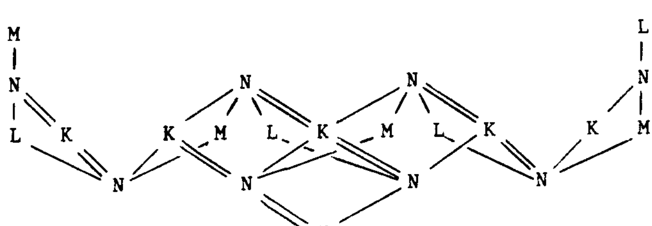
$\Omega^3(K)$:



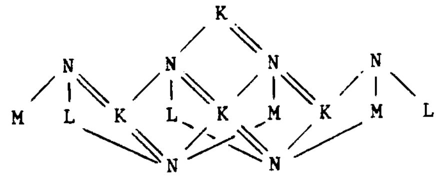
$\Omega^4(K)$:



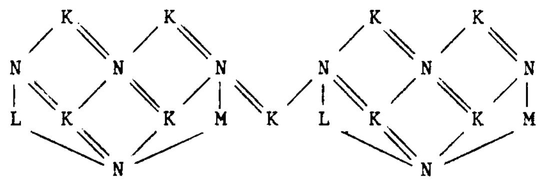
$\Omega^5(K)$:



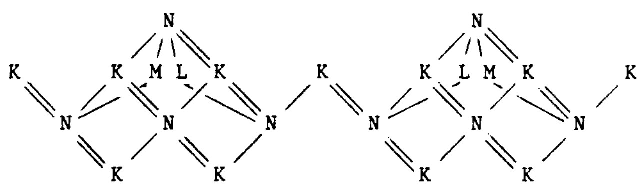
$\Omega^6(K)$:



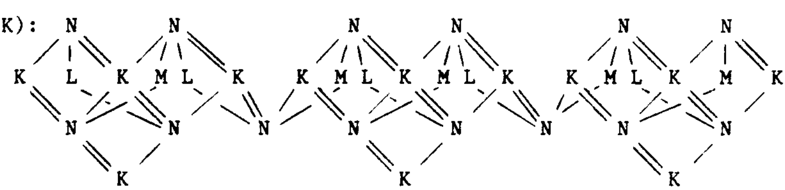
$\Omega^7(K)$:



$\Omega^8(K)$:

