# The center of the generic division ring and twisted multiplicative group actions

Esther Beneish
Department of Mathematics
Eastern Washington University
Cheney, WA 99004
email: ebeneish@mail.ewu.edu

**Abstract:**

The problem we study is whether the center $C_n$, of the division ring of $n \times n$ generic matrices is stably rational over the base field F.

Procesi and Formanek [F], have shown that $C_n$ is stably isomorphic to the fixed field under the action of $S_n$ of $F(G_n)$, the quotient field of the group algebra $F[G_n]$ of a specific $S_n$-lattice denoted by $G_n$. In [B1] we showed that if p is a prime, $C_p$ is stably isomorphic to $F(ZS_p \otimes_{ZN} G_p)^{Sp}$, where N is the normalizer of a p-Sylow subgroup of $S_p$. In this article we further reduce the problem by reformulating it in terms of a lattice induced from a p-Sylow subgroup H of $S_p$. Let A be the root lattice, and let L=F(ZG/H). We show that there exits an element $\alpha \in Ext^1_{Sp}(ZS_p \otimes_{ZH} A, L^*)$ such that $L_\alpha(ZS_p \otimes_{ZH} A)^{Sp}$ is stably isomorphic to the center of the division ring of $p \times p$ generic matrices over F. The extension $\alpha$ corresponds to an element of the relative Brauer group of L over $L^H$. [1]

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**Introduction:**

The problem we study is whether the center $C_n$, of the division ring of n×n generic matrices is stably rational over the base field F. This is a major open question with connections to important problems in other fields such as geometric invariant theory and Brauer groups.

Given a finite group G, a ZG-lattice M, and a field F, we let F(M) denote the quotient field of the group algebra F[M] of the abelian group M written multiplicatively. It was shown in [F] that $C_n$ is stably isomorphic to $F(G_n)^{Sn}$, the fixed field under the action of $S_n$ of $F(G_n)$, where $G_n$ is a specific $ZS_n$-lattice which we define below. If M and M' are G-faithful ZG-lattices, their corresponding fields F(M) and F(M') are stably isomorphic and the isomorphism respects the G-action, if and only if, M and M' are in the same flasque class. Thus $C_n$ is stably equivalent to $F(M)^{Sn}$ for any $ZS_n$-lattice in the flasque class of $G_n$; flasque classes of ZG-lattices are defined in section 1.

Let p be a prime and let N be the normalizer in $S_p$ of a p-Sylow subgroup. In [B1] we showed that $G_p$ and $ZS_p \otimes_{ZN} G_p$ are in the same flasque class, which implies that $C_p$ is stably isomorphic to $F(ZS_p \otimes_{ZN} G_p)^{Sp}$. In [B2], we show that the flasque class of $G_p$ depends mostly on the structure of $\hat{G}_p$ as a $\hat{Z}$N-lattice, where $\hat{Z}$ denotes the p-adic completion of Z, and $\hat{G}_p = G_p \otimes \hat{Z}$. These results together with the decomposition of $\hat{G}_p$ into indecomposable $\hat{Z}$N-modules from [B2], are used to find a family of $ZS_p$-lattices whose corresponding fixed fields are stably isomorphic to $C_p$, the center of the division ring of p×p generic matrices, Theorem 1.5. This family is a subset of the flasque class of $G_n$.

Let G be a finite group, let M be a ZG-lattice, and let K be a field on which G acts as automorphisms. We denote by K(M) the quotient field of the group algebra K[M]. Given an element $\alpha \in \mathrm{Ext}^1_G(M,K^*)$, we have an α-twisted action of G on K(M) which will be denoted by $K_\alpha(M)$. α-twisted action will be defined in section 2.

In Theorem 2.1, we further reduce the problem by reformulating it in terms of a lattice induced from a p-Sylow subgroup H of $S_p$. Let A be the root lattice. We find a field extension L of F, on which $S_p$ acts faithfully as F-automorphisms, and an element α in $\mathrm{Ext}^1_{Sp}(ZS_p \otimes_{ZH} A, L^*)$, such that $L_\alpha(ZS_p \otimes_{ZH} A)^{Sp}$ is stably isomorphic to the center of the division ring of p×p generic matrices over F. Moreover $L^{Sp}$ is stably rational over F. The theorem says that if $L_\alpha(ZS_p \otimes_{ZH} A)^{Sp}$ is stably rational over F, then so is $C_p$. Since E is quasi-permutation, $L(ZS_p \otimes_{ZH} A)^{Sp}$ is rational over $L^{Sp}$, however there are no known analogous results for $L_\alpha(ZS_p \otimes_{ZH} A)^{Sp}$.

