TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 6, June 1996

REAL CONNECTIVE K-THEORY AND THE QUATERNION GROUP

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ABSTRACT. Let ko be the real connective K-theory spectrum. We compute ko_*BG and ko^*BG for groups G whose Sylow 2-subgroup is quaternion of order 8. Using this we compute the coefficients $t(ko)_*^G$ of the G fixed points of the Tate spectrum $t(ko)$ for $G = Sl_2(3)$ and $G = Q_8$. The results provide a counterexample to the optimistic conjecture of Greenlees and May [9, Conj. 13.4], by showing, in particular, that $t(ko)^G$ is not a wedge of Eilenberg-Mac Lane spectra, as occurs for groups of prime order.

1. INTRODUCTION

Let G be a group whose Sylow 2-subgroup is quaternion of order 8. The real connective K-homology of the classifying space BG is of interest because a number of geometric invariants of manifolds with fundamental group G lie in ko_*BG. In particular, Stolz shows in [14] that for fundamental groups which are finite 2-groups the question of positive scalar curvature reduces to $ko_*(BG)$. Using this, the Gromov-Lawson conjecture has been settled for manifolds whose fundamental group has periodic cohomology [5, 6, 7, 15].

Further, the periodic K-theory of BG is determined by the real, complex and quaternionic representations of G. The connective K-theory provides more refined invariants of G whose algebraic meaning would be interesting to discern.

The approach we use applies quite generally to the computation of the ko-homology and cohomology of BG, and is as follows. We localize at 2 throughout, let H^* be $H^*(\ ;\mathbf{Z}/2)$, and let $\mathcal{A}$ and $\mathcal{A}(1)$ be the mod 2 Steenrod algebra and its subalgebra generated by Sq^1 and Sq^2.

1. Compute the periodic K-theory KO_*BG in terms of the representation ring of G, using the result of Stolz and Rosenberg quoted as Theorem 2 below.
2. Decompose BG stably using the results of Martino and Priddy quoted as Theorem 1.
3. Describe the corresponding summands of H^*BG as $\mathcal{A}(1)$-modules.
4. Compute the E_2-term of the Adams spectral sequence
$$\mathrm{Ext}_{\mathcal{A}}(H^*(ko \wedge X), \mathbf{Z}/2) \cong \mathrm{Ext}_{\mathcal{A}(1)}(H^*X, \mathbf{Z}/2) \Longrightarrow ko_*X$$
for each summand X of BG.

Received by the editors August 10, 1994.
1991 *Mathematics Subject Classification.* Primary 19L41, 19L47, 19L64, 55N15, 55R35, 55Q91, 55M05.
Key words and phrases. Quaternion group, classifying space, connective K-theory, Tate cohomology.

5. Determine the differentials and extensions in the Adams spectral sequence of each summand using the Bockstein spectral sequence of $ko \wedge X$ and the fact that, after inverting the Bott element, ko_* becomes KO_*, which is known by steps 1 and 2.
6. Write X as a convenient direct limit, $X = \varinjlim X_n$, compute each $ko^* X_n$ using S-duality, and compute $ko^* X = \varprojlim ko^* X_n$.

In the last section we provide some background information on Tate theories and compute $t(ko)_*^G$ using the *norm sequence*

$$\cdots \to ko_n BG_+ \to ko^{-n} BG_+ \to t(ko)_n^G \to ko_{n-1} BG_+ \to \cdots .$$

The authors wish to thank Stephan Stolz for numerous helpful conversations. and for interesting us in these matters, and Peter Gilkey for sharing his calculations of the eta invariant at a crucial moment.

Some of the results here are a part of the first author's thesis [3].

2. Decomposition of BG and group representations of Q_8

Theorem 1 (J. Martino and S. Priddy [10]). *Let G be a finite group with Sylow 2-subgroup the quaternion group Q_8. Then the complete 2-local stable decomposition of BG into indecomposable summands is either*

1. $BG \simeq BSl_2(3)$, *or*
2. $BG \simeq BSl_2(3) \vee \Sigma^{-1} BS^3/BN \vee \Sigma^{-1} BS^3/BN$,

where N is the normalizer of a maximal torus in S^3.

Note that BQ_8 itself is the larger of these. Since Q_8 acts freely on the 3-sphere, its cohomology is periodic with period 4. Explicitly,

$$\begin{aligned} H^*(BQ_8) &= \mathbf{Z}/2[x,y,w]/(x^2+xy+y^2, x^2y+xy^2), \\ & \quad |x| = |y| = 1, \text{ and } |w| = 4, \text{ and} \\ H^*(BSl_2(3)) &= \mathbf{Z}/2[w] \otimes \wedge(\tilde{x}), \ |w| = 4, \ |\tilde{x}| = 3. \end{aligned}$$

The inclusion of the Sylow 2-subgroup $Q_8 \to Sl_2(3)$ induces an isomorphism in cohomology in dimensions 3 and 4 mod 4, with $\tilde{x}$ mapping to $x^2y = xy^2$. The two copies of $\Sigma^{-1}BS^3/BN$ are embedded as

$$\{x^\epsilon w^k \mid \epsilon = 1,2 \text{ and } k \geq 0\}$$

and

$$\{y^\epsilon w^k \mid \epsilon = 1,2 \text{ and } k \geq 0\}.$$

The Bocksteins are given by $\beta_1 x = x^2$, $\beta_1 y = y^2$, and $\beta_3 \tilde{x} = w$ by [13]. Thus, $\Sigma^{-1}BS^3/BN$ is composed of mod 2 Moore spaces and $BSl_2(3)$ is composed of mod 8 Moore spaces.

The real representations of Q_8 determine the periodic real K-theory of BQ_8 by the following result.

Theorem 2 (S. Stolz and J. Rosenberg [15]). *Let G be a finite p-group. Then $KO_*(BG)$ splits as a KO_*-module into the direct sum of copies of*

	0	1	2	3	4	5	6	7	(mod 8)
($\mathbb{R}$)	0	$(\mathbf{Z}/2)$	$(\mathbf{Z}/2)$	$\mathbf{Z}/p^\infty$	0	0	0	$\mathbf{Z}/p^\infty$	
($\mathbb{C}$)	0	$\mathbf{Z}/p^\infty$	0	$\mathbf{Z}/p^\infty$	0	$\mathbf{Z}/p^\infty$	0	$\mathbf{Z}/p^\infty$	
($\mathbb{H}$)	0	0	0	$\mathbf{Z}/p^\infty$	0	$(\mathbf{Z}/2)$	$(\mathbf{Z}/2)$	$\mathbf{Z}/p^\infty$	

one for each nontrivial irreducible real, complex, or quaternionic representation, respectively. Here $(\mathbf{Z}/2)$ *is* $\mathbf{Z}/2$ *if* $p = 2$ *and is* 0 *otherwise.*

Note that, when $p = 2$, these three KO_*-modules are $KO_*(\mathbf{RP}^\infty)$, $KU_*(\mathbf{RP}^\infty)$, and $KSp_*(\mathbf{RP}^\infty)$.

