Research notes

Volume $\underline{IX}$

Contents

A question of Parlis

The question is this: does the homomorphism of Σ_3 to Σ_6, given by the regular representation, extend to a homomorphism of $GL(3,2)$ to $GL(6,2)$. The answer is no. Let k be the field of two elements. The algebra $k\Sigma_3$ has two simple modules: the trivial one-dimensional module k; a two-dimensional module W. The six-dimensional module given by the regular representation of Σ_3 is just $k\Sigma_3$ and this has composition factors k, k, W, W.

The algebra $kGL(3,2)$ has four simple modules: the trivial one-dimensional module k; the 'standard' module V, three-dimensional; the dual module V^* where $GL(3,2)$ acts on the standard vector space by inverse transpose; an eight dimensional 'Steinberg' module. If the answer to the question were yes then $kGL(3,2)$ would have a six-dimensional module whose restriction to $k\Sigma_3$ had composition factors k, k, W, W – and, moreover, would only have k as a submodule once, by inspection of the $k\Sigma_3$-module $k\Sigma_3$. Let M be such a six-dimensional module. If M has k as a composition factor then it occurs at least three times so the same would hold on restriction to $k\Sigma_3$, which is not the case. Hence, the only modules that can occur as composition factors of M are V and V^*. The only possibilities are the following:

$$V \oplus V,\quad V \oplus V^*,\quad V^* \oplus V^*,\quad U,\quad U^*.$$

Here

$$0 \to V^* \to U \to V \to 0$$

and

$$0 \to V \to U^* \to V^* \to 0$$

are non-split extensions.

Now the restriction of V and V^* to $k\Sigma_3$ are equal and are $k \oplus W$. Hence, in the first three cases, if M were isomorphic to any of these, we would have $k \oplus k$ as a submodule of the restriction of M to $k\Sigma_3$, which is not so. Let's see that $M \simeq U$ is not possible (and $M \simeq U^*$ can be dealt with similarly) so our claim will be proved.

However, $GL(3,2)$ permutes the seven non-zero vectors of the standard vector space and using these as a basis of a module gives us $k \oplus U$. It is easy to look at this permutation action restricted to Σ_3 and see that as a module this is $k \oplus k \oplus k \oplus V \oplus V$. Hence, the restriction of U to $k\Sigma_3$ is isomorphic with $k \oplus k \oplus V \oplus V$.

Invariant forms on G-algebras

Let A be an algebra (with unit) over the field k. Let $\mathcal{F}$ be the k-vector space of commutative, associative bilinear forms on A so, as we saw in volume $\overline{VII}$, p. 26,

$$\mathrm{Hom}_{A\otimes A^o}(A, A^*) \simeq \mathcal{F}$$

where $A^* = D(A) = \mathrm{Hom}_k(A, k)$ and the form $(\, , \,)$ is mapped to the homomorphism sending each $a \in A$ to $\varphi_a = (a, \,)$ in A^*.

Now suppose that A is a G-algebra so we have an induced action of G on $\mathcal{F}$: $g(\, , \,) = (g^{-1}(\,), g^{-1}(\,))$. Suppose that t is the associated linear functional for $(\, , \,)$ so $t(a) = (a, 1)$ and $(a, b) = t(ab)$. Then gt is associated with the image of $(\, , \,)$ under g. Indeed, $(gt)(ab) = (gt)(ba)$ as $(gt)(ab) = t(g^{-1}(a)g^{-1}(b)) = t(g^{-1}(b)g^{-1}(a)) = gt(ba)$. Moreover, gt defines a bilinear form $[\;\;]$ by $[a, b] = (gt)(ab) = t(g^{-1}(a)g^{-1}(b)) = (g^{-1}a, g^{-1}b)$. Thus, the invariant forms correspond to the linear functionals t with $t(ga) = t(a)$ for all $a \in A$, $g \in G$. Our goal is a description of the invariant forms; we denote these by $\mathcal{F}^G$ (fixed-point notation). (Note: now drop assumption $1 \in A$.)