**Section 1:**

Let G be a finite group. An equivalence relation is defined in the category $L_G$ of ZG-lattices as follows. The ZG-lattices M and M' are said to be equivalent if there exists permutation modules P and P', such that $M \oplus P \cong M' \oplus P'$. The set of equivalence classes forms an abelian monoid under the direct sum. Lattices equivalent to 0 are said to be stably permutation. The equivalence class of a lattice M will be denoted by [M].

For any integer n, $H^n(G,M)$ will denote the n-th Tate cohomology group of G with coefficients in M. A ZG-lattices M is flasque if $H^{-1}(H,M)=0$ for all subgroups H of G. A flasque resolution of a ZG-lattice M is a ZG-exact sequence

$$0 \to M \to P \to E \to 0$$

with P permutation, and E flasque. It follows directly form [EM, Lemma 1.1], that any ZG-lattice M has a flasque resolution. The flasque class of M is [E], and will be denoted by $\phi(M)$. By [CTS, Lemma 5, section 1], $\phi(M)$ is independent of the flasque resolution of M. Lattices whose flasque class is 0 are said to be quasi-permutation. For more on flasque classes see [CTS, section 1].

We now define the $ZS_n$-lattice $G_n$ mentioned in the introduction. Let U be the $ZS_n$-lattice with Z-basis $\{u_i : 1 \leq i \leq n\}$ with $S_n$-action given by $gu_i = u_{g(i)}$ for all $g \in S_n$. Let A be the root lattice, equivalently defined by the exact sequence

$$0 \to A \to U \to Z \to 0$$
$$u_i \to 1$$

Then $G_n = A \otimes_Z A$, [F, Theorem 3].

Throughout the rest of these notes we will adopt the following notation unless otherwise specified.

- $G = S_p$, where p is a prime.
- H = p-Sylow subgroup of G. Thus H is cyclic of order p.
- a will denote a primitive (p-1)st root of 1 mod p.
- N = Normalizer of H in G. Thus $N = H \rtimes C$, is the semi-direct product of H by a cyclic group C, of order p-1. H will be generated by h, C by c, and we have $chc^{-1} = h^a$.
- $\hat{Z}$ = p-adic completion of Z.

- For any finite group G and any ZG-lattice M, $\hat{M}$ will denote the p-adic completion of M, and for any prime q, $M_q$ will denote the localization of M at q.

Since ZN/H ≅ ZC ≅ $Z[x]/(x^{p-1} - 1)$ as ZN-lattices, the decomposition of $\hat{Z}$N/H into indecomposables is given by

$$\hat{Z}N/H \cong \bigoplus_{k=0}^{p-2} Z[x]/(x-\vartheta^k) \cong \bigoplus_{k=0}^{p-2} Z_k$$

where $\vartheta$ is a primitive (p-1st root of 1 in $\hat{Z}$ which is congruent to a mod p, and $Z_k$ is the $\hat{Z}$N-module of $\hat{Z}$-rank 1 on which H acts trivially, and such that $c1=\vartheta^k$.

The restriction from G to N of U is isomorphic to ZH, and the isomorphism being $u_i \to h^i$, with $c.h=h^a$. $\hat{U}$ is an $\hat{Z}$N-indecomposable module by [CR, Theorem 19.22].

For k=0,…,p-2, we set $U_k = \hat{U} \otimes Z_k$. Since $\hat{Z}N \cong \hat{Z}N \otimes_{\hat{Z}H} \hat{Z}H \cong \hat{Z}N/H \otimes \hat{U}$, we have

$$\hat{Z}N \cong \bigoplus_{k=0}^{p-2} U_k$$

For k=0,…,p-2, $A_k$ will denote the $\hat{Z}$N-lattice $\hat{Z}H(h-1)^k$. Under this notation $A_1 = \hat{A}$, by [B1, Theorem 3.2]. We also set $X_k = Z_k/pZ_k$.