Q_8 has three nontrivial 1-dimensional representations and one irreducible 2-dimensional representation. By the Frobenius-Schur indicator, the 1-dimensional representations are all real, and the 2-dimensional representation is quaternionic. (Equivalently, we could observe that the Wedderburn decomposition of the real group ring $\mathbb{R}[Q_8]$, is $\mathbb{R}^4 \oplus \mathbb{H}$.) Comparing the splittings of KO_*BQ_8 given by the preceding two theorems, the following corollary is immediate.

Corollary 3. *As* KO_**-modules,*

$$\begin{aligned} KO_*(\Sigma^{-1}BS^3/BN) &\cong KO_*(\mathbf{RP}^\infty), \\ KO_*(BSl_2(3)) &\cong KO_*(\mathbf{RP}^\infty) \oplus KSp_*(\mathbf{RP}^\infty), \\ KO_*(BQ_8) &\cong \bigoplus_3 KO_*(\mathbf{RP}^\infty). \oplus KSp_*(\mathbf{RP}^\infty) \end{aligned}$$

3. The Bockstein differentials in the Adams spectral sequence

Recall that $H^*ko = \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbf{Z}/2$ [16]. As a result, the Kunneth isomorphism

$$H^*(ko \wedge X) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbf{Z}/2 \otimes H^*X \cong \mathcal{A} \otimes_{\mathcal{A}(1)} H^*X$$

and the adjunction

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(1)} M, N) \cong \mathrm{Hom}_{\mathcal{A}(1)}(M, N)$$

give the ordinary mod 2 Adams spectral sequence the form

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(H^*X, \mathbf{Z}/2) \Longrightarrow ko_{t-s}X.$$

Recall also that for $X = S^0$ the spectral sequence collapses at E_2 to give

$$\begin{aligned} E_2 = E_\infty &= \mathbf{Z}/2[h_0, h_1, w, b]/(h_0h_1, h_1^3, h_1w, w^2 + h_0^2 b), \\ & |h_0| = (0,1),\ |h_1| = (1,1),\ |w| = (4,3),\ |b| = (8,4), \end{aligned}$$

where $|x| = (t-s, s)$ specifies the geometric degree $t-s$ and the Adams filtration s. We will refer, ambiguously, to b and to a generator $\beta \in ko_8(S^0)$ detected by b, as 'the Bott element', since multiplication by its image in $KO_8(S^0)$ is the Bott periodicity isomorphism.

As an $\mathcal{A}(1)$-module, $H^*(\Sigma^{-1}BS^3/BN)$ is a direct sum,

$$H^*(\Sigma^{-1}BS^3/BN) = \bigoplus_{n \geq 1} H^*M^{4n-2}(2),$$

where $M^n(k) = S^{n-1} \cup_k e^n$ is the mod k Moore space with top cell in dimension n. It is well known, and easy to calculate, that, as an $E_2(S^0)$-module,

$$\begin{aligned} E_2(M^1(2)) &= E_\infty(M^1(2)) \\ &= \langle x, y \mid h_0x = 0,\ h_1^2 x = h_0 y,\ h_1^2 y = wx,\ wy = 0 \rangle \\ &\Longrightarrow ko_*(M^1(2)), \end{aligned}$$

with $|x| = (t-s, s) = (0,0)$ and $|y| = (2,1)$. It follows that $E_2(\Sigma^{-1}BS^3/BN)$ is the direct sum of copies of $E_2(M^{4n-2}(2))$ generated by x_n and y_n for $n \geq 1$ with $|x_n| = (4n-3, 0)$ and $|y_n| = (4n-1, 1)$. (See Figure 1.)

