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Complexity and Multiple Complexes

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1. Introduction

Let G be a finite group and let R be a commutative ring with unit. One of the most basic problems in group cohomology is that of finding suitable projective resolutions for RG-modules. Indeed, much machinery has been developed over the years in order to circumvent this seemingly impossible problem. The ideal situation is the one in which the group G is cyclic or abelian. It has long been known that for cyclic groups there exists a canonical periodic (even minimal) projective resolution for the trivial RG-module R. For an abelian group, which is a direct product of cyclic groups, a minimal projective resolution for R can be found by taking the tensor product of the minimal resolutions over the cyclic factors. However, for noncommutative groups or for modules with nontrivial G-action, the problem is far more difficult. In most cases minimal projective resolutions cannot be written as product of periodic complexes.

The main result of this paper shows that if $R=k$ is a field of characteristic $p>0$, and if M is any finitely generated kG-module then there exists a projective resolution of M that is a tensor product of periodic complexes. The constructed resolutions are usually not minimal, but do have the same rate of growth (complexity) as the corresponding minimal resolutions. The Theorem (3.4) sheds some light on questions raised by Alperin in [1]. The principal ingredients in the construction are cohomology elements whose varieties are in a short of "general position" with respect to each other and the variety of M. There is a lot of freedom in choosing these elements, so the constructed resolutions are by no means unique. The techniques are also used in Sect. 4 to obtain an integral version of the main theorem for RG-lattices.

As an application of the results we present in Sect. 5 a new proof of G. Carlsson's theorem [12] that a finite group acting freely on a product of n spheres of dimension r must have p-rank at most n for all primes p. It seems likely that other problems of this type can be reduced to questions in modular representation theory. In Sect. 6 we discuss some of these questions and show that any kG-module can be resolved by a finite complex with otherwise com-

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pletely reducible homology. Section 7 is devoted to a brief discussion of general multiple complexes and minimal projective resoltutions. An example of a minimal multiple-complex resolution over M_{11} is given.

Notation. Throughout the paper G is a finite group and R is a commutative ring with unit. All RG-modules are assumed to be finitely generated left RG-modules. If M and N are RG-modules then $M \otimes N$ denotes the RG-module $M \otimes_R N$ with diagonal G-action, unless otherwise indicated.

2. Complexity

In [2] Jon Alperin introduced the term "complexity of a module" to describe the polynomial rate of growth of the cohomology of a module (see also [27]). The use of this term was motivated by the observation that in many cases one can find a minimal projective resolution for a module that is the total complex of a bounded n-complex (see [1]). Briefly an n-complex is an array of modules indexed by n-tuples of integers with boundary homomorphisms d_j, $j=1, \ldots, n$, running parallel to the axes. It is assumed that in any line parallel to an axis we have a complex, i.e. $d_j \circ d_j = 0$, and that $d \circ d = 0$ where $d = \sum d_j$ is the total boundary operator. By a bounded n-complex we mean one such that the dimension or rank of each module is bounded. These ideas are discussed in detail in Sect. 7. Formal definitions of complexity can be given as follows.

Definition 2.1. Let k be a field of characteristic $p > 0$, and let M be a kG-module. The complexity $c_{kG}(M)$ is the least integer s satisfying any one of the following equivalent conditions.

(a) $\lim_{m \to \infty} (\mathrm{Dim}_k P_m)/m^s = 0$ where

$$\ldots \to P_1 \to P_0 \to M \to 0$$

is a minimal projective resolution of M.

(b) There exists a number λ with $\mathrm{Dim}_k(P_m) \leq \lambda m^{s-1}$ for all m sufficiently large.

(c) $\lim_{m \to \infty} (\mathrm{Dim}_k \mathrm{Ext}^m_{KG}(M, M))/m^s = 0.$

The equivalence of the conditions is proved, for example, in [10]. For R a commutative ring we use the following definition.

Definition 2.2. Let R be a commutative ring with unit and let M be an RG-module. The complexity of M is

$$c_{RG}(M) = \mathrm{Max}\{c_{(R/\mathscr{P})G}((R/\mathscr{P}) \otimes_R M)\}$$

where the maximum is taken over all maximal ideals $\mathscr{P}$ in R.

Alperin and Evens have shown that for a kG-module the complexity is the maximum of the complexities of the restrictions to elementary abelian p-subgroups of G. Webb [29, 30] has proved an integral version of the Alperin-Evens theorem. In particular Webb showed that for ZG-lattices, Definition (2.2) is equivalent to conditions (a) or (b) of (2.1) with k replaced by Z. One consequence of Theorem 4.4 is that the same is true for any integral domain R that satisfies the conditions of Remark 4.6.

Let R be a commutative ring. A 1-complex (or simply a complex) of RG-modules is a pair (X, ∂) where $X=\sum X_j$ is a direct sum of RG-modules indexed by the integers and $\partial\colon X\to X$ is an RG-homomorphism such that $\partial(X_j)\subseteq X_{j-1}$ and $\partial\circ\partial=0$. We say that (X, ∂) is nonnegative if $X_j=\{0\}$ whenever $j<0$.

Definition. A nonnegative 1-complex (X, ∂) is said to be exact if the corresponding sequence $\ldots\to X_2\to X_1\to X_0$ is exact; that is, if $H_i(X, \partial)=\{0\}$ for all $i>0$. We say that (X, ∂) is periodic if for some positive integer s and all $j\geqq 0$ there exists an isomorphism $\mu_j\colon X_{j+s}\to X_j$ such that $\partial_j\mu_j=\mu_{j-1}\partial_{j+s}$ for all $j\geqq 1$.

If (X, ∂) and (Y, ∂') are two complexes then their tensor product is the complex $(Z, \partial'')=(X, \partial)\otimes(Y, \partial')$ where $Z_j=\sum\limits_i X_i\otimes Y_{j-i}$ and

$$\partial''(x_i\otimes y_l)=\partial x_i\otimes y_l+(-1)^i x_i\otimes\partial' y_l \qquad \text{for } x_i\in X_i,\ y_l\in Y_l.$$

Note that the tensor is taken over R. The tensor product of complexes is associative.

One connection between complexity and n-complexes is reflected in the following.

