**MONOMIAL ACTIONS OF THE SYMMETRIC GROUP.**

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**Abstract:**

Let F be a field, and let G be a finite group. A rational extension of F on which G acts purely monomially, is of the form F(M) for some ZG-lattice M, where F(M) is the quotient field of the group algebra of the abelian group M. It follows from work of Endo-Miyata, Swan, and Lenstra, that G-faithful ZG-lattices are in the same flasque class if and only if they have G-isomorphic corresponding fields. We investigate the stable rationality of $F(M)^G$ over F, when G is $S_p$, the Symmetric group on p letters and p is a prime. Thus the study of flasque classes of $ZS_p$-lattices plays a fundamental role in this investigation. Let N be the normalizer of a p-sylow sylow subgroup of $S_p$ We show that there are classes of $ZS_p$-lattices for which induction restriction from N to $S_p$, does not affect the flasque class. We also present sufficient conditions for the flasque class of a $ZS_p$-lattice to be zero, which implies that the corresponding fixed field is stably rational over F. In particular we study the flasque class of a specific lattice, $G_p$, which has the property that $F(G_p)^{Sp}$ is stably isomorphic to the center of the division ring of generic matrices, [F]. Let $\hat{M}$ denote the p-adic completion of a $ZS_p$-lattice M. We show that any $ZS_p$-lattice M having the property that $M_q \cong (G_p)_q$ as $Z_qG$-lattices for all primes q≠p, and that $\hat{M} \cong \hat{G}_p$ as $\hat{Z}$N-lattices, is in the flasque class of $G_p$. For a finite group G, lattices in the same genus are not in general in the same flasque class, however they are for $G=S_n$. We extend this to a larger class of $ZS_p$-lattices containing the genus of $G_p$.[1]

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**Introduction:**

This paper is in part a continuation of the investigation started in [B3], of the stable rationality of the center of the ring of $n\times n$ generic matrices over the complex numbers. Let G be a finite group, and let K be a field on which G acts as a group of automorphisms, possibly trivially. Let $K(x_1,\ldots,x_n)$ be a rational extension of K. A monomial action of G on $K(x_1,\ldots,x_n)$ is defined by $gx_i=k_ix_1^{a_{i1}}\ldots x_n^{a_{in}}$, where $g\in G$, and the matrix $(a_{ij})$ is an invertible matrix with integers entries, and $k_i\in K$. The action is said to be purely monomial if $k_i = 1$ for all i. If a group G acts on $K(x_1,\ldots,x_n)$ purely monomially, and if we let M be the ZG-lattice with Z-basis $\{m_1,\ldots,m_n\}$, where g acts on $x_i$ as on $m_i$, then $K(x_1,\ldots,x_n)$ is isomorphic K(M), the quotient field of the group algebra K[M], and the isomorphism respects the G-actions. Conversely if M is a ZG-lattice, then K(M) has a purely monomial G-action.

Let F be a field and let $C_n$ denote the center of the division ring of $n\times n$ generic matrices over F. It was shown in [F], that $C_n$ is stably isomorphic over F, to the fixed field under the action of $S_n$ of $F(G_n)$, where $G_n$ is a specific $ZS_n$-lattice, which we define below. Thus the question of the stable rationality of $C_n$ is a special case of the problem of finding invariants of fields on which a group acts purely monomially. Let G = $S_p$, the symmetric group on p symbols where p is a prime. In this paper we consider purely monomial actions on rational extensions of F, that is fields of the form F(M), where M is a ZG-lattice and G acts trivially on F. It follows directly from work of Endo-Miyata [EM1], Lenstra [L], and Swan [S], that for any finite group G and any G-faithful ZG-lattices M and M', F(M) and F(M') are stably isomorphic as G-fields if and only if M and M' are in the same flasque class, Theorem 1.1. Thus the study of flasque classes of ZG-lattices is the starting point in this investigation.

We present our results on this question in section 1. Let N be the normalizer of a p-sylow subgroup of G. Thus N is the semi-direct product of a group of order p by a cyclic group of order p-1. In Theorem 1.4, which is a generalization of [B3, Proposition 2.4], we describe a class of ZG-lattices M for which induction-restriction from N to G preserves the flasque class; the flasque class of M will depend on the localization of M at all primes $q \neq p$.

In Theorem 1.6, we apply these results to the flasque class of $G_p$. We let $\hat{M}$ denote the p-adic completion of a ZG-lattice M. We show that any ZG-lattice M having the property that $M_q \cong (G_p)_q$ as ZG-lattices, and that $\hat{M} \cong \hat{G}_p$ as $\hat{Z}$N-lattices, is in the flasque class of $G_p$. Thus $F(M)^G$ is stably isomorphic to $C_p$, the center of the division ring of $p \times p$ generic matrices. For a finite group G, lattices in the same genus are not in general in the same flasque class, however they are if G is the symmetric group. Theorem 1.6 extends this to a larger class of ZG-lattices containing the genus of $G_p$. In Theorems 1.7 and 1.8, we present sufficient conditions for flasque classes to be 0, which implies that the corresponding fixed fields are stably rational over F.