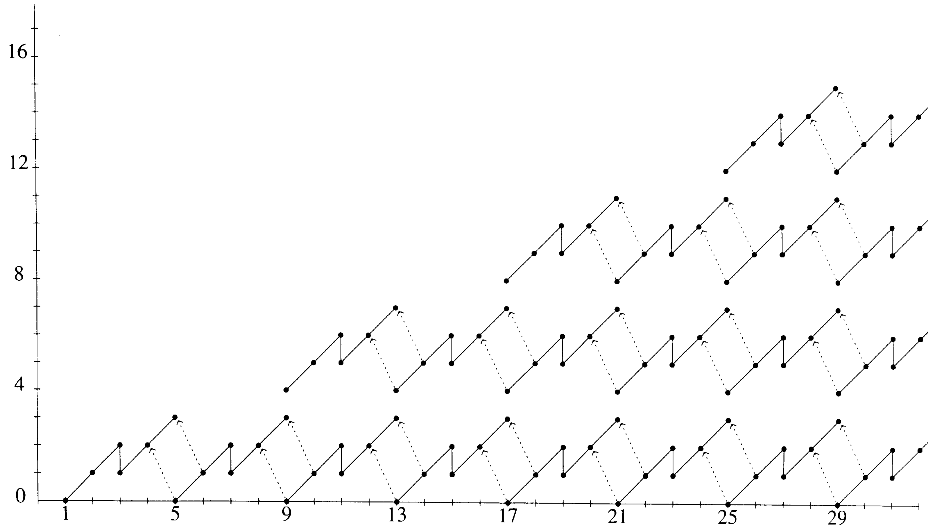

FIGURE 1. E_2 for $ko_*\Sigma^{-1}BS^3/BN$

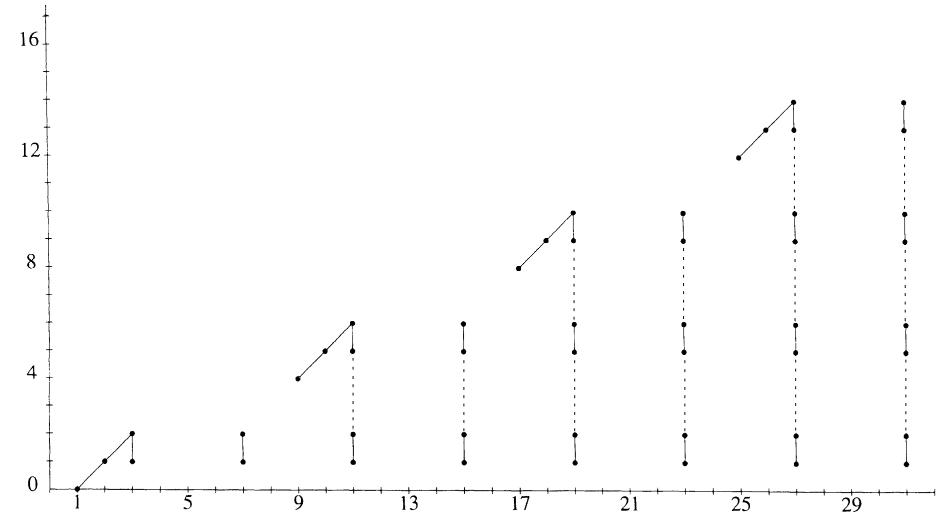

FIGURE 2. $E_3 = E_\infty = E^0(ko_*\Sigma^{-1}BS^3/BN)$

As an $\mathcal{A}(1)$-module, $H^*(BSl_2(3))$ is a direct sum of $\mathbf{Z}/2$'s, one in each dimension congruent to 3 or 4 mod 4. Thus, $E_2(BSl_2(3))$ is the corresponding sum of copies of $E_2(S^0)$, one starting in each dimension congruent to 3 or 4 mod 4. The theorem of [11] tells us that, for $r \geq 2$, there is a 1-1 correspondence between infinite h_0-towers in E_r of the Adams spectral sequence, i.e. sets $\{h_0^i x | i \geq 0\}$ with $h_0^i x \neq 0$ for all i, and the E_r-term of the homology Bockstein spectral sequence, and that, under this correspondence, the Adams d_r connecting these towers corresponds to the Bockstein β_r. We will compute the cohomology Bockstein spectral sequence and dualize. For ko, the Bockstein spectral sequence is well known ([12]) to collapse at E_2, with $E_2 = E_\infty$ spanned by the equivalence classes of $Sq^{4n}\iota$, where $\iota \in H^0 ko$ is the generator. The Bockstein spectral sequence of $BSl_2(3)$ has $E_1 = E_3$, $\beta_3(\tilde{x}w^k) = w^{k+1}$, and $E_4 = E_\infty = 0$. Tensoring these, we find that the Bockstein spectral sequence for $ko \wedge BSl_2(3)$ has

$$E_2 = E_3 = \{Sq^{4n}\iota | n \geq 0\} \otimes \wedge(\tilde{x}) \otimes \mathbf{Z}/2[w]$$

with

$$\beta_3(Sq^{4n}\iota \otimes \tilde{x}w^k) = Sq^{4n}\iota \otimes w^{k+1}$$

and $E_4 = E_\infty = 0$. Thus, the towers in the Adams spectral sequence will be annihilated by Adams d_3's. Examining E_3 we find that this d_3 must be

$$d_3(x_{4n}) = \alpha h_0^3 x_{4n-1} + \beta w x_{4n-5}$$

with $\alpha, \beta \in \{0,1\}$ and $x_i \in E_3^{0,i}$. Comparison with the Adams spectral sequence for $ko_* M^{4n}(8)$ shows that $\alpha = 1$. In order to determine β, it will help to examine the isomorphisms

$$\begin{aligned} \mathrm{Ext}_{\mathcal{A}}(H^*(ko \wedge X), \mathbf{Z}/2) &\cong \mathrm{Ext}_{\mathcal{A}}(H^*(ko) \otimes H^*(X), \mathbf{Z}/2) \\ &\cong \mathrm{Ext}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}/2) \end{aligned}$$

more carefully. The first isomorphism is induced by the usual Kunneth isomorphism $H^*(ko \wedge X) \cong H^*ko \otimes H^*X$, in which the tensor product has the diagonal $\mathcal{A}$ action. The second isomorphism applies generally to extended modules, and requires that the tensor product be given the 'extended' $\mathcal{A}$ action in which $\mathcal{A}$ acts only on the left factor, $H^*ko \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbf{Z}/2$. The two isomorphisms can thus be combined only because of a third isomorphism θ between the diagonal and extended actions. For convenience of notation, note that if we let $\widehat{H^*X}$ denote H^*X with trivial $\mathcal{A}$ action, i.e., $\mathcal{A}$ action pulled back along the augmentation $\mathcal{A} \to \mathbf{Z}/2$, then the diagonal action on $H^*ko \otimes \widehat{H^*X}$ is the extended action. The isomorphism

$$\theta : H^*ko \otimes \widehat{H^*X} \to H^*ko \otimes H^*X$$

is $\theta(a\iota \otimes x) = \sum a'\iota \otimes a''x$, with inverse $\theta^{-1}(a\iota \otimes x) = \sum a'\iota \otimes \chi(a'')x$, where $\psi(a) = \sum a' \otimes a''$. These matters are discussed nicely in [1, §III.16] and [2, 3.1]. To apply the theorem of [11] to towers in $\mathrm{Ext}_{\mathcal{A}(1)}(H^*(X), \mathbf{Z}/2)$, the convenient Bockstein is therefore $\widehat{\beta}_3 = \theta^{-1}\beta_3\theta$. We find

$$\begin{aligned} \widehat{\beta}_3(\iota \otimes \tilde{x}w^k) &= \iota \otimes w^{k+1}, \text{ and} \\ \widehat{\beta}_3(Sq^4\iota \otimes \tilde{x}w^{k-1}) &= Sq^4\iota \otimes w^k + \iota \otimes w^{k+1}. \end{aligned}$$

Dualizing, we see that

$$d_3(x_{4n}) = h_0^3 x_{4n-1} + w x_{4n-5}.$$

Thus, in E_4, we have generators x_{4n-1} and y_{4n+1} for $n \geq 1$ with $|x_{4n-1}| = (4n-1, 0)$ and $|y_{4n+1}| = (4n+1, 1)$, and with the relation $wx_{4n-1} = h_0^3 x_{4n+3}$ (and hence $h_0^2 b x_{4n-1} = h_0^6 x_{4n+7}$) for every $n > 0$. See Figure 3, in which we have replaced the class $b^i x_{4n-1}$ by the sum $b^i x_{4n-1} + h_0^{4i} x_{8i+4n-1}$ to simplify the diagram.