**Lemma 1.1:**

There exists a ZN-exact sequence

$$0 \to U \to Z \oplus A^* \to L \to 0$$

where $L = Z/p^rZ$ for all integers $r \geq 1$.

**Proof:**

Dualizing the defining sequence of the ZG-lattice A, we get

$$0 \to Z \to U \to A^* \to 0$$

since U is permutation, and hence isomorphic to its dual. The map $U \to A^*$ is the composition of restriction with the isomorphism from U to U*. We denote it by Res. The map $U \to Z \oplus A^*$ is given by $u_i \to p^{r-1} + \text{Res}\, u_i$. The result follows directly.

**Theorem 1.2:**

There exists a $\hat{Z}$N-exact sequence

$$0 \to \hat{Z}N \to \hat{G}_p \oplus \hat{A} \to Z_1/p^rZ_1 \to 0.$$

**Proof:**

In [B2, Theorem 2.5], we show that the decomposition of $\hat{G}_p$ into indecomposable $\hat{Z}$N-modules is

$$\hat{G}_p \cong \bigoplus_{k=0}^{p-2}{}_{k\neq 1} U_k \oplus Z_1.$$

By [B1, Theorem 3.2] $\hat{G}_p \cong \hat{A}^*\otimes Z_1$. Thus tensoring the sequence of Lemma 1.1 by $Z_1$ we obtain

$$0 \to U_1 \to Z_1 \oplus A_1 \to Z_1/p^rZ_1 \to 0$$

Adding $\bigoplus_{k=0}^{p-2}{}_{k\neq 1} U_k$ to the first two terms of the sequence we get

$$0 \to \bigoplus_{k=0}^{p-2}{}_{k\neq 1} U_k \oplus U_1 \to \bigoplus_{k=0}^{p-2}{}_{k\neq 1} U_k \oplus Z_1 \oplus A_1 \to Z_1/p^rZ_1 \to 0$$

But $\hat{Z}N \cong \bigoplus_{k=0}^{p-2} U_k$ which proves the result.

**Lemma 1.3:**

Let a be a primitive (p-1)st root of 1 mod p. The map

$$i\colon ZC \to ZC$$
$$1 \to c - a$$

is an injection of ZN-modules whose cokernel is $L_1 \oplus L_2$, where $L_1 = Z_1/p^rZ_1$ for some $r \geq 1$, and $L_2$ is a finite cohomologically trivial ZN-module of order prime to p.

**Proof:**

The map i is injective since c-a is not a zero divisor, so its cokernel is finite. A computation shows that coker(i) is cyclic of order $a^{p-1}$ -1. Since a is a primitive (p-1)st root of 1 mod p, $a^{p-1}$ - 1 is divisible by p, and the p-primary component of coker(i) is $L_1$.

For primes $q \neq p$ we have

$$0 \to Z_qC \xrightarrow{i} Z_qC \to (L_2)_q \to 0$$

Let $C_q$ be any subgroup of N of q-power order. We may assume that $C_q$ is contained in C. Thus $H^m(C_q, (L_2)_q) = 0$ for all integers m, which proves the result.

**Lemma 1.4:**

Let G be a finite group, and R a Dedekind domain of characteristic 0. Suppose there exits RG-exact sequences

$$0 \to V \to E \to L \to 0$$
$$0 \to V' \to E' \to L \to 0$$

where E and E' are RG-lattices, and V and V' are RG-projectives. Then

$$E \oplus V' \cong E' \oplus V.$$

Furthermore if $G=S_n$, then E and E' are in the same flasque class.

**Proof:**

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & E & \to & L & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & V & \to & M & \to & E' & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & V' & \to & V' & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since projectives are injectives in the category of RG-lattices, and since E and E' are RG-lattices, the middle sequences split and we have

$$V \oplus E' \cong V' \oplus E.$$

Since $G=S_n$ and since V and V' are RG-projective, they are stably permutation by [EM, Theorem 3.3], therefore E and E' are in the same flasque class.

**Theorem 1.5:**

Let p be a prime, let r be a positive integer and let $S= ZG\otimes_{ZN}(Z_1/p^rZ_1)$. Let

$$0 \to ZG \to M \to S \to 0$$

be any extension of S by ZG. Then the center of the division ring of $p\times p$ generic matrices over an F is stably isomorphic to $F(M)^G$.

