Research Notes

Volume X

As usual these notes are not assumed correct, complete or chronological!

Contents

On simple modules for $SL(2,q)$

(This is joint work with D. Mason)

We are going to generalize the results of Mason from the case $p=2$ to arbitrary primes. Let us fix some notation. We set $G = SL(2,q)$, where $q = p^n$, and let F be a splitting field of characteristic p for G. Our main result is the following: We now assume $q > p$.

Theorem 1 If V is an FG-module then the following statements are equivalent:

1) V is simple of dimension a power of p;

2) There is an elementary abelian p-subgroup of G such that the restriction of V to this subgroup affords the regular representation.

We shall first prove that 1) implies 2) and then establish the reverse conclusion. Now G acts on the algebra of polynomials in two variables, X and Y, in the usual way: the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

maps X to $aX + cY$ and Y to $bX + dY$. The space M of homogeneous polynomials of degree $p-1$ is a simple FG-module of dimension p (the so-called basic Steinberg representation of G). Let $\sigma_1, \dots, \sigma_r$ be distinct automorphisms of the field of q elements so the tensor product

$$\sigma_1(M) \otimes \dots \otimes \sigma_r(M)$$

of the twists of M is a simple FG-module of dimension p^r; each simple module of dimension a power of p is of this form so it suffices to produce an elementary abelian subgroup of G of order p^r such that the restriction of this tensor product to this subgroup is a projective module. Let U be the subgroup of G consisting of the lower unitriangular matrices so U is a Sylow p-subgroup of G and is elementary abelian of order p^n. We shall prove a preliminary result

<u>Lemma 2</u>. If $g_1, \dots, g_r$ are elements of U then the tensor product is regular on $\langle g_1, \dots, g_r \rangle$ if, and only if,

$$\det(\sigma_i(\lambda_j)) \neq 0.$$

where $g_j = \begin{pmatrix} 1 & 0 \\ \lambda_j & 1 \end{pmatrix}$, $1 \le j \le r$.

Before proving this result, let us see that it provides all we require in order to establish that 1) implies 2) in the theorem. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the distinct automorphisms of the field of q elements and let $X_1, \dots, X_n$ be generators of U so that the n lower left entries of the X_i are a basis of the field of q elements. The linear independence of automorphisms implies that

$$\det(\alpha_i(X_j)) \neq 0.$$

Hence, by the Laplace expansion of the determinant (using $r \times r$ minors instead of the usual expansion by row elements) for the r rows given by $\sigma_1, \dots, \sigma_r$ there are r of the X_i such that the minor for these and $\sigma_1, \dots, \sigma_r$ is not zero. Hence, the lemma gives the desired projectivity. (This can also be done by complexity theory.)

We turn to the lemma. Let D be the linear transformation of M given by $D = Y\,\partial/\partial X$ so the kernel of D is one-dimensional and is spanned by Y^{p-1}. Moreover, D is an endomorphism of M

as a FU-module. Indeed, if

$$g = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

is in U then

$$gD(X^iY^j) = g(iX^{i-1}Y^{j+1}) = i(X+\alpha Y)^{i-1}Y^{j+1},$$
$$Dg(X^iY^j) = D((X+\alpha Y)^iY^j) = i(X+\alpha Y)^{i-1}Y^{j+1}.$$

Similarly, D defines an endomorphism of the FU-module $\sigma(M)$ for an automorphism σ of the field of q elements: this is the same vector space but g acts by sending X to $\sigma(\alpha)X+Y$ and Y to Y.
Finally, let D_i be the endomorphism of

$$\sigma_1(M) \otimes \cdots \otimes \sigma_r(M),$$

as FU-module where

$$D_i(m_1 \otimes \cdots \otimes m_r) = m_1 \otimes \cdots \otimes Dm_i \otimes \cdots \otimes m_r,$$

with the obvious notation.

In the tensor product

$$\sigma_1(M) \otimes \cdots \otimes \sigma_r(M)$$

set

$$w_1 = XY^{p-2} \otimes Y^{p-1} \otimes \cdots \otimes Y^{p-1}$$
$$w_2 = Y^{p-1} \otimes XY^{p-2} \otimes Y^{p-1} \otimes \cdots$$

and so on for $w_3, \ldots, w_r$. Also let

$$w_0 = Y^{p-1} \otimes \cdots \otimes Y^{p-1}.$$

Thus, if

$$g = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in U$$

we have

$$g w_i = w_i + \sigma_i(\lambda) w_0, \qquad 1 \le i \le r$$
$$g w_0 = w_0$$

and $W = \langle w_0, w_1, \ldots, w_r \rangle$ is an $r+1$-dimensional FU-submodule of the tensor product.

Now suppose that $\det(\sigma_i(\lambda_j)) = 0$. Hence, there exist $\gamma_1, \dots, \gamma_r$ in F, not all zero, such that

$$\sum_i \gamma_i \sigma_i(\lambda_j) = 0$$

for $j = 1, \dots, r$. Then

$$\begin{aligned} g_j\left(\sum_{i=1}^{r} \gamma_i w_i\right) &= \sum \gamma_i \left(w_i + \sigma_i(\lambda_j) w_0\right) \\ &= \sum \gamma_i w_i + \left(\sum \gamma_i \sigma_i(\lambda_j)\right) w_0 \\ &= \sum \gamma_i w_i \end{aligned}$$

for each j. Thus, the tensor product has at least two dimensions of fixed points for $\langle g_1, \dots, g_r \rangle$ as w_0 is also left fixed so the tensor product, being of dimension p^r, cannot afford the regular representation.

Next, suppose that the tensor product does not afford the regular representation. Hence, there are at least two dimensions of fixed points for $\langle g_1, \dots, g_r \rangle$ and so there is a fixed point which is not a multiple of w_0. If we can produce such a fixed point that also has the property of being in W then the above calculation shows that the determinant is indeed zero.

To do this we set $\mathcal{D}$ to be the algebra of linear transformations of the tensor product generated by the identity I together with $D_1, \dots, D_r$.

<u>Lemma 3</u>. The socle of the tensor product, as a $\mathcal{D}$-module, is Fw_0 and the second socle is W.

Let's complete our proof before establishing the lemma. Let f be a fixed point, $f \notin W$. Suppose that f is not in the k-th socle but is in the $(k+1)$-st. Hence, there is an element of the radical of D whose product with f is in the h-th socle but not the $(h-1)$st and this product will again be a fixed point.

The lemma can be proved by an inspection: since D is nilpotent on M with one dimensional kernel there is a basis of M of the form $m, Dm, \ldots, D^{p-1}m$ and we can use this to construct a basis of the tensor product which is easy to deal with. However, there is a short-cut. Let $E = \langle e_1, \ldots, e_r \rangle$ be an elementary abelian p-group of order p^r. Since $D_i^p = 0$, $D_i D_j = D_j D_i$, there is a homomorphism of FE onto $\mathcal{D}$ sending each e_i to $I + D_i$. Now, regarding the tensor product as an FE-module, there is only one dimension of fixed points, namely Fw_0, as this space is easily seen to be the common annihilator of $D_1, \ldots, D_r$. Hence, the tensor product is a free module of rank one for FE so the map of FE to $\mathcal{D}$ is an isomorphism. Hence, the second socle is of dimension $r+1$; but W is included in this so W/Fw_0 is annihilated by all the D_j so we are done.

We turn to the other half of our theorem. Let V be an FG-module which affords the regular representation of the subgroup U_0 of U. Let T be the natural two-dimensional module for G restricted to U. Now if $U_0 = U$ then V is projective of dimension p^n so V is simple as $q > p$ by known results (e.g. see Andersen et al). Hence, let U_1 be a subgroup of U containing U_0 as a subgroup of index p.

The key step is the next result.

Proposition 4. If the fixed-point space of U_0 on $M \otimes V$ is not free as a module for U_1/U_0 then the restriction of V to U_1 has a subquotient isomorphic with the restriction of T to U_1.

Let us now prove the rest of the theorem assuming this proposition. We proceed by induction on $|U : U_0|$; we have already dealt with the case that this is 1. If the first possibility in the proposition holds then $M \otimes V$ affords the regular representation of U_1 so $M \otimes V$ is simple, by induction, so certainly V is as well. Thus, we may assume that T is a subquotient as stated. Furthermore, we can apply the same argument to each Galois conjugate of M and so we may assume, as well, that each Galois conjugate of T appears as a subquotient of the restriction of V to U_1. We proceed to argue assuming all of this holds.

We first establish a preliminary result.

Lemma 5 The common annihilator in FU of T and all its Galois conjugates is $\mathrm{rad}^2(FU)$.

Indeed, suppose $U = \langle x_1, \ldots, x_n \rangle$ and set $X_i = x_i - 1 \in FU$ so $X_1, \ldots, X_n$ give a basis of $\mathrm{rad}(FU)/\mathrm{rad}^2(FU)$. Since T is two-dimensional and so $\mathrm{rad}^2(FU)$ annihilates it, we need only prove that if $\alpha_1, \ldots, \alpha_n$ are in the field of q^n elements then

$$\alpha_1 X_1 + \cdots + \alpha_n X_n$$

does not annihilate T and all its conjugates unless all the α_i are zero. However, suppose

$$x_i = \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix}$$

so $\alpha_1 X_1 + \dots + \alpha_n X_n$ is represented on T by the matrix

$$\begin{pmatrix} 0 & 0 \\ \alpha_1\lambda_1 + \dots + \alpha_n\lambda_n & 0 \end{pmatrix}$$

and on the Frobenius of T by

$$\begin{pmatrix} 0 & 0 \\ \alpha_1\lambda_1^p + \dots + \alpha_n\lambda_n^p & 0 \end{pmatrix}$$

and so on. Hence, we must see that the equations

$$\begin{gathered} \alpha_1\lambda_1 + \dots + \alpha_n\lambda_n = 0 \\ \alpha_1\lambda_1^p + \dots + \alpha_n\lambda_n^p = 0 \\ \vdots \end{gathered}$$

are inconsistent. It suffices to prove that the determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^p & \lambda_2^p & \cdots & \lambda_n^p \\ & \vdots & & \end{vmatrix}$$

is non-zero. But it is easily seen to be the product of all factors of the form

$$\gamma_1\lambda_1 + \dots + \gamma_n\lambda_n$$

where the γ_i's run over all n-tuples from $GF(p)$ (except the n-tuple of all zeroes). Since $\lambda_1, \dots, \lambda_n$ are linearly independent the lemma is proved.

Now let us continue the argument. We now have that the annihilator in FU_1 of V is contained in $\text{rad}^2(FU)$. This implies that the annihilator in FU_1 of V is contained in $\text{rad}^2(FU_1)$. Indeed, $\text{rad}^2(FU) \cap (FU_1) \subseteq \text{rad}^2(FU_1)$. For, if $U_1 = \langle x_1, \dots x_t \rangle$ with the above notation, then

$X_1, \dots, X_t$ are linearly independent modulo $\mathrm{rad}^2(FU)$ and give a basis of $\mathrm{rad}(FU_1)/\mathrm{rad}^2(FU_1)$. But now, V is a free module of rank one for FU_0. Any element of FU_1 induces a linear transformation of V commuting with FU_0, as U is abelian, so agrees with an endomorphism of V as FU_0-module. But again, FU_0 is commutative so each element of FU_1 agrees with an element of FU_0 on V, that is, FU_1 is the sum of FU_0 and the annihilator in FU_1 of V. In particular,

$$FU_1 = FU_0 + \mathrm{rad}^2(FU_1).$$

Hence, as $\mathrm{rad}^2(FU_1) \cap FU_0 \leq \mathrm{rad}^2(FU_0)$, we have

$$\dim_F (FU_1/\mathrm{rad}^2 FU_1) \leq \dim (FU_0/\mathrm{rad}^2 FU_0),$$

a contradiction to $U_0 < U_1$.

We are now left with the proof of Proposition 4. Again we require some preliminary results.

<u>Lemma 6.</u> As a module for U_1, M is uniserial and the top two-dimensional quotient of M^* is isomorphic to the restriction of T to U_1.

<u>Lemma 7.</u> There is an isomorphism of $\mathrm{Hom}_{FU_1}(W, M \otimes V)$ onto $\mathrm{Hom}_{FU_1}(W \otimes M^*, V)$, for any FU_1-module W, which takes φ to Φ where

$$\varphi(w) = \sum m_i(w) \otimes v_i,$$

for a basis $v_1, \dots, v_d$ of V and $m_i(w) \in M$, and

$$\Phi(w \otimes \mu) = \sum \mu(m_i(w))\, v_i$$

for each $\mu \in M^*$, $w \in W$.

The fact that these vector spaces are isomorphic is standard; we require the details of the map.

Let's prove the proposition using the lemmas. We take W to be the fixed-points of U_0 on $M \otimes V$. Let φ be the inclusion of W into $M \otimes V$. Since W is not free as a module for U_1/U_0 its socle, as U_1-module, is of dimension at least two. Now $\mathrm{soc}(M) \otimes V \cong V$ as FU_0-module so its socle is one-dimensional. Hence, there exists $w \in \mathrm{soc}(W)$, $w \notin \mathrm{soc}(M) \otimes V$. Express, as in Lemma 7,

$$w = \sum m_i(w) \otimes v_i$$

so there is $m_j(w) \notin \mathrm{soc}(M)$. Hence, there is $\mu \in \mathrm{rad}(M^*)$ with $\mu(m_j(w)) \neq 0$. Thus, again with the notation of Lemma 7,

$$\Phi(w \otimes \mu) = \sum \mu(m_i(w)) \otimes v_i \neq 0$$

as $\mu(m_i(w)) \neq 0$ and the v_i are linearly independent. In particular,

$$\Phi(\mathrm{soc}(W) \otimes \mathrm{rad}(M^*)) \neq 0.$$

Now $\mathrm{soc}(W) \otimes M^*$, as an FU_1-module, is isomorphic with a direct sum of copies of M^*. Hence, there is a summand of $\mathrm{soc}(W) \otimes M^*$ which is isomorphic with M^* such that Φ does not annihilate its radical; Lemma 6 concludes the proof.