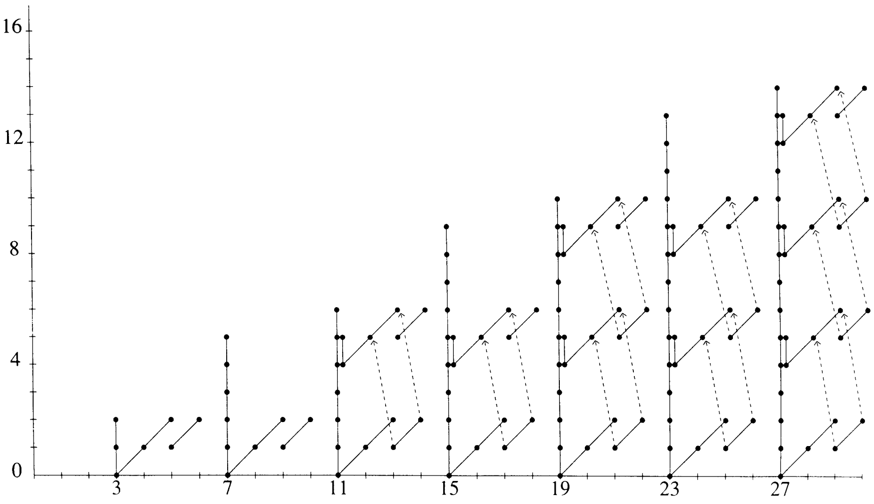

FIGURE 3. E_4 for $ko_*BSl_2(3)$

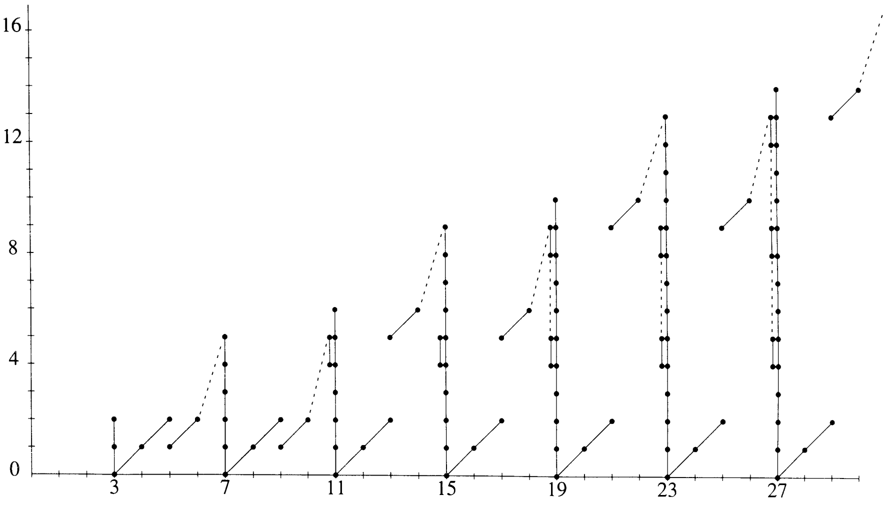

FIGURE 4. $E_5 = E_\infty = E^0(ko_*(BSl_2(3)))$

4. THE ko-HOMOLOGY OF $\Sigma^{-1}BS^3/BN$

Theorem 4. *In the Adams spectral sequence for $ko_*\Sigma^{-1}BS^3/BN$,*

$$d_2(x_{n+1}) = h_1 y_n$$

and $E_3 = E_\infty$. Furthermore, all possible extensions are nonzero. Therefore, for $n \geq 0$,

$$ko_{8n+\epsilon}\Sigma^{-1}BS^3/BN = \begin{cases} \mathbf{Z}/2 & = & \langle \beta^n x_1 \rangle & \epsilon = 1 \\ \mathbf{Z}/2 & = & \langle \eta\beta^n x_1 \rangle & 2 \\ \mathbf{Z}/2^{2n+2} & = & \langle y_{2n+1} \rangle & 3 \\ \mathbf{Z}/2^{2n+2} & = & \langle y_{2n+2} \rangle & 7 \\ 0 & & & 4,5,6,8 \end{cases}$$

with $wy_n = 4y_{n+1}$, $\beta y_n = 4y_{n+2}$, *and* $\eta^2\beta^n x_1 = 2^{2n+1}y_{2n+1}$ *for all* $n > 0$.

Proof. Recall that

$$ko_*(\Sigma^{-1}BS^3/BN)[b^{-1}] = KO_*\Sigma^{-1}BS^3/BN \cong KO_*\mathbf{RP}^\infty$$

and is therefore 0 in dimensions congruent to 4,5, or 6, mod 8. We will show that this forces $d_2(x_{2n}) = h_1y_{2n-1}$ for each $n > 0$. Similarly, the fact that $KO_*\mathbf{RP}^\infty$ has order 2 in dimensions 1 and 2 mod 8, and is 0 in dimensions congruent to 0 mod 8 will imply that $d_2(x_{2n+1}) = h_1y_{2n}$.

We proceed inductively. First, consider $x_2 \in E_2^{5,0}$. If $d_2(x_2) \neq h_1y_1$, then b^nx_2 survives to E_∞ for each n. Since $KO_5 = 0$, b^nx_2 must be a boundary for large enough n. Choose such an n, and let $u_i = b^{n-i}x_{2i+2}$ and $v_i = b^{n-i}h_1y_{2i+1}$ in order to simplify the notation. In order for $u_0 = b^nx_2$ to be a boundary, it must be $d_{r_1}h_1u_{t_1} = u_0$ for some r_1 and t_1. The fact that u_0 is not a multiple of h_1 implies that $h_1u_{t_1}$ must not be an h_1 multiple in E_{r_1}. Hence, u_{t_1} must have supported an earlier nonzero differential $d_{r_2}u_{t_1} = v_{s_2}$, $r_2 < r_1$, and we must have had $h_1v_{s_2} = 0$ in E_{r_2} to allow $h_1u_{t_1}$ to survive to E_{r_1}. This requires a still earlier differential $d_{r_3}h_1u_{t_2} = h_1v_{s_2}$ with $r_3 < r_2 < r_1$. To allow v_{s_2} to survive to E_{r_2}, we find that $h_1u_{t_2}$ must not be an h_1 multiple in E_{r_3}. Continuing in this way, we find an infinite sequence of ever shorter differentials. This contradiction shows that b^nx_2 cannot be the target of a differential, and we conclude that $d_2(x_2) = h_1y_1$.

The remaining even differentials $d_2(x_{2n}) = h_1y_{2n-1}$ follow by induction on n, since the inductive hypothesis implies that this is the only nontrivial differential x_{2n} can support. The possibility that x_{2n} could be the target of a differential is ruled out exactly as for x_2.

The same argument also shows that b^nx_1 cannot be the target of any differential. Since $KO_*\Sigma^{-1}BS^3/BN \cong KO_*\mathbf{RP}^\infty$ as KO_*-modules, it follows that b^nx_1 and $b^nh_1x_1$ survive to nonzero classes in E_∞. The remaining classes in stems congruent to 0,1 and 2, mod 8 must vanish upon inverting the Bott element, and, hence, the same argument as for the even differentials establishes the odd differentials, $d_2(x_{2n+1}) = h_1y_{2n}$, for $n > 0$.

This leaves no room for further differentials, so E_∞ is as shown in Figure 2. The fact that each $KO_i\Sigma^{-1}BS^3/BN$ is cyclic, together with the fact that the Bott element acts monomorphically on E_∞, implies that all possible extensions are nontrivial. Finally, the ko_*-module structure is evident from the Adams spectral sequence. QED

5. The ko-homology of $BSl_2(3)$)

Theorem 5. *In the Adams spectral sequence for* $ko_*(BSl_2(3))$

$$d_4(y_{4n+1}) = h_1bx_{4n-9}$$

and $E_5 = E_\infty$. *As a* ko_**-module,* $ko_*(BSl_2(3))$ *is generated by elements* y_5, y_9, *and* x_{4n+3} *for* $n \geq 0$. *Furthermore, for* $n \geq 0$,

$$ko_{8n+\epsilon}BSl_2(3) = \begin{cases} \mathbf{Z}/2^{4n+3} \oplus \mathbf{Z}/2^{2n} & = \langle x_{8n+3}\rangle \oplus \langle \bar{x}_{8n+3}\rangle \qquad \epsilon = 3 \\ \mathbf{Z}/2 & = \langle \eta x_{8n+3}\rangle \qquad 4 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & = \langle \eta^2 x_{8n+3}\rangle \oplus \langle \beta^n y_5\rangle \qquad 5 \\ \mathbf{Z}/2 & = \langle \eta\beta^n y_5\rangle \qquad 6 \\ \mathbf{Z}/2^{4n+6} \oplus \mathbf{Z}/2^{2n} & = \langle x_{8n+7}\rangle \oplus \langle \bar{x}_{8n+7}\rangle \qquad 7 \\ \mathbf{Z}/2 & = \langle \eta x_{8n+7}\rangle \qquad 8 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & = \langle \eta^2 x_{8n+7}\rangle \oplus \langle \beta^n y_9\rangle \qquad 9 \\ \mathbf{Z}/2 & = \langle \eta\beta^n y_9\rangle \qquad 10 \end{cases}$$

with relations $\eta^2\beta^n y_5 = 2^{4n+5}x_{8n+7}$, $\eta^2\beta^n y_9 \equiv 2^{2n+1}\bar{x}_{8n+11} \bmod \langle 2^{4n+6}\rangle$, *and* $\bar{x}_n \equiv \beta x_{n-8} - 16x_n \bmod \langle 32, \beta^2\rangle$ *for all* $n \geq 0$.