Since A is a G-algebra so is $A \otimes A^o$ so we can form the algebra $(A\otimes A^o)G$, where if $a, b, c, d \in A$, $g, h \in G$,

$$(a\otimes b)g \cdot (c \otimes d)h = (a\, g(c) \otimes g(d)\, b)\, gh.$$

Now A is a $(A\otimes A^o)G$-module, extending the action of $A \otimes A^o$ and G; indeed, let

$$(a \otimes b)g \cdot \alpha = a\, g(\alpha)\, b$$

so that

$$((a\otimes b)g\ .\ (c\otimes d)h)\,\alpha = (a\,g(c)\otimes g(d)b)\,gh\,\alpha$$
$$= a\,g(c)\,gh(\alpha)\,g(d)\,b$$
$$(a\otimes b)g\ .\ ((c\otimes d)h\ .\ \alpha) = a\,g(c\,h(\alpha)\,d)\,b$$
$$= a\,g(c)\,g(h(\alpha))\,g(d)\,b.$$

Moreover, A^* is also a $(A\otimes A^\circ)G$-module. Indeed, if $g\in G$, $\varphi\in A^*$ then $(g\varphi)(a) = \varphi(g^{-1}a)$, while

$$((a\otimes b)\varphi)(\alpha) = \varphi(b\alpha a)$$

is alright as

$$((a\otimes b\ .\ c\otimes d)\varphi)(\alpha) = (ac\otimes db\ .\ \varphi)(\alpha)$$
$$= \varphi(db\alpha ac)$$
$$(a\otimes b\ ((c\otimes d)\varphi))(\alpha) = (c\otimes d)\varphi(b\alpha a)$$
$$= \varphi(db\alpha ac).$$

Checking these define an action of $(A\otimes A^\circ)G$ we have

$$((a\otimes b)g)(\varphi)\alpha = ((a\otimes b)(g\varphi))\alpha$$
$$= g\varphi(b\alpha a)$$
$$= \varphi(g^{-1}(b)\,g^{-1}(\alpha)\,g^{-1}(a))$$

so

$$(((a\otimes b)g\ .\ (c\otimes d)h)\varphi)\alpha = (((a\,g(c)\otimes g(d)b)gh)\varphi)\alpha$$
$$= \varphi(h^{-1}(d)\,h^{-1}g^{-1}(b)\,h^{-1}g^{-1}(\alpha)\,h^{-1}g^{-1}(a)\,h^{-1}(c))$$

while

$$((a\otimes b)g(((c\otimes d)h)\varphi))\alpha$$
$$= (((c\otimes d)h)\varphi)(g^{-1}(b)\,g^{-1}(\alpha)\,g^{-1}(a))$$
$$= \varphi(h^{-1}(d)\,h^{-1}g^{-1}(b)\,h^{-1}g^{-1}(\alpha)\,h^{-1}g^{-1}(a)\,h^{-1}(c)).$$

We can now state a result and a consequence:

<u>Prop.</u> $\mathcal{F} \simeq \mathrm{Hom}_{A\otimes A^\circ}(A, A^*)$ as kG-modules

<u>Cor</u> $\mathcal{F}^G \simeq \mathrm{Hom}_{(A\otimes A^\circ)G}(A, A^*)$.

First of all, the above discussions show that we do have actions of G on $\mathcal{F}$ and on $\operatorname{Hom}_{A\otimes A^\circ}(A, A^*)$. Indeed, the first action was explicitly given. The second action is also a formality. In fact, if R is a G-ring and U, V are RG-modules then $\operatorname{Hom}_R(U,V)$ is invariant under G: if $\rho \in \operatorname{Hom}_R(U,V)$ then $g\rho \in \operatorname{Hom}_R(U,V)$ where $g\rho(-) = g(\rho(g^{-1}-))$. Indeed, if $r \in R$, $u \in U$,

$$\begin{aligned} r((g\rho)(u)) &= r(g(\rho(g^{-1}u))) \\ &= g\, g^{-1}(r)\, \rho(g^{-1}u) \\ &= g\, \rho(g^{-1}(r)\, g^{-1}u) \\ &= g\, \rho(g^{-1} r u) \\ &= (g\rho)(ru). \end{aligned}$$

The fixed-points are just the maps that commute with the elements of G. Hence, the Corollary is immediate once the Proposition is demonstrated.