We now prove Lemma 6. Since the standard two-dimensional module for G is self-dual we need only prove the assertion for M. (The matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ conjugates each element of $SL(2,q)$ to the transpose of its inverse.) If $1 \neq g = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \in U$ then the matrix of g with respect to the basis $X^{p-1}, X^{p-2}Y, \dots, Y^{p-1}$ of M is

$$\begin{pmatrix} 1 & & & & \\ (p-1)\lambda & 1 & & & \\ & (p-2)\lambda & 1 & & \\ & & \ddots & & \\ & & & \lambda & 1 \end{pmatrix}$$

that is, lower unitriangular with subdiagonal entries $(p-1)\lambda, \ldots, \lambda$. Since none of these is zero the matrix is similar to a Jordan block so the vector space M is uniserial with respect to this single element of U. To conclude the proof we need only see that the dual of the two-dimensional submodule of M is isomorphic to the restriction of T to U. But this two-dimensional submodule is clearly isomorphic with T which is self-dual, as the two-dimensional standard module for G is self-dual.

Last we prove Lemma 7. We shall show that there are two isomorphisms of FU_1-modules

$$\mathrm{Hom}_F(W, M) \otimes V \simeq \mathrm{Hom}_F(W, M \otimes V)$$
$$\mathrm{Hom}_F(W, M) \otimes V \simeq \mathrm{Hom}_F(W \otimes M^*, V)$$

which send, respectively, $\Psi \otimes v$, for $\Psi \in \mathrm{Hom}_F(W, M)$, $v \in V$, to the maps which sends each $w \in W$ to $\Psi(w) \otimes v$ and to the maps which sends $w \otimes \mu$, for $\mu \in M^*$, to $\mu(\Psi(w)v)$. It is easy to see, by choosing bases for example, that these are such maps which are F-isomorphisms. It remains to see these maps commute with the action of any $g \in U_1$.

Now the first map sends $\Psi \otimes v$ to $\Psi(\ ) \otimes v$. Application of g to $\Psi \otimes v$ yields $g(\Psi(g^{-1}(\ ))) \otimes gv$ which goes to $g(\Psi(g^{-1}(\ ))) \otimes gv$ as desired.

For the second, we send $\Psi \otimes v$ to the map $w \otimes \mu \to \mu(\Psi(w))v$
Hence, $g \cdot \Psi \otimes v = g\Psi g^{-1} \otimes gv$ goes to the map

$$w \otimes \mu \to \mu(g(\Psi(g^{-1}(w))))gv$$

while, on the other hand, the map $w \otimes \mu \to \mu(\Psi(w))v$ goes, under application of g, to, in three stages

$$w \otimes \mu \to g^{-1}w \otimes g^{-1}\mu \to (g^{-1}\mu)(\Psi(g^{-1}w))v \to g\left[(g^{-1}\mu)(\Psi(g^{-1}w))v\right]$$

which is

$$\mu(g(\Psi(g^{-1}(w))gv$$

as desired.

Now let's go on to make some further observations. First, let's point out that we have proven a statement stronger than we have made explicit. We introduce some notation. If V and V' are simple FG-modules then we write V/V' if V' is the tensor product of V and another simple module in the usual way. We have already established

Proposition 8. If $U_0 \leq U$ is regular on the simple FG-module V and $U_0 \leq U_1$ another subgroup of U then there is V' with V/V', such that U_1 is regular on V'.

Indeed, we have proven this for the case that the order of U_1 is p times the order of U_0: to complete the proof we need only apply the obvious induction argument. This leaves unanswered the first conjecture of Scott's letter: if $U_1 \leq U_2$, V_0/V_2 where U_2 is regular on V_2 find V' with V'/V_2 as well.

Now let's state the new result:

Proposition 9. If U_0 is regular on the simple FG-module V and V/V' then there is U_1, $U_0 \le U_1 \le U$ with U_1 regular on V'.

This is, in fact, easier to prove than the previous result. The proof proceeds by using $U_1 \le U$ with U_1 regular on V', U_1 not necessarily containing U_0, and then "changing" U_1. Hence, choose such a subgroup U_1 of U which we have already proved to exist. Let $\mathfrak{A}$ be the annihilator of V' in FU so, as before, we have $FU = FU_1 + \mathfrak{A}$, a vector space direct sum. Next, observe that $FU/\mathfrak{A}$ is isomorphic with V' as FU-modules: since V' is regular for FU_1 it is certainly cyclic so there is an FU-homomorphism of FU onto V' and the kernel is $\mathfrak{A}$. Let π be the projection of FU onto FU_1 "along" $\mathfrak{A}$. Since FU_1 is now projective for $\pi(FU_0)$, we have that if $x_1, \ldots, x_s$ are a basis of U_0 then the elements $\pi(x_i - 1)$, $1 \le i \le s$, are linearly independent modulo $\mathrm{rad}^2(FU_1)$ (by Theorem 6.2, p125, of J. Carlson's paper, The varieties and the cohomology ring of a finite groups, J. Algebra 85 (1983), 104–143). Hence, since using elements $g-1$, g running over a basis of U_1, gives a basis of $\mathrm{rad}(FU_1)/\mathrm{rad}^2(FU_1)$, we can choose $x_{s+1}, \ldots, x_t \in U_1$ so that the elements $\pi(x_j - 1)$, $1 \le j \le t$, are also a basis of this vector space. Again, this means that FU_1 is free (of rank one), for $\langle x_j \mid 1 \le j \le t \rangle$, by Theorem 6.2 again, so the proposition is proved using this group in place of U_1.

Now Geoff has made a second conjecture. Suppose U_0 is regular on V, $U_0 \leq U_2$ with U_2 regular on V'', V/V'' and we have V' with V/V', V'/V'' then we can choose U_1, $U_0 \leq U_1 \leq U_2$ with U_1 regular on V'. Moreover we can do this by obtaining U_1 from U_0 by adding to U_0 some elements of any basis of U_2, fixed in advance, which contains a basis of U_0. We have shown above that this conjecture holds when $U = U_2$ as then V'' is the Steinberg module.

Let us proceed to prove the second conjecture. The set-up:

$$\begin{array}{ll} V'' & U_2 \\ V' & U_1\ ?? \\ V & U_0 \end{array}$$

(and we have a fixed basis of the elementary abelian group U_2 containing a basis of U_0). Choose $U_1 \leq U$ that works for V' so, as usual, $FU = FU_1 + \mathfrak{o}\mathfrak{l}$. We will modify U_1 to get another "correct" U_1. Let $A = FU_1$ and let B be the image of FU_2 in A (using the usual projection). We assert that $B = A$; suppose, on the contrary, $B < A$. Now B is also a local algebra so $\operatorname{rad} B < \operatorname{rad} A$, even the image of $\operatorname{rad} B$ in $\operatorname{rad} A/(\operatorname{rad} A)^2$ is proper, by the generation properties of A. Now $\operatorname{rad} A/(\operatorname{rad} A)^2$ and $\operatorname{soc}^2 A/\operatorname{soc} A$ are paired into $\operatorname{soc} A$ by multiplication so we deduce the existence of an ideal I of A, $\operatorname{soc} A < I \leq \operatorname{soc}^2 A$ with $\operatorname{rad} B . I = 0$. $\therefore \operatorname{rad} B . IV' = 0$, that is, $\operatorname{rad} B$ annihilates a space of at least two dimensions (we are using that V' is a free module for A). Hence, $B = F1 + \operatorname{rad} B$ yields that U_2 fixes at least two dimensions of vectors of V' and so the same holds for V'', as V'/V'', a contradiction.

Now V' is a free module for FU_0 so the image of the part of the fixed basis lying in U_0, in A consists of elements of the form $1+r$, $r \in \mathrm{rad}\, A$, where the elements r are linearly independent modulo $\mathrm{rad}^2 A$. Since $B = A$ we can proceed as before to find a subset of the basis of U_2, containing the basis in U_0, whose images in A, when 1 is subtracted from each, give a basis for $\mathrm{rad}\, A / \mathrm{rad}^2 A$. Hence, again quoting Carlson, we are done.

Next, we turn to the first conjecture. We begin with a generalization of Lemma 5:

<u>Lemma 10</u>: If $U' \leq U$, $V' = \sigma_1(M) \otimes \dots \otimes \sigma_r(M)$ (with the usual notation) and V' affords the regular representation of U' then the annihilator in FU' of

$$\sigma_1(T) \oplus \dots \oplus \sigma_r(T)$$

is $\mathrm{rad}^2(FU')$.

To see this just copy the proof of Lemma 5 using U' in place of U. The critical determinant that arises is then that of Lemma 2 !!

Now let's turn to the first conjecture. The set-up:

$$\begin{array}{lll} V'' & U_2 & \\ ?? \; V' & U_1 & |U_1 : U_0| = p \\ V & U_0 & \end{array}$$

(the restriction $|U_1 : U_0| = p$ is no limitation). We must produce V' of the form $\sigma(M) \otimes \mathbf{V} / V''$.

If there is an automorphism σ such that $\sigma(M) \mid V''$, $\sigma(M) \nmid V$ and $(\sigma(M) \otimes V)^{U_0}$ is free as a module for U_1/U_0 then $\sigma(M) \otimes V = V'$ clearly suffices. Hence, we assume otherwise: for each such σ, the module $(\sigma(M) \otimes V)^{U_0}$ is not free for U_1/U_0 so, by Proposition 4, V_{U_1} has a subquotient isomorphic with $\sigma(T)_{U_1}$.

Now if τ is an automorphism so that $\tau(M) \mid V$ then the tensor product structure of V also gives V_{U_1} with a subquotient isomorphic with $\tau(T)_{U_1}$ (consider the submodule, for U_1 of the factor $\tau(M)$ tensored with the fixed points in the other factors). Hence, the annihilator in FU_1 of V is contained in $\mathrm{rad}^2(FU_2)$, by Lemma 10. Now proceed exactly as before in the argument following the proof of Lemma 5 to reach a contradiction.

Hence, we have established this conjecture too!

However, this is all too heavy-handed: a direct determinantal argument applies! Consider the second conjecture:

$$\begin{array}{cc} V'' & U_2 \\ V' & ?? \\ V & U_0 \end{array}$$

with a fixed basis. Consider the non-zero determinant corresponding to V'' and U_2 so the rows correspond to certain automorphisms and the columns to the basis of U_2. Considering V' means picking a subset of the rows, which will be of course linearly independent. The columns corresponding to U_0, restricted to these rows will also be linearly independent since they are already on the rows corresponding to V, by assumption. Hence, looking at the matrix whose rows correspond to V' (with all columns) and since its row rank equals its column rank we can supplement the columns corresponding to V_0 by other columns to get a square matrix of non-zero determinant and so conjecture 2 is proved! Arguing similarly for columns and rows gives conjecture 1. Note that the rank argument is an alternative to the Laplace expansion.

Virtual permutations

The study of the symmetric groups Σ_X on an infinite set X is quite old (R. Baer, Studia Math 5 (1934), 15-17; J. Schreier and S. Ulam, Studia Math 4 (1934), 131-41; Karrass and Solitar, Math Z. 66 (1956), 64-9). For example, there is no alternating group but the group of finitary permutations (almost everywhere the identity) is a normal subgroup; let $\bar{\Sigma}_X$ be the corresponding quotient. We need to observe the fact that $\bar{\Sigma}_X$ has outer automorphisms (this is quite easy) and to interpret this event in a natural way.

Definition 1. A function α with domain Y cofinite in X and range Z cofinite in X which is one-to-one and onto from Y to Z is called an almost permutation of X.

If α and α' are almost permutations of X we say they are equivalent if they agree on a cofinite subset of X (that is, on a cofinite subset of the intersection of their domains of definition. This is easily seen to be an equivalence relation.

Definition 2 An equivalence class of almost permutations of X is called a virtual permutation of X.

The virtual permutations form a group under multiplication. The product is well-defined, the inverse of a new permutation containing the almost permutation α is the class of the inverse of α, the function whose domain is the range of α, whose range is the domain of α and which is inverse to α on these sets. Call this group N_X. We also have a natural homomorphism of Σ_X to N_X and the kernel is the finitary permutations so we have a natural embedding of $\bar{\Sigma}_X$ in N_X. Now let σ be a one-to-one map of X to X with equal to X less exactly one element so σ is an almost permutation. Let σ also denote the corresponding element of N_X.

<u>Theorem</u> The group N_X is the semi-direct product of $\bar{\Sigma}_X$ by the infinite cyclic subgroup generated by σ.

After we establish this result, a little more work will show that σ induces an automorphism of $\bar{\Sigma}_X$ which is <u>not</u> inner.

<u>Lemma 1</u>. If f is an almost of X then there exist permutations π_1, π_2, π_3 of X and non-negative integers m, n with

$$f = \pi_3 \circ \sigma^n \circ \pi_2 \circ \bar{\sigma}^m$$

Here $\bar{\sigma}$ is the inverse of σ so is defined on $\sigma(X)$ and has range X.

<u>Proof</u>. Let $f = f_2 \circ f_1$ where f_1 has domain equal to that of f but range X while f_2 has domain X with range equal that of f. Suppose that $|X - f_2(X)| = n$. Hence, there is a permutation α so that $\alpha \circ f_2(X) = \sigma^n(X)$ so there is another permutation β with $\alpha \circ f_2 \circ \beta = \sigma^n$. Thus, $f_2 = \alpha^{-1} \circ \sigma^m \circ \beta^{-1}$. Similarly, suppose that the domain of f_1 has cocardinality m so there is a permutation γ so that

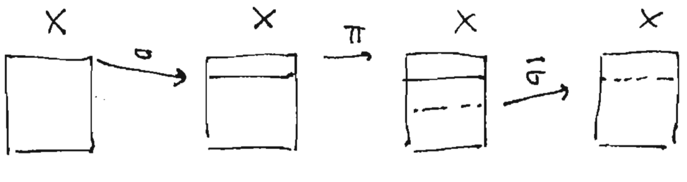
$f_1 \circ \gamma$ and $\bar{\sigma}^m$ have the same domain. Hence, there is another permutation δ with $\delta \circ f_1 \circ \gamma = \bar{\sigma}^m$ so $f_1 = \delta^{-1} \circ \bar{\sigma}^m \circ \gamma^{-1}$ and the lemma is proved.