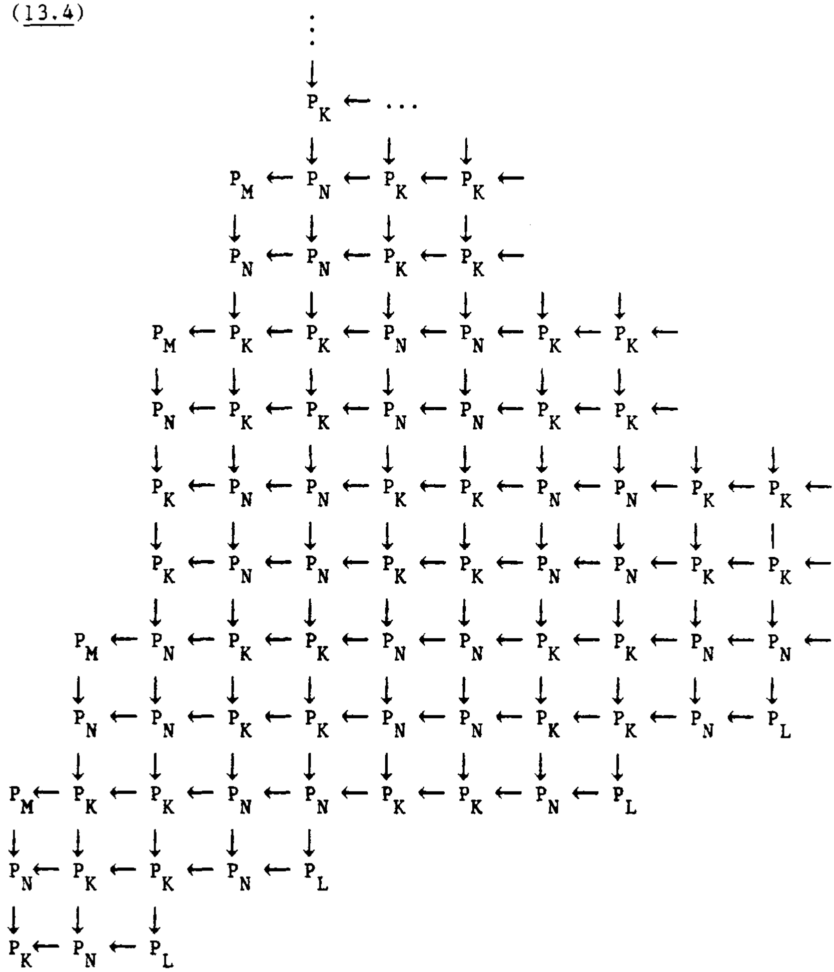


$\Omega^9(K)$:



The diagrams have a natural periodicity of order 8. This can be seen more clearly by calculating more of these diagrams. From the calculation of the diagrams it can be shown that the minimal projective resolution of K is the total complex of the double complex sketched in (13.4).

(13.4)

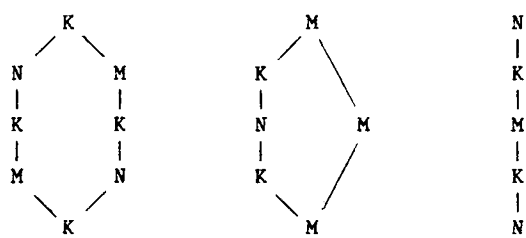


14. Other examples.

Space does not permit us to explore all of the examples that can be addressed using diagrammatic methods. In this section we list without detail a few other calculations that we have made. Other possible examples might be found among the calculations in [10].

(14.1). $G = PSL(3,3)$, characteristic of $K = 2$. The principal block of KG has three irreducible modules K, M, N of dimensions 1, 12, 26 respectively. The diagrams for their projective covers are exactly the same as those given at the beginning of Section 12 for the modules for M_{11}. Hence there is an equivalence of categories and all of the results of Section 12 hold in this case.

(14.2). $G = A_7$, characteristic of $K = 2$. The principal block of KG irreducible modules K, M and N of dimensions 1, 14 and 20. The projective covers have the following diagrams



Note that uniserial diagrams have unique representations and string diagrams are rigid. It can be shown that

$$\mathrm{Ext}^*_{KG}(K,K) \cong K[\alpha, \beta_1, \beta_2]/(\beta_1\beta_2)$$

where $\deg \alpha = 2$, and $\deg \beta_i = 3$;

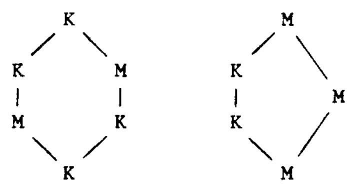
$$\mathrm{Ext}^*_{KG}(M,M) = K\langle\gamma_1,\gamma_2\rangle/(\gamma_2^2,\ \gamma_1^2\gamma_2 - \gamma_2\gamma_1^2)$$

where $\deg \gamma_i = i$; and

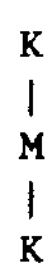
$$\mathrm{Ext}^*_{KG}(N,N) = K[\mu_2,\ \mu_3]/(\mu_2^2)$$

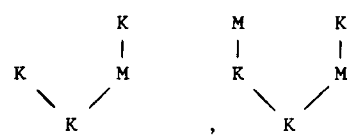
where $\deg \mu_i = i$. Notice that the cohomology ring of M is isomorphic to that of the 44-dimensional module for M_{11} (see Theorem 12.14). The module N is periodic with period 3.