Section 2 is devoted to finding the decomposition of $\hat{G}_p$ into indecomposables -$\hat{Z}$N-modules. The main result, Theorem 2.3, gives a characterization of all invertible ZG-lattices in the flasque class of $G_p$ up to stable isomorphism, that is isomorphism up to permutation modules.

**Section 1:**

Let G be a finite group. Flasque classes of ZG-lattices play an important role in this problem; they are defined as follows. The ZG-lattices M and M' are said to be equivalent if there exist permutation modules P and P' such that $M \oplus P \cong M' \oplus P'$. This defines an equivalence relation on the category of ZG-lattices. The equivalence class of a ZG-lattice M will be denoted by [M]. The set of equivalence classes of ZG-lattices forms an abelian monoid under the direct sum. Lattices equivalent to 0 are said to be stably permutation. These are lattices M for which there exist permutation modules P and R such that $M \oplus P \cong R$.

For all $n \in Z$, $H^n(G, M)$ will denote the nth Tate cohomology group of G with coefficients in M. A ZG-lattice M is flasque $H^{-1}(H,M)=0$ for all subgroups H of G. A flasque resolution of a ZG-lattice M is an exact sequence

$$0 \to M \to P \to E \to 0$$

with P permutation and E flasque. It follows directly from [EM1, Lemma 1.1], that for any ZG-lattice M, there exist such a flasque resolution. The flasque class of M is defined to be [E]. Lattices whose flasque class is 0 are said to be quasi-permutation. These are lattices for which there exists a ZG-exact sequence

$$0 \to M \to P \to R \to 0$$

with P and R permutation.

Throughout the rest of this paper we adopt the following notation unless otherwise specified.

- $G = S_p$ , where p is prime.
- H = p-sylow subgroup of G, so H is cyclic of order p.
- a = primitive (p-1)st root of 1 mod p.
- $N = N_G(H)$ will be the normalizer of H in G, so $N = H \rtimes C$, where C is cyclic of order p-1.
- H will be generated by h, C by c, and we have $chc^{-1} = h^a$.
- For any finite group G, and for any ZG-lattice M, we will denote by $\phi(M)$ or $\phi_G(M)$ the flasque class of M.
- For any finite group G and any ZG-lattice M, $\hat{M}$ will denote the p-adic completion of M, and for any prime q, $M_q$ will denote the localization of M at q.

**Definition:**

Let Q be the field of rational numbers. Let G be a finite group, and let M be a ZG-lattice. The Q-class of M is the set of ZG-lattices M' such that QM = QM'.

**Definition:**

Let L and K be fields and let G be a finite subgroup of their automorphisms groups. L and K are said to be isomorphic (stably isomorphic) as G-fields if they are isomorphic (stably isomorphic) and the isomorphism respects their G-actions.

**Theorem 1.1:**

Let G be a finite group and let M and M' be G-faithful ZG-lattices. Then F(M) and F(M') are stably isomorphic as G-fields if and only if M and M' are in the same flasque class.

**Proof:**

By [B3, Lemma 2.1], $\phi(M) = \phi(M')$ implies that F(M) and F(M') are stably isomorphic as G-fields. To prove the converse, assume that F(M) and F(M') are stably isomorphic as G-fields. Then so are $F(ZG \oplus M)$ and $F(ZG \oplus M')$. Let L = F(ZG). The following is an adaptation of the proof of [L, Theorem 1.7].

There exist G-trivial indeterminates $y_i$, which are algebraically independent over L(M) and G-trivial intederminates $z_i$, which are algebraically independent over L(M'), such that

$$L(M)(y_1,\ldots,y_t) \cong L(M')(z_1,\ldots,z_r).$$

Let $R_1 = L[M][y_1,\ldots,y_t]$ and $R_2 = L[M'][z_1,\ldots,z_r]$. By [S, Lemma 8], there exists elements $a_1 \in R_1^G$ and $a_2 \in R_2^G$ such that $R_1[a_1^{-1}]$ and $R_2[a_2^{-1}]$ are isomorphic as L-algebras. By [S, Lemma 7], we have ZG-exact sequences

$$0 \to R_1^* \to R_1[a_1^{-1}]^* \to P \to 0$$

$$0 \to R_2^* \to R_2[a_2^{-1}]^* \to S \to 0$$

where P and S are ZG- permutation. Thus we obtain ZG-exact sequences

$$0 \to R_1^*/ L^* \to R_1[a_1^{-1}]^*/L^* \to P \to 0$$

$$0 \to R_2^* / L^* \to R_2[a_2^{-1}]^*/L^* \to S \to 0$$

Since $R_1^*/ L^* \cong M$ and $R_2^*/ L^* \cong M'$, we have that $\phi(M) = \phi(M')$ by [CTS, Lemma 8, section 1].