<u>Lemma 2</u> If π is a permutation of X then there is a permutation ρ of X such that $\sigma \circ \pi \circ \bar{\sigma}$ defines the same virtual permutation as does ρ.

<u>Proof</u>. $\sigma \circ \pi \circ \bar{\sigma}$ has domain and range of cocardinality one and it is one-to-one and onto from its domain to range.

<u>Lemma 3</u>. If π is a permutation of X then there is a permutation ρ of X such that $\bar{\sigma} \circ \pi \circ \sigma$ and ρ define the same virtual permutation.

<u>Proof</u>. If π preserves $\sigma(X)$ then $\bar{\sigma} \circ \pi \circ \sigma$ is a permutation. If π does not preserve $\sigma(X)$ then we have the following picture:



We have that $\bar{\sigma} \circ \pi$ is defined on a set of cocardinality two so the range of $\bar{\sigma} \circ \pi$, which is the range of $\bar{\sigma} \circ \pi \circ \sigma$, is of cocardinality one, that is, the cocardinality of the domain of $\bar{\sigma} \circ \pi \circ \sigma$.

<u>Proof of theorem</u>. With the notation of the first lemma we have that f and $\sigma^{n-m} f$ represent the same element for a

suitable permutation π. The above two lemmas give the desired normality. Finally, different powers of σ cannot "differ" by a permutation. Indeed, σ^n and a permutation, for $n>0$ cannot represent the same class as they cannot agree on a cofinite set as the permutation does not map a cofinite set to a set of larger cocardinality.

<u>Remarks</u>.

1. To finish the last argument introduce the following concept earlier. If α is an almost permutation define

$$\Delta(\alpha) = |X - \text{range}\,\alpha| - |X - \text{domain}\,\alpha|.$$

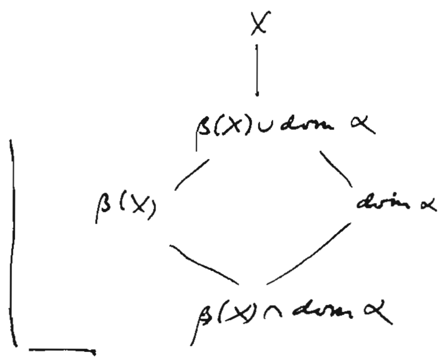
Then $\Delta(\alpha)$ depends only on its class as is easy to see. $\Delta(\sigma) = +1$, $\Delta(\pi) = 0$ if π is a permutation and $\Delta(\pi\sigma^i) = i$, so in view of our results, π is a homomorphism to $\mathbb{Z}^+$.

2. Seems easy to just prove directly that Δ is a homomorphism and so deduce our main result. Need only calculate $\Delta(\alpha\circ\beta)$ for almost permutations α, β where each has domain or range all of X. If each has range (or each has domain) all of X this is easy. Say β has domain X and α has range X. Picture:

$$\begin{tikzcd}
X \arrow[rd, "\beta"] & & X & & X \\
\text{dom}(\alpha\circ\beta) \arrow[rd] & \beta(X) \arrow[rrru] & & & \text{range}(\alpha\circ\beta) \\
& \beta(X)\cap\text{dom}\,\alpha \arrow[rrru] & & &
\end{tikzcd}$$

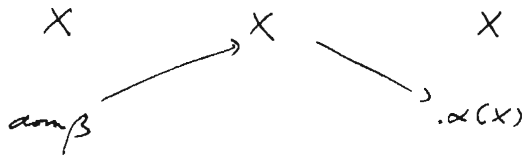
Hence, we have

$$
\begin{aligned}
\Delta(\alpha\circ\beta) &= -\,|X - \mathrm{dom}(\alpha\circ\beta)| + |X - \mathrm{range}(\alpha\circ\beta)| \\
&= -\,|\beta(X) - \beta(X)\cap\mathrm{dom}\,\alpha| + |\mathrm{dom}\,\alpha - (\beta(X)\cap\mathrm{dom}\,\alpha)| \\
&= -\,|(\beta(X)\cup\mathrm{dom}\,\alpha) - \mathrm{dom}\,\alpha| + |(\beta(X)\cup\mathrm{dom}\,\alpha) - \beta(X)| \\
&= -\,\;|(\beta(X)\cup\mathrm{dom}\,\alpha) - \mathrm{dom}\,\alpha| \quad |X - (\beta(X)\cup\mathrm{dom}\,\alpha)| \\
&\qquad + |(\beta(X)\cup\mathrm{dom}\,\alpha) - \beta(X)| + |X - (\beta(X)\cup\mathrm{dom}\,\alpha)| \\
&= -\,|X - \mathrm{dom}\,\alpha| + |X - \beta(X)|
\end{aligned}
$$



as required.

Similarly, suppose that β has range X and that α has domain X. Picture:



This is also easy.

Let's now show the automorphism given by σ is outer. We can replace σ by any other element in its coset (i.e. any element with $\Delta(\ ) = 1$). Assume $X = \{1, 2, \ldots\} \cup F$ and let $\sigma : n \to n+1$, $f \to f$, all $f \in F$, all $n \in \mathbb{N}$. It suffices to show there is no virtual permutation represented by a permutation π which is fixed by conjugation by σ, other than the identity. Suppose otherwise so $\bar{\sigma}\circ\pi\circ\sigma$ and π agree on a cofinite set. But

$$\bar{\sigma}\circ\pi\circ\sigma\,(n) = \pi(n+1) - 1$$

so, for all large n, $\pi(n+1) = \pi(n) + 1$. Also, if $f \in F$ then $\pi\circ\sigma(f) = \sigma\circ\pi(f)$, almost always, that is, $\pi(f) = \sigma(\pi(f))$ so $\pi(f) \in F$. That is, the set F_0 of all $f \in F$ such that $\pi(f) \in F$ is cofinite in F. We can now get that $\Delta(\pi) \neq 0$

The module Π affords the regular representation of the group
$$\langle \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix} ; \ 1 \le i \le r \rangle$$
of order p^r containing K. Hence, Π is certainly free as a module for FK.

Next, assume conversely that Π is a free FK-module. It suffices to show that there is a subgroup L of U containing K such that L has order p^r with Π affording the regular representation of L. For then the r by s matrix under consideration will be an r by s submatrix of a non-singular r by r matrix, again by our previous theorem.

However, there is a subgroup H of U, H of order p^r such that Π affords the regular representation of H: the n by n matrix $(\sigma_i(\lambda_j))$ is non-singular so the r by n matrix corresponding to $\sigma_1, \ldots, \sigma_r$ has rank r so it has r linearly independent columns. Let $\mathfrak{a}$ be the annihilator of Π in U, an ideal, so $FU = FH + \mathfrak{a}$, a vector space direct sum, since FU induces endomorphisms of the regular FH-module Π which are all induced by FH, as before in our results. Since Π is cyclic we now have that $FU/\mathfrak{a} \cong \Pi$ as FU-modules, so we can work with $FU/\mathfrak{a}$ instead of Π. We need only show that there is a subgroup L of U containing K such that $FU/\mathfrak{a}$ is the regular FL-module.

Of course, $FU/\mathfrak{a} \cong FH$ as FH-modules, so we can use the theory of the group algebra of an elementary abelian p-group. Indeed, if $H = \langle h_1, h_2, \ldots, h_r \rangle$ then $FU/\mathfrak{a}$ is a group algebra of the group with basis $h_i + \mathfrak{a}$, $1 \le i \le r$. Since $FK \cap \mathfrak{a} = 0$ (as $FU/\mathfrak{a}$ is free as a FK-module) we have a subgroup of the group of units of $FU/\mathfrak{a}$ generated by the elements $g_1 + \mathfrak{a}, \ldots, g_s + \mathfrak{a}$, this is group is elementary abelian of order p^s and (again) $FH/\mathfrak{r}$ is a free module. Hence, we can apply a result we have mentioned before: Theorem 6.2, p125, of the paper by J. Carlson, entitled "The varieties and the cohomology rings of a finite groups," J. Algebra 85 (1983), 104-145. We deduce that the elements $g_i - 1 + \mathfrak{r}$ of $FU/\mathfrak{r}$ are in $\mathrm{rad}(FU/\mathfrak{r})$ and linearly independent modulo $\mathrm{rad}^2(FU/\mathfrak{r})$. Hence, after suitable renumbering, we have that the elements $g_i - 1 + \mathfrak{r}$ and $h_j - 1 + \mathfrak{r}$, $1 \le j \le s - r$ give a basis of $\mathrm{rad}(FU/\mathfrak{r}) / \mathrm{rad}^2(FU/\mathfrak{r})$. Since these elements commute, their p-th powers are zero, they are a basis of the radical quotient given while $FU/\mathfrak{r}$ has dimension p^r it follows that $FU/\mathfrak{r}$ is a group algebra on the subgroup $\langle g_1, \ldots, g_s, h_1, \ldots, h_{s-r} \rangle = L$. This is clearly the desired subgroup.

We now get a number of consequences just as in Geoff's letter!

Now let's turn to the question on $SL_2(q)$ of Geoff's about freeness on other simple modules. Let $V = T \otimes S$ be a simple with S the "Steinberg factor" (usual tensor product of Steinberg's theorem, in this case Brauer-Nesbitt). Want freeness on V to imply freeness on S. Need only show that there is an $SL_2(q)$ module R with $k \mid R \otimes T$. By algebraic conjugacy only need to do this for T one of the basic simple modules of dimensions $2, 3, \ldots, p-1$. Hence, only need splitting of Clebsch-Gordan sequence (where V_i is of dimension i, the representation on homogeneous polynomials of degree $i-1$)

$$0 \to V_{i-1} \to V_2 \otimes V_i \to V_{i+1} \to 0$$

But the module $V_2 \otimes V_i$ is self-dual so we're O.K. (actually get V_2 self-dual then splitting of $V_2 \otimes V_2$ gives V_3 self-dual, etc.) For if non-split then $\mathrm{soc}(V_2 \otimes V_i) = V_{i-1}$, $\mathrm{soc}((V_2 \otimes V_i)^*) = V_{i+1}$.

Another approach is to get a backwards map

$$V_{i+1} \to V_2 \otimes V_i$$

(backwards to the multiplication map sending $V_2 \otimes V_i$ to V_{i+1}). Hence, we only need a commutative diagram

$$\begin{tikzcd}
V_{i+1} \arrow[r] & \overbrace{V_2 \otimes \cdots \otimes V_2}^{i} \arrow[dl] \\
V_2 \otimes V_i \arrow[u] &
\end{tikzcd}$$

where the undefined map is multiplication on the last $i-1$ factors and tracing V_{i+1} all the way around gives the identity. But this is just the characteristic zero argument that the symmetric tensors, a subspace of a tensor power, are the symmetric power, a quotient space of tensor power, which follows using the central idempotent of the symmetric group for the principal character. This works here in this range, sending (if V_1 has basis X, Y)

$$X^a Y^b \to \frac{1}{i!} \sum \underbrace{X \otimes \cdots \otimes X}_{a} \otimes \underbrace{Y \otimes \cdots \otimes Y}_{b} \qquad (a+b=i)$$

where the sum means all results using place permutations of Σ_i.

Restricting the Steinberg module

We shall investigate Geoff Mason's factorization question. Let G be a finite group of Lie type and characteristic p, k an algebraically closed field of characteristic p, P a parabolic subgroup of G with Levi decomposition $P = L.U$. Let St_G be the Steinberg kG-module (St_L for L, etc.).

<u>Proposition 1</u>. $Res^G_P St_G \cong Ind_L^P St_L$.

<u>Remarks</u> 1. This gives the desired factorization in this case, as $Ind_L^P St_L \cong Ind_L^P ((Res^P_L \hat{St}_L) \otimes k) \cong \hat{St}_L \otimes Ind_L^P k$ where $\hat{St}_L$ is the lift of St_L to P.

2. Probably this result is well-known!

3. This should be easily provable by the alternating sum - building - formula for the Steinberg module: look at $GL(n, q)$ and parabolics containing a Borel subgroup in terms of flags, for example.

4. This raises a lot of questions: What is the corresponding result for the basic and "partial" Steinberg modules? Is there an analogous result for the Steinberg complex of Webb and the Brown complex? Are these complexes related to the partial Steinberg modules? Is there a homotopy theoretic proof?

<u>Proof</u>. $Res^G_P St_G$ is a projective kP-module with the simple kP-module $\hat{St}_L$ as a submodule (by Steve Smith's theorem - see Geoff Mason's letter) so the injective envelope (= projective cover) of $\hat{St}_L$ is a summand of $Res^G_P St_G$. Hence, it suffices to prove that $Ind_L^P St_L$ is such a cover. However, $Ind_L^P St_L$ is projective, its dimension is $|P|_p$ so it is indecomposable and $\hat{St}_L$ is a homomorphic image since

$\mathrm{Hom}_{kP}(\mathrm{Ind}_L^P St_L, \hat{St}_L) = \mathrm{Hom}_{kL}(St_L, \mathrm{Res}_L^P \hat{St}_L) \simeq \mathrm{Hom}_{kL}(St_L, St_L) \neq 0.$
Hence, the proposition is proved.

One part of the argument suggests a line to follow as a digression. Here let $G = L.U$, a semi-direct product of a normal p-subgroup U and a complement L.

Proposition 2. If S is a simple kL-module and Q_S is its projective cover then Ind_L^G is the projective cover of the lift $\hat{S}$ of S to G.