**Proof:**

As above we let $G=S_p$, and let H be a p-Sylow subgroup of G. Let $i_2$ be the injection of ZG/H into $ZG \cong ZG\otimes_{ZH}ZH$ , defined by $i_2(\bar{g}_i) = \sum_{j=1}^{p} g_i \otimes h^j$ where $\{g_i\}$ is a transversal for H in G .

Let $i_1$ be any injective endomorphism of ZG/H with the property that the p-primary component of its cokernel is isomorphic to S.

Form the commutative diagram.

0 0

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & ZG/H & \xrightarrow{i_2} & ZG & \rightarrow & ZG/H\otimes A & \rightarrow & 0 \\
 & & {\scriptstyle i_1}\downarrow & & \downarrow & & & & \\
0 & \rightarrow & ZG/H & \rightarrow & E & \rightarrow & ZG/H\otimes A & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{coker}(i_1) & \rightarrow & \mathrm{coker}(i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\qquad (*)
$$

The vertical middle sequence becomes

$$0 \rightarrow ZG \rightarrow E \rightarrow S \oplus S' \rightarrow 0 \quad (1)$$

where S' is finite of order prime to p.

**Step 1:**

We show that $\hat{Z}G\otimes_{\hat{Z}N}\hat{G}_p \oplus \hat{Z}G\otimes_{\hat{Z}N}\hat{A} \cong \hat{E}$.

Tensoring the sequence

$$0 \rightarrow \hat{Z}N \rightarrow \hat{G}_p \oplus A_1 \rightarrow Z_1/p^rZ_1 \rightarrow 0$$

of Theorem 1.2, by $\hat{Z}G$ over $\hat{Z}N$, we get

$$0 \rightarrow \hat{Z}G \rightarrow \hat{Z}G\otimes_{\hat{Z}N}\hat{G}_p \oplus \hat{Z}G\otimes_{\hat{Z}N}\hat{A} \rightarrow S \rightarrow 0 \qquad (2)$$

Tensoring sequence (1) by $\hat{Z}$, and applying Lemma 1.4 to sequences (1) and (2) we get

$$\hat{Z}G\otimes_{\hat{Z}N}\hat{G}_p \oplus \hat{Z}G\otimes_{\hat{Z}N}\hat{A} \oplus \hat{Z}G \cong \hat{E} \oplus \hat{Z}G.$$

By the Krull-Schmit-Azumaya theorem , we have

$$\hat{Z}G\otimes_{\hat{Z}N}\hat{G}_p \oplus \hat{Z}G\otimes_{\hat{Z}N}\hat{A} \cong \hat{E}$$

**Step 2:**

We show that $G_p$ and E are in the same flasque class.

The defining sequence of the ZG-lattice A is

$$0 \rightarrow A \rightarrow U \rightarrow Z \rightarrow 0$$

$$u_i \rightarrow 1$$

For all primes $q \neq p$, this sequence splits, with splitting map $1 \rightarrow (1/p)\Sigma u_i$. Thus

$$U_q \cong A_q \oplus Z_q$$

and

$$U_q\otimes A_q \cong A_q\otimes A_q \oplus A_q$$

Since $G_p = A\otimes A$, we have

$$U_q\otimes A_q \cong (G_p)_q \oplus A_q$$

As ZN-modules, $U \cong ZH \cong ZN/C$, and $A \cong ZH(h-1)$. We also have an isomorphism of ZC-modules $A \cong ZC$ given by $h^i(h-1) \rightarrow c^i$ for $i=1,\ldots,p-1$. Therefore

$$Z_qN/C \otimes Z_qC \cong (G_p)_q \oplus A_q$$

which implies

$$Z_qN \cong (G_p)_q \oplus A_q$$

and

$$Z_qG \cong Z_qG \otimes_{ZN} (G_p \oplus A).$$

On the other hand, since H is of order p, $A_q$ is $Z_qH$-projective for all primes $q \neq p$, the horizontal sequences in (*) split when localized at the primes q, therefore $Z_qG \cong E_q$. Thus we have $E_q \cong Z_qG \otimes_{ZN} (G_p \oplus A)$ for all primes $q \neq p$.