Remark. Note that the elements ηx_n and $\eta^2 x_n$ are annihilated by β. Thus, the map $ko_*BSl_2(3) \to KO_*BSl_2(3)$ has a nontrivial kernel, unlike $ko_*B\mathbf{Z}/2 \to KO_*B\mathbf{Z}/2$ or $ko_*\Sigma^{-1}BS^3/BN \to KO_*\Sigma^{-1}BS^3/BN$.

Proof. The isomorphism of Corollary 3,

$$KO_*BSl_2(3) \cong KO_*\mathbf{RP}^\infty \oplus KSp_*\mathbf{RP}^\infty,$$

implies that $KO_{4n} = 0$, $KO_{4n+1} \cong KO_{4n+2} \cong \mathbf{Z}/2$, and that η^2 times KO_{4n+1} is nonzero.

We show first that this implies that $b^n y_5$ and $h_1 b^n y_5$ survive to nonzero elements of E_∞ for all $n \geq 0$. Clearly, their filtration is too high to support nontrivial differentials. On the other hand, if $b^n y_5$ is the target of a differential, then the same sort of infinite regress as in Theorem 4 leads to a contradiction. Therefore, $b^n y_5$ maps to the nonzero class in $KO_{8n+5}BSl_2(3)$, and hence, η^2 times it is nonzero. It follows that $h_1 b^n y_5$ is also nonzero in E_∞.

Now, all other classes in dimensions $8n+4$, $8n+5$, and $8n+6$ must vanish upon inverting b. We show that this implies $d_4(y_{8n+5}) = h_1 b x_{8n-5}$ by induction. Suppose $n = 1$. If $d_4(y_{13}) \neq h_1 b x_3$ then, for some n, we must have $b^n y_{13}$ the target of a differential. Say $d_r(h_1 b^{n-i} y_{8i+13}) = b^n y_{13}$. Since $b^n y_{13}$ is not an h_1 multiple, $h_1 b^{n-i} y_{8i+13}$ must be indecomposable under multiplication by h_1 in E_r. Hence $b^{n-i} y_{8i+13}$ supports an earlier differential than d_r, and we are led to a contradiction as before. The induction now proceeds as in Theorem 4: $d_4(y_{8n+5}) = h_1 b x_{8n-5}$ is the only possible nonzero differential on y_{8n+5} by the inductive hypothesis. The alternative possibility, that $b^k y_{8n+5}$ is the target of a differential for some k, leads to a contradiction.

Now, the remaining survivors, $b^n y_9$ and $h_1 b^n y_9$, and the remaining differentials $d_4(y_{8n+1}) = h_1 b x_{8n-9}$, follow by exactly the same arguments.

In the resulting E_5-term, the indecomposables which might support differentials lie in dimensions $4n-1$. The only remaining elements in dimensions $4n-2$ must be nonzero in E_∞. Hence, $E_5 = E_\infty$.

The hidden η-extensions are propagated by multiplication by β once $\eta^2 y_5$ and $\eta^2 y_9$ are determined. The first, $\eta^2 y_5$, is inherited from the Moore space $M^4(8) \to BSl_2(3)$. The second, $\eta^2 y_9$, is detected by considering

$$BSl_2(3) \leftarrow B^8 \to B^8/B^4 = M^8(8)$$

where B^n is the n-skeleton of $BSl_2(3)$. (See Figure 4.)

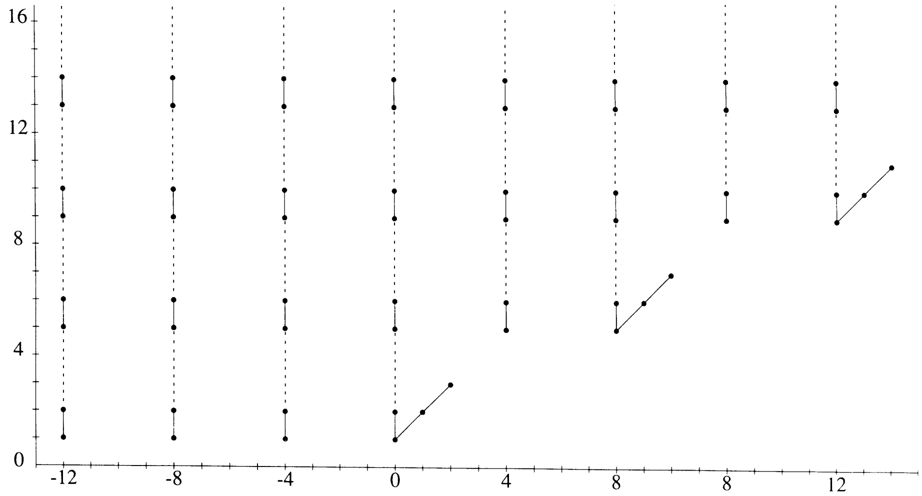

FIGURE 5. $E_\infty = E^0(ko^{-*}\Sigma^{-1}BS^3/BN)$

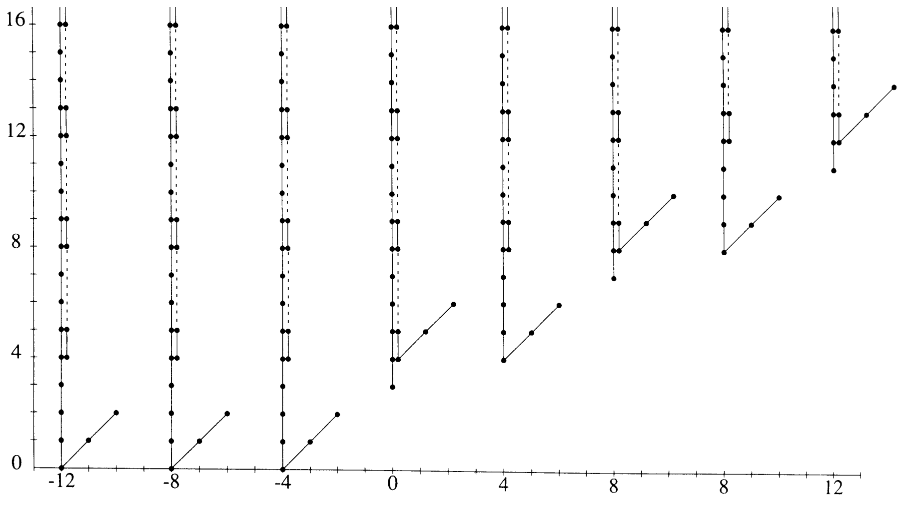

FIGURE 6. $E_\infty = E^0(ko^{-*}BSl_2(3))$

It remains to determine the extensions in dimensions $4n-1$. Multiplication by the Bott element is a monomorphism in dimensions $4n-1$ in E_∞, and hence in ko_*. Since $KO_{4n-1}BSl_2(3) = \mathbf{Z}/2^\infty \oplus \mathbf{Z}/2^\infty$, $ko_{4n-1}BSl_2(3)$ has rank ≤ 2.