Proof. Let $S_1, S_2, \ldots, S_n$ be the distinct simple kL-modules and let $\hat{S}_i$ be the lift of S_i to G. Let Q_i be the projective cover of S_i. If $\dim_k S_i = d_i$ then, as kL-modules

$$kL \simeq \bigoplus d_i Q_i$$

so

$$kG = \mathrm{Ind}_L^G kL \simeq \bigoplus d_i \,\mathrm{Ind}_L^G Q_i \,.$$

But $\mathrm{Ind}_L^G Q_i$ is projective and has $\hat{S}_i$ as an image (as in the proof of Proposition 1) so the projective cover P_i of $\hat{S}_i$ is a summand of $\mathrm{Ind}_L^G Q_i$ so $\bigoplus d_i P_i$ is a summand of kG. We must have equality so we have equality "term by term."

Remark. It is easy to see how to adapt this if k is not algebraically closed using the dimension of the endomorphism algebra of S_i which is the same as for $\hat{S}_i$. However, there is even a better result, an easy sort of "dual" to Green's induction theorem.

We no longer need k algebraically closed. We keep $G = L.U$.

Proposition 3. If X is an indecomposable kL-module then $\mathrm{Ind}_L^G X$ is an indecomposable kG-module.

Proof. First, we observe that if Y is any kG-module then $\mathrm{rad}(kU).Y$ is a kG-submodule of Y contained in $\mathrm{rad}(kG).Y$. For $\mathrm{rad}\, kU$ annihilates any simple kG-module, by Clifford's theorem, so $\mathrm{rad}\, kU \subseteq \mathrm{rad}\, kG$. If $g \in G$ then

$$g.\ \mathrm{rad}(kU)Y = g(\mathrm{rad}(kU))g^{-1} gY = \mathrm{rad}\, k(gUg^{-1}).Y = \mathrm{rad}(kU).Y$$

so our first claim is valid.

To prove the proposition it is therefore sufficient to see that if $Y = \mathrm{Ind}_L^G X$ then $\hat{X} \cong Y/\mathrm{rad}(kU).Y$, where $\hat{X}$ is the lift of X to G. Indeed, if $Y = Y_1 \oplus Y_2$ then $Y/(\mathrm{rad}\, U)Y \cong Y_1/(\mathrm{rad}\, U)Y_1 \oplus Y_2/(\mathrm{rad}\, U)Y_2$, each summand being non-zero, if Y_1 and Y_2 are non-zero, as $\mathrm{rad}\, U \subseteq \mathrm{rad}\, kG$.

Now we have a vector space epimorphism $Y = kG \otimes_{kL} X \to \hat{X}$ sending each $g \otimes x$ to x and this is clearly a kG-homomorphism. Since U acts trivially on $\hat{X}$ it does on the quotient of Y by the kernel, that is, if $u \in U$ then $u-1$ annihilates this quotient, that is, $\mathrm{rad}\, kU$ does so we have an epimorphism of $Y/(\mathrm{rad}\, kU)Y$ to $\hat{X}$. We just need a suitable inverse map! We have a vector space map of $\hat{X}$ to Y sending each $x \in \hat{X}$ to $1 \otimes x \in Y$. We need only prove this is a kG-homomorphism when composed with the natural map of Y onto $Y/\mathrm{rad}(kU).Y$, as the rest is clear. To check that this backwards map ($x \to 1 \otimes x$ composed with the natural map) commutes with the action of G let $l \in L$, $u \in U$. We first must see that $u(1 \otimes x)$ is the image of ux, that is, $u(1 \otimes x) \equiv 1 \otimes ux = 1 \otimes x$. But $u(1 \otimes x) = u \otimes x = (1 + (u-1)) \otimes x \equiv 1 \otimes x$. Also $l(1 \otimes x) = l \otimes x = 1 \otimes lx$ the image of lx.

<u>Remark</u>. The proof shows that if $X_1 \neq X_2$ then $Y_1 \neq Y_2$ (with the obvious notation) so we have a sort of equivalence less than a Morita equivalence.

<u>Remark</u>. Proposition 1, in terms of characters, appears in Carter's latest book with a proof using the alternating sum due to Howlett.

Now let's return from the diversion. Let's consider the "factor" that comes in Proposition 1 (see Remark 1) and its structure in one case in the hope this leads to an insight. We consider the permutation module of $GL(n,p)$ over $\mathbb{F}_p$ acting on the vectors in the standard vector space for $GL(n,p)$, so this module is of dimension p^n. Our result generalizes a theorem Mike O'Nan told us years ago.

<u>Proposition 3.</u> The permutation module has a filtration the successive subquotients being the truncated symmetric powers of the standard vector space.

We mean the following. Let V be the standard module. The truncated symmetric algebra on V is the quotient of the symmetric algebra on V by the p-th powers so this has homogeneous terms of degree $0, 1, \ldots, n(p-1)$ and total dimension p^n. (If $p=2$ this is also the exterior algebra - which is the case O'Nan told us about.)

To prove this we let E be a (multiplicative) elementary abelian p-group of order p^n so we can let $GL(n,p)$ "be" the automorphism group of E so it acts as automorphisms of the group algebra $\mathbb{F}_p E$ and this is just our permutation module! Now $\mathbb{F}_p E$ is isomorphic with the truncated symmetric algebra in n variables (generated by the elements $e_i - 1$ as $e_1, \dots, e_n$ is a basis of E). Hence, the successive quotients of the radical series of $\mathbb{F}_p E$ (the augmentation powers) are the homogeneous terms of the truncated symmetric algebra. It remains only to see that $GL(n,p)$ acts properly on $\operatorname{rad}(\mathbb{F}_p E)/\operatorname{rad}^2(\mathbb{F}_p E)$. But $e_1 - 1, \dots, e_n - 1$ give a basis for this space so the formula

$$xy - 1 = (x-1) + (y-1) + (x-1)(y-1)$$

is all that we need.

<u>Remarks.</u> 1. For $GL(n,q)$, $q = p^e$ we could get information using $GL(n,q) \leq GL(ne,p)$ about the standard module.

2. Question: Is there some sort of Steinberg factorization, at least up to composition factors that applies in the previous remark? (The case $GL(2,q)$ should be easily accessible.)

3. The number of orbits of a Sylow p-subgroup of $GL(n,p)$ on V is easily seen to be $1 + (p-1)n$ so the fixed point space of this Sylow subgroup on the permutation module is of this dimension. If the permutation module were semisimple then, as each subquotient considered is simple, the fixed point space would be also $1 + n(p-1)$ so we can't easily get non-semisimplicity! But that should be the case. It looks like, for $p = 2$, after splitting the two trivial modules that the rest is uniserial and one should be able to prove this by considering the action of $GL(n,2)$ on fixed points of the Sylow 2-subgroup.

Now let's try another approach. This doesn't do what is needed but it comes close. So now let G be a universal Chevalley group over k so G is semisimple, simply connected and so on. Let α be a positive root so $L = \langle X_\alpha(t), X_{-\alpha}(t) \rangle = SL(2,k)$ is the derived group of the Levi complement of a (minimal) parabolic subgroup (put α as a basic root for the appropriate Borel subgroup). Let St_G be the Steinberg module (in the sense of algebraic groups) for G so it has dimension the appropriate power of p, restricts to the basic Steinberg for $G(q)$ and is an "induced" module (dual of a Weyl module) for the highest weight which is $p-1$ times the sum of the fundamental weights. (For the last, see Jantzen's book II 3.19 (4).) If $G = SL(2)$ then these induced modules are just the symmetric powers of the standard 2-dimensional module (again see Jantzen p209 or Green, LNM 830, 4.4 Example 1+(4.1c)

For the purposes of generalizing from $SL(2)$, Mason and I need certain connections to exist between $St_G|_L$ and St_L. For example, if the latter divided the former this would do. Or, if $St_G|_L$ had a filtration and St_L divided each subquotient then we'd also be fine. This brings to mind the good filtration theory of Donkin + Mathieu (S. Donkin, Rational representations of algebraic groups, Springer Verlag, Lect. Notes in Math. v1140, 1985; Olivier Mathieu, Filtrations of G-modules, Annales Sci. L'Ecole Normal Superieure, v23 (4th series) 1990, 625-45). Suppose the factors of the good filtration of $St_G|_L$, which exists as part of the theory, were suitable. This doesn't occur as Donkin points out for $p=2$, $G=SL(3)$. However in that case we are still OK. For the Steinberg module for $SL(3)$

is eight-dimensional, can be realized as matrices of trace 0 under conjugation so it is easy to see that the restriction to $SL(2)$ is $E \oplus E^* \oplus (E \otimes E^*)$ ($\simeq E \oplus E \oplus (E \otimes E)$) where E is the standard module for $SL(2)$. The picture:

$$E = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \qquad E \otimes E^* = \begin{pmatrix} -t & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad E^* = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(t so trace is 0)

$$SL(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

[As a side remark note that lots of modules for $SL(2)$ are divisible by St_L. $St_L = V_p$ the p-dimensional symmetric power of degree $p-1$ and $V_p \otimes V_2^{(p)} \simeq V_{2p}$, by multiplication, where the upper p denotes Frobenius. We also have the modular version of Clebsch-Gordan.]

It is also true that we would be fine if St_L virtually divided $St_G|_L$ in the sense that there were modules V_1 and V_2 with $St_G|_L \oplus (V_1 \otimes St_L) \simeq V_2 \otimes St_L$. Applying the Frobenius, working in the Green ring and tensor the expressions for $St_G|_L$, $St_G^{(p)}|_L$, ... and passing to the finite groups would give us what we need. [Also, it's more formal as $A \oplus B \simeq C$, $A' \oplus B' \simeq C'$ imply $A \otimes A' \oplus (B \otimes C') \oplus (B' \otimes C) \simeq (B \otimes B') \oplus (C \otimes C')$ if we have suitable cancellation. For, with obvious notation,

$$(aa' + bc' + b'c) + ab' = ac' + bc' + b'c = cc' + b'c$$
$$(bb' + cc') + ab' = cb' + cc'.]$$

However, Donkin has proven what we have just asked for and more. To state his result we have G as above, and let H be the derived group of a Levi factor of G (so $H = L$ is allowed). Then there are "tilting" modules for H (in particular, modules; tilting, in this context is equivalent to having filtrations both by induced modules and by Weyl modules), M_1 and M_2 so that

$$St_G/_H \otimes (M_1 \otimes St_H) \cong M_2 \otimes St_H.$$

This is an easy consequence of a (preliminary) preprint of his on tilting modules.

<u>Remarks</u> 1. Using Smith's result on other parabolics together with this result of Donkin, we can say more about comparing Steinberg modules and freeness properties, it would seem.

2. Perhaps some of this is relevant to characterizing the partial Steinberg modules (using Ronan-Smith "sheaf" methods?).

3. A number of questions face us. Is there a "semi-freeness" property relevant, in particular, for the root subgroups acting on partial Steinbergs? Is there an independence between different root subgroups? This might help inductively in building up a sheaf. Are there strengthened forms of Smith's theorem in our case so that we can give a filtration of the restriction of St, for example, to parabolics in terms of modules described by the St, of the Levi factor, as lifted to the parabolic that we are in the process of studying!

Let us study the "semi-free" question for root subgroups. Let G be our universal Chevalley group over k, X a root subgroup and M a tensor product of d conjugates of St_1. Let $Y \leq X$ be of order p^d such that St_1 is a free kY-module. We want to know if $M^X = M^Y$? If the answer is positive and we pass to $G(q)$ get positive answer to the question of semi-freeness so we say a module V for a p-gp E is semi-free with respect to $F \leq E$ if $V^E = V^F$ and V is a free kF-module. [note: in this case the socle series = radical series - we have p-groups so these are the same for free modules - of V as kF-module equals the same series for kE. This is easy to prove as $\Delta E \cdot \Delta F = \Delta F \cdot \Delta E$ (augmentation ideals).]

Note that if this occurs and $Y' \leq X$ also of order p^d and acting freely then X is semi-free with respect to Y', as $M^{Y'} \geq M^X$ and by dimension counting (which gives $\dim M^Y = \dim M^{Y'}$), and that $M^{Y'} = M^Y$. We next observe that it suffices to deal with the case $d = 1$. For say we have semi-freeness there and that, with the obvious notation,

$$M = S_1 \otimes \dots \otimes S_d,$$

where $\dim S_i = p^N$ when N is the number of positive roots. Now $\dim M^Y = p^{(N-1)d}$ $(\dim M / |Y|)$ so we need only show that $\dim M^X \geq p^{(N-1)d}$ and to do this it suffices to prove that $\dim S_i^X \geq p^{N-1}$. But each $y \in Y$, $y \neq 1$ acts freely on each S_i (conjugacy, plus the field of p elements and the projectivity of St on $G(p)$) (We must work with points over the algebraic closure of the field of p elements.) Hence, we have $St^X = St^{\langle y \rangle}$ is of dimension p^{N-1} as needed.

To deal with the case $d=1$ we need only consider St_1. Let's first consider type A so $G = SL_n$ for some $n \geq 2$. Let $X = X_{1,n}$ (matrix notation) so G acts on the standard vector space with basis $x_1, \dots, x_n$, $X_{1,n}(t): x_1 \to x_1 + t x_n$, $x_i \to x_i$, $i > 1$. If, as we remember, $St_1 \subseteq k[x_1, \dots x_n]$ then it suffices to prove that $k[x_1, \dots, x_n]^{X_{1,n}(1)} = k[x_1, \dots, x_n]^{X_{1,n}}$. But $k[x_1, \dots x_n]^{X_{1,n}(1)} = k[x_2, \dots, x_n] \otimes k[x_1, x_n]^{X_{1,n}(1)}$. A typical invariant is a Dickson invariant (a determinant we have seen) $x_1^p x_n - x_1 x_n^p$. But this is not invariant under $X_{1,n}$. (There are, if memory serves, results that show this is the only sort of thing to worry about so presumably this leads to a proof of what the invariants of $X_{n,1}(1)$ are.) Hence, we must see that such invariants do not show up when dealing with St_1, as the hoped for equality is quite false.