<u>Proof</u>. Let θ be the isomorphism of $\mathcal{F}$ onto $\operatorname{Hom}_{A\otimes A^\circ}(A, A^*)$ defined by the rule that θ maps $F \in \mathcal{F}$ to $\theta_F \in \operatorname{Hom}_{A\otimes A^\circ}(A, A^*)$ where $\theta_F(a) = F(a, -)$. We have

$$\begin{aligned} \theta_{gF}(a) &= g\,F(a, -) \\ &= F(g^{-1}(a), g^{-1}(-)) \\ (g\,\theta_F)(a) &= g(\theta_F(g^{-1}(a))) \\ &= g\; F(g^{-1}(a), -) \\ &= F(g^{-1}(a), g^{-1}(-)) \end{aligned}$$

by the action of G on A^* as $F(g^{-1}(a), -) \in A^*$.

Correction to "Projective Complexes" (vol VIII)

On page 81 we assert there is a backwards map from M_0 to M_1 on line 9. The diagram preceding the statement needs to be enlarged to give the explanation. It should be

$$\begin{tikzcd}
M_1 \arrow[r, "\partial"] \arrow[d, "\partial"] & M_0 \arrow[d, "1"] \\
M_0 \arrow[r, "1"] & M_0 \\
M_1 \arrow[u, "\partial"] \arrow[r, "\partial"] & M_1 \arrow[u]
\end{tikzcd}$$

so the backwards map comes from the projectivity of the 1-complex used on the first column on the right.

Weights for Lie-type groups

As a consequence of results in his thesis, Cabanes verified my weight conjecture for groups of Lie type in the natural characteristic (and implicitly for the block-by-block version also). Let's give a short proof. Let G be a finite group of Lie type and characteristic p, $B \supseteq U$ as usual and k an algebraically closed field of characteristic p. We shall establish a sequence of inequalities.

First, where "weight" means projective weight, we have

(1) $\quad$ # weights $\leq$ # summands in a decomposition of $(k_U)^G$ into indecomposable modules.

Indeed, if W is a weight and V is the kG-module which is the Green correspondent of W then $V \mid (k_U)^G$ (we proved this before for arbitrary finite groups G). Next,

(2) $\quad$ # summands in $(k_U)^G \leq$ # simple kG-modules.

For, if S is a simple kG-module then by the Curtis-Richen theory, $\dim_k \mathrm{Hom}_{kU}(k, S_U) = 1$, so $\dim_k \mathrm{Hom}_{kG}((k_U)^G, S) = 1$, so $(k_U)^G/\mathrm{rad}((k_U)^G)$ is multiplicity-free and involves each simple kG-module. Now next, if S is still any simple kG-module then it has a Lie weight (see, e.g., Carter-Lusztig, PLMS 32 (1976) 374-84) which is a pair (X, J) where X is a linear character of B (i.e. one-dimensional kB-module) and J is a subset of the fundamental reflections. This arises by letting S_0 be the fixed-point space of U on S, so S_0 is one-dimensional, and X is the action of B on S_0 and the parabolic P_J containing B and corresponding to J stabilizes S_0 so χ extends to P_J. Hence, we have a one-to-one map of simple kG-modules to linear characters of parabolics (as P_J determines J and the restriction, to B, of the extension of χ is χ again, of course). Hence,

(3) $\qquad$ # simple kG-modules $\leq$ # linear characters of parabolic subgroups.

Now, if P is a parabolic subgroup, R is its unipotent radical (i.e. $R = O_p(P)$) then P/R is of Lie type so it has a Steinberg module St_p. This is simple and projective so as a kP-module is a weight. Moreover, if λ is a linear character then $St \otimes \lambda$ is another distinct one, as the character of St does not vanish on any p'-element. Hence,

(4) $\qquad$ # linear characters of parabolics $\leq$ # weights.

Thus, the string of inequalities, beginning and ending with the # weights, forces that all the inequalities are equalities. We deduce a number of conclusions:

(a) The number of weights equals the number of simple modules.

(b) The number of weights equals the number of linear characters of parabolic subgroups.