The E_∞-term shows that $2^{4n+3}ko_{8n+3} = 0$ and $2^{4n+6}ko_{8n+7} = 0$. An element detected in filtration 0 will generate a subgroup of this order, which is therefore a direct summand. Dividing by this subgroup must yield a cyclic group, forcing the extensions. QED

6. The ko-cohomology of BG

The preceding calculations of ko-homology are easily dualized to obtain the following.

Theorem 6. *There are classes* $z_n \in ko^{4n}\Sigma^{-1}BS^3/BN$, $n \geq 0$, *which generate* $ko^*\Sigma^{-1}BS^3/BN$ *as a* ko_**-module, with relations* $\eta z_n = 0$ *if* $n \neq 0$, *and* $\beta z_n = \omega z_{n-1} = 4z_{n-2}$. *Thus*

$$ko^i\Sigma^{-1}BS^3/BN = \begin{cases} \widehat{\mathbf{Z}}_2, & i \equiv 0 \bmod 4, \\ \mathbf{Z}/2, & i \equiv -1, -2 \bmod 4, i < 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 7. *There are classes* $z_n \in ko^{4n}BSl_2(3)$, $n > 0$, *which generate* $ko^*BSl_2(3)$ *as a* ko_**-module, with relations* $\eta\beta z_n = 0$ *if* $n > 2$, *and* $\omega z_{n+1} \equiv 8z_n \bmod (8, \beta)$. *Further,*

$$ko^iBSl_2(3) = \begin{cases} \widehat{\mathbf{Z}}_2 \oplus \widehat{\mathbf{Z}}_2, & i \equiv 0 \bmod 4, \\ \mathbf{Z}/2, & i \equiv -1, -2 \bmod 4, \\ 0, & \textit{otherwise.} \end{cases}$$

Note: We will see that z_1 and z_2 each generate a copy of ko_*, while the other generators generate a copy of $ko_*/(\eta\beta)$.

We will prove these two results in tandem as follows. We can filter $B = BSl_2(3)$ and $X = \Sigma^{-1}BS^3/BN$ so that

$$\begin{gathered} B^0 = X^0 = *, \\ B^n/B^{n-1} = M^{4n}(8), \\ X^n/X^{n-1} = M^{4n-2}(2). \end{gathered}$$

For convenience of indexing when we compute the coefficients of the Tate theory, we will henceforth index cohomology by the negative of the usual index. To calculate $ko^{-*}BSl_2(3)$ and $ko^{-*}\Sigma^{-1}BS^3/BN$ we use duality:

$$\begin{aligned} ko^{-q}B &= ko^{-q} \lim_{\rightarrow} B^n \\ &= \lim_{\leftarrow} ko^{-q}\ B^n \\ &= \lim_{\leftarrow} ko_q\ DB^n \end{aligned}$$

and similarly for $X = \Sigma^{-1}BS^3/BN$. It will emerge from the calculation of the inverse systems, that they are Mittag-Leffler and hence there are no $\lim\limits_{\leftarrow}{}^1$-terms to consider. The finiteness of DX^n and DB^n implies the convergence of the inverse limit of the Adams spectral sequences.

The filtration of DX^n induced by that of X^n has filtration quotients

$$DX^{-n}/M^{4n+3}(2) = DX^{-n+1}$$

for $n < 0$. Thus, the inverse limit Adams spectral sequence for $ko^{-*}X$ has

$$E_2 = \bigoplus_{n=-\infty}^{-1} E_2(M^{4n+3}(2)).$$

Let x_n and y_n generate the n^{th} summand here, as in §3, so that $|x_n| = (4n+2, 0)$ and $|y_n| = (4n+4, 1)$ for each $n < 0$. The geometric boundary map for this filtration of DX^n, namely, the map

$$M^{-4n-1}(2) \longrightarrow DX^n \longrightarrow M^{-4n-4}(2)$$

is dual to the geometric boundary map

$$M^{4n+2}(2) \longleftarrow X^n \longleftarrow M^{4n+5}(2).$$

Since the latter map induces the differential $d_2(x_{n+1}) = h_1 y_n$, we see that the diagram

$$\begin{array}{ccc} S^{4n+4} & \xrightarrow{\eta^2} & S^{4n+2} \\ \downarrow & & \uparrow \\ M^{4n+5}(2) & \longrightarrow & M^{4n+2}(2) \end{array}$$

must commute after smashing with ko. Hence the dual diagram

$$\begin{array}{ccc} S^{-4n-4} & \xleftarrow{\eta^2} & S^{-4n-2} \\ \uparrow & & \downarrow \\ M^{-4n-4}(2) & \longleftarrow & M^{-4n-1}(2) \end{array}$$

must also. This implies that the geometric boundary map for DX^n also induces the differential

$$d_2(x_{n+1}) = h_1 y_n .$$

There are no further differentials possible. The resulting E_∞-term is shown in Figure 5. Note that no elements of filtration zero survive to E_∞.

The inverse limit Adams spectral sequence for $ko^{-*}B$ has

$$E_2 = \bigoplus_{n=-\infty}^{-1} E_2(S^{4n}) \oplus E_2(S^{4n+1})$$

with the first differential again implied by the Bockstein, leaving an E_4-term generated by x_{4n} and y_{4n+2}, $-\infty < n < 0$, with $|x_{4n}| = (4n, 0)$ and $|y_{4n+2}| = (4n+2, 1)$, and with the relation $wx_{4n} = h_0^3 x_{4n+4}$. Applying duality as we did for X, we conclude that

$$d_4(y_{4n+2}) = h_1 b x_{4n-8}$$

and that there are no further differentials possible. The extensions are resolved in the same way as they were for $ko_* BSl_2(3)$. The resulting E_∞-term is shown in Figure 6.