We now define the lattice $G_p$ mentioned in the introduction. Let U be the ZG-lattice with Z-basis $\{u_i : 1,....,p\}$ with G-action given by $gu_i = u_{g(i)}$ for g in G. Let A be the kernel of the map $U \rightarrow Z$ which sends the $u_i$ to 1. Then $G_p$ is defined to be $A\otimes_Z A$ and by [F, Theorem 3], $F(G_p)^G$ is stably isomorphic to the center of the ring of $p\times p$ generic matrices.

**Corollary 1.2:**

The classes $[G_p]$ and $[ZG\otimes_{ZN}G_p]$ of the ZG-lattices $G_p$ and $ZG\otimes_{ZN}G_p$ are equal. In particular $G_p$ and $ZG\otimes_{ZN}G_p$ are in the same flasque class.

**Proof:**

By [B3, Theorem 2.6], $F(G_p)$ and $F(ZG\otimes_{ZN}G_p)$ are isomorphic as G-fields. By Theorem 1.1 this implies that the flasque classes of $G_p$ and $ZG\otimes_{ZN}G_p$ are equal. By [CTS, Lemma 8, section 1] this is equivalent to the existence of ZG-exact sequences

$$0\rightarrow G_p \rightarrow E \rightarrow P \rightarrow 0$$

and

$$0\rightarrow ZG\otimes_{ZN}G_p \rightarrow E \rightarrow R \rightarrow 0.$$

with P and R permutation. By [BL, Proposition 3, section 3.1] $G_p$ is invertible, and hence so is $ZG\otimes_{ZN}G_p$. Therefore these sequences split by [CTS, Lemma 9, section 1], and we have

$$E \cong G_p\oplus P \cong ZG\otimes_{ZN}G_p \oplus R.$$

Hence $[G_p] = [ZG\otimes_{ZN}G_p]$.

**Lemma 1.3:**

Let M and M’ be ZG-lattices in the same Q-class. If there exists a ZG-exact sequence

$$0 \to M \to M' \to L \to 0.$$

where L is cohomlogically trivial, then [M] = [M’]. Consequently $\phi(M) = \phi(M')$.

**Proof:**

There exists a ZG-exact sequence

$$0 \to Pr \to Fr \to L \to 0$$

with Fr free. Since both Fr and L are cohomlogically trivial, so is Pr. By [BK, Theorem 8.10, Chapter VI], Pr is ZG-projective. We now form the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & M & \to & M' & \to & L & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M & \to & E & \to & Fr & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Pr & \to & Pr & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where E is the pullback of the maps $M' \to L$ and $Fr \to L$. Since Pr and Fr are projective, and since projectives are injectives in the category of ZG-lattices, we get $M \oplus Fr \cong M' \oplus Pr$ . Now by [EM2, Theorem 3.3] Fr and Pr are stably permutation since $G=S_p$, thus M and M’ are in the same class, and a fortiori in the same flasque class.

The following theorem is a generalization of [B3, proposition 2.4]. It describes a class of lattices for which induction-restriction from N to G preserves the flasque class.

**Remark:**

- Note that since $G=S_p$, every ZG-lattice is in the Q-class of a stably permutation lattice.
- Let G be any finite group, and let H be a subgroup of G. Let R be a ring, and let M be a RG-module. Let $\{g_i : i,\ldots,[G:H]\}$ be a transversal for H in G. The map

$$RG/H\otimes_R M \to RG\otimes_{RH}\mathrm{Res}^G_H\, M$$
$$\Sigma_{i,j}a_{ij}\,\bar{g}_i\otimes m_j \quad \to \Sigma_{i,j}a_{ij}g_i\otimes g_i^{-1}m_j$$

for $a_{ij}\in Z$ and $m_j\in M$, is an isomorphism of RG-modules.

**Theorem 1.4:**
Let $N=H\rtimes C$ be the normalizer of a p-Sylow subgroup H of G. Let M be a ZG-lattice and let R be a stably permutation ZG-lattice in the Q-class of M. If for each prime $q\neq p$, the q-primary component of R/M is cohomologically trivial then

$$\phi(M) = \phi(ZG\otimes_{ZN}M).$$

Consequently F(M) and $F(ZG\otimes_{ZN}M)$ are stably isomorphic as G-fields.