Before attacking the case $d=1$ for type A by another means, let's reflect on what we are after. We will be looking at a module for an elementary abelian p-subgroup X and Y will be a subgroup of X. We will want that M is a free module for Y and $M^Y = M^X$ — is semifreeness. One might ask for a stronger condition and ask that we have $kX = kY + \mathfrak{a}$, where $\mathfrak{a}$ is the annihilator of M in kX, a condition that does not hold in the classical situation: consider $p=2$, SL_3 and use the restriction to SL_2 to get at X (using previous discussions). So we will hope for an intermediate situation, something that will do later things we have in mind.

We return to $G = SL_n$ the universal Chevalley group of type A_{n-1} over the algebraic closure k of the field of p elements, $X = X_{1,n}$, a root subgroup, $Y = X(p)$, the elements over the field of p elements in X, U the group of unitriangular (upper) matrices, $U(p)$ as you would guess. Let M be the basic Steinberg module, $M = St_1$, of dimension $p^{\frac{n(n+1)}{2}}$. We want to show that M as an X-module is semifree with respect to Y (as this gives the desired semifreeness in general as we have seen above).

If $x \in X$ then the linear transformation L_x induced by x on M is an endomorphism of M considered as a module for U since $x \in Z(U)$. Hence, as L_x commutes with all endomorphisms, by definition, we have that L_x is in the center of the algebra of U-endos of M. We _would like_ that L_x is in the center of the algebra of $U(p)$-endomorphisms of M! (For L_x is certainly in this algebra). This would be the case _if_ the two algebras coincide, that is, if the endomorphism algebra of M as U-module is of dimension at least $p^{n(n+1)/2}$.

Let's _assume_ that this _is_ _the case_ and proceed from there. The first consequence is that there is $z \in Z(kU(p))$ such that $x - z$ annihilates M. Indeed, choose a free basis of M as $U(p)$-module: elements m_u, $u \in U(p)$ such that $u' m_u = m_{u'u}$ for any $u' \in U$. The endomorphisms consist of the space spanned by the maps $R_{u'}$, where $R_{u'} \cdot m_u = m_{uu'}$ so we can make sense out of all R_a, $a \in kU(p)$. The central ones, $a \in Z(k(U))$ then coincide with the (obviously defined) L_a so $L_x = L_a$, $a \in Z(k(U))$ just as we desired. _Remark_: this is under an _assumption_.

Lemma. If Q is a normal subgroup of the p-group P then kP has a kP-submodule which, as a kQ-module, is isomorphic with kQ.

Proof. Let N be a free kQ-module of rank one and let $Q \wr P/Q$ act on $N \oplus \cdots \oplus N$ ($|P/Q|$ copies) in the usual way. Now P embeds in $Q \wr P/Q$ in the usual way, Q embedding in the base group projecting onto each factor and the image of P covering the quotient of $Q \wr P/Q$ by the base group. For these reasons, P has just one dimension of fixed points on the sum (one in each summand under Q and these permuted by P). By dimension counting we have a free module for kP. The "diagonal" copy of N is invariant under $Q \wr P/Q$ so is a kP-module and the way Q embeds in the base means there is one dimension of fixed points under Q (the "diagonal" N intersects the fixed points of Q in a one-dimensional space, the fixed points of Q being of dimension $|P/Q|$.

Next, let $U_0 \le U$ be the subgroup with each first row and last column entry 0 (except the diagonal). Let E be the "complementary" subgroup so $U = U_0 E$ is a semi-direct product as is $U(p) = U_0(p) E(p)$ with $E(p)$ extra-special (we're assuming $n \geq 3$ as we may). Let V, V_0 be for the lower triangular group correspondingly so $V_0 E$ is another maximal unipotent subgroup and there is an element w of the Weyl group with $w U_0 w^{-1} = V_0$, $w E w^{-1} = E$, $w^2 = 1$. We have the "picture"

<u>Lemma</u> If $\alpha \in k[V_0(p)E(p)] - kE(p)$, $\beta \in kU(p) - kE$ and $\varepsilon \in kE_0(p)$ and $\alpha+\beta+\varepsilon$ annihilates M then $\alpha=\beta=0$.

It suffices to prove that $\alpha=0$ and then to use the element w just discussed to deduce, then, that $\beta=0$ as well.

<u>Proof</u>. Let M' be a $kU_0(p)$-submodule of M which is free, of rank one, as a $kE(p)$-module. Now $M^{E(p)}$ is, by Steve Smith's theorem as Geoff has shown, the basic Steinberg module for $\langle U_0(p), V_0(p)\rangle$ so the fixed point space of one of these Sylow subgroups on $M^{E(p)}$ is one-dimensional and this one-dimensional space generates $M^{E(p)}$ under the other, by Curtis-Richen theory, as U_0, V_0 are "opposites." In particular, $M^{E_0(p)} \cap M'$ is one-dimensional being $(M')^{E_0(p)}$ and we have the picture

$$\begin{tikzcd}[column sep=small]
M^{E(p)} \arrow[dr, no head] & & M' \arrow[dl, no head] \\
& {} &
\end{tikzcd}$$

Let m be a non-zero vector in this intersection. We claim m is a generator of M as $V_0(p)E(p)$-module. Choose $a \in kV_0(p)$ so that am and m are linearly independent. Now aM' is also a $kE(p)$-module, as $E(p)$ is normalized by $V_0(p)$ so $(aM')^{E(p)} \ni am$ so $aM' \cap M' = 0$ or else the intersection would have a non-zero vector fixed by $E(p)$. Continuing in this way, using a basis of $M^{E(p)}$ and the fact that m generates $M^{E(p)}$ as $V_0(p)$-module (since m is fixed by $U_0(p)$ as $U_0(p)$ certainly leaves invariant the E fixed points on M') we get a direct sum of "translates" of M' under $kV_0(p)$ adding up to M, as the direct sum has the correct dimension. Thus, as M is free as a module for $V_0(p)E(p)$, we have that the elements $ve\cdot m$, $v\in V_0(p)$, $e\in E(p)$ are a basis of M.

Now consider $\alpha+\beta+\varepsilon$. The elements β, ε leave M' invariant, M' free for $E(p)$. Hence the elements $ve\cdot m$, $v\neq 1$ are linearly independent and with the elements $e\cdot m$ are the basis of M we are using. Now αm is a linear combination of the elements $ve\cdot m$, $e\neq 1$ as α is such a linear combination of elements ve, so $(\alpha+\beta+\varepsilon)m=0$ implies $\alpha m=0$ so $\alpha=0$.

Let's return to the situation we had: $x-z$ annihilates M where $z\in Z(kU(p))$. Write $z=\beta+\varepsilon$ where $\beta\in kU(p)-kE(p)$, $\varepsilon\in kE(p)$. Then $w(x-z)w^{-1}$ also annihilates M, $wxw^{-1}=x$ by definition of x, so $z-wzw^{-1}$ annihilates M. But $wzw^{-1}=w\beta w^{-1}+w\varepsilon w^{-1}$ and $w\beta w^{-1}$ is in $kV_0(p)E(p)-E(p)$ so we deduce that $w\beta w^{-1}=0$ so $\beta=0$ and hence $z\in kE(p)$, $z\in Z(kU(p))$. Thus z is a linear combination of $U(p)$ class sums lying in $E(p)$.

To get at semi-pureness we need to see how such elements act on $M^{X(p)}$. Let x_0 be a generator for $X(p)$. The elements in $X(p)$ are, of course, no problem. Any element of $E(p)-X(p)$ is conjugate, in $E(p)$, to the product of itself by x_0 (or itself by any power of x_0). Hence, let σ be a class sum of $E(p)-X(p)$ so $\sigma x_0=\sigma$ as multiplication by x_0 permutes the elements of the class. Thus $\sigma(x_0-1)=0$. $\therefore$ σ annihilates $(x_0-1)M$ which certainly contains $M^{X(p)}$. Thus, z must act on $M^{X(p)}$ by scalar multiplication and x is unipotent.

Remarks. 1. Above goes over for $G(q)$ and the appropriate Steinberg.
2. Some other goals: characterization starting perhaps with St_1;
a proof of Steve Smith's theorem in our case via a characterization and
strengthening his result via filtrations using some kind of
semifreeness for unipotent radicals; semifreeness for more than
one root group to be used in characterizations and strengthening.
3. The following can't be used for semifreeness on tensor products
but is indicative.

<u>Lemma</u>. If G is a group, U, V are kG-modules then
$\operatorname{codim}(\operatorname{ann}_{kG} U) \cdot \operatorname{codim}(\operatorname{ann}_{kG} V) \geq \operatorname{codim}(\operatorname{ann}_{kG} U \otimes V)$.

<u>Proof</u>. Say the first two codimensions are m, n and $x_1, \dots, x_m$ and $y_1, \dots, y_n$ are in G and give bases for kG modulo these annihilators. Let X_i be the linear transformation induced by x_i on U, Y_j for y_j on V. Say $g \in G$ so there are scalars $\lambda_1, \dots, \lambda_m$ such that

$$gu = \sum \lambda_i X_i u$$

for all $u \in U$, and scalars $\mu_1, \dots, \mu_n$ so that

$$gv = \sum \mu_j Y_j v$$

for all $v \in V$. Hence

$$g \cdot u \otimes v = \sum \lambda_i \mu_j (X_i \otimes Y_j)(u \otimes v)$$

and the proof is complete (tensor product linear transformation acts as usual).

4. Seem only to need that any $kU(p)$ endomorphism of St_1 is a $kU(p)X$ endomorphism.

Let $G(q)$, $q=p^e$, $U(q)$, etc... be as above with the obvious notation. We present an equivalent formulation of the result we need, the "if" above, to get the semifreeness for most subgroups.

<u>Proposition</u> The following assertions are equivalent:

1) Every $U(p)$ endomorphism of St_1 is a $U(q)$-endomorphism;

2) Every $U(q)$ endomorphism of $St_1 \otimes St_1^{(p)} \otimes \cdots \otimes St_1^{(p^{e-1})}$ is a linear combination of tensor products of $U(q)$ endomorphisms of $St_1, St_1^{(p)}, \dots, St_1^{(p^{e-1})}$.

We can express the latter statement by

$$End_{U(q)}(St_1 \otimes \cdots \otimes St_1^{p^{e-1}}) = End_{U(q)}(St_1) \otimes \cdots \otimes End_{U(q)}(St_1^{(p^{e-1})}).$$

The key step is the following calculation: Here $St = St_1 \otimes \cdots \otimes St_1^{p^{e-1}}$, the "usual" Steinberg on $G(q)$.

<u>Lemma</u>. $\dim_k End_{kU(q)}(St) = q^{n(n+1)/2}$.

<u>Proof</u>. Let $B(q)$ be the Borel corresponding to $U(q)$, $B(q) = T(q)U(q)$, $T(q)$ the torus. Let $St(B)$, $St(T)$ be the usual Steinberg modules for this parabolic and Levi complement. Then

$$\begin{aligned}
Hom_{kU(q)}(St, St) &\simeq Hom_{kB(q)}(Res^{G(q)}_{B(q)} St,\ Ind^{B(q)}_{U(q)} St) \\
&\simeq Hom_{kB(q)}(St(B) \otimes Ind^{B(q)}_{T(q)} k,\ Ind^{B(q)}_{U(q)} St) \\
&\simeq Hom_{kB(q)}(Ind^{B(q)}_{T(q)} St(T),\ Ind^{B(q)}_{U(q)} St) \\
&\simeq Hom_{kT(q)}(St(T),\ Res^{B(q)}_{T(q)} Ind^{B(q)}_{U(q)} St) \\
&\simeq Hom_{kT(q)}(St(T),\ Ind^{T(q)}_{T(q)\cap U(q)} Res^{U(q)}_{T(q)\cap U(q)} St)
\end{aligned}$$

by Mackey's theorem, as $B(q) = T(q)\,U(q)$,

$$\simeq Hom_{kT(q)}(St(T),\ Ind_1^{T(q)}(\overbrace{k \oplus \cdots \oplus k}^{\dim St}))$$

so the lemma holds as the right hand side is $\dim St$ copies of the regular representation and $kT(q)$ is semisimple.

<u>Remark</u>: Easier proof! St is free of rank one for $kU(q)$.

Now suppose that 1) holds. Then dim $\mathrm{End}_{U(q)}(St_1) = p^{n(n+1)/2}$ so dim $\mathrm{End}_{U(q)}(St_1^{(p^i)}) = p^{n(n+1)/2}$ (as St_1 and $St_1^{(p^i)}$ are conjugate $U(q)$ modules) so

$$\dim\left(\mathrm{End}_{U(q)}(St_1) \otimes \cdots \otimes \mathrm{End}_{U(q)}(St_1^{(p^{e-1})})\right) = \overbrace{p^{n(n+1)/2} \times \cdots \times p^{n(n+1)/2}}^{e}$$
$$= q^{n(n+1)/2}$$
$$= \dim \mathrm{End}_{U(q)}(St),$$

by the Lemma, so 2) holds.

Finally, suppose that 2) is valid. Then

$$q^{n(n+1)/2} = \dim \mathrm{End}_{U(q)}(St) = \dim(\mathrm{End}_{U(q)} St_1) \times \cdots \times \dim(\mathrm{End}_{U(q)} St_1^{(p^{e-1})})$$
$$= (\dim(\mathrm{End}_{U(q)} St_1))^e$$

so $\dim(\mathrm{End}_{U(q)} St_1) = p^{n(n+1)/2} = \dim(\mathrm{End}_{U(p)} St_1)$ so 1) holds too!