Proposition 2.4. *Let R be a commutative ring and let M be an RG-lattice (an RG-module that is projective as an R-module). Suppose that (X^i, ∂^i), $i=1, \ldots, t$ are exact periodic nonnegative 1-complexes of finitely generated RG-modules such that the tensor complex $(X^1, \partial^1)\otimes\ldots\otimes(X^t, \partial^t)$ is a projective resolution of M. Then $c_{RG}(M)\leqq t$.*

Proof. Let $(\tilde{X}, \partial)=(X^1, \partial^1)\otimes\ldots\otimes(X^t, \partial^t)$. By hypothesis $H_*(\tilde{X})=H_0(\tilde{X})\cong M$. That is

$$\ldots\to\tilde{X}_1\xrightarrow{\partial}\tilde{X}_0\to M\to 0$$

is a projective resolution of M. Let $g(a_1, \ldots, a_t)$ be the minimum of the cardinalities of the R-generating sets for $X^1_{a_1}\otimes\ldots\otimes X^t_{a_t}$. Note that this is a finite number. Hence because of the periodicity, the number $B=\max\{g(a_1, \ldots, a_t)\}$ is also finite. If $\mathscr{P}\subseteq R$ is a maximal ideal then

$$\mathrm{Dim}_{R/\mathscr{P}}(R/\mathscr{P})\otimes_R(X^1_{a_1}\otimes\ldots\otimes X^t_{a_t})\leqq B, \quad \text{and} \quad \mathrm{Dim}_{R/\mathscr{P}}((R/\mathscr{P})\otimes\tilde{X}_n)\leqq s_n B$$

where $s_n=\binom{n+t-1}{t-1}$ is the number of distinct t-tuples $(a_1, \ldots, a_t)$ of nonnegative integers with $\sum a_i=n$. Since $((R/\mathscr{P})\otimes\tilde{X}, 1\otimes\partial)$ is an $(R/\mathscr{P})G$-projective resolution of $(R/\mathscr{P})\otimes M$ and since s_n is a polynomial of degree $t-1$ in the variable n, we are done.

Interestingly, Lewis has shown that if G has a series $\{0\}=N_0\subseteq N_1\subseteq\ldots\subseteq N_n=G$ of normal subgroups such that each N_i/N_{i-1} has periodic Z-cohomology, then there exist periodic 1-complexes (X^i, ∂^i) $i=1, \ldots, n$ whose tensor product $(\tilde{X}, \partial)$ is a projective resolution for Z (see Theorem 3.6 of [18]). This result extends easily to RG-lattices. For example, if G is a group of order

p^n, R is a commutative ring in which p is not a unit, and M is an RG-lattice, then M has a projective resolution that is a tensor product of n periodic 1-complexes. One need only construct such a resolution $(\tilde{X}, \partial)$ for Z and take the tensor product $M \otimes (\tilde{X}, \partial) = (M \otimes \tilde{X}, 1 \otimes \partial)$.

3. The Main Theorem

Let K be an algebraically closed field for characteristic $p>0$. For a finite group G let $H(G, K) = H^*(G, K)$ if $p=2$ and $H(G, K) = \sum_{n \geqq 0} H^{2n}(G, K)$ if p is an odd prime. It is well known that $H(G, K)$ is a finitely generated commutative graded ring and we may consider its maximal ideal spectrum $V_G(K)$. This homogeneous affine variety was described by Quillen in [23] and [24]. If M is a KG-module the ring $\mathrm{Ext}^*_{KG}(M, M)$ is a finitely generated module over $H(G, K)$. Let $J(M)$ denote the annihilator in $H(G, K)$ of $\mathrm{Ext}^*_{KG}(M, M)$. Then $J(M)$ is a graded ideal and its associated variety $V_G(M)$ is called the variety of the module M. That is $V_G(M)$ is the collection of all maximal ideals of $H(G, K)$ that contain $J(M)$, and it is a closed homogeneous subvariety of $V_G(K)$.

Before stating some properties of the varieties we need some further notation. Suppose that k is a field of characteristic $p>0$. If M is a kG-module and P is a projective cover of M, then $\Omega(M)$ is defined to be the kernel of the covering $P \to M$. Inductively $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for $n>1$. Similarly $\Omega^{-1}(M)$ is the cokernel of an injection $M \to Q$ where Q is an injective hull for M, and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$. Since kG is a self-injective ring, $\Omega^n(M)$ has no projective (or injective) submodules.

Suppose that $\ldots \to P_1 \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} k \to 0$ is a minimal projective reolution of k. Then P_n is the projective cover of $\Omega^n(k)$, and $\partial_n(P_n) \subseteq \mathrm{Rad}(P_{n-1})$. So any $\zeta \in \mathrm{Ext}^n_{kG}(k, k) \cong H^n(G, k)$ is represented by a unique cocycle $\zeta' \in \mathrm{Hom}_{kG}(P_n, k)$ and by a unique homomorphism $\hat{\zeta} \in \mathrm{Hom}_{kG}(\Omega^n(k), k)$. For $\zeta \in H^n(G, k)$, $\zeta \neq 0$, we define L_ζ to be the kernel of $\hat{\zeta}\colon \Omega^n(k) \to k$. If $\zeta = 0$ then let $L_\zeta = \Omega^n(k) \oplus \Omega(k)$.

Most of the following results are provided in [4, 10, 11], and [12]. We refer the reader to [5] for a comprehensive treatment.

Proposition 3.1. *Let M and N be KG-modules.*

i) $V_G(M) = \{0\}$ *if and only if M is projective.*

ii) $\dim V_G(M) = c_{KG}(M)$.

iii) $V_G(M \oplus N) = V_G(M) \cup V_G(N)$.

iv) $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.

v) $V_G(\Omega^n(M)) = V_G(M)$ *for all n.*

vi) *For $\zeta \in H^n(G, K)$, $V_G(L_\zeta) = V_G(\langle \zeta \rangle)$. That is, $V_G(L_\zeta)$ is the hypersurface of $V_G(K)$ consisting of all maximal ideals that contain ζ.*

vii) $\dim V_G(K)$ *is the p-rank of G* [23].

Lemma 3.2. *Let k be a field of characteristic $p>0$ and let K be its algebraic closure. Suppose that $\zeta \in H^{2m}(G, k)$, $\zeta \neq 0$. There exists an exact, periodic, nonnegative 1-complex (X, ∂) such that $H_0(X) = k$ and for all j, $V_G(K \otimes_k X_j) \subseteq V_G(\langle 1 \otimes \zeta \rangle)$, the hypersurface defined by $1 \otimes \zeta \in K \otimes H(G, k) \cong H(G, K)$.*

Proof. Let $\hat{\zeta}$: $\Omega^{2m}(k)\to k$ represent ζ, and let (P, ε) be a minimal projective resolution of k. Then we have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & & & & \\
0 \arrow[r] & \Omega^{2m}(k) \arrow[r] \arrow[d, "\hat{\zeta}"'] & P_{2m-1} \arrow[r] \arrow[d] & P_{2m-2} \arrow[r] \arrow[d, equal] & \ldots \arrow[r] & P_0 \arrow[r, "\varepsilon"] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r, "\mu"] \arrow[d] & B \arrow[r, "\nu"] \arrow[d] & P_{2m-2} \arrow[r] & \ldots \arrow[r] & P_0 \arrow[r, "\varepsilon"] & k \arrow[r] & 0 \\
 & 0 & 0 & & & & &
\end{tikzcd} \tag{3.3}$$

where $L=L_\zeta$ is the kernel of $\hat{\zeta}$ and the bottom row is the pushout of the middle row along $\hat{\zeta}$. In particular, $B=\Omega^{-1}(L)$ and $K\otimes B\cong\Omega^{-1}(K\otimes L)\oplus(\text{proj})$ $\cong\Omega^{-1}(L_{1\otimes\zeta})\oplus(\text{proj})$. So $V_G(K\otimes B)=V_G(\langle 1\otimes\zeta\rangle)$ by Lemma 4.1(v) and (vi). Also $V_G(K\otimes P_i)=\{0\}$. Define the 1-complex (X, ∂) by splicing the sequence (3.3) to itself an infinite number of times. That is, for all $j\geqq 0$ $X_{2mj+i}=P_i$ if $0\leqq i\leqq 2m-2$, and $X_{2mj-1}=B$ for all $j\geqq 1$. Here $\partial_{2mj}=\mu\varepsilon$, $\partial_{2mj-1}=\nu$ for $j>0$, and $\partial_{2nj+i}=\partial_i$ for $j\geqq 0$ and $0\leqq i\leqq 2m-2$. Hence (X, ∂) has the required properties.