**Proof:**
Since QM = QR for some stably permutation ZG-lattice R, we have a ZG-exact sequence

$$0\to M \to R \to L \to 0, \qquad (1)$$

with R/M = L finite. Since L is a finite abelian group, it is isomorphic to the direct sum of its primary components. So $L = \oplus_{q\ \mathrm{prime}}L_q$. Let $L' = (\oplus_{q\ \mathrm{prime},q\neq p}\ L_q)$. Now consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & L_p & \to & L_p & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & M & \to & R & \to & L & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M & \to & M' & \to & L' & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here M' is the pullback of the maps R $\to$ L and L' $\to$ L. Since the q-primary component of L is cohomologically trivial for each prime $q \neq p$, L' is cohomologically trivial by [BK, Corollary 10.2, and Theorem 10.3, Chapter III] and by Lemma 1.3, $\phi_G(M) = \phi_G(M')$. Thus it suffices to assume that $L = L_p$. Let $S = Z/p^rZ$ where $p^r$ is the exponent of L. Tensoring sequence (1) by ZG over ZN we obtain

$$0 \to ZG\otimes_{ZN}M \to ZG\otimes_{ZN}R \to SG\otimes_{SN}L \to 0. \quad (2)$$

Now let $I_{G/N}$ be the kernel of the augmentation map $Z_pG/N \to Z_p$, and let $I = I_{G/N}/p^rI_{G/N}$. Since [G:N] is prime to p we have that

$$Z_pG/N = Z_p \oplus I_{G/N}, \text{ and } SG/N = S \oplus I,$$

By Mackey's subgroup theorem [CR, Theorem10.13], $Res^G_H ZG/N = Z \oplus ZH^m$ for some positive integer m, hence $I_{G/N} \cong Z_pH^m$ as a $Z_pH$-lattice by [CR, Theorem 36.1]. Therefore $I \cong SH^m$ as an SH-module and hence as a ZH-module, where the ZH-action is via the ring map

$$ZH \to SH.$$

Now $I\otimes L \cong SH^m\otimes L \cong (SH\otimes_S L)^m$ and by the remark preceding the theorem, $SH\otimes_S L \cong SH\otimes_S Res^H_{\{1\}}L$ as SH-modules. Now as an abelian group, $L \cong \oplus_{i=1,\ldots,r}(Z/p^iZ)^{a_i}$. Therefore $I\otimes L \cong (SH\otimes L)^m \cong \oplus_{i=1,\ldots,r} ((Z/p^iZ)H)^{ma_i}$ as a SH-module and hence as a ZH-module. For each i=1,…,r, there exists a ZH-exact sequence

$$0 \to ZH \to ZH \to ((Z/p^iZ)H) \to 0$$

where the map $ZH \to ZH$ is multiplication by $p^i$. Thus we get a ZH-exact sequence

$$0 \to ZH^k \to ZH^k \to I\otimes L \to 0,$$

where $k = m\Sigma_i a_i$.

Taking its cohomology, we see that $I\otimes L$ is cohomologically trivial as a ZH-lattice. For any integer n and any subgroup K of G, $H^n(K,I\otimes L)$ injects into $H^n(H,I\otimes L)$ since $I\otimes L$ is of p-power-order by [BK, Corollary 10.2, and Theorem 10.3, Chapter III]. Since $H^n(H,I\otimes L)=0$, $I\otimes L$ is ZG- cohomologically trivial.

We now write sequence (2) as

$$0\to ZG\otimes_{ZN}M \to ZG\otimes_{ZN}R \to L \oplus I\otimes L \to 0. \quad (2)$$

and consider the exact sequence

$$0\to M'\to M''\to I\otimes L \to 0 \text{ with } M'' \text{ free}. \quad (3)$$

Since M” and $L\otimes I$ are cohomologically-trivial so is M’, thus M’ is ZG-projective by [BK, Theorem 8.10, Chapter VI], and by [EM2, Theorem 3.3] it is stably permutation. We add the sequences (1), (3) and form a commutative diagram with (2),

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & ZG\otimes_{ZN}M & \to & ZG\otimes_{ZN}R & \to & L\oplus I\otimes L & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & ZG\otimes_{ZN}M & \to & M_3 & \to & R\oplus M'' & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & M'\oplus M & \to & M\oplus M' & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $M_3$ is the pullback of the maps $ZG\otimes_{ZN}R \to L\oplus I\otimes L$ and $R\oplus M''\to L\oplus I\otimes L$.

Since M” and, R and consequently $ZG\otimes_{ZN}R$, are stably permutation, we have by [CTS, Lemma 8, section 1]

$$\phi(M) \oplus \phi(M') = \phi(ZG\otimes_{ZN}M).$$

Since M’ is stably permutation, $\phi(M') = 0$, hence $\phi(M) = \phi(ZG\otimes_{ZN}M)$. The last statement follows by Theorem 1.1.

**Theorem 1.5:**

Let M, M' and R be ZG-lattices and assume that R is stably permutation. Suppose that $M_q \cong M_q' \cong R_q$ for all primes $q \neq p$. If $\hat{M} \cong \hat{M}'$ as $\hat{Z}$N-lattices, then $\phi_G(M) = \phi_G(M')$.