From Step 1, we have $\hat{Z}G \otimes_{\hat{Z}N} \hat{G}_p \oplus \hat{Z}G \otimes_{\hat{Z}N} \hat{A} \cong \hat{E}$ which implies that

$$E_p \cong Z_pG \otimes_{ZN} (G_p \oplus A)$$

by [CR, Proposition 30.17]. Thus E and $ZG \otimes_{ZN} (G_p \oplus A)$ are in the same genus. By [BL, Proposition 2.2] they are in the same flasque class, since $G=S_p$. Since A is quasi-permutation, this implies that E and $ZG \otimes_{ZN} G_p$ are in the same flasque class. By [B2, Corollary 1.2] $G_p$ and $ZG \otimes_{ZN} G_p$ are in the same flasque class , thus so are E and $G_p$.

**Step 3:**

We show that $G_p$ and M are in the same flasque class.

Since the horizontal sequences in diagram (*) split, S' is cohomologically trivial. Hence there exits a ZG-exact sequence

$$0 \rightarrow Pr \rightarrow Fr \rightarrow S' \rightarrow 0 \qquad (3)$$

with Fr free, Pr cohomologically trivial. By [Br, Theorem 8.10] Pr is ZG-projective, and by [EM, Theorem 3.3], it is stably permutation. Adding sequences (3) and the sequence from the hypothesis of the theorem, namely

$$0 \rightarrow ZG \rightarrow M \rightarrow S \rightarrow 0$$

we get

$$0 \rightarrow Pr \oplus ZG \rightarrow Fr \oplus M \rightarrow S' \oplus S \rightarrow 0 \quad (4)$$

Applying Lemma 1.4 to (1) and (4), we get that $Fr \oplus M$ and E are in the same flasque class which implies that so are M and E, since Fr is free. Hence $G_p$ and M are in the same flasque class.

By [B2, Theorem 1.1] this implies that $F(G_p)^G$ and $F(M)^G$ are stably isomorphic. The result follows by [F, Theorem 3] .

**Section 2:**

Given a finite group G, and a ZG-lattice M, and a field L on which G acts as automorphisms, we may form the field L(M), and this field has a G-action induced from the action of G on M. However there exist other G-actions on L(M). These actions were found by Saltman, [S], and called $\alpha$-twisted actions. They are defined as follows.

Let $\alpha \in Ext^1_G(M, L^*)$, where $L^*$ is the multiplicative group of L. Let the equivalence class of

$$0 \to L^* \to M' \to M \to 0$$

in $Ext_G(M, L^*)$ be $\alpha$. Writing M and M' as multiplicative abelian groups, we have,

$$M' = \{ x.m: x \in L^*, m \in M\}$$

and the G-action on M' is given by $g{*}x.m = g(x)d_g(m).gm$, where $d : G \to Hom_Z(M,L^*)$ is the derivation corresponding to $\alpha$. In particular, for x=1, we have

$$g{*}m = d_g(m).gm,$$

Thus we obtain an $\alpha$-twisted action on L(M). Denote by $L_\alpha(M)$ the field L(M) with the corresponding G-action.

**Definition:**

Let L and K be fields and let G be a finite subgroup of their automorphisms groups. Then L and K are said to be isomorphic (stably isomorphic) as G-fields if they are isomorphic (stably isomorphic) and the isomorphism respects their G-actions.

The following remark is needed in the proof of Theorem 2.1.

**Remark:**

Recall that N is the normalizer of a p-Sylow subgroup H of G. Thus $N = H \rtimes C$, is the semi-direct product of H by a cyclic group C, of order p-1. H is generated by h, C by c, and we have $chc^{-1} = h^a$, where a is a primitive (p-1)st root of 1 mod p.

Let $n_h = \Sigma_i h^i$ be the norm of H. The kernel of the ZH-map $ZH \to ZH(h-1)$, multiplication by h-1, is $n_h ZH$. Thus $A \cong ZH(h-1) \cong ZH/n_h ZH$ as ZH-modules.

**Theorem 2.1:**

Let $L = F(ZG/H)$. There exists an $\alpha$-twisted action of $G$ on $L(ZG/H\otimes A)$ such that $L_\alpha(ZG/H\otimes A)^G$ is stably isomorphic to $C_p$. The extension $\alpha$ corresponds to an element of the relative Brauer group $Br(L/L^H)$. If $L_\alpha(ZG/H\otimes A)^G$ is stably rational over F, then $C_p$ is stably rational over F.