Before stating the main theorem, a note concerning change of field is in order. Suppose that K is the algebraic closure of k. As noted earlier $H(G, K)$ $\cong K\otimes_k H(G, k)$ by universal coefficients. If M is a kG-module and $J_k(M)$ is the annihilator in $H(G, k)$ of $\operatorname{Ext}^*_{kG}(M, M)$ then it can be seen that $J_K(K\otimes M)$ $=K\otimes J_k(M)$. Clearly $K\otimes J_k(M)\subseteq J_K(K\otimes M)$. On the other hand if $\zeta\in J_K(K\otimes M)$ and the degree of ζ is n, then ζ is represented by a homomorphism $\hat{\zeta}$: $K\otimes_k\Omega^n(k)\to K$, and $\hat{\zeta}\otimes I$: $K\otimes(\Omega^n(k)\otimes M)\to K\otimes M$ must factor through a projective. Hence, $\hat{\zeta}\otimes I=\sum_{g\in G} gfg^{-1}$ for some

$$f\in\operatorname{Hom}_K(K\otimes\Omega^n(k)\otimes M, K\otimes M)\cong K\otimes\operatorname{Hom}_k(\Omega^n(k)\otimes M, M)$$

(see [15]). But the operation $f\to\sum_{g\in G} gfg^{-1}$ commutes with $K\otimes_k -$ and so by a standard basis argument $\zeta\in K\otimes J_k(M)$.

Theorem 3.4. *Let k be a field of characteristic $p>0$. Suppose M is a kG-module with complexity $c_{kG}(M)=r$. Then there exist exact, periodic, nonnegative 1-complexes (X^i, ∂_i), $i=1, \ldots, r$, such that the tensor product complex $(\tilde{X}, \partial)=(X^1, \partial^1)$ $\otimes\ldots\otimes(X^r, \partial^r)$ is a projective resolution of M.*

Proof. Let K be the algebraic closure of k. By Proposition 3.1(ii) the Krull dimension of $H(G, K)/J_K(K \otimes M) \cong K \otimes_k (H(G, k)/J_k(M))$ is exactly $r = c_{kG}(M)$. By standard theorems on Krull dimension [20] there exist homogeneous elements $\zeta_1, \ldots, \zeta_r \in H(G, k)$ such that $H(G, k)/J_k(M)$ is a finitely generated module over the ring generated by their images. Hence also $H(G, K)/J_K(K \otimes M)$ is finitely generated as a module over the ring generated by $1 \otimes \zeta_i + J_K(K \otimes M)$. Therefore

$$V_G(M) \cap V_G(\langle 1 \otimes \zeta_1 \rangle) \cap \ldots \cap V_G(\langle 1 \otimes \zeta_r \rangle) = \{0\}. \tag{3.5}$$

For each $i = 1, \ldots, r$, let (Z^i, ∂^i) be an exact, periodic, nonnegative 1-complex of kG-modules with $H_0(Z^i) = k$ and $V_G(K \otimes Z^i_j) \subseteq V_G(\langle 1 \otimes \zeta_i \rangle)$ for all j (see Lemma 3.2). Then $(X^1, \partial^1) = (M \otimes Z^1, 1 \otimes \partial^1)$ is an exact, periodic, nonnegative 1-complex with $H_0(X^1) = M \otimes H^0(Z^1) \cong M$ and

$$V_G(X^1_j) = V_G(M) \cap V_G(Z^1_j) \subseteq V_G(M) \cap V_G(\langle 1 \otimes \zeta_1 \rangle)$$

by Lemma (3.1)(iv). Consequently if we let $(X^i, \partial^i) = (Z^i, \partial^i)$, $i = 2, \ldots, r$, then the product complex $(\tilde{X}, \partial)$ is a projective resolution by (3.5) with $H_0(\tilde{X}) = M$.

4. An Integral Version of the Main Result

Let R be the ring of integers in an algebraic number field. In this section we extend Theorem 3.4 to obtain projective resolutions for RG-lattices as tensor products of exact periodic 1-complexes. In addition, if M is an RG-module that is free as an R-module then a free RG-resolution can be obtained by this method. Recall that we are considering only finitely generated RG-modules.

Lemma 4.1. *An RG-module M is projective if and only if the following two conditions are satisfied.*

i) *M is projective as an R-module.*

ii) *For each maximal ideal $\mathscr{P} \subseteq R$ with $|G| \in \mathscr{P}$, $(R/\mathscr{P}) \otimes_R M$ is a projective $(R/\mathscr{P})$ G-module.*

Proof. See Theorem 78.1 of [15].

Lemma 4.2. *Suppose that $\mathscr{P}$ is a maximal ideal of R such that $R/\mathscr{P}$ has characteristic p. Assume that $|G| = p^a q$ and $(p, q) = 1$. If $\zeta \in H^{2m}(G, R/\mathscr{P})$, then $\zeta^{p^{a+1}}$ is in the image of the natural homomorphism $H(G, R) \to H(G, R/\mathscr{P})$.*

Proof. This is a straightforward generalization of Lemma 4.5 of [7].

If M is an RG-module, let

$$\ldots \to F_1 \to F_0 \to M \to 0$$

be a resolution of M by free RG-modules, and denote by $\tilde{\Omega}^m(M)$ the m^{th} kernel of this resolution. By the extended Schanuel Lemma (see (1.4) of [28]) if $\tilde{\Omega}(M)$ and $\tilde{\Omega}_1(M)$ are defined using different free solutions, then there exist free modules F, F' such that $\tilde{\Omega}(M) \oplus F = \tilde{\Omega}_1(M) \oplus F'$. We may choose a large enough resolution for R so that

$$\tilde{\Omega}^m(M) \otimes \tilde{\Omega}^n(R) \cong \tilde{\Omega}^{m+n}(M) \oplus (\text{free}).$$

We now introduce modules $\tilde{L}_\xi$ analogous the modules L_ζ of the last section. Let $\xi \in H^{2m}(G, R)$. Let

$$\ldots \to F_1 \to F_0 \to R \to 0$$

be a free RG-resolution of R. Then ξ is represented by a homomorphism $\hat{\xi}$: $\tilde{\Omega}^{2m}(R) \to R$. By making the resolution large enough, that is by adding extra free modules if necessary, we may assume that $\hat{\xi}$ is surjective even when $\xi = 0$. Let $\tilde{L}_\xi$ denote the kernel of $\hat{\xi}$. Let $\mathscr{P}$ be a prime ideal in R and let K be the algebraic closure of $R/\mathscr{P}$. If the image of ξ is ζ under the natural homomorphism from $H^{2m}(G, R)$ to $H^{2m}(G, K)$, then $K \otimes \tilde{L}_\xi \cong L_\zeta \oplus(\text{proj})$.