**Proof:**

Let J= $\{q \in Z^+$ : q prime, q divides |G| and $q \neq p\}$. By [CR, Lemma 31.4] there exists a ZG-exact sequence

$$0 \rightarrow M \rightarrow R \rightarrow L \rightarrow 0$$

with L finite of order prime to each element of J, since $M_q \cong R_q$ for all primes $q \in J$. Let $L = L_p \oplus L'$, where $L_p$ is the p-primary component of L. Then L' has order prime to the order of G, and hence L' is cohomologically trivial by [BK, Corollary 10.2, Chapter III]. By Theorem 1.4, we have $\phi_G(M) = \phi_G(ZG \otimes_{ZN} M)$ and by a similar argument $\phi_G(M') = \phi_G(ZG \otimes_{ZN} M')$.

By [CR, Theorem 30.17] $M_p \cong M_p'$ as $Z_p$N-lattices, hence M and M' are in the same genus as ZN-lattices, and thus $ZG \otimes_{ZN} M$ and $ZG \otimes_{ZN} M'$ are in the same genus as ZG-lattices. By [BL, Proposition 2, section 2.3], $[ZG \otimes_{ZN} M] = [ZG \otimes_{ZN} M']$, therefore $\phi_G(ZG \otimes_{ZN} M) = \phi_G(ZG \otimes_{ZN} M')$, and $\phi_G(M) = \phi_G(M')$.

We now apply these results to the flasque class of the ZG-lattice $G_p$.

**Theorem 1.6:**

Let M be a ZG-lattice such that $M_q \cong (G_p)_q$ for all primes $q \neq p$, and such that $\hat{G}_p \cong \hat{M}$ as $\hat{Z}$N-lattices. Then the flasque classes of M and $G_p$ are equal.

**Proof:**

By Theorem 1.5, it suffices to show that there exist a stably permutation ZG-lattice R such that $(G_p)_q \cong R_q$ for all primes $q \neq p$. Recall that $G_p = A \otimes_Z A$, where the defining sequence of the ZG-lattice A is

$$0 \to A \to U \to Z \to 0$$

$$u_i \to 1$$

For all primes $q \neq p$, this sequence splits, with splitting map $1 \to (1/p)\Sigma u_i$. Thus

$$U_q \cong A_q \oplus Z_q$$

and

$$U_q \otimes A_q \cong A_q \otimes A_q \oplus A_q$$

Since $G_p = A \otimes A$, we have

$$U_q \otimes A_q \cong (G_p)_q \oplus A_q$$

Thus

$$U_q \otimes A_q \oplus Z_q \cong (G_p)_q \oplus U_q$$

From the definition of U, one sees directly that $U \cong ZG/S_{n-1}$ and that as an $S_{n-1}$-module A is permutation. Thus $U \otimes A$ is ZG-permutation.

Now for any ZG-lattice M let $M^* = \mathrm{Hom}_Z(M,Z)$ be its dual. If M is permutation then M* is isomorphic to M by [CR, Corollary 10.29]. Therefore

$$U_q \cong A_q \oplus Z_q \cong A_q^* \oplus Z_q$$

and hence $\hat{A}_q \cong \hat{A}_q^*$ for all primes $q \neq p$. By [CR, Proposition 30.17] this implies tha $A_q \cong A_q^*$ for all primes $q \neq p$.

Now by [B3, Proposition 1.1], the lattice $B = A^* \otimes A$ has the property that

$$U \otimes A \oplus Z \cong B \oplus U$$

and hence it is stably permutation. Furthermore $B_q \cong (G_p)_q$ for all primes $q \neq p$, and the result is proved by taking B=R.

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The following result gives a sufficient condition for an invertible lattice to be stably permutation. The point of interest is that this is a condition on the restriction to N of the Q-class of the lattice, and on its localization at primes $q \neq p$.

**Theorem 1.7:**

Let M be an invertible ZG-lattice. If $Res_N^G M$ is in the Q-class of a projective ZN-lattice, then $\phi_G(ZG\otimes_{ZN}M)=0$, that is $ZG\otimes_{ZN}M$ is stably permutation. If also, for all primes $q\neq p$, the q-primary component of R/M is cohomologically trivial, where R is a stably permutation ZG-lattice in the Q-class of M, then $\phi_G(M)=0$. Thus $F(M)^G$ is stably rational over F.