**Proof:**

Let $i_1$ be the map

$$ZG/H \to ZG/H$$
$$\bar{1} \to \bar{c} - \bar{a}$$

Since $ZG/H \cong ZG\otimes_{ZN}ZC$, the map $i_1$ is the map i of Lemma 1.3 induced up to G, and thus it is injective. As in Theorem 1.5, we consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & i_2 & \downarrow & & & & \\
0 & \to & ZG/H & \to & ZG & \to & ZG/H\otimes A & \to & 0\\
 & i_1 & \downarrow & & \downarrow & & & & \\
0 & \to & ZG/H & \to & M & \to & ZG/H\otimes A & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{coker}(i_1) & \to & \mathrm{coker}(i) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Let $\{g_i\}$ be a transversal for H in G. Set $b_i = i_1(\bar{g}_i)$, and as in Theorem 1.5 $i_2(\bar{g}_i) = \sum_{j=1}^{p} g_i \otimes h^j$. Thus

$$M \cong ZG/H \oplus ZG /\{(b_i , -\sum_{j=1}^{p} g_i \otimes h^j) : i=1,...(p-1)! , j= 1,\dots,p-1\}.$$

From this isomorphism we obtain a G-surjection of rings

$$F[ZG/H \oplus ZG] \to F[M].$$

We let $y_i$ and $x_{ij}$ denote the elements of the Z-basis of $ZG/H \oplus ZG$, corresponding to and $g_i\otimes h^j$ respectively, when $ZG/H \oplus ZG$ is viewed as a multiplicative abelian groups. Thus the $y_i$ and $x_{ij}$ are independent commuting indeterminates over F. Let $m_i$ be the monomial in the $y_i$, corresponding to $b_i$. Then $F[ZG/H \oplus ZG] = F[y_i^{\pm 1}, x_{ij}^{\pm 1}]$, and the kernel of the above surjection is

$$I = \langle m_i \prod_{j=1}^{p} x_{ij}^{-1} - 1 : i=1,...(p-1)! \rangle$$

by [P, Lemma 1.8]. Thus $F[M] \cong F[y_i^{\pm 1}, x_{ij}^{\pm 1}]/I$ . Let

$$\bar{y}_i = y_i \bmod I$$

$$\bar{x}_{ij} = x_{ij} \bmod I \text{ for } j = 1,..., p-1.$$

Then $\bar{x}_{ip} = \overline{m}_i \prod_j \bar{x}_{ij}$ and $g\bar{y}_i = \overline{gy}_i$. The set $\{ \bar{y}_i, \bar{x}_{ij} : i=1,...,(p-1)!, j=1,...,p-1\}$ is algebraically independent over F, since its cardinality, p!, is equal to the Krull dimension of F[M]. Thus $F(M) = F(\bar{y}_i, \bar{x}_{ij}$ ): i=1,...,(p-1)!, j=1,...,p-1). We have a G-isomorphism

$$F[y_i] \to F[\bar{y}_i] \subseteq F[M]$$

$$y_i \to \bar{y}_i$$

Set $L = F(\bar{y}_i)$, then $L \cong F(ZG/H)$ and $F(M) \cong L(\bar{x}_{ij})$.

Let M* be the subgroup of F(M)* generated by L* and M. By the remark preceding the theorem $A \cong ZH/n_h ZH$ as a ZH-module, hence $M^*/L^* \cong ZG/H \otimes A$. We have an G-exact sequence

$$\alpha: \quad 0 \to L^* \to M^* \to ZG/H \otimes A \to 0.$$

Clearly $F(M) \cong F(M^*) = L_\alpha(ZG/H \otimes A)$as G-fields, where by F(M*) we mean the smallest subfield of F(M) generated by F and M*.

For the next statement, $\alpha \in \mathrm{Ext}^1_G(ZG/H \otimes A, L^*) \cong \mathrm{Ext}^1_H(A, L^*)$ by Shapiro's Lemma.

Taking the cohomlogy of the ZH-sequence

$$0 \to A \to ZH \to Z \to 0$$

we have $\mathrm{Ext}^1_H(A, L^*) \cong \mathrm{Ext}^2_H(Z, L^*) \cong H^2(H, L^*) = \mathrm{Br}(L/L^H)$.

By e.g. [B1, Lemma 2.1] $L^G$ is stably rational over F since ZG/H is G-faithful permutation, and the last statement follows.

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