Lemma 4.3. (*cf.* [10]). *Suppose that $\xi \in H^{2m}(G, R)$ and $\chi \in H^{2n}(G, R)$. Then there exists a short exact sequence*

$$0 \to \tilde{\Omega}^{2m}(\tilde{L}_\chi) \oplus F_1 \to \tilde{L}_{\xi\chi} \oplus F_2 \to \tilde{L}_\xi \to 0$$

where F_1 and F_2 are free RG-modules.

Proof. Tensor the short exact sequence $0 \to \tilde{L}_\chi \to \tilde{\Omega}^{2n}(R) \xrightarrow{\hat{\chi}} R \to 0$ with $\tilde{\Omega}^{2m}(R)$. We obtain the middle row of the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{\Omega}^{2m}(\tilde{L}_\chi) \oplus F_1 \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & \tilde{L}_\xi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{\Omega}^{2m}(\tilde{L}_\chi) \oplus F_1 \arrow[r] & \tilde{\Omega}^{2m+2n}(R) \oplus F_2 \arrow[r, "1\otimes\chi"] \arrow[d, "\gamma"] & \tilde{\Omega}^{2m}(R) \arrow[r] \arrow[d, "\hat{\xi}"] & 0 \\
 & & R \arrow[r, equal] \arrow[d] & R \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Here $\gamma = \hat{\xi}(1 \otimes \hat{\chi})$ is cohomologous to $\xi\chi$ and hence its kernel B is isomorphic to $\tilde{L}_{\xi\chi} \oplus F_2$.

We may now prove the main result of the section.

Theorem 4.4. *Let M be an RG-lattice, that is, an RG-module that is projective as an R-module. Let $n = c_{RG}(M)$ (see Definition 2.2). Then there exist exact, periodic, nonnegative 1-complexes (X^i, ∂^i), $i = 1, \ldots, n$, such that the product $(X^1, \partial^1) \otimes \ldots \otimes (X^n, \partial^n)$ is a projective resolution for M. Moreover if M is free as an R-module, then we can find such complexes so that the product is a free resolution of M.*

Proof. If $\mathscr{P}$ is a maximal ideal in R then let $\overline{R/\mathscr{P}}$ denote the algebraic closure of $R/\mathscr{P}$. There are only a finite number of maximal ideals $\mathscr{P} \subseteq R$ with $|G| \in \mathscr{P}$ and for each we may choose elements $\zeta_1(\mathscr{P}), \ldots, \zeta_n(\mathscr{P}) \in H(G, R/\mathscr{P})$, all homogeneous with even degrees, so that

$$V_G((\overline{R/\mathscr{P}}) \otimes M) \cap V_G(\langle\zeta_1(\mathscr{P})\rangle) \cap \ldots \cap V_G(\langle\zeta_n(\mathscr{P})\rangle) = \{0\}.$$

If for a particular $\mathscr{P}$, $c_{(R/\mathscr{P})G}((R/\mathscr{P}) \otimes M)$ is less than n, then we allow some of the elements $\zeta_1(\mathscr{P}), \ldots, \zeta_n(\mathscr{P})$ to be zero. This does not affect the argument except that we must consider each such as the zero element in some positive even degree. Now replace each $\zeta_i(\mathscr{P})$ (if necessary) by a suitable power so that for each $i = 1, \ldots, n$ the elements $\zeta_i(\mathscr{P})$ have the same degree for all maximal ideals $\mathscr{P} \subseteq R$ with $|G| \in \mathscr{P}$. By the same process and Lemma 4.2 we may also ensure that every $\zeta_i(\mathscr{P})$ lifts to an element of $H^*(G, R)$. Moreover since $H^m(G, R)$, $m > 0$, is the direct sum of its $\mathscr{P}$-primary components, for each $i = 1, \ldots, n$, there exists an element $\xi_i \in H^{m_i}(G, R)$ such that the image of ξ_i in $H^{m_i}(G, R/\mathscr{P})$ is $\zeta_i(\mathscr{P})$ for all $\mathscr{P}$ containing $|G|$. In particular $\overline{R/\mathscr{P}} \otimes \tilde{L}_{\xi_i} \cong L_{\zeta_i(\mathscr{P})} \oplus (\text{proj})$.

We now follow the recipe of the last section. As in Lemma 3.2, we obtain for each i, a periodic 1-complex (X^i, ∂^i) with the property that for every $\mathscr{P}$ containing $|G|$, $V_G((\overline{R/\mathscr{P}}) \otimes X^i_m)$ is contained in $V_G(\langle\zeta_i(\mathscr{P})\rangle)$ for all m. Also every X^i_m is free as an R-module. In this case, $X^i_{jm_i+k} \cong F_k$, $0 \leqq k < m_i - 1$, and $X^i_{m_i-1} \cong F_{m_i-1}/\tilde{L}_{\xi_i}$. As in the proof of Theorem 3.4, take the tensor product $(\tilde{X}, \partial) = M \otimes (X^1, \partial^1) \otimes \ldots \otimes (X^n, \partial^n)$. Each $\tilde{X}_m$ is projective by Lemma 4.1 and the hypothesis. This proves the first statement.

Now suppose that M is free as an R-module. In the complex constructed above all modules are free except possibly for those isomorphic to

$$N = M \otimes \left(\bigotimes_{i=1}^{n} (F_{m_i-1}/\tilde{L}_{\xi_i}) \right)$$

which we only know to be projective. Let $\tilde{K}_0(RG)$ denote the reduced Grothendieck group of projective RG-modules. This is a finite abelian group (see Prop. 9.1 of [25]). Suppose that $[N]$ has order t in $\tilde{K}_0(RG)$. We claim that if we replace ξ_1 by ξ_1^t then $[N]$ becomes zero in $\tilde{K}_0(RG)$. First observe that if an element of $\tilde{K}_0(RG)$ is represented by a tensor product of modules, then we may replace any of the tensor multiplicands by its translate under $\tilde{\Omega}^j$ without affecting the resulting element of $\tilde{K}_0(RG)$. Thus

$$[N] = \left[M \otimes \left(\bigotimes_{i=1}^{n} \tilde{L}_{\xi_i} \right) \right].$$

By Lemma 4.3 and induction

$$\left[M \otimes L_{\xi_1^r} \otimes \left(\bigotimes_{i=2}^{n} L_{\xi_i} \right) \right] = r[N] \qquad \text{for all } r \geqq 1.$$

This proves the claim. That is if ξ_1 is replaced by ξ_1^t then $[\tilde{X}_m] = 0$ in $\tilde{K}_0(RG)$ for all m. It is still not certain that we have a free resolution. That is, the module N may be only stably free. However by adding suitable free modules to one of the 1-complexes (X^i, ∂^i) we can make it free.

Corollary 4.5. *Let n be the maximum value of the p-rank of G for all primes p. Then there exist n nonnegative periodic 1-complexes whose tensor product is a free resolution for the trivial RG-module R.*

Proof. This is a direct consequence of Proposition 3.1 (ii) and (vii) and the above theorem.