**Proof:**

We have a ZN-exact sequence

$$0\to M \to Pr \to L \to 0$$

with Pr projective and L finite. We will now show that L is cohomologically trivial. Let q be a prime. By Swan's theorem [CR, Theorem 32.11], $QPr \cong QN^m$ for some positive integer m. So QM is QS-free for any q-Sylow subgroup S of N. Since M is invertible and N is metacyclic, M is in the genus of a free ZS-lattice for each S, by [B2, Theorem 1.1]. Hence M is ZN-cohomologically trivial by [BK, Corollary 10.2 and Theorem 10.3, Chapter III]. By [BK, Theorem 8.10, Chapter VI] M is ZN-projective. Therefore $ZG\otimes_{ZN}M$ is ZG-projective, and by [EM2, Theorem 3.3] it is stably permutation, so $\phi(ZG\otimes_{ZN}M) = 0$. If also, for all primes $q\neq p$, the q-primary component of R/M is cohomologically trivial, where R is a stably permutation ZG-lattice in the Q-class of M, then $\phi_G(M)= \phi_G(ZG\otimes_{ZN}M)$ by Theorem 1.4. Thus $\phi_G(M) = 0$. The last statement follows from [B3, Lemma 2.1].

**Theorem 1.8:**

If M is a ZN-lattice such that $M^H = M$ then $\phi(ZG\otimes_{ZN}M) = 0$.

**Proof:**

The structure of M as a ZN-lattice is given by its structure as a ZC-lattice, since $ZC \cong ZN/H$. A ZC-flasque resolution for M, is also a ZN-flasque resolution. Let

$$0 \to M \to P \to E \to 0$$

be this resolution. So P is ZC-permutation, and since E is flasque and C is cyclic, E is ZC-invertible by [EM2, Theorem 1.5]. Hence there exist an invertible module E' and a permutation module R, such that $E \oplus E' = R$. Adding E' to the last two terms of the sequence we get

$$0 \to M \to E' \oplus P \to R \to 0$$

By [CTS, Lemma 7, section 1], $\phi(M) = \phi(E')$. By [EM1, Proposition 3.1 and Theorem 3.3] there exist ZC-permutation modules V and V' and a projective ideal I in ZC such that

$$E' \oplus V \cong V' \oplus I.$$

So $\phi(E') = \phi(I)$. By Swan's theorem [CR1, Theorem 32.11] we have $I_q \cong Z_qC \cong Z_qN/H$ for all primes q. This implies that $ZG\otimes_{ZN}(E'\oplus V)$ is in the genus of a ZG-permutation module and by [BL, Proposition 2, section 2.3] $[ZG\otimes_{ZN}E']=0$, therefore $\phi(ZG\otimes_{ZN}M) = 0$.

**Corollary 1.9:**

If M is any ZN-lattice then $\phi(ZG\otimes_{ZC}M) = 0$.

**Proof:**

The same arguments as in the proof of Theorem 1.8 hold since we are inducing from C which is cyclic, up to G.

**Section 2:**

We will use the same notation as in section 1. So G is $S_p$, the symmetric group on p letters, and p is a prime. N is the normalizer of a p-Sylow subgroup H of G, thus N is the semi-direct product of H by a cyclic group C of order p-1. H is generated by h, C by c, and we have $chc^{-1} = h^a$, where a was defined to be a primitive (p-1)st root of 1 mod p.

In section 1 we defined the ZG-lattice A by the exact sequence

$$0 \to A \to U \to Z \to 0$$

Let A* = Hom(A,Z) be its dual. It is immediate from its definition that $U \cong ZG/S_{p-1}$, and that $Res^G_N U \cong ZH$. This isomorphism is given by

$$U \to ZH$$
$$u_i \to h^i \quad \text{for } i=1,\dots,p$$

where H acts in the natural way and $c.h^i = h^{ai}$. It follows that $A \cong ZH(h-1)$ as a ZN-module. Moreover as a ZH-module A is the augmentation ideal of the group ring ZH, and since H is cyclic $A_p \cong A_p^*$ as $Z_pH$-modules. Thus $Z_pN \otimes_{Z_pH} A_p \cong Z_pN \otimes_{Z_pH} A_p^*$. For each k=1,..,p-1 we have

$$A \cong ZH(h-1) \cong ZH(h-1)^k \text{ as ZH-modules}$$
$$h^i(h-1) \to h^i(h-1)^k.$$

Since $ZN/H \cong ZC \cong Z[x]/(x^{p-1}-1)$ as ZN-lattices, the decomposition of $\hat{Z}N/H$ into indecomposables is given by

$$\hat{Z}N/H \cong \oplus_{k=1,\dots,p-1} \hat{Z}[x]/(x-\theta^k) \cong \oplus_{k=1,\dots,p-1} Z_k$$

where θ is a primitive (p-1)st root of 1 in $\hat{Z}$ which is congruent to a mod p, and $Z_k$ is the trivial $\hat{Z}N$-module of $\hat{Z}$ rank 1 with trivial H action, and such that $c1=\theta^k$.

**Notation:**

- For k=1,...,p-1, we set $A_k = \hat{Z}H(h-1)^k$.

- For k=1,...,p-1, we set $G_k = \hat{A}^*\otimes \hat{Z}H(h-1)^k$.
- For k=1,…,p-1, we set $X_k = Z_k/pZ_k$.