(14.3). $G = S_4$, characteristic of $K = 2$. The group algebra has two simple modules K and M of dimensions 1 and 2. The projective covers are representations of the diagrams



Here we must be careful because the uniserial diagram (14.4) does not have a unique representation. However the diagrams (14.5) do have unique representations. This permits the calculation of

(14.4)


(14.5)


the cohomology. We get that

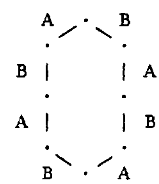
$$\mathrm{Ext}^*_{KG}(M,M) \cong K\langle\gamma_1,\gamma_2\rangle/(\gamma_2^2,\gamma_1^2\gamma_2-\gamma_2\gamma_1^2) , \quad \deg \gamma_i = i .$$

Again the latter is the same as that for the 44-dimensional module for M_{11} (Theorem 12.14).

(14.6) $G = D_8$, (dihedral group of order 8), characteristic of $K = 2$. The diagrams for modules in this case have been treated thoroughly by Ringel [14]. Since G is a

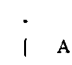
2-group, the vertices in a diagram may be taken to represent basis elements of the module and the edges to represent multiplication by actual elements of the group ring. Ringel's method is to write $G = \langle x,y \mid x^4 = y^2 = (xy)^2 = 1\rangle$ and let $A = 1 + y$, $B = 1 + xy$. Then

$$KG \cong K\langle A,B\rangle/(A^2,B^2,(AB)^2 - (BA)^2) .$$



(14.7)

As a projective module, KG is the representation of diagram (14.7). Note that the edges, but not the vertices, are labeled. Uniserial diagrams do not have unique representations, but Ringel's classification of the modules deals with this problem adequately. Ringel does not compute cohomology. We can make some calculations using the methods of this paper. For example, let $M = KG/(1+x^2, 1+xy)$. This is the induced module $K_H{\uparrow}^G$ where $H = \langle x^2,xy\rangle$, and is represented by the diagram (14.8). Then



$$\mathrm{Ext}^*_{KG}(M,M) \cong K\langle\gamma_0,\gamma_1\rangle/(\gamma_0^2, \gamma_0\gamma_1^2 - \gamma_1^2\gamma_0)$$

(14.8)

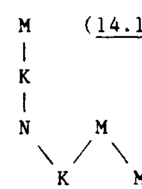
where $\deg \gamma_i = i$. Here γ_0 is the obvious nilpotent D-endomorphism of M . As noted in [9], this appears to be the source of the noncommutativity of the cohomology rings $\mathrm{Ext}^*_{KG}(M,M)$ for M as in (12.14), (14.2) and (14.3). All of the modules are direct summands of modules that are induced from the M given here for some D_8 contained in the group in question.

(14.9) Almost split sequences. In Section 3 we saw how to construct modules which are pushouts (or dually, pullbacks) provided all of the modules in the pushout diagram have module diagrams and the maps are D-homomorphisms. The almost split sequence of a module L is the pushout of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(L) & \longrightarrow & P_L & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta} & & & & \| & & \\ 0 & \longrightarrow & \Omega^2(L) & \longrightarrow & & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

where P_L is the projective cover of L and the cohomology class of θ is in the $\overline{\mathrm{Hom}}_{KG}(L,L)$ - socle of $\mathrm{Ext}^1_{KG}(L,\Omega^2(L))$ [3]. Here $\overline{\mathrm{Hom}}$ denotes the Hom group modulo those maps that factor through a projective. If, for example, all maps in $\mathrm{Hom}_{KG}(\Omega(L), \Omega(L))$ and in $\mathrm{Hom}_{KG}(\Omega(L), \Omega^2(L))$ are linear combinations of D-homomorphisms then it can be seen that θ can be taken to be a D-homomorphism, and sometimes the middle term of the almost split sequence can be constructed as an amalgamation. The one problem that arises is that the amalgamation might not satisify condition (2.1, iii) (see Propositon 3.5).