This leaves us with the problem of establishing 2). One thing (and further calculations of this sort) is that

$$\mathrm{Hom}_{U(q)}(St_1, St_1^{(p^i)}) \cong k$$

if $0 < i < e$. For

$$\mathrm{Hom}_{U(q)}(St_1, St_1^{(p^i)}) \cong (St_1^* \otimes St_1^{(p^i)})^{U(q)}$$
$$\cong (St_1 \otimes St_1^{(p^i)})^{U(q)}$$

which is the fixed points of $U(q)$ on the restriction of a simple module to $U(q)$; so this is one-dimensional by Curtis' Theorem. In fact, this gives us that each of the modules for $U(q)$

$$St_1 \otimes St_1^{(p)} \otimes \cdots \otimes St_1^{(p^i)}$$

has the property that it has a simple socle and also a simple radical quotient and there are no maps between this and $St_1^{(p^{i+1})}$, in either direction except the ones with image the socle and kernel the radical. This is the key. The next result then finishes this saga?

A good bet now is that this is not going to work. If it did then $\mathrm{End}_{kU(q)}(St_1) \simeq kU(p)$. Since the endomorphism algebra is a group algebra we will then have $\mathrm{End}_{kU(q)}(St_1^{(p)}) \simeq kU(p)$ as well. (Look at it in terms of intertwining matrices. A group basis goes to a group basis under field automorphisms.) Then, we would obtain

$$\begin{aligned} kU(q) &\simeq \mathrm{End}_{kU(q)}(St) \\ &\simeq \mathrm{End}_{kU(q)}(St_1 \otimes \cdots \otimes St_1^{(p^{e-1})}) \\ &\simeq \mathrm{End}_{kU(q)}(St_1) \otimes \cdots \otimes \mathrm{End}_{kU(q)}(St_1^{(p^{e-1})}) \\ &\simeq kU(p) \otimes \cdots \otimes kU(p) \\ &\simeq k[U(p) \times \cdots \times U(p)] \end{aligned}$$

a most unlikely isomorphism.

Perhaps we can look at $SL(3,4)$, with k the field of four elements. This argument seems to carry over there.

Tilting modules for SL_2 in characteristic two.

We shall work with the algebraic group $SL_2(k)$, k an algebraically closed field of characteristic two. We use the usual notation we have used (for $SL_2(2^n)$) so $V_1 = St_1$ is the standard 2-dimensional module and V_i is the $(i-1)$st Frobenius twist, $V_I = \bigotimes_{i\in I} V_i$ where $I \subseteq \mathbb{N} = \{1, 2, \dots\}$ and $V_\emptyset = k$. If S is an initial segment of $\mathbb{N}$, so $S = \{1, 2, \dots, s\}$ for some $s \in \mathbb{N}$ and $I \subseteq S$ then $V_{I,S} = V_I \otimes V_S$ (as in our previous study of simply generated modules for $SL_2(2^n)$). Recall that a tilting module is one that has a filtration by "induced" modules (that is, symmetric powers of V_1) and has a filtration by Weyl modules (the duals of the induced modules)

<u>Theorem</u>. The modules $V_{I,S}$ are indecomposable, self-dual, tilting with a simple socle (V_{S-I} for $V_{I,S}$). They are exactly the indecomposable summands of the tensor powers of V_1.

<u>Proof</u>. Suppose $I \subsetneqq S$. Now $V_{I,S} = V_I \otimes V_I \otimes V_{S-I}$ so as k is a submodule of $V_I \otimes V_I$ we certainly have V_{S-I} in the socle. However, this is the socle on restriction to $SL_2(2^s)$, as $I \subsetneqq S$, the restriction being the projective cover, so this is the socle and so is indecomposable as well (and self dual as each V_i and V_I is too). If $I = S$ this last part fails but comparing $V_{S,S}$ and $V_{S,S\cup\{s+1\}} \cong V_{S,S} \otimes V_{s+1}$ completes this part of the argument. To see these are tilting modules we need only prove the second statement, as V_1 is certainly alright, by work of Donkin on filtrations of tensor products.

However, the second part is just an old analysis (see my paper on $SL(2,2^n)$) and can be done by induction. We just illustrate the first cases.

$$
\begin{array}{ll}
V_1 & V_{\emptyset,\{1\}} \\
V_1 \otimes V_1 & V_{\{1\},\{1\}} \\
V_1 \otimes V_1 \otimes V_1 = V_1 \oplus V_1 \oplus V_{12} & \text{new}: V_{12} = V_{\emptyset,\{1,2\}} \\
V_1 \otimes V_{12} & V_{\{1\},\{1,2\}} \\
V_1 \otimes V_1 \otimes V_{12} = (V_1 \oplus V_1 \oplus V_{12}) \otimes V_2 & \text{new}: V_{12} \otimes V_2 = V_{\{2\},\{1,2\}} \\
V_1 \otimes V_{12} \otimes V_2 & V_{\{1,2\},\{1,2\}} \\
V_1 \otimes V_{12} \otimes V_{12} = (V_1 \otimes V_1 \otimes V_1) \otimes (V_2 \otimes V_2) & \\
\quad = (V_1 \oplus V_1 \oplus V_{12}) \otimes (V_2 \otimes V_2) & \\
\quad = V_{\{2\},\{1,2\}} \oplus V_{\{2\},\{1,2\}} \oplus V_1 \otimes V_2 \otimes V_2 \otimes V_2 & \\
+ .\ V_1 \otimes V_2 \otimes V_2 \otimes V_2 \simeq V_1 \otimes (V_2 \oplus V_2 \oplus V_{23}) & \text{new}: V_{123} = V_{\emptyset,\{1,2,3\}} \\
 & \text{etc.}
\end{array}
$$

Examples. Let's start by describing the "induced" modules, by "degree" (of symmetric power) heuristically using Clebsch–Gordan (though Carter–Cline or later should prove it).

Degree	0	1	2	3	4	5	6	7	8	9
module	0	1	$\begin{matrix}0\\2\end{matrix}$	12	$\begin{matrix}2\\0\\3\end{matrix}$	$\begin{matrix}1\\13\end{matrix}$	$\begin{matrix}0\\3\\23\end{matrix}$	123	$\begin{matrix}23\\3\\0\\4\end{matrix}$	$\begin{matrix}13\\1\\14\end{matrix}$

Degree	10	11	12	13	14	15	16	17
module	$\begin{matrix}3 & & \\ \vert & & \\ 0 & - & 2 \\ \vert & & \vert \\ 4 & - & 24\end{matrix}$	$\begin{matrix}12\\124\end{matrix}$	$\begin{matrix}2 & - & 24 \\ \vert & & \vert \\ 0 & - & 4 \\ & & \vert \\ & & 34\end{matrix}$	$\begin{matrix}1\\14\\134\end{matrix}$	$\begin{matrix}0\\4\\34\\234\end{matrix}$	1234	$\begin{matrix}234\\34\\4\\0\\5\end{matrix}$	$\begin{matrix}134\\14\\1\\15\end{matrix}$

Now some of our modules with filtrations suggested.

$$1 \otimes 1 \otimes 2 = \begin{array}{c} 2 \\ 0 \\ 3 \\ \hline 0 \\ 2 \end{array}$$

$$1 \otimes 1 \otimes 2 \otimes 2 = \begin{array}{c} 0 \\ 2 \\ 0 \end{array} \otimes \begin{array}{c} 0 \\ 3 \\ 0 \end{array} = \begin{array}{ccc} 0 & 3 & 0 \\ 2 & 23 & 2 \\ 0 & 3 & 0 \end{array}$$

$$1 \otimes 1 \otimes 2 \otimes 2 \otimes 3 = \begin{array}{ccc|cc} 3 & 0 & 4 & 0 & 3 \\ 23 & 2 & 24 & 2 & 23 \\ 3 & 0 & 4 & 0 & 3 \end{array}$$

<u>Theorem</u>. Every tilting module is isomorphic with a direct sum of various $V_{I,S}$.

<u>Proof</u>. The highest weight for V_1 is $1 \cdot \lambda$ and each tensoring with V_1 produces a module with highest weight one more multiple of λ and produces one "new" indecomposable summand so the i-th power of V_1 gives the i-th $V_{I,S}$ and it has highest weight $i\lambda$. The result of Donkin finishes the proof.

Let's produce a table listing the indecomposable tilting modules first, the degree, i.e. multiple that occurs in the highest weight, and the corresponding simple composition factors of the indecomposable module. The pattern is clear and should be easily made formal, if desired.

$V_{S,I}$	Degree	Simple
0	0	0
1	1	1
$1 \otimes 1$	2	2
$1 \otimes 2$	3	12
$1 \otimes 1 \otimes 2$	4	3
$1 \otimes 2 \otimes 2$	5	13
$1 \otimes 1 \otimes 2 \otimes 2$	6	23
$1 \otimes 2 \otimes 3$	7	123
$1 \otimes 1 \otimes 2 \otimes 3$	8	4
$1 \otimes 2 \otimes 2 \otimes 3$	9	14
$1 \otimes 1 \otimes 2 \otimes 2 \otimes 3$	10	24
$1 \otimes 2 \otimes 3 \otimes 3$	11	124
$1 \otimes 1 \otimes 2 \otimes 3 \otimes 3$	12	34

For odd characteristic the same technique works: the tensor powers of the two-dimensional module keep producing tilting modules and a unique new highest weight — <u>hence</u> a unique new indecomposable summand which then must involve this new weight. Presumably the structures can be worked out here as well.

Of course we can carry on to SL_n, e.g., or any situation where the Weyl modules for the fundamental are closed — as a set — under the taking of duals.

We can also give a slightly different proof of weak – i.e. virtual – divisibility for SL_2. For characteristic two all the tilting modules are divisible by the (basic) Steinberg (so Krull-Schmidt gives divisibility, in fact) while for odd characteristic the tilting modules for weights $a\lambda_1$, $a < p-1$ are of dimension $a+1$ and not of dimension divisible by p while the rest have dimension divisible by p and, inductively are virtually divisible by the Steinberg. The divisibility of the dimension by p follows from restriction to $SL_2(p)$ and the projectivity of the Steinberg there and from the tensor product construction, using the Steinberg and other simples to get tilting modules.

This last discussion gives the splitting of the "Clebsch-Gordan" sequence for SL_2. That is, if V_i is the i-dimensional induced module, i.e. polynomials of degree $i-1$, then the exact sequence

$$0 \to V_{i-1} \to V_2 \otimes V_i \to V_{i+1} \to 0$$

splits if $i < p$. For the extension, if non-split is not self dual so get two tilting modules with the same highest weight.

Tilting modules for SL_4 in characteristic two

We shall use the tensor product method (also described by Donkin in an early paper) to do a few cases and apply the results to the semi-free question. Let k have characteristic two, $G = SL_4(k)$, V be the standard four-dimensional module, $V^{(2)}$ the first "twist" (and use similar notation elsewhere) so V, $V \wedge V$, $V \wedge V \wedge V \simeq V^*$ are the fundamental modules.

<u>Lemma</u> $V \otimes V$ is uniserial with composition factors $V \wedge V$, $V^{(2)}$, $V \wedge V$ in that order.

<u>Proof</u>. Let X_1, X_2, X_3, X_4 be a basis of V so $V \otimes V$ has a submodule isomorphic with $V \wedge V$ with quotient the homogeneous polynomials in the X_i of degree two. The latter has a submodule of squares, X_i^2, $1 \leq i \leq 4$, so it is clear that the composition factors are as asserted. Now

$$\mathrm{Hom}_{kG}(V \otimes V, V^{(2)}) \simeq \mathrm{Hom}_{kG}(V, V^* \otimes V^{(2)})$$

and $V^* \otimes V^{(2)}$ is simple, by the Steinberg tensor product theorem, so $V \otimes V$ has no quotient isomorphic with $V^{(2)}$. Similarly, it has no submodule isomorphic with $V^{(2)}$ so the lemma is valid.

Notice that we similarly have a structure for $V^* \otimes V^*$ in terms of $V \wedge V$, $(V^*)^{(2)}$, $V \wedge V$, as $V \wedge V \simeq V^* \wedge V^*$. (We thus have the structure of the tilting modules for highest weights $2\omega_1$, $2\omega_3$.) (Next, we start an analysis for the tilting module for $\omega_1 + \omega_2$ which we do not conclude.)

<u>Lemma</u>. The module $V \otimes (V \wedge V)$ has V^* as a summand.

<u>Proof</u>. Multiplication in the exterior algebra on V gives an epimorphism of $V \otimes (V \wedge V)$ to $V \wedge V \wedge V \cong V^*$. We shall give a suitable "backwards" map. If $u, v, w \in V$ we map

$$u \wedge v \wedge w \to u \otimes (v \wedge w) + v \otimes (w \wedge u) + w \otimes (u \wedge v).$$

Does this make sense? (If it is then it is certainly a kG-homomorphism and composition with multiplication by three, that is, by one!) Since we do have a map from $V \otimes V \otimes V$ we need only see that if any two of u, v, w coincide then the right hand side is zero and that the right hand side is skew symmetric with respect to transpositions among u, v, w. This is all easy to check.

Note that we similarly have $V \mid V^* \otimes (V \wedge V)$ $(\cong V^* \otimes (V^* \wedge V^*))$.

<u>Lemma</u> $V \otimes (V \wedge V) \otimes (V \wedge V \wedge V) \cong (V \otimes V) \oplus (V^* \otimes V^*) \oplus St_1(G)$.

<u>Proof</u>. We have $V^* \mid V \otimes (V \wedge V)$ so $V^* \otimes V^* \mid V \otimes (V \wedge V) \otimes V^*$. Similarly, $V \otimes V \mid V \otimes ((V \wedge V) \otimes V^*)$ so the indecomposable modules $V \otimes V$ and $V^* \otimes V^*$ are summands. But St_1 is also a summand – it's the tilting module for weight $\omega_1 + \omega_2 + \omega_3$ so we're done by dimension considerations.