**Theorem 2.1:**

For each k=1,...,p-1 we have $\hat{A}^*\otimes Z_k \cong A_k$ and $G_k \cong \hat{A}^*\otimes A_k$.

**Proof:**

By [B3, Lemma 1.2] there exits a ZN-exact sequence

$$0\to A\to A^*\to X\to 0 \qquad (1)$$

Tensoring by $Z_k$, and using the fact that for each k, the sequence

$$0\to ZH(h-1)^{k+1}\to ZH(h-1)^k \to X_k \to 0$$

is exact, with the map $ZH(h-1)^k \to X_k$ given by $(h-1)^k \to 1$, we get the following commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
0\to & \hat{A}\otimes Z_k & \to & \hat{A}^*\otimes Z_k & \to & X_k & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0\to & \hat{A}\otimes Z_k & \to & M & \to & \hat{Z}H(h-1)^k & \to 0 \\
 & & & \uparrow & & \uparrow & \\
 & & & \hat{Z}H(h-1)^{k+1} & \to & \hat{Z}H(h-1)^{k+1} & \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

where M is the pullback of the maps $\hat{A}^*\otimes Z_k \to X_k$ and $\hat{Z}H(h-1)^k \to X_k$. We now show that $Ext^1_H(A,A)= 0$ which will imply that $Ext^1_H(\hat{A},\hat{A})$ since H is of order p, and hence that $Ext^1_H(\hat{A}\otimes Z_k, \hat{Z}H(h-1)^{k+1}) = 0$. Consider the ZH-sequence

$$0 \to A \to ZH \to Z \to 0$$

We have

$$\ldots\to Ext^1_H(ZH,A) \to Ext^1_H(A,A) \to Ext^2_H(Z,A) \to Ext^2_H(ZH,A) \to \ldots$$

Since ZH is ZH-free this becomes

$$0 \to Ext^1_H(A,A) \to Ext^2_H(Z,A) \to 0.$$

But $Ext^2_H(Z,A) = H^2(H, A)$. Since H is cyclic $H^2(H, A) = H^0(H,A)$, and $H^0(H,A)=0$, since $A^H=0$. The middle vertical sequence splits since $Ext^1_N(\hat{A}\otimes Z_k, \hat{Z}H(h-1)^{k+1})$ injects into $Ext^1_H(\hat{A}\otimes Z_k, \hat{Z}H(h-1)^{k+1}) \cong Ext^1_H(\hat{A},\hat{A})=0$. Similarily $Ext^1_N(\hat{Z}H(h-1)^{k+1}, \hat{A}\otimes Z_k)=0$, so

$$\hat{A}\otimes Z_k \oplus \hat{Z}H(h-1)^k \cong \hat{A}^*\otimes Z_k \oplus \hat{Z}H(h-1)^{k+1}.$$

It is clear that $\hat{A}\otimes Z_k$ and $\hat{A}\otimes Z_{k+1}$ are not isomorphic as $\hat{Z}N$-lattices. Since all these modules are indecomposable the Krull-Schmidt-Azumaya implies that $\hat{Z}H(h-1)^k \cong \hat{A}^*\otimes Z_k$. Thus $A_k \cong \hat{A}^*\otimes Z_k$. Tensoring by $\hat{A}^*$, we get $G_k = \hat{A}^*\otimes \hat{Z}H(h-1)^k \cong \hat{A}^*\otimes \hat{A}_k$.

**Remark:**

Under the above notation we have $\hat{A} \cong \hat{Z}H(h-1) \cong \hat{A}^*\otimes Z_1$. Recall that $G_p$, the ZG-lattice defined in section 1, having the property that the center $C_p$ is stably isomorphic to $F(G_p)^G$, is equal to $A\otimes A$. Therefore $\hat{G}_p \cong \hat{A}^*\otimes Z_1\otimes \hat{A}^*\otimes Z_1 \cong \hat{A}^*\otimes \hat{A}^*\otimes Z_2 \cong \hat{A}^*\otimes A_2 \cong G_2$.

We will denote the ZN-lattice $ZH(h-1)^2$ by A'.

**Theorem 2.2:**

1)Set $J = A^*\otimes A'$. Then J and $G_p$ are in the same genus as ZN-modules. Furthermore the fields $F(J \oplus A')$ and $F(ZN)$ are isomorphic as N-fields.

2)The field $F(ZG\otimes_{ZN}(G_p \oplus A'))^G$ is stably rational over F.

3)$\phi_G(ZG\otimes_{ZN}A') = [G_p]$.