(c) There is a (natural) one-to-one correspondence between simple kG-modules and linear characters of parabolic subgroups.

(d) The indecomposable summands of $(k_U)^G$ occur with multiplicity one, have simple tops (+ dually simple socles) and this sets up a one-to-one correspondence with the simple kG-modules

Relative resolutions

R. Knox has told me that he has established the existence of relative projective covers and minimal relative resolutions. We shall prove that here, possibly proving more than Knox did; we don't have any details of his work.

Let us begin with the

<u>Lemma</u> A summand of a contractible complex is contractible.

So we have a complex of modules with the usual boundaries
$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$
and there are homomorphisms (of abelian groups)
$$\leftarrow C_{n+1} \xleftarrow{s_n} C_n \xleftarrow{s_{n-1}} C_{n-1} \leftarrow$$
such that
$$1_{C_n} = s_{n-1} \circ \partial_n + \partial_{n+1} \circ s_n$$
for each n $(\in \mathbb{Z})$.

Now suppose we have a summand, so we have projections, injections and the restriction of ∂ as follows:

$$\begin{tikzcd}
C_{n+1} \arrow[r, shift left, "\partial_{n+1}"] \arrow[d, shift right, "\pi_{n+1}"'] & C_n \arrow[l, shift left, "s_n"] \arrow[r, shift left, "\partial_n"] \arrow[d, shift right, "\pi_n"'] & C_{n-1} \arrow[l, shift left, "s_{n-1}"] \arrow[d, shift right, "\pi_{n-1}"'] \\
D_{n+1} \arrow[r, "\partial_{n+1}'"] \arrow[u, shift right, "i_{n+1}"'] & D_n \arrow[r, "\partial'"] \arrow[u, shift right, "i_n"'] & D_{n-1} \arrow[u, shift right, "i_{n-1}"']
\end{tikzcd}$$

Define $t_n = \pi_{n+1} s_n i_n$, we claim that
$$1_{D_n} = t_{n-1} \partial_n' + \partial_{n+1}' t_n .$$

Hence, it suffices to show that

$$i_n = i_n t_{n-1} \partial_n' + i_n \partial_{n+1}' t_n .$$

But

$$RHS = i_n \pi_n s_{n-1} i_{n-1} \partial_n' + i_n \partial_{n+1}' \pi_{n+1} s_n i_n$$

and

$$\begin{aligned} i_n \partial_{n+1}' \pi_{n+1} s_n i_n &= i_n \pi_n \partial_{n+1} s_n i_n \\ &= i_n \pi_n (1_{C_n} - s_{n-1} \partial_n) i_n \\ &= i_n \pi_n i_n - i_n \pi_n s_{n-1} \partial_n i_n \\ &= i_n \pi_n i_n - i_n \pi_n s_{n-1} i_{n-1} \partial_n' \end{aligned}$$

so substituting in the equation for RHS, we get $RHS = i_n \pi_n i_n = i_n$ as required.

Of course, we are interested in relativization so we fix a subgroup H of a group G and a coefficient field k. If M is a kG-module then a relative projective resolution (everything "relative" is with respect to H) of M is a complex, where each P_i is relatively projective,

$$\cdots \to P_1 \to P_0 \to M \to 0$$

which is exact and which is H-split, that is, the map of each P_i to $\partial P_i \subseteq P_{i-1}$ (setting $M = P_{-1}$) splits as a kH-homomorphism.

Now we can obviously define H-homotopies and H-contractible (better: relative homotopy, relative contractible) so the previous lemma holds in the obviously extended form. And then, the above complex is H-contractible, that is, relatively contractible.

Our main result is as follows.

Proposition If

$$P: \quad \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

and

$$Q: \quad \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \to 0$$

are relative projective resolutions of the kG-module M then each is a direct sum of a relative projective resolution of M and a relative projective resolution of 0 so that the first summands are isomorphic.

Note that, as a consequence, we have the existence of relative projective covers. For if $R_0 \to M \to 0$ is H-split, exact and R_0 is relatively projective then we can expand this sequence to a resolution.