Now let $L = SL_2(k) \leq G = SL_4(k)$, the "upper left hand corner." The natural 2-dimensional module for L is just $St_1(L)$. We easily see that $V|_L \cong St_1(L) \oplus k \oplus k$, $V^*|_L \cong St_1(L) \oplus k \oplus k$ and $V \wedge V|_L \cong St_1(L) \oplus St_1(L) \oplus k \oplus k$. Hence, we have nicely

<u>Lemma</u> We have the isomorphism
$$St_1(G)|_L \simeq 8\, St_1(L) \oplus 8(St_1(L) \otimes St_1(L)) \oplus 2(St_1(L) \otimes St_1(L) \otimes St_1(L)).$$

This is an easy calculation and application of the Krull-Schmidt theorem. Now $St_1(L) \otimes St_1(L)$ is uniserial with composition factors k, $St_1(L)^{(2)}$, k and
$$St_1(L) \otimes St_1(L) \otimes St_1(L) \simeq St_1(L) \oplus St_1(L) \oplus (St_1(L) \otimes St_1(L)^{(2)})$$
Hence, semipureness is not valid in the sense we have been using - though the summands each is semipure.

Let's also examine the restriction to $H = SL_3(k)$. It has the standard module W of dimension three. Then
$$V|_H \cong W \oplus k$$
$$V \wedge V|_H \simeq W \oplus W^*$$
$$V^*|_H \simeq W^*$$
as is easy to see. (To see the second isomorphism use the fact that $V \wedge V$ is self-dual). Then, it is easy to calculate that
$$St_1(G)|_H \simeq W \otimes St_1(H) \oplus W^* \otimes St_1(H) \oplus St_1(H) \oplus St_1(H).$$

On the Steinberg module for SL_3.

We are going to determine the restriction of $St_1(G)$, $G = SL_3(k)$, k algebraically closed of characteristic p, to $L = SL_2(k)$. Let $V_1 = k, V_2, \ldots, V_p \simeq St_1(L)$ be the standard modules, with $\dim_k V_i = i$. Our result is

<u>Proposition</u>. We have the isomorphism
$$St_1(G)|_L \simeq V_p \otimes V_p \oplus 2(V_1 \oplus \ldots \oplus V_{p-1}) \otimes V_p.$$

The key step is the next result.

<u>Lemma</u>. We have the isomorphism of kG modules
$$S^{p-1}(V) \otimes S^{p-1}(V^*) \simeq S^{p-2}(V) \otimes S^{p-2}(V^*) \oplus St_1(G)$$

Here V is the standard three dimensional module for G. Let's first observe that the proposition does follow from the lemma. Consider $S^n(V)$ in the usual way so that it has a basis consisting of all monomials $X^iY^jZ^k$, $i+j+k = n$. We may assume that L acts on X, Y fixing Z so $S^n(V)$ has a direct decomposition according to powers of Z, as an L-module. Hence, if $W = V_2$ we have
$$S^n(V)|_L \simeq S^n(W) \oplus \ldots \oplus S^1(W) \simeq V_{n+1} \oplus \ldots \oplus V_1.$$
Now $V_L \simeq W \oplus k$ so V_L is self dual so $S^n(V^*)|_L \simeq S^n(V)|_L$. The lemma now gives the proposition by a direct calculation using the Krull-Schmidt theorem. It remains now to establish the lemma.

First, there are $(p-1)^2$ "contractions"

$$V^{\otimes(p-1)} \otimes (V^*)^{\otimes(p-1)} = V\otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^* \to V^{\otimes(p-2)} \otimes (V^*)^{\otimes(p-2)}$$

(sending a factor $V \otimes V^* \to k$) so let σ be their sum so σ is a kG-homomorphism. It's definition means it defines a kG-homomorphism

$$S^{p-1}(V) \otimes S^{p-1}(V^*) \to S^{p-2}(V) \otimes S^{p-2}(V^*).$$

It suffices to see that this is surjective! Indeed, suppose we have this established. By tilting theory

$$V^{\otimes(p-1)} \otimes (V^*)^{\otimes(p-1)} = St_1 \oplus \dots$$

by highest weights (as $(p-1)(\lambda_1+\lambda_2)$ is highest and unique and St_1 is the corresponding tilting module as it is self-dual - so is "Weyl" and "induced"). Now St_1 is simple and the highest weight $(p-1)\rho = (p-1)(\lambda_1+\lambda_2)$ "lives" in $S^{p-1}(V) \otimes S^{p-1}(V^*)$, as is easy to see, so this tensor product has St_1 as a submodule. But $(p-1)\rho$ does not "live" in $S^{p-2}(V) \otimes S^{p-2}(V^*)$ and the difference in the dimensions between $S^{p-1}(V) \otimes S^{p-1}(V^*)$ and $S^{p-2}(V) \otimes S^{p-2}(V^*)$ is the dimension of St_1. Hence, the surjectivity is all we need, as claimed.

As above let X, Y, Z be a basis of V with X a generator under the "upper" Sylow subgroup so if X^*, Y^*, Z^* is the dual basis then X^* is fixed under the upper Sylow while the lower fixes X and X^* is a generator under it. The same holds for X^{p-1} and $(X^*)^{p-1}$ since we're in characteristic p and degree $< p$. Hence, it follows that $X^{p-1} \otimes X^{*p-1}$ generates $S^{p-1}(V) \otimes S^{p-1}(V^*)$. Similarly, $X^{p-2} \otimes X^{*p-2}$ generates $S^{p-2}(V) \otimes S^{p-2}(V^*)$. But $X^{p-1} \otimes X^{*p-1}$ is mapped to $(p-1)^2 X^{p-2} \otimes X^{*p-2}$ so we're done.

Strongly embedded subgroups

Let G be a finite group and k a field of characteristic p.

Proposition. If H is a proper subgroup of G then the following are equivalent:

1) H is strongly p-embedded;

2) For every kG-module M and every positive integer n, the restriction map

$$H^n(G, M) \to H^n(H, M)$$

is an isomorphism.

(This remark is motivated by a result of O. Mislin in Comm. Math. Helv. 65 (1990), 454-61).

Proof. We may assume $p \mid |G|$ so also $p \mid |H|$ by hypothesis. (For 1) $\Rightarrow$ 2) is standard by stable equivalence so we need only show 2) $\Rightarrow$ 1).)
Let Q be a p-subgroup of H so

$$H^*(Q, k) \simeq H^*(G, \mathrm{Ind}_Q^G k) = H^*(H, \mathrm{Res}_H^G \mathrm{Ind}_Q^H k)$$

$$\simeq \bigoplus_{H\backslash G/Q} H^*(H, \mathrm{Ind}_{tQt^{-1}\cap H}^H k)$$

$$= H^*(H, \mathrm{Ind}_Q^H k) \oplus \cdots \oplus H^*(H, \mathrm{Ind}_{tQt^{-1}\cap H}^H k) \oplus \cdots$$

But $H^*(H, \mathrm{Ind}_Q^H k) \simeq H^*(Q, k)$ so we deduce that if $t \notin H$

$$H^*(H, \mathrm{Ind}_{tQt^{-1}\cap H}^H k) = 0.$$

If $t \in N(Q)$, $t \notin H$ then

$$H^*(H, \mathrm{Ind}_{tQt^{-1}\cap H}^H k) \simeq H^*(Q, k) \neq 0,$$

a contradiction, so $N(Q) \subseteq H$, as required.

Further questions (where $H \leq G$, k as above).

1. What happens if we assume just
$$H^*(G, M) \xrightarrow{res} H^*(H, M)$$
to be an isomorphism when $M \in B_0(G)$?
2. Or, assume $Ext^*(M_1, M_2) \xrightarrow{res} Ext^*(M_1, M_2)$ always an isomorphism when $M_1, M_2 \in B_0(G)$.
3. Or further assume that restriction gives a stable equivalence for $B_0(G)$.

Notice that there are two important cases for 3. There is H weakly p-embedded, by a theorem of Broué, and there is the isomorphic block situation. (The latter is not included in the former: Let $H = GL(3, 11^2)$, $G = H\langle\varphi\rangle$ with φ the Frobenius of order two and let $p = 5$ so the Sylow 5-subgroups of H are in $GL(3, 11)$ and fixed by φ. The normalizer = centralizer (by eigenvalues) of
$$\left\langle \begin{pmatrix} \alpha & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\rangle$$
where α is a (primitive) fifth root of unity is a p-local subgroup that has the desired (non-)properties.)

We want to indicate a previous, longer, proof of the proposition. Apply the argument to $N_G(P)$ instead of G where P is a Sylow p-subgroup of H and so of G, by Mislin's theorem (as that gives H controlling strong fusion on a weaker hypothesis) and get that P intersects any Sylow p-subgroup of G not in H, in the identity. The stabilizer K of the set of Sylow p-subgroups of H is then strongly p-embedded and
$$K \geq H \geq O^{p'}(K) = O^{p'}(H).$$

Next, $K = H\, C_K(P)$, by the Frattini argument and the strong control (even though H need not be normal!). Finally, apply the double coset argument to $H^*(H,k)$ using elements of $C_K(P)$ as double coset representatives!

One more comment, a question suggested by discussion with D. Benson. Let P be a Sylow p-subgroup of G. If a family of subgroups of P "tests" $H^*(G,k)$ (i.e. $\alpha \in H^*(P,k)$ comes from $H^*(G,k)$ if, and only if, α_Q comes from $N_G(Q)$ for all Q in the family) then does this family control strong fusion?

Partial Steinberg modules and complexity.

First, let's improve the result for $SL(2,q)$ relating freeness and the rank of a certain Van der Monde type determinant. As usual

$$X(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

and the matrix representing $X(\lambda)$ on $V_p = St_1$ is

$$\begin{pmatrix} 1 & \lambda & & & \\ & 1 & 2\lambda & & \\ & & 1 & \ddots & \\ & & & \ddots & (p-1)\lambda \\ & & & & 1 \end{pmatrix}$$

Hence, the shifted element

$$1 + \alpha_1 (X(\lambda_1) - 1) + \dots + \alpha_r (X(\lambda_r) - 1)$$

is represented by

$$\begin{pmatrix} 1 & \alpha_1\lambda_1 + \dots + \alpha_r\lambda_r & & & \\ & 1 & 2(\alpha_1\lambda_1 + \dots + \alpha_r\lambda_r) & & \\ & & 1 & \ddots & \\ & & & \ddots & (p-1)(\alpha_1\lambda_1 + \dots + \alpha_r\lambda_r) \\ & & & & 1 \end{pmatrix}$$

so it acts freely if, and only if, $\alpha_1\lambda_1 + \dots + \alpha_r\lambda_r \neq 0$ (where $(\alpha_1, \dots, \alpha_r) \neq (0, \dots, 0)$ is assumed. Hence, by the intersection theorem for rank varieties, the same element acts freely on

$$St_1^{(p^{e_1})} \otimes \dots \otimes St_1^{(p^{e_s})}$$

if, and only if, for some j, $1 \le j \le s$,

$$\alpha_1 \lambda_1^{p^{e_j}} + \dots + \alpha_r \lambda_r^{p^{e_j}} \neq 0.$$

(That is, the element fails to be on the intersection of the rank varieties we have just calculated.) Hence, by Dade's theorem on shifted elements, the whole subgroup $\langle X(\lambda_1), \dots, X(\lambda_r) \rangle$ is free if, and only if, the s equations in r unknowns

$$\lambda_1^{p^{e_1}} x_1 + \dots + \lambda_r^{p^{e_1}} x_r = 0$$
$$\vdots$$
$$\lambda_1^{p^{e_s}} x_1 + \dots + \lambda_r^{p^{e_s}} x_r = 0$$

have only the trivial solution, that is, the corresponding matrix has full rank. (all of this over the field of q elements, of course).

Now we wish to turn our thoughts to generalizing the characterization of partial Steinberg modules for $SL(3,q)$ to bigger groups (including $SL(n,q)$, e.g.). Our first goal is a recognition lemma:

<u>Lemma</u> Let G be a finite group of Lie type of characteristic p and k an algebraically closed field of characteristic p. Let S and V be kG-modules with S simple and $\dim_k S = \dim_k V$. Furthermore, assume that whenever P is a parabolic subgroup of G with unipotent radical U then S^U and V^U are isomorphic kP-modules. It follows that $S \simeq V$.

<u>Remark</u>: S. Smith has proved a similar result where the hypothesis on $\dim_k V$ is replaced by a "generation" hypothesis, namely, that V is generated by all the subspaces V^U as U runs over all such radicals.

<u>Proof</u>. Let S' be a simple submodule for V. With the given notation, $S^U = V^U \supseteq S'^U \neq 0$ so $S^U \simeq S'^U$ as S^U is simple by S. Smith's theorem. We now shall prove that S, S' have the same Curtis-Richen weight in Curtis' sense (cf. end of his article in LNM #131). But it is easy to see this weight is determined by the largest parabolic subgroup, containing the Borel subgroup B, for which the S. Smith corresponding simple module is one-dimensional. For the unipotent radicals are included opposite to the inclusion of parabolics so fixed point spaces of unipotent radicals "go up."

Now, with J. Carlson's work on $SL(2,q)$ using complexity in mind, we can formulate a conjecture. Namely, a kG-module is a partial Steinberg (and say G is of type A, D or E if necessary) if

$$p^{C^*(V)} \dim V = |G|_p$$

where $C^*(V)$ is a new invariant, a variation of complexity – a sort of layered complexity.

Let's give an example of one of these new invariants: Say $Q \triangleright R$ are p-groups and M is a kR-module. Then

$$H^*(Q/R, H^*(R, M))$$

is a graded commutative algebra so has a growth! We can also use more subgroups and consider

$$H^*(Q/R, H^*(R/S, H^*(S,M))),$$

presumably the E^2 term of a triple complex spectral sequence!!

In our case above, perhaps we want to use filtrations of the Sylow p-subgroup by unipotent radicals of parabolic subgroups. In fact, let's propose a conjecture based on the (usual) complexity. First, a couple of definitions, the first "standard" from Mason's letters.