Remark 4.6. The properties of R used in the proof of Theorem 4.4 are the following.

a) Lemma 4.1 holds for RG-modules.
b) There are only finitely many maximal ideals $\mathscr{P} \subseteq R$ with $|G| \in \mathscr{P}$.
c) $\tilde{K}_0(RG)$ is a finite group (Only necessary for the last statement).

Other rings such as localizations and completions of algebraic number rings also satisfy these properties and hence the theorem is also true for these rings.

5. Group Actions on Products of Spheres

Using the methods developed in the previous section we give a proof of G. Carlsson's theorem [13] concerning group actions on products of spheres. In some sense our proof is equivalent to Carlsson's but has been made more conceptual by using the language of varieties for modules. W. Browder also has a very short proof using the method of exponents in spectral sequences.

Theorem 5.1 [13]. *Suppose that G acts freely on a finite CW-complex X with the homotopy type of a product $(S^r)^n$ of n spheres of the same dimension r, with trivial action on homology. Then for any prime p the p-rank of G is at most n. Moreover the complex of cellular chains on X is G-chain homotopic to tensor product of n complexes, the homology of each of which is the homology of a sphere.*

Proof. Denote by C_* the complex

$$0 \to C_s \xrightarrow{\partial_s} C_{s-1} \to \dots \xrightarrow{\partial_1} C_0 \to 0$$

of cellular chains on X, and choose a diagonal approximation $C_* \to C_* \otimes C_*$. The truncaton $C_r \to C_{r-1} \to \dots \to C_0 \to 0$ is a complex of free ZG-modules whose homology is Z in degree zero and which is exact elsewhere. Also $\ker \partial_r / \operatorname{Im} \partial_{r+1}$ is a direct sum of n copies of Z (with trivial G-action) corresponding to the n spheres. These summands correspond to elements $\xi_1, \dots, \xi_n \in H^{r+1}(G, Z)$. These elements are, in fact, the transgressions of the fundamental classes of the spheres in the Serre spectral sequence $H^*(BG, H^*(X)) \Rightarrow H^*(X/G)$ of the fibration $X \to X \times_G EG \to BG$, as pointed out in [13], but we do not need to know this. For each sphere we have a surjection $\hat{\xi}_i$: $\ker \partial_r \to Z$ whose kernel we denote by $\tilde{L}_{\xi_i}$. In this way we obtain for each $i = 1, \dots, n$ a map μ_i of chain complexes as in the diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & & & \\
 & & & & \tilde{L}_{\xi_i} \arrow[d] & & & & \\
0 \arrow[r] & C_s \arrow[r] & \dots \arrow[r] & C_{r+1} \arrow[r] & C_r \arrow[r] \arrow[d] & C_{r-1} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & C_0 \arrow[r] \arrow[d, equal] & 0 \\
 & & & 0 \arrow[r] & C_r/\tilde{L}_{\xi_i} \arrow[r] \arrow[d] & C_{r-1} \arrow[r] & \dots \arrow[r] & C_0 \arrow[r] & 0 \\
 & & & & 0 & & & &
\end{tikzcd}$$

Let $C_*^{(i)}$ denote the bottom row. Note that the homology of $C_*^{(i)}$ is the homology of S^r. Iterating the diagonal approximation $n-1$ times we obtain a map of complexes $\mu\colon C_* \to \bigotimes_{i=1}^{n} C_*^{(i)}$ as the composition

$$C_* \to C_* \otimes \dots \otimes C_* \xrightarrow{\mu_1 \otimes \dots \otimes \mu_n} \bigotimes C_*^{(i)}.$$

Since, for both complexes we know the comultiplication on the homology, we easily see that μ is a homology isomorphism. This proves the last statement of the Theorem. The modules appearing in $\bigotimes C_*^{(i)}$ are all free except possibly for the nr^{th} term which is $L = \bigotimes_{i=1}^{n} (C_r/\tilde{L}_{\xi_i})$. We claim that L is also projective. Add a finite, exact, free chain complex F_* to C_* to make the map $C_* \oplus F_* \to \bigotimes C_*^{(i)}$ surjective. The long exact sequence of homology shows that the kernel is a finite exact chain complex all of whose terms, except possibly the nr^{th} term is projective. This implies that the nr^{th} term and hence also L are cohomologically trivial. Because L is Z-free it must be projective.

For a prime p let K be the algebraic closure of the prime field of characteristic p. Let $\zeta_i \in H^{r+1}(G, K)$ be the image of ξ_i under mod-p reduction. Then $K \otimes \tilde{L}_{\xi_i} \cong L_{\zeta_i} \oplus (\text{proj})$. The construction given above implies that $\bigotimes_{i=1}^{n} \Omega^{-1}(L_{\zeta_i})$ is a projective module. By Proposition 3.1 its variety is $V_G(\langle\zeta_1\rangle) \cap \dots \cap V_G(\langle\zeta_n\rangle) = \{0\}$. Hence the dimension of $V_G(K)$, which is equal to the p-rank of G, is at most n.

It has been conjectured that Theorem 5.1 still holds without the restriction that the spheres have the same dimension. This seems to be difficult to prove by the above method. The problem is that we have no obvious way of knowing that the fundamental classes of the spheres get as far as transgressing in the Serre spectral sequence so that the maps $C_* \to C_*^{(i)}$ exist. On the other hand it seems possible that the p-rank condition is sufficient as well as necessary, in the following sense.

Conjecture 5.2. The following conditions on a finite group G are equivalent.

a) G has a free action on a finite CW-complex X with the homotopy type of a product of n spheres with trivial action on homology.

b) For all primes p, the p-rank of G is at most n.

Remarks 5.3. (1) The case $n=1$ in the above conjecture is known to be true [26]. It is also known that we cannot demand that X be homeomorphic to a sphere. For example if $G=S_3$, the symmetric group on 3 letters, then it follows from Milnor [21] that G has no free action on a sphere of any dimension.

(2) Oliver [22] has shown that we cannot demand that the spheres have the same dimension. Hence the converse to Theorem 5.1 is not true. His example is the alternating group $G=A_4$, which he shows has no free action on a CW-complex homotopic to a product of any number of spheres of the same dimension, with trivial action on homology. However, he also shows that A_4 does act freely on $S^2 \times S^3$.

(3) We might try to prove that (b) implies (a) in (5.2) using Corollary 4.5 with $R=Z$, or, more precisely, using an integral analog of the proof of Lemma 6.2 (to follow) with $M=R$. This gives a finite multiple complex whose total complex has the homology of a product of n spheres. The problem would be to find a CW-complex realization of the total complex. Oliver's example demonstrates that the elements $\xi_1, \ldots, \xi_n \in H(G, Z)$ must be chosen very carefully. The first obstruction to realizing the complex comes from the Steenrod algebra. It seems we would need that, for each $j>0$ and each p dividing $|G|$, the ideal generated by the images of those ξ_i's lying in degrees at most j is invariant under the action of the Steenrod algebra. Higher obstruction probably also exist.

(4) In the next section we obtain another proof [17] that (a) implies (b) in (5.2) whenever $n \leqq 3$.