Additively, $ko^{-*}B = \bigoplus \Sigma^{4n} A$ summed over all integers n, where A is the graded module $\hat{Z}_2 \oplus \hat{Z}_2, Z/2, Z/2, 0$ with generators x, y, ηx, and $\eta^2 x$, in degrees 0,0,1, and 2 respectively. The action of the Bott generator $\beta \in ko^{-8}$ is more interesting. In negative degrees, i.e. $ko^n B$ with $n < 0$, multiplication by β is an isomorphism, as suggested by the right hand side of Figure 6. In positive degrees however, multiplication by β annihilates the η multiples and introduces complicated relations among the torsion-free summands, as can be deduced from the hidden extensions in Figure 6. If we let x_n be the filtration 0 generator of $\Sigma^{4n} A$ in degree $4n$, then

$$4\beta x_n \equiv 64 x_{n+2} \bmod \langle 128, \beta^2 \rangle.$$

We shall see in the next section that the norm sequence relating ko_*, ko^{-*} and the Tate theory $t(ko)_*$ works to insert $ko_* BG$ and the ko-homology of a mod 8 Moore space coming from the disjoint basepoints in $ko_q BG_+$ and $ko^{-q} BG_+$, into this picture in such a way that it becomes perfectly uniform.

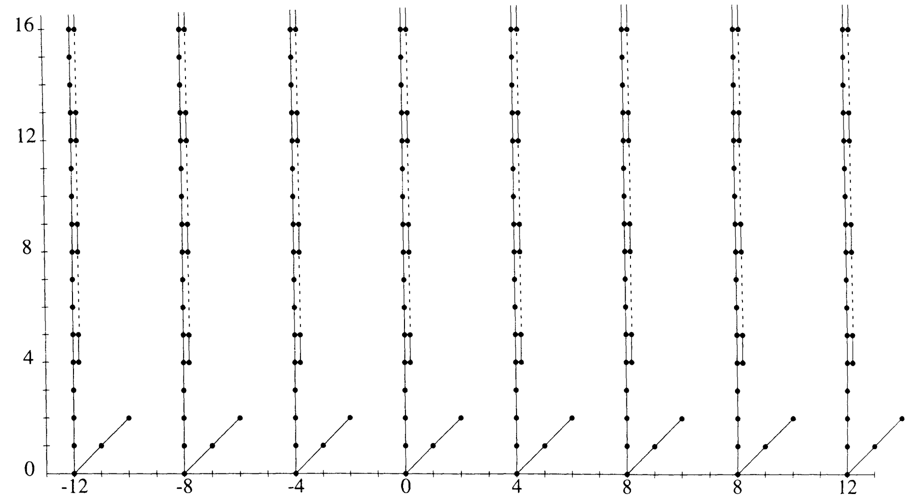

FIGURE 7. $E_\infty = E^0(t(ko)_*^{Sl_2(3)})$

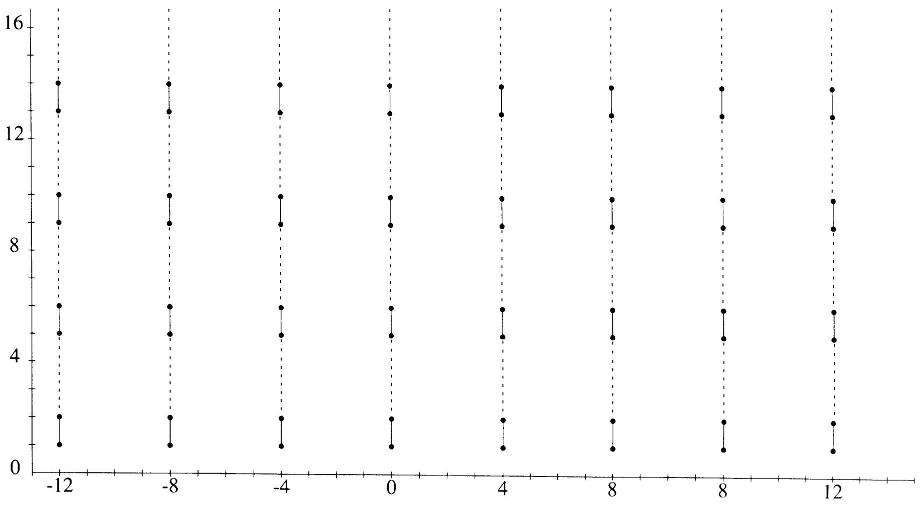

FIGURE 8. E_∞ for the part of $E^0(t(ko)_*^{Q_8})$ coming 'from X'

7. Coefficients of the Tate theory $t(ko)^G$

Greenlees and May [9] have recently given a definition of a spectrum $t(k)$, which they call the *Tate spectrum* of k, for any G-spectrum k. Their definition agrees with previous, ad hoc, constructions of candidates for general 'Tate theories' and enjoys many good properties. Since generalized Tate theories are not yet widely known, we will discuss a few of these properties, then explain the conjecture which we disprove and the context which made it a reasonable conjecture. Finally, we suggest a minimal modification of the conjecture consistent with the results here. For a comprehensive treatment of this material, see [9]. Note that the construction takes place in the category of G-spectra, though some, but not all, of its good properties can be seen on the non-equivariant level by taking G-fixed points.

When G acts freely on the unit sphere of a representation V, there is a particularly simple description of the G-fixed points $t(k)^G$,

$$t(k)^G = \lim_{\leftarrow} BG^{Ad(G)-nV} \wedge \Sigma k,$$

where $Ad(G)$ is the adjoint representation of G on the tangent space at the identity and BG^α is the Thom spectrum of a virtual representation α. When G is $Z/2$, Z/p for an odd prime p, or S^1 this becomes

$$t(k)^{Z/2} = \lim_{\leftarrow} \mathbb{R}P^\infty_{-n} \wedge \Sigma k,$$

$$t(k)^{Z/p} = \lim_{\leftarrow} \mathbb{L}^\infty_{-n} \wedge \Sigma k,$$

$$t(k)^{S^1} = \lim_{\leftarrow} \mathbb{C}P^\infty_{-n} \wedge \Sigma^2 k$$

where the extra suspension in the last case comes from $Ad(S^1) = \mathbb{R}$. (Here $\mathbb{R}P^\infty_{-n}$, $\mathbb{L}^\infty_{-n}$ and $\mathbb{C}P^\infty_{-n}$ are the usual stunted real projective space, lens space and complex projective space, respectively.) Thus, Tate theories provide a vast generalization of Mahowald's remarkable root invariant inverse system and spectral sequence, which led to Lin's theorem and the proof of the Segal conjecture. We shall exploit this isomorphism for $G = Q_8$. Let QP^∞_{-n} be the Thom space of the virtual representation $-n\mathbb{H}$, where $\mathbb{H}$ is the quaternions with the usual Q_8 action. Then

$$t(k)^{Q_8} = \lim_{\leftarrow} QP^\infty_{-n} \wedge \Sigma k.$$

James periodicity applied to the finite stunted projective spaces QP^n_m will allow us to determine $t(k)^{Q_8}_*$ immediately from the calculations already done.

Perhaps the most remarkable property of Greenlees and May's construction of Tate spectra is the fact that $t(k)$ is a ring spectrum when k is, and that the natural map $k \to t(k)$ is a ring map. In particular, this shows that the root invariant spectral sequence, properly graded, is multiplicative, a fact which is not apparent from ad hoc constructions of Tate spectra.