Proof. Let f be a chain map of P to Q extending the identity map of M and let g be a chain map of Q to P also extending the identity map of M. Let $(fg)^\infty Q$ and $(gf)^\infty P$ be the 'eventual' images under powers of fg and gf – this makes sense as the images under powers stabilize at each dimension after a finite number of steps. These clearly give the desired decomposition and the lemma applies. (Other summand can be characterized at dimension n by kernels of $(fg)^r, (gf)^r$ for large r.)

On a remark of Robinson

Let R be the usual type of ring.

<u>Proposition</u>. If M is an RG-lattice with character χ and $\chi = \chi_1 + \dots + \chi_s$ where each χ_i is an irreducible character and no two χ_i and χ_j have a common composition factor modulo p then

$$M = M_1 + \dots + M_s$$

is a direct sum of RG-lattices where M_i has character χ_i.

D. Robinson derived the same conclusion assuming that there was a p'-subgroup H such that the character of M_H, that is χ_H, was multiplicity free.

<u>Proof</u>. There is, as M has character χ, a submodule N of M, of finite colength, such that

$$N = N_1 + \dots + N_s$$

is a direct sum of RG-modules and N_i has character χ_i. Moreover, M is isomorphic with a submodule of N of finite colength in N. However, our hypothesis readily implies that each submodule of N of finite colength is the direct sum of a submodule of N_1, a submodule of N_2, and so on.

Blocks with one simple and normal defect group

We assume D is the defect group of the block B of G, D is normal and $\ell(B) = 1$. We want to relate the structure of D and B. First, let's use Külshammer's reduction (Comm. Alg 13 (1985), 147–168). We are, as usual, over an algebraically closed field k of characteristic p and we shall translate his crossed product back to a group-theoretic situation. Hence, we have $G = D.\hat{Q}$ where $\hat{Q}$ is a p'-group, $D\,C(D) = D\,Z$, $Z = \hat{Q} \cap C(D)$ where Z is a cyclic central subgroup of $\hat{Q}$. We also have a faithful one-dimensional module θ for Z corresponding with B such that there is a unique simple module T for $\hat{Q}$ (& hence for G) "over" θ.

$$\begin{tikzcd}[column sep=small]
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
D \arrow[dr, dash] & & \hat{Q} \;\; T \arrow[dl, dash] \\
 & 1 & Z \;\; \theta \arrow[l, dash]
\end{tikzcd}$$

Hence, since $\hat{Q}$ is a p'-group, we have that $\hat{Q}$ is of "central type," a situation first studied by DeMeyer and Janusz and recently by Howlett and Isaacs (in the mod Z when they proved $Q = \hat{Q}/Z$ is solvable, using the classification of simple groups.)

We shall examine a special case: henceforth we assume Q is <u>abelian</u>. This is because Q acts on D and there are special things that can be done with group actions in the abelian case that have not been generalized to the non-abelian case. We set Q^* to be the dual group of Q, $Q^* = \mathrm{Hom}(Q, k^*)$. Now Q acts on kD: if $g \in Q$, $v \in kD$ then $g v g^{-1} \in kD$. Hence, kD is the direct sum of "eigenspaces" kD_λ, $\lambda \in Q^*$.

Now, let P_T be the projective cover of T; our aim is to describe the endomorphism algebra of P_T and so determine B up to Morita equivalence. As a $k\hat{Q}$-module, we have a direct sum, $P_T = T + $ and P_T so, as $P_T = kG.T$, we get $P_T = kD.T$ (since $k\hat{Q}.T = T$). But dim $P_T = |D|$ dim T, by Alperin - Collins - Sibley, so $P_T = kD.T$ is a tensor product at the vector space level.

Suppose that $v \in kD$ is an eigenvector for $\lambda \in Q^*$. It follows that vT is a $k\hat{Q}$-module, indeed, if $t \in T$, $\xi \in \hat{Q}$,

$$g . vt = gvg^{-1}. gt = \lambda(\xi)\, v . \xi t$$

so $vT \simeq \lambda \otimes T$ as $k\hat{Q}$-module. But $\lambda \otimes T \simeq T$ (central type remember) so there is a linear transformation L_λ of T such that for all q, t

$$L_\lambda q L_\lambda^{-1} . t = \lambda(q)\, \xi\, t .$$

That