(5) The problem of free actions of finite groups on products of spheres is discussed also in the papers by Browder [9], Conner [14], Heller [16, 17] and Lewis [19].

6. Completely Reducible Homology Complexes

In this section K is a field of characteristic $p>0$ and G is a finite group. We wish to consider the purely algebraic question of when a module can be resolved by a finite complex whose positive homology groups are all completely reducible KG-modules.

Definition 6.1. A complex $C=(C_i, \partial)$ is a CRH-complex if it is finite and satisfies the following.

(1) Each C_i is a finitely generated projective KG-module.

(2) $C_i=0$ if $i<0$ and

(3) $H_i(C)$ is a completely reducible KG-module for all $i>0$.

Lemma 6.2. *Let M be a simple KG-module and let B be any positive integer. There exists a* CRH-*complex $C=(C_i, \partial)$ with $H_0(C) \cong M$ and $H_i(C)=\{0\}$ if $1 \leqq i \leqq B$.*

Proof. Let n be the p-rank of G. As in the proof of Theorem 4.3, choose homogeneous elements $\zeta_i, \ldots, \zeta_n \in H(G, K)$ so that $V_G(\zeta_1) \cap \ldots \cap V_G(\zeta_n) = \{0\}$. As in the proof of Lemma 4.2 we have for each i an exact sequence

$$0 \to K \to \Omega^{-1}(L_{\zeta_i}) \to P_{2m_i-2} \to \ldots \to P_0 \to K \to 0$$

where $2m_i$ is the degree of ζ_i. Let $X^{(i)} = (X_j^{(i)}, \partial^{(i)})$ be the finite complex

$$0 \to \Omega^{-1}(L_{\zeta_i}) \to P_{2m_i-2} \to \ldots \to P_1 \to P_0 \to 0.$$

So $H_0(X^{(i)}) = K = H_{2m_i-1}(X^{(i)})$, $H_j(X^{(i)}) = 0$ if $j \neq 0$, $2m_i - 1$. Let $C = X^{(1)} \otimes \ldots \otimes X^{(n)} \otimes M$. Then

$$H_j(C) = \sum H_{i_1}(X^{(1)}) \otimes \ldots \otimes H_{i_n}(X^{(n)}) \otimes M,$$

the sum being over all indices such that $i_1 + \ldots + i_n = j$. So $H_j(C)$ is a direct sum of copies of M and is completely reducible. Moreover $H_0(C) = K \otimes M \cong M$.

Let $s = \min\{\deg(\zeta_i) - 1\}$. Then $H_j(C) = 0$ for $0 < j < s$. If $s \leqq B$ then replace each ζ_i by $\zeta_i' = \zeta_i^{r_i}$ for some r_i such that $\deg(\zeta_i') = r_i \deg(\zeta_i) > B + 1$. In the same way form the complex C' beginning with the ζ_i''s. In this case we have that $H_j(C) = 0$ for $1 \leqq j \leqq B$.

Theorem 6.3. *Let M be any KG-module and let B be any positive integer. There exists a* CRH-*complex C such that $H_0(C) \cong M$ and $H_i(C) = \{0\}$ for $0 < i \leqq B$.*

Proof. We proceed by induction on the composition length of M. Let N be a simple submodule of M. Then we have an exact sequence $0 \to N \to M \to M/N \to 0$. Choose CRH-complexes $C' = (C_i', \partial)$, $C'' = (C_i'', \partial'')$ such that $H_0(C'') = M/N$, $H_j(C'') = \{0\}$ for $0 < j \leqq B$, $H_0(C') = N$, and $H_j(C') = \{0\}$ for $0 < j < s + 2$ where s is the largest integer such that $C_s'' \neq \{0\}$. Let $C_i = C_i' \oplus C_i''$ for all i. We shall show that there exist boundary maps ∂_i: $C_i \to C_{i-1}$ such that the complex $C = (C_i, \partial)$ is a CRH-complex satisfying the conditions of the Theorem. Let μ_i: $C_i' \to C_i$, ν_i: $C_i \to C_i''$ be the injection and projection homomorphisms $\mu_i(c') = (c', 0)$, $\nu_i(c', c'') = c''$, $c' \in C_i'$, $c'' \in C_i''$. Then we have the following diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & C_0' \arrow[r, "\mu_0"] \arrow[d, "\varepsilon'"] & C_0 \arrow[r, "\nu", "0"'] & C_0'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d] & M \arrow[r, "\beta"] \arrow[d] & M/N \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Since C_0'' is projective there exists f: $C_0'' \to M$ with $\beta f = \varepsilon''$. So define ε: $C_0 \to M$ by

$$\varepsilon(c', c'') = \alpha \varepsilon'(c') + f(c'').$$

Because ε', ε'' are onto, so is ε. Moreover the diagram, with ε filled in, is commutative.

Suppose that by induction we have defined $\partial_i\colon C_i \to C_{i-1}$ for $i=1, \ldots, t$ such that $\nu_{i-1}\partial_i = \partial_i''\nu_i$ and $\partial_i\mu_i = \mu_{i-1}\partial_i'$. Then we have the following diagram

$$\begin{tikzcd}
 & \arrow[d] & & \arrow[d] & \\
0 \arrow[r] & C_{t+1}' \arrow[r, "\mu"] \arrow[d, "\partial'"] & C_{t+1} \arrow[r, "\nu"] & C_{t+1}'' \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C_t' \arrow[r, "\mu"] \arrow[d, "\partial'"] & C_t \arrow[r, "\nu"] \arrow[d, "\partial"] & C_t'' \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C_{t-1}' \arrow[r, "\mu"] \arrow[d] & C_{t-1} \arrow[r, "\nu"] \arrow[d] & C_{t-1}'' \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

which is commutative with exact rows and columns. Note that if $t>s$ then $C_t''=\{0\}=C_{t+1}''$ and we can let $\partial_{t+1}=\partial_{t+1}'$. So we may assume that $t\leqq s$ in which case the left hand column is exact. Now define $\theta\colon C_{t+1}'' \to C_{t-1}'$ by $\theta(c'')=\mu^{-1}\partial_t(0, \partial''(c''))$. Here, of course, $\nu\partial(0, \partial''(c''))=\partial''\partial''(c'')=0$ and hence θ is well defined. Also $\partial'\theta(c'')=0$. So by the exactness of the left column there exists a homomorphism $\psi\colon C_{t+1}'' \to C_t'$ such that $\partial'\psi=\theta$. Now define $\partial_{t+1}\colon C_{t+1} \to C_t$ by $\partial_{t+1}(c', c'')=(\partial'(c')-\psi(c''), \partial''c'')$. Then

$$\begin{aligned}
\partial_t\partial_{t+1}(c', c'') &= \partial_t(\partial'c'-\psi(c''), \partial''c'')=\partial_t(\partial'c'-\psi(c''), 0)+\partial_t(0, \partial''c'') \\
&= -(\partial'\psi(c''), 0)+(\theta(c''), 0) \\
&= 0.
\end{aligned}$$

The fact that ∂_{t+1} makes the appropriate diagram commute is obvious. So we have that $C=(C_i, \partial)$ is a complex and that there exists an exact sequence of chain maps

$$0 \to C' \xrightarrow{\mu} C \xrightarrow{\nu} C'' \to 0.$$

Therefore we have a long exact sequence in homology

$$\begin{aligned}
&\ldots \to H_1(C'') \to H_0(C') \to H_0(C) \to H_0(C'') \to 0 \\
&\ldots \to H_{n+1}(C'') \to H_n(C') \to H_n(C) \to H_n(C'') \to H_{n-1}(C') \to \ldots .
\end{aligned}$$

Hence $H_0(C) \cong M$ as desired and

$$\begin{aligned}
H_n(C) &= H_n(C'') \quad \text{for } 1\leqq n\leqq s \\
H_n(C) &= H_n(C') \quad \text{for } n>s.
\end{aligned}$$

This proves the Theorem.