For an easy example, let L be the representation of diagram (12.2,iii). Then $\Omega^2(M)$ has diagram (12.2,ii) and the middle term of the almost split sequence has diagram (14.10).



(14.10)

15. Extending the Algebra.

It is clear that the diagrammatic methods, as presented here, do not work well for p-groups. The problem is that $\mathrm{Dim}\ \mathrm{Ext}^1_{KG}(K,K) > 1$ whenever G is a noncyclic p-group. For cyclic p-groups it is very easy to calculate cohomology using diagrams. If the p-group G has an automorphism of order prime to p then the following technique may be used. Consider the group $\hat{G}$ which is the split extension of G by the cyclic group T generated by the automorphism. The irreducible $K\hat{G}$-modules are precisely the one-dimensional KT modules $M_1,\ldots,M_r$. Then $M_{i|G} \cong K$ and by Shapiro's Lemma $\mathrm{Ext}^n_{KG}(K,K) \cong \mathrm{Ext}^n_{K\hat{G}}(K,K_G^{\uparrow\hat{G}}) \cong \bigoplus_{i=1}^{r} \mathrm{Ext}^n_{K\hat{G}}(K,M_i)$. The groups $\mathrm{Ext}^1_{K\hat{G}}(K,M_i)$ are smaller and easier to handle. For example, if G is elementary abelian of order p^n, then T may be taken to be the group generated by the Singer cycle, which is an automorphism of order $p^n - 1$. This is obtained by regarding G as the additive group of the field with p^n elements and T as the multiplicative group of nonzero elements.

Many p-groups, however, have no automorphism of order prime to p. In this case, one may resort to extending the group algebra to an algebra which is not a group algebra. We illustrate this technique by the example of the semi-dihedral group of order 2^n over a field K of characteristic 2.

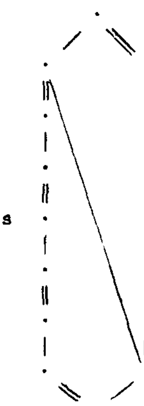
The semi-dihedral group $SD(2^n)$ of order 2^n is given by generators and relations as follows:

$$G = \langle g,h \mid g^{2^{n-1}} = h^2 = 1 ,\ hgh^{-1} = g^{2^{n-2}-1} \rangle .$$

Let $a = gh + 1$, $b = h + 1 \in KG$. Then

$$KG \cong K\langle a,b\rangle / \langle b^2,\ (ba)^m b - a^2 \rangle$$

where $m = 2^{n-2} - 1$ (see (1.5) of [15]). In fact our method works for any value of $m \geq 1$. From this presentation it can be seen that the projective module KG is a representation of the diagram (15.1). The case shown is that of $G = SD(16)$, $m = 3$. The general case is similar. In fact, none of the calculations depend on the value of m . So we shall continue to draw only the case $m = 3$. In the diagram all of the vertices are labeled with the unique simple KG-module K . Because $\mathrm{Ext}^1_{KG}(K,K)$ has dimension 2 we have chosen a basis for the Ext group so that

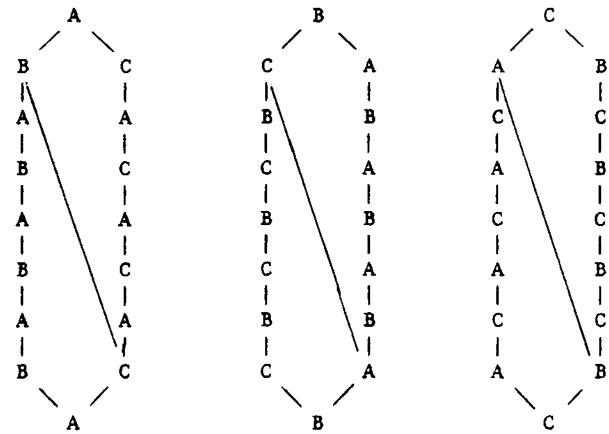


$$\vert \cong KG/\langle a^2,b\rangle , \qquad \Vert \cong KG/\langle a,b^2\rangle \qquad (\underline{15.1})$$

In some sense the single bond corresponds to multiplication by a while the double represents multiplication by b .