**Proof:**

From the ZN-exact sequence

$$0\to ZH(h-1)^{k+1} \to ZH(h-1)^k \to X_k \to 0$$

we have that $A_q' \cong Z_qH(h\text{-}1)^2 \cong A_q$ for all primes q different from p, since $X_k$ is of order p. Dualizing the defining sequence for A, we get ZN-sequence

$$0 \to Z \to U \to A^* \to 0$$

Tensoring by A' over Z we get

$$0 \to A' \to ZN \to J \to 0 \quad (1)$$

We have $J_q \cong A^*_q \otimes A_q \cong A_q \otimes A_q \cong A^*_q \otimes A'_q \cong (G_p)_q$ for all primes q different from p, and by Thereom 2.1 $\hat{G}_p \cong \hat{J}$, therefore J and $G_p$ are in the same genus. This implies that J is invertible by [B1, Theorem A]. By applying [B3, Lemma 2.1] to sequence (1) we get $F(J \oplus A')$ and F(ZN) are isomorphic as N-fields. This proves 1).

Tensoring (1) by ZG over ZN we obtain

$$0 \to ZG \otimes_{ZN} A' \to ZG \to ZG \otimes_{ZN} J \to 0. \quad (2)$$

By [B3, Lemma 2.1], F(ZG) and $F(ZG \otimes_{ZN} (J \oplus A'))$ are stably isomorphic as G-fields. Since J and $G_p$ are in the same genus as ZN-lattices, $ZG \otimes_{ZN} J$ and $ZG \otimes_{ZN} G_p$ are in the same class, by [BL, Propostion 2, section 2.3]. Therefore F(ZG) and $F(ZG \otimes_{ZN} (G_p \oplus A'))$ are stably isomorphic as G-fields. Again by [B3, Lemma 2.1] $F(ZG \otimes_{ZN} (G_p \oplus A'))^G$ is stably rational over F. This proves 2).

Finally from sequence (2) we have $\phi(ZG \otimes_{ZN} A') = [ZG \otimes_{ZN} J] = [ZG \otimes_{ZN} G_p] = [G_p]$.

**Notation:**

For k=1,...,p-1, let $U_k = \hat{U} \otimes Z_k$. Under this notation $\hat{U} = U_{p-1}$.

**Theorem 2.3:**

The decomposition of $\hat{G}_p$ into indecomposable $\hat{Z}$N-modules is given by

$$\hat{G}_p \cong \oplus_{k=1,\ldots,p-2} U_k \oplus Z_1$$

Any invertible ZG-lattice in the flasque class of $G_p$ will have the same $\hat{Z}$N-decomposition up to stable isomorphism, that is isomorphism modulo permutation modules.

**Proof:**

We have $U \cong ZN/C$ as a ZN-lattice , and $U\otimes A \cong ZN\otimes_{ZC}Res^N{}_C A$. A Z-basis for A is $\{h^i-1: i=1,\ldots,p-1\}$and this basis is permuted by C. It follows directly that A is ZC-free. Thus

$$U\otimes A \cong ZN\otimes_{ZC}Res^N{}_C A. \cong ZN\otimes_{ZC}ZC \cong ZN.$$

Let $B=A^*\otimes A$. By Theorem 2.2, $\hat{B}$ is isomorphic to $G_1$ as a $\hat{Z}$N- module. By [B3, Proposition 1.1], we have $B \oplus U \cong U\otimes A \oplus Z$ as ZG-lattices. Thus as $\hat{Z}$N-lattices

$$G_1 \oplus U_0 \cong \hat{Z} \oplus \hat{Z}N$$

Tensoring by $Z_1$ over Z, we get

$$G_2 \oplus U_1 \cong Z_1 \oplus \hat{Z}N$$

or equivalently

$$\hat{G}_p \oplus U_1 \cong Z_1 \oplus \hat{Z}N.$$

Since $\hat{Z}N\cong\hat{Z}H\otimes\hat{Z}C\cong\hat{U}\otimes(\oplus_{k=1,\ldots p-1}Z_k)\cong\oplus_{k=1,\ldots p-1}U_k$, we have

$$\hat{G}_p\cong\oplus_{k=1,\ldots p-2}U_k \oplus Z_1.$$

For the second statement note that since $G_p$ is invertible by [BL, Proposition 3, section 3.1], $\phi(G_p) = -[G_p]$. If I is an invertible ZG-lattice in the flasque class of $G_p$, then $\phi(I) =-[I]= \phi(G_p) =-[G_p]$, hence $[G_p] = [I]$. Thus there exist permutation ZG-lattices P and R such that

$$G_p \oplus P \cong I \oplus R.$$

and the result follows.

It is of interest to note that all summands in this decomposition are $\hat{Z}$N-projective except $Z_1$ which is H-trivial. However, by Theorems 1.7 and 1.8 we know that there can be no ZG-lattice M in the flasque class of $G_p$ such that $Res^G_N \hat{M}$ is the sum of a projective lattice and an H-trivial lattice, since $\phi(G_p) \neq 0$ for $p \geq 5$ by [BL, Corollary 1, section 3.2].

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