Remarks 6.4. In the proof of Lemma 6.2 the degrees of the cohomology classes ζ_i could have been chosen so that all of the homology modules $H_j(C)$, $j>0$, were simple KG-modules. Likewise the induction step in the proof of Theorem 6.3 can be altered to obtain a CRH-complex with $H_0(C)=M$ and $H_j(C)$ simple for all $j>0$.

Definition 6.5. Let C be a CRH-complex. Let $d(C)=\sum_{i=1}^{\infty} \operatorname{Dim} H_i(C)$. If M is a KG-module let $h(M)=\min\{d(C)\}$ where C runs through the collection of all CRH-complexes with $H_0(C)=M$.

Let s be the p-rank of G. By the proof of Lemma 6.2 we know that $h(K)\leqq 2^s-1$. The proof that (a) implies (b) in Conjecture 5.2 would follow easily from the following.

Conjecture 6.6. $h(K)=2^s-1$ where s is the p-rank of G.

In fact the weaker conjecture $h(K)\geqq 2^{s-1}$ suffices for this purpose. We may verify this conjecture in a few special cases. In essence the proof is very similar to that in [17], but the change in point of view, i.e. using projective resolutions, may ultimately prove useful.

Proposition 6.7. *Let G be an elementary abelian p-group of order p^n. If $n=1$ or 2 then $h(K)=2^n-1$. If $n=3$ then $h(K)\geqq 4$.*

The case $n=1$ is obvious because any CRH-complex C with $H_0(C)=K$ and $d(C)=0$ would necessarily be exact in positive degrees, and hence not finite. To deal with the cases $n=2$, 3 we need the following.

Lemma 6.8. *Let M be a KG-module and let $C=(C_i, \partial)$ be a* CRH-*complex with $H_0(C)\cong M$. Suppose that $P^{(i)}=(P_j^{(i)}, \varepsilon_i)$ is a minimal projective resolution of $H_i(C)$ for $i>0$. Then there exists a projective resolution $F=(F_j, \varepsilon)$ of M and an injective chain map $\mu\colon C\to F$ such that $F_s=\sum_{i>0} P^{(i)}_{s-i-1}\oplus C_s$.*

Proof. Let r be the greatest integer such that $C_r\neq\{0\}$. To construct F we find a sequence of complexes $X^{(r)}, X^{(r-1)}, \ldots, X^{(1)}=F$ with the following properties.

(a) There exist injective chain maps

$$C\xrightarrow{\nu_{r+1}} X^{(r)}\xrightarrow{\nu_r} X^{(r-1)}\to\ldots\to X^{(1)}\cong F$$

(b) $H_j(X^{(i)})=0$ if $j\geqq i$.

(c) For each i, $X_s^{(i)}=\sum_{j\geqq i} P^{(j)}_{s-j-1}\oplus C_s$.

We may consider that $C=X^{(r+1)}$. Assume by induction that $X^{(t)}$ has been constructed with the above properties. Then $X^{(t)}$ is exact in degree t and all larger degrees. Moreover $X_s^{(t)}=C_s$ for $s\leqq t$ because $P_l^{(j)}=\{0\}$ whenever $l<0$. So we have the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\ldots\longrightarrow & X^{(t)}_{t+2} & \longrightarrow & X^{(t)}_{t+1} & \longrightarrow & X^{(t)}_t & \longrightarrow\ \partial_t(C_t) & \longrightarrow & 0\\
 & & & & & & \downarrow & & \\
 & & & & & & \operatorname{Ker}\partial_{t-1} & & \\
 & & & & & & \downarrow & & \\
\ldots\longrightarrow & P_2^{(t-1)} & \longrightarrow & P_1^{(t-1)} & \longrightarrow & P_0^{(t-1)} & \longrightarrow H_{t-1}(M) & \longrightarrow & 0\\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Here the two rows and the column are exact. Using standard techniques we may fill in the middle row with a complex $\tilde{X}^{(t-1)}$ where $\tilde{X}_s^{(t-1)} = X_s^{(t)} \oplus P_{s-t}^{(t-1)}$. This is, in fact, a projective resolution of Ker ∂_{t-1}. The complex $X^{(t-1)}$ is the splice of $\tilde{X}^{(t-1)}$ with the first $t-1$ terms of C, at the module Ker ∂_{t-1}. Then $X^{(1)} = F$ has all of the desired properties.

Proof of Proposition 6.7. Let C be a CRH-complex with $H_0(C) = K$ and $d(C) = h(K)$. Let r be the least index such that $C_r \neq \{0\}$. Because KG is a self-injective algebra we may assume that $H_r(C) \neq \{0\}$, since otherwise we may replace C by C' where $C'_r = 0$, $C'_{r-1} = C_{r-1}/\partial C_r$, $C'_j = C_j$ for $j \neq r, r-1$. Note that for each i, $H_i(C)$ is either zero or a direct sum of copies of K. Choose a basis $a_0, \ldots, a_s$ for $H_*(C)$ so that $\deg(a_0) = 0$, $\deg(a_i) = d_i$, $d_s = r$ and $s = d(C)$. It is possible that $d_i = d_j$ if $i \neq j$, but the number of elements in the set $\{i \mid d_i = t\}$ is Dim $H_t(C)$. By Lemma 6.8, C can be embedded in a projective resolution (F, ε') of K in such a way that $F_v = \left(\sum_{i=1}^{r} P_{v-d_i-1} \right) \oplus C_v$ where (P, ε) is a minimal projective resolution of K. Now recall that

$$\begin{aligned} \operatorname{Dim} P_m &= \operatorname{Dim} \operatorname{Ext}_{KG}^m(K, K) \cdot |G| \\ &= \binom{m+n-1}{n-1} \cdot p^n \end{aligned}$$

since G is elementary abelian of order p^n. Since (P, ε) is minimal projective resolution of K, there is a surjective chain map $(F, \varepsilon') \to (P, \varepsilon)$. In particular Dim $F_j \geqq$ Dim P_j for all j. Setting $j = r+1$ we have that

$$\begin{aligned} (\operatorname{Dim} F_{r+1})/p^n &= \sum_{i=1}^{s-1} (\operatorname{Dim} P_{r-d_i})/p^n + 1 \\ &\geqq (\operatorname{Dim} P_{r+1})/p^n. \end{aligned} \tag{6.9}$$

Suppose that $n = 2$, $s = 2$ and $d_1 = a$. Then (6.9) says that

$$1 + (\operatorname{Dim} P_{r-a})/p^n = 1 + \binom{r-a+1}{1} \geqq \binom{r+2}{1}$$

or $r + 2 - a \geqq r + 2$. Since this is impossible we conclude that $s \geqq 3$. This proves the case $n = 2$.