We extend the algebra in such a way that the Ext^1 groups become one dimensional. That is, we construct an algebra of dimension $3|G|$ with 3 simple modules A, B, and C whose projective covers are representations of the diagrams in (15.2). Such an algebra can be easily created by taking a vector space whose basis consists of the diagram endomorphisms of the disjoint union of the above diagrams, with nonempty connected images. An arbitrary diagram endomorphism is identified with the appropriate sum of basis elements and multiplication is defined by bilinear extension of composition of diagram endomorphisms. In the present case it is easier to write down the algebra by inspection. We assume that K contains a primitive cube root of

(15.2)



unity, ω . Then the algebra can be given as follows

$$R = K\langle a,b,c\rangle/\langle b^2,\ (ba)^m b - a^2,\ c^3 - 1\ ,\ cac^2 - \omega a,\ cbc^2 - \omega^2 b\rangle\ .$$

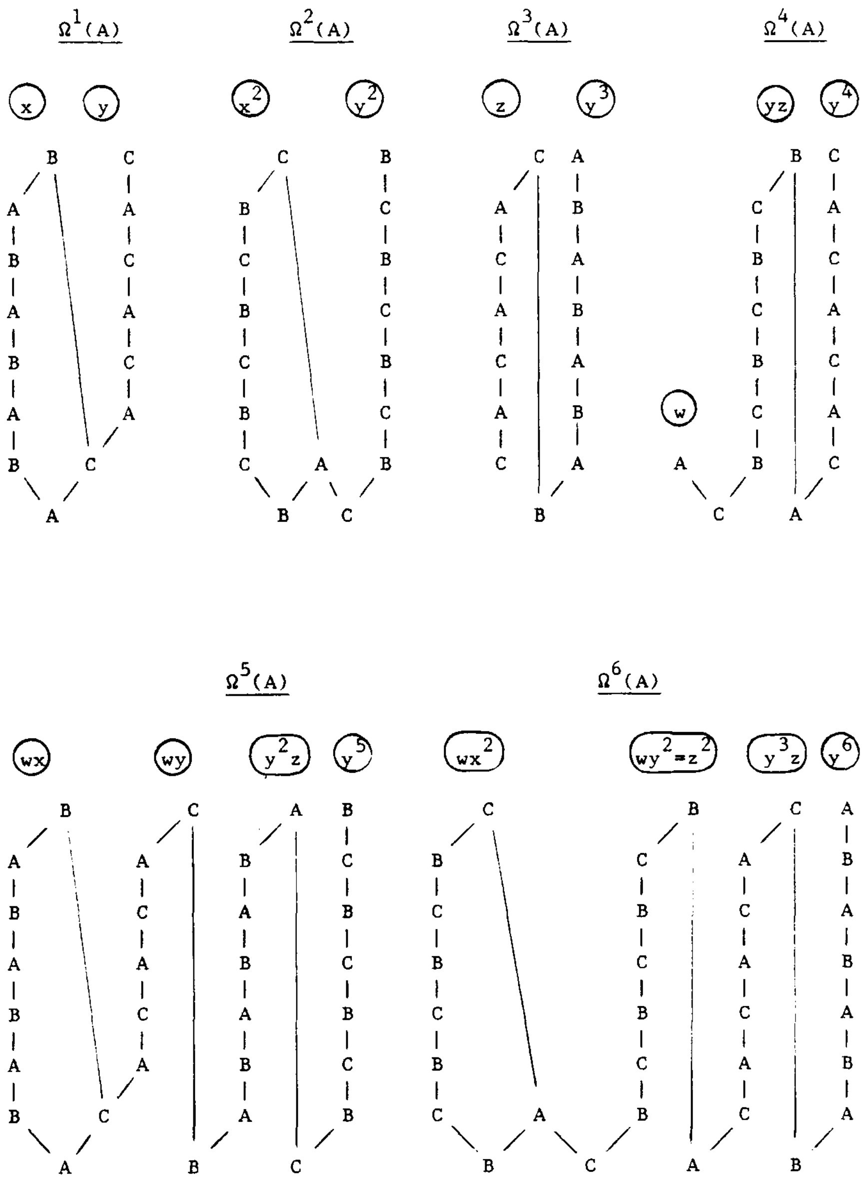
In the representations A, B and C , the element c acts by multiplication by 1, ω, and ω^2 , respectively.