Def. A module V for an elementary abelian p-group E is semi-free if there is a subgroup F of E so that V_F is a free kF-module and $V^F = V^E$. In this case, the breadth of V is the dimension of V^F and $|F|$ is the depth of V (so $\dim_k V$ is their product).

Def A kG-module V is semi-free with respect to the collection $\mathcal{H}$ of subgroups of G if V^{H_2} is a semi-free H_1/H_2 module whenever $H_1, H_2 \in \mathcal{H}$, $H_2 \triangleleft H_1$ and H_1/H_2 is elementary abelian.

Conjecture. If V is a partial Steinberg module for $SL(n,q)$, $q=p^e$, the t fold tensor product of conjugates of St_1, then V is semi-free with respect to the radicals of parabolic subgroups. Moreover, if U_1, U_2 are such subgroups, $U_2 \lhd U_1$ and U_1/U_2 is elementary abelian, say of order q^w, then V^{U_2} is semi-free of depth p^{wt} for U_1/U_2.

Conversely, if V' is a $kSL(n,q)$-module, $q>p$, $\dim V' = p^{t\frac{n(n-1)}{2}}$ while V' has the semi-free properties of the preceding paragraph, then V' is a (t fold) partial Steinberg module.

The converse part seems to be easier! Namely, we should be able to use the above lemma and the first Alperin-Mason paper. For that paper should be enough to keep track of all the Curtis weights. We need to go between the usual Lie weights and the Curtis weights and can probably refer to Nelson Tinberg's paper, "Weights and admissible pairs," Comm. in Alg. v 12 (1984), 1257-1263, or proceed directly. I believe the parabolic belonging to St_1, the basic Steinberg, is the Borel subgroup (in the Curtis-Richen sense) and the linear character is easy to guess. Let

$$\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \in SL(n,q)$$

so we can determine the linear characters of B corresponding to the fundamental representations: V, $V\wedge V$, $V\wedge V\wedge V$, ... (where V is the standard module. We get λ_1 corresponds to α_n (i.e. the value on the diagonal matrix is α_n, λ_2 to $\alpha_{n-1}\alpha_n$ etc.) so St_1 corresponds to

$$((\alpha_2 \cdots \alpha_n)(\alpha_3 \cdots \alpha_n) \cdots \alpha_n)^{p-1},$$

that is

$$(\alpha_2^{1}\alpha_3^{2} \cdots \alpha_{n-1}^{n-2}\alpha_n^{n-1})^{p-1}.$$

The first part of the conjecture seems harder. But if we work in a "shifted" sense then things are much simpler. So let's make the definitions of shifted semi-free and shifted semi-free with respect to a collection of subgroups just as above where we allow F to be a shifted subgroup of kE. (Note: there are two senses of "shifted" which doesn't matter. Evens, in his book allows any image of a subgroup of E under any algebra automorphism of the algebra kE.) Then we have the Shifted Conjecture which seems more amenable. The second part should go as above since Carlson proved the shifted form of the theorem of characterization in the first Alperin-Mason paper.

Here we showed versus the connection with complexity.

Lemma If V is a kE-module for an elementary abelian group E of order p^r then V is shifted semi-free if, and only if,

$$p^{c_E(V)} \dim_k V = p^r \dim_k V^E.$$

Proof. Now there is a shifted subgroup F of order $p^{r-c_E(V)}$ by the usual theory, and $V^F \supseteq V^E$ so if the equality holds then V is certainly shifted semifree. On the other hand, suppose V is such for a subgroup F, $|F| = p^f$. Then the extension spectral sequence (of Lyndon-Hochschild-Serre) yields $H^*(E,V) \cong H^*(E/F, V^F)$ so $c_E(V) = r - f$ so $\dim V = p^{r-c_E(V)} \dim V^E$ as required. (Remember, $\dim V = |F| \dim V^F$.)

Now, we want to turn to the first part of the twisted conjecture. We shall describe what to do in the easiest case, that when we consider the unipotent radical of a maximal parabolic, which is then elementary abelian. The general case should go by induction and by consideration of direct products of special linear groups, we hope!

So we have a partial Steinberg module V of dimension $p^{t\left(\frac{n(n-1)}{2}\right)}$ the t fold tensor product. We let E be the abelian unipotent radical of order p^w, as just above. We have $q = p^e$. Our problem is to see that

$$C_E(V) = ew - tw.$$

We proceed by eliminating the possibilities that $C_E(V)$ is too small or too big. The previous lemma then means we are done. Now, if $C_E(V) < ew - tw$ then there is a shifted subgroup F, $|F| > p^{tw}$ which is free on V so $\dim V^F < p^{t\frac{n(n-1)}{2} - tw}$ which means V^E is no bigger and this is too small by Smith's theorem (presumably it gives a partial Steinberg, a t fold product, for $N(E)/E$). (This is O.K. e.g. if E is the "last column.")

Now, let's first finish all the argument when $t = 1$. We must eliminate the possibility that $C_E(V) > (e-1)w$. But then the same holds for the e conjugates of V and so the complexity of their tensor product is not zero, it is too big as tensor products have the intersection variety and this will be of positive dimension.

Now for general t, if $C_E(V) > ew - tw$ we can take the tensor product with the "remaining" $e-t$ basic Steinberg twists and use the fact that we know the dimension of the variety when $t = 1$ to again contradict the projectivity of the tensor product of all e twists of St_1.

This approach seems quite hopeful! Perhaps we can even calculate the complexity of St_1! We don't know off-hand what are the maximal elementary abelian p-subgroups of $SL(n,q)$ but the normal such subgroups (normal in a Sylow p-subgroup) are known by work of Weir. Hence, we guess that if $q = p^e$ as above then

$$C_{SL(n,q)}(St_1) = (e-1)[n^2+1/4].$$

Let's turn to the first part of our first (non-shifted) conjecture. Let E be a unipotent radical of order q^w as before, E a product of w root subgroups. We want to know the variety of the kE-module $(St_1)_E$. First, let's observe what is the variety of the kX-module $(St_1)_X$ where X is a root subgroup. By Smith's theorem, applied to the right Levi complement, $(St_1)_X$ is just the restriction of the $SL(2,q)$-module $V_p = St_1(SL(2,q))$ to X plus another summand. Hence, a non-free shifted element on V_p is still non-free on St_1. On the other hand, the Lin-Lin theorem implies that free on V_p is free on St_1 (using the connection between tensor products and intersections of varieties.

Now we can derive the result for any partial Steinberg again using the tensor product – intersection of varieties connection. This gives the Theorem B of Alperin-Mason. The variety is a subspace! Now to get the first part of the conjecture it would also be enough to see that the variety of St_1 on E is a subspace for then, by dimension considerations, it will be the subspace generated by the subspaces for each of the root subgroups. The linearity of the variety easily gives the conjecture as we can choose the desired subgroup F as a direct product of groups one from each root subgroup.

We turn to the question of semi-freeness (as opposed to shifted semi-freeness). Let R be an abelian radical for $SL(n,k)$, k algebraically closed of characteristic p, R the direct product of root groups $X_1, \dots, X_r$. Let $V = St_1^{(p^{e_1})} \otimes \dots \otimes St_1^{(p^{e_s})}$ where $0 < e_1 < \dots < e_s < e$ where $q = p^e$ and the finite group being studied is $SL(n,q)$. Let $t = rs$ and suppose

$$g_i = X_1(\lambda_{1i})\, X_2(\lambda_{2i}) \cdots X_r(\lambda_{ri}),$$

$1 \le i \le t$, are elements of $R(q) \le SL(n,q)$. We need to know necessary and sufficient conditions that $\langle g_1, \dots, g_t \rangle$ be of order p^t and act freely on V.

<u>Theorem</u>. If the rank variety of $R(q)$ on V is the product of the rank varieties of the $X_i(q)$ then the following two conditions are equivalent:

1) $\langle g_1, \dots, g_t \rangle$ is of order p^t and free on V;

2) The $t \times t$ matrix

$$\begin{pmatrix} \lambda_{11}^{p^{e_1}} & \lambda_{12}^{p^{e_1}} & \cdots & \lambda_{1t}^{p^{e_1}} \\ \vdots & & & \\ \lambda_{11}^{p^{e_s}} & \lambda_{12}^{p^{e_s}} & \cdots & \lambda_{1t}^{p^{e_s}} \\ \lambda_{21}^{p^{e_1}} & \lambda_{22}^{p^{e_2}} & \cdots & \lambda_{2t}^{p^{e_2}} \\ & & \vdots & \\ \lambda_{r1}^{p^{e_s}} & \lambda_{r2}^{p^{e_s}} & \cdots & \lambda_{rt}^{p^{e_s}} \end{pmatrix}$$

is non-singular.

For $g \in R(q)$ let $\bar{g} = g - 1 + \mathrm{rad}^2 kR(q)$, so $\overline{gh} = \bar{g} + \bar{h}$, as usual. Assume 1) holds. The freeness holds, by Dade's theorem on shifted elements, exactly when all the shifted elements

$$\alpha_1 \bar{g}_1 + \dots + \alpha_t \bar{g}_t$$

represent shifted elements which act freely on V (i.e. pull back modulo $\mathrm{rad}^2(kR(q))$ and add 1 to get a shifted element – in the broad sense of shifted element a la Evans' book) whenever $(\alpha_1, \dots, \alpha_t) \neq (0, \dots, 0)$ is in k^t. But

$$\bar{g}_i = \overline{X_1(\lambda_{1i})} + \dots + \overline{X_r(\lambda_{ri})}$$

so, by our results on root groups and by our hypothesis on direct products, this freeness is equivalent to not all

$$\alpha_1 \overline{X_j(\lambda_{j1})} + \alpha_2 \overline{X_j(\lambda_{j2})} + \dots + \alpha_t \overline{X_j(\lambda_{jt})}$$

being in the j-th variety (i.e. given $(\alpha_1, \dots \alpha_t)$ there is such a j). That is, by the first page of this section, exactly when not all the numbers

$$\alpha_1 \lambda_{j1}^{p^{e_1}} + \dots + \alpha_t \lambda_{jt}^{p^{e_1}}$$
$$\vdots$$
$$\alpha_1 \lambda_{j1}^{p^{e_s}} + \dots + \alpha_t \lambda_{jt}^{p^{e_s}}$$

are zero. Hence, we deduce that if 1) holds then the product of the matrix in 2) and the vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix}$ is not zero whenever some α_j is not zero. Thus 2) holds.

Conversely, all goes right through, except we must see that the condition 2) implies that the order of $\langle g_1, \dots g_t \rangle$ is p^t and not smaller. Hence, suppose

$$g_1^{a_1} \cdots g_t^{a_t} = 1,$$

a_i in the field of p elements. We wish to deduce that $a_1 = \dots = a_t = 0$. We do have immediately that

$$a_1 \bar{g}_1 + \dots + a_t \bar{g}_t = 0$$

so that for each i, $1 \le i \le r$

$$a_1 \lambda_{i1} + \dots + a_t \lambda_{it} = 0$$

Since each a_j is in the field of p elements this implies that

$$a_1 \lambda_{i1}^{p^{e_u}} + \dots + a_t \lambda_{it}^{p^{e_u}}$$

whenever $1 \le u \le s$. Hence, $\begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix}$ is in the kernel of the matrix in 2) and we have the desired conclusion.

so that for each i, $1 \le i \le r$

$$a_1 \lambda_{i1} + \cdots + a_t \lambda_{it} = 0$$

Since each a_j is in the field of p elements this implies that

$$a_1 \lambda_{i1}^{p^{e_u}} + \cdots + a_t \lambda_{it}^{p^{e_u}}$$

whenever $1 \le u \le s$. Hence, $\begin{pmatrix} a_1 \\ \vdots \\ a_t \end{pmatrix}$ is in the kernel of the matrix in 2) and we have the desired conclusion.

Next, we turn to the question of the rank variety in the case $p = 2$, $n = 3$. Let

$$X = X_{13} = \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} \qquad Y = X_{23} = \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix}$$

and $R = \langle X, Y \rangle$ a "radical." Let

$$g = X(\lambda) Y(\mu) = \begin{pmatrix} 1 & & \lambda \\ & 1 & \mu \\ & & 1 \end{pmatrix}$$

The action of g on $V \otimes V^*$ is as follows (remember $p = 2$):

$$\begin{pmatrix} 1 & 0 & \lambda \\ & 1 & \mu \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ 0 & 1 & \\ \lambda & \mu & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & & & & \lambda & & \\ 0 & 1 & 0 & & & & & \lambda & \\ \lambda & \mu & 0 & & & & \lambda^2 & \lambda\mu & \lambda \\ & & & 1 & 0 & 0 & \mu & & \\ & & & 0 & 1 & 0 & & \mu & \\ & & & \lambda & \mu & 1 & \lambda\mu & \mu^2 & \mu \\ & & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & 0 \\ & & & & & & \lambda & \mu & 1 \end{pmatrix}$$

Hence, $\sum \alpha_i (X(\lambda_i) - 1) + \sum \beta_j (Y(\mu_j) - 1)$ is represented by (with $L = \sum \alpha_i \lambda_i$, $M = \sum \beta_j \mu_j$)

$$\begin{pmatrix} 0 & 0 & 0 & & & & L & & \\ 0 & 0 & 0 & & & & & L & \\ L & M & 0 & & & & * & * & L \\ & & & 0 & 0 & 0 & M & & \\ & & & 0 & 0 & 0 & & M & \\ & & & L & M & 0 & * & * & M \\ & & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & L & M & 0 \end{pmatrix}$$

We want: this matrix has rank < 4 if, and only if, $L = M = 0$. For $V \otimes V^* \cong St_1 \oplus k$ (is eight plus one-dimensional) so this condition of rank less than four is the condition for being on the variety. But if $L = M = 0$ this is clear while if $L \neq 0$ or $M \neq 0$ it is easy to get four linearly independent rows.

Conclusion:

Prop The rank varieties are as conjectured in $SL(3, 2^e)$.