Now suppose that $n = 3$, and $s = 3$. Let $d_1 = a$, $d_2 = b$. Then (6.9) implies that

$$1 + \binom{r-a+2}{2} + \binom{r-b+2}{2} \geqq \binom{r+3}{2}$$

or that $f(r) = 2 + (r-a+2)(r-a+1) + (r-b+2)(r-b+1) - (r+3)(r+2) \geqq 0$. Note that $f(r)$ is a quadratic polynomial with positive leading coefficient. It can be seen that $f(a+b) < 0$ and, if $a \geqq b$, $f(a) < 0$. Since $r \geqq a$ we must have that $r > a+b$. This leads to a contradiction. For if C were a complex satisfying all of these conditions, then the dual complex $C^* = (C', \partial^*)$ with $C'_i = C^*_{r-i}$ would also be a CRH-complex with $H_0(C^*) = K$. But for C^*, $d'_1 = r - a$, $d'_2 = \mathrm{r} - \mathrm{b}$ and

$d'_3 = r$. But now $d'_1 + d'_2 = (r-a) + (r-b) > r$ since $r > a+b$. Therefore $s \geqq 4$ as required.

7. Multiple Complexes and Minimal Resolutions

Let Z_+ denote the set of non-negative integers. In Z^n_+ let $e_i = (0, \ldots, 1, \ldots, 0)$ denote the vector with 1 in the i^{th} position and zeros elsewhere. Let K be a field of characteristic $p > 0$. Following [1], we define an n-complex of KG-modules to be a pair $(X, \{d_j\})$ consisting of a KG-module $X = \sum_{b \in Z^n} X_b$ and a collection of boundary homomorphisms, $d_j\colon X \to X$, $j = 1, \ldots, n$. We assume that

(1) $d_j(X_b) \subseteq X_{b-e_j}$ for all $b \in Z^n$;
(2) $d_j \circ d_j = 0$ for all j; and
(3) $d_i \circ d_j + d_j \circ d_i = 0$ for all i, j.

This implies that if $Y_{b,j} = \sum_{m \in Z} X_{b+me_j}$ then $(Y_{b,j}, d_j)$ is a complex. For $b = (b_1, \ldots, b_n) \in Z^n$, let $|b| = \sum_{j=1}^{n} b_j$. Conditions (2) and (3) guarantee that $(\tilde{X}, d)$ is complex, where $\tilde{X}_m = \sum_{|b|=m} X_b$ and $d = \sum_{j=1}^{n} d_j$. We call this the total complex of $(X, \{d_j\})$. In general we shall restrict ourselves to nonnegative n-complexes, meaning those for which $X_b = \{0\}$ whenever $b \notin Z^n_+$.

A 1-complex (X, d_1) is said to be almost periodic if there exists a positive integer s such that, for all but a finite number of integers i, there exist isomorphisms ψ_i, ψ_{i+1} such that

$$\begin{array}{ccc} X_{i+s+1} & \xrightarrow{d_1} & X_{i+s} \\ \Big\downarrow{\scriptstyle \psi_{i+1}} & & \Big\downarrow{\scriptstyle \psi_i} \\ X_{i+1} & \xrightarrow{d_1} & X_i \end{array}$$

commutes. It is almost exact if the sequence $X_{i+1} \xrightarrow{d_1} X_i \xrightarrow{d_1} X_{i-1}$ is exact for all but a finite number of integer i. An n-complex $(X, \{d_j\})$ is almost periodic or almost exact if for every $b \in Z^n$, $j = 1, \ldots, n$, the 1-subcomplex $(Y_{b,j}, d_j)$ is respectively almost periodic or almost exact. An n-complex $(X, \{d_j\})$ has finite type if there are only a finite number of isomorphism classes of modules X_b and maps $d_{j,b}\colon X_b \to X_{b-e_j}$. Finally a module M is said to have almost periodic, almost exact or finite n-complex type if there exists an n-complex having the prescribed property and whose total complex is a minimal projective resolution of M.

In [1] Alperin asked such questions as for what groups do modules have bounded (dimensions of X_b's) n-complex type. Were it not for the minimality requirement, Theorem 3.4 would imply that any KG-module M with $c_{KG}(M) \leqq n$ had bounded, almost periodic, almost exact, finite n-complex type. Unfortunately

$$
\begin{tikzcd}
 & & & & & & P_{10} \arrow[d] & P_{10} \arrow[l] \arrow[d] & \arrow[l] \\
 & & & & & & P_1 \arrow[d] & P_{44} \arrow[l] \arrow[d] & \arrow[l] \\
 & & & & P_{10} \arrow[d] & P_{10} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & \arrow[l] \\
 & & & & P_1 \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_1 \arrow[l] \arrow[d] & \arrow[l] \\
 & & P_{10} \arrow[d] & P_{10} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_1 \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & \arrow[l] \\
 & & P_1 \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_1 \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & \arrow[l] \\
P_{10} \arrow[d] & P_{10} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_1 \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_{44} \arrow[l] \arrow[d] & P_1 \arrow[l] \arrow[d] & \arrow[l] \\
P_1 & P_{44} \arrow[l] & P_{44} \arrow[l] & P_1 \arrow[l] & P_{44} \arrow[l] & P_{44} \arrow[l] & P_1 \arrow[l] & P_{44} \arrow[l] & \arrow[l]
\end{tikzcd}
$$

Fig. 1

the resolutions constructed in the proof of (3.4) are seldom minimal. For example, if $p>2$ and $r=c_G(M)\geqq 2$ then $\tilde{X}_0\cong M\otimes X_0^1\otimes \dots \otimes X_0^r$ is the tensor product of M with projective covers for K. Hence $\tilde{X}_0$ is too large to be a projective cover for M. Consequently, the questions asked by Alperin remain, at least technically, unanswered.

Using diagrammatic methods [6], the authors have succeeded in making some progress on these questions for very restrictive classes of groups and modules. We end this paper with one example in which K has characteristic 2, $G=M_{11}$ (the Mathieu group on 11 letters) and $M=K$ the trivial KG-module. The display in Fig. 1 gives the first few stages of a minimal projective resolution for K in the form of an almost periodic, almost exact 2-complex of finite type. Here the symbol P_i denotes the projective cover for the simple module of dimension i. The 2-complex has the additional desirable feature that all of the modules (X_b's) are indecomposable. Such a situation is impossible to achieve in general; we need only consider the minimal resolution of the trivial module of the quaternion group in characteristic 2 for a counterexample. Moreover each row of the complex is exact except at or near the end. Away from the end each row is part of the minimal projective resolution of the nonsplit extension of the 44-dimensional simple module by itself. As can be seen this module is periodic of period 3. The details of the calculation can be derived from [6].

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