The following may be proved by the same methods used in the examples of sections 11 and 12.

Theorem 15.3. Every representable string diagram for R has a D-unique representation. The modules $\Omega^i(A)$, $i=1$,..., 6 are the unique representation of the diagrams in (15.4). Here edges which are not incident to the bottom vertex of a cycle are assumed to be normalized.

Restricting back to KG provides diagrams for the modules $\Omega^i(K)$, $i=1$,..., 6 . We have given names in circles to the

(15.4)



Ω^1(A)
Ω^2(A)
Ω^3(A)
Ω^4(A)
x y
x^2 y^2
z y^3
w yz y^4
Ω^5(A)
Ω^6(A)
wx wy y^2z y^5
wx^2 wy^2=z^2 y^3z y^6

cohomology elements displayed by the resulting diagram. For example the element $x \in \mathrm{Ext}^1_{KG}(K,K)$ is represented by the cocyle $\hat{x}\colon \Omega^1(K) \to K$ which takes the top constituent marked B in $\Omega^1(A)$ to K by the identity map. Of course the restriction of A, B and C to KG are all isomorphic to K . The computation of the cup products is performed as in sections 11 and 12.

It can be checked from the diagrams that $L_{wy} \cong L_w \oplus L_y$. By Lemma 9, $V_G(wy) = V_G(w) \cup V_G(y)$ and $V_G(w) \cap V_G(y) = \{0\}$. Therefore y generates the periodicity of L_w . By Theorem 9.3, we have a short exact sequence

$$0 \longrightarrow \Omega^j(L_w) \longrightarrow \Omega^{j+4}(L_w) \longrightarrow \Omega^j(K) \longrightarrow 0$$

for all $j \geq 0$. Applying Theorem 9.5 we obtain

$$P_{K,K}(t) = \frac{1}{(1-t^4)} (1 + 2t + 2t^2 + 2t^3 + t^4 P_{L_w,K}(t)) ,$$

$P_{L_w,K}(t) = 2/(1-t)$, and hence

$$P_{K,K}(t) = \frac{1 + t}{(1-t)(1-t^4)} .$$

Now using the usual methods (see Theorem 12.10) we may show that the cohomology ring is as given in [11]. Note that Evens and Priddy call the elements x, y, z and w by the names x, x + y, P and u_3 respectively.

Theorem 15.5. If G is a semi-dihedral group of order 2^n , $n \geq 4$ and K is a field of characteristic 2, then

$$H^*(G,K) = \mathrm{Ext}^*_{KG}(K,K) \cong K[x,y,z,w]/(x^3,xy,xz,z^2+wy^2)$$

where $\deg x = \deg y = 1$, $\deg z = 3$, $\deg w = 4$.

Remark 15.6. In some sense this cohomology ring narrowly misses being noncommutative. In analogy with (12.15), the minimal projective resolution of K is the total complex, of an almost periodic double complex, in which 2-deg(x) = $(0,1)^-$, 2-deg(y) = $(1,0)^+$, 2-deg(z) = $(2,1)^+$, and 2-deg(w) = $(2,2)^+$. It is only because multiplication by x annihilates almost everything that x can afford to reverse the orientation on the double complex.

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