The properties which one would expect of a Tate theory are, first, that it vanishes on free G-spaces (or spectra), second, that it represents classical Swan-Tate homology if k is ordinary homology and G is finite, and, third, that there is a long exact *norm sequence*

$$\cdots \to t(k)^G_{n+1} \to k_n BG_+ \to k^{-n} BG_+ \to t(k)^G_n \to \cdots$$

in which the middle map is a generalized norm map which is simply multiplication by the Euler characteristic of G on the components coming from the disjoint basepoint of BG_+.

Roughly speaking, for K-theory, the norm sequence exhibits k-cohomology as a lattice in the Tate theory $t(k)$, with quotient the suspension of the k-homology. For periodic K-theory, and 2-groups, this is a sum of copies of the sequence $\widehat{\mathbb{Z}}_2 \to \mathbb{Q}_2 \to \mathbb{Q}_2/\widehat{\mathbb{Z}}_2$, in which the connective K-theory is imbedded as a sum of copies of $\widehat{\mathbb{Z}}_2 \to \widehat{\mathbb{Z}}_2 \to \mathbb{Z}/2^n$ for various n. More precisely, Greenlees and May have shown that $t(K)$ and $t(KO)$ are rational [9]. (In general, Greenlees and Sadofsky have shown that if k is a v_n-spectrum, the $t(k)^G$ is a v_{n-1}-spectrum [8].) The norm sequence for periodic K-theory then takes the form shown in the following table, in which the maps of the norm sequence are the obvious natural maps. Let RO,

RU, and RSp be the real and complex representation rings and the quaternionic representation module. Let JO and JSp be the kernels of the augmentation, and let $\widehat{(\ )}$ denote JO-adic completion.

n mod 8	$KO^{-n}BG_+$	$\to$	$t(KO)^G_n$	$\to$	$KO_{n-1}BG_+$
0	$\widehat{RO}$		$\widehat{JO} \otimes \mathbb{Q}$		$\widehat{JO} \otimes \mathbb{Q}/\mathbb{Z}$
1	$\widehat{RO/RU}$		0		$\mathbb{Z}$
2	$\widehat{RU/RSp}$		$\widehat{RU/RSp} \otimes \mathbb{Q}$		$(\widehat{RU/RSp} \otimes \mathbb{Q}/\mathbb{Z}) \oplus \widehat{RO/RU}$
3	0		0		$\mathrm{Torsion}(\widehat{RU/RSp})$
4	$\widehat{RSp}$		$\widehat{JSp} \otimes \mathbb{Q}$		$\widehat{JSp} \otimes \mathbb{Q}/\mathbb{Z}$
5	$\widehat{RSp/RU}$		0		$\mathbb{Z}$
6	$\widehat{RU/RO}$		$\widehat{RU/RO} \otimes \mathbb{Q}$		$(\widehat{RU/RO} \otimes \mathbb{Q}/\mathbb{Z}) \oplus \widehat{RSp/RU}$
7	0		0		$\mathrm{Torsion}(\widehat{RU/RO})$

Since rational spectra split into a wedge of Eilenberg-Mac Lane spectra, it follows that $t(KO)^G$ is the wedge of rational Eilenberg-Mac Lane spectra determined by the right hand column above. Greenlees and May conjectured that the following analog of this fact holds for $t(ko)^G$.

Conjecture 8 ([9], Conj. 13.4)**.** *With the wedge taken over all integers i,*

$$\begin{aligned} t(ko) \quad \simeq \quad & \bigvee K(\widehat{JO}, 8i) \vee K((\widehat{RU/RSp})/torsion, 8i+2), \\ & \vee K(\widehat{JSp}, 8i+4) \vee K((\widehat{RU/RO})/torsion, 8i+6). \end{aligned}$$

The conjecture holds for G of prime order by results of Davis and Mahowald [4]. However, the conjecture fails for $G = Q_8$ or $Sl_2(3)$ for the following reason. It would imply that $t(ko)^G_* \longrightarrow t(KO)^G_*$ is a monomorphism. However, the Bott element acts invertibly on $t(KO)^G_*$, but annihilates some elements of $t(ko)^G_*$, so this is impossible. Also, if $t(ko)^G$ is to split into Eilenberg-Mac Lane spectra, then η will act trivially on $t(ko)^G_*$. Our calculation shows that there are nontrivial η-multiples in $t(ko)^G_*$. In fact, these are the elements which are annihilated by the Bott element. Thus, eliminating the elements annihilated by the Bott element leads to the following minimal modification to the conjecture consistent with our calculations here.

Conjecture 9. *There exist a spectrum $t(ko)/(\ker\beta)$ and maps*

$$t(ko) \longrightarrow t(ko)/(\ker\beta) \longrightarrow t(KO)$$

factoring the natural map. Further, $t(ko)/(\ker\beta)$ splits into Eilenberg-Mac Lane spectra as in Conjecture 8.

Our calculation which disproves Conjecture 8 and suggests our conjecture is as follows.

Theorem 10. *If $G = Q_8$ or $Sl_2(3)$ then, 2-locally, the coefficients of the Tate G-spectrum of ko are*

$$t(ko)^{Q_8}_* = t(ko)^{Sl_2(3)}_* \oplus \bigoplus_{n=-\infty}^{\infty} \Sigma^{4n}(\hat{Z}_2 \oplus \hat{Z}_2)$$

and

$$t(ko)_*^{Sl_2(3)} = \bigoplus_{n=-\infty}^{\infty} \Sigma^{4n} A$$

where A is the graded module $(\hat{Z}_2 \oplus \hat{Z}_2, Z/2, Z/2, 0)$, with generators x, y, ηx, and $\eta^2 x$.

Proof. We may assemble the Adams spectral sequences converging to

$$\begin{aligned} ko^{-*}\, BG_+ &= \pi_*\, F(BG_+, ko), \text{ and} \\ ko_*\, \Sigma BG_+ &= \pi_*\, ko \wedge \Sigma BG_+ \end{aligned}$$

in standard fashion to get a spectral sequence converging to $\pi_* t(ko)^G$ with differentials derived from the differentials in the two halves and from the boundary map

$$ko \wedge BG_+ \xrightarrow{\tau} F(BG_+, ko).$$

However, the differentials implied by τ are difficult to determine in general. When $G = Q_8$, periodicity implies them as follows. We use the description of the Tate spectrum as

$$t(ko)^{Q_8} = QP_{-\infty}^{\infty} \wedge \Sigma ko = \lim_{\leftarrow} QP_{-n}^{\infty} \wedge \Sigma ko$$

as in [9, Thm 16.1], where QP_{-n}^{∞} is the Thom complex of $-n$ times the quaternionic line bundle over BQ_8. The differentials in the resulting Adams-Atiyah-Hirzebruch spectral sequence are then periodic by James periodicity applied to the finite stunted quaternionic projective spaces. As a consequence, the differentials which hold in the Adams spectral sequence for $ko^{-n}BG$, $G = Q_8$ or $BSl_2(3)$, for $n < 0$, hold for all n in the spectral sequence for $t(ko)_*^G$. A similar periodicity applies to the extension questions. The resulting E_∞-terms for $t(ko)_*^{BSl_2(3)}$ and each of the two other identical pieces of $t(ko)_*^{Q_8}$ are displayed in Figures 7 and 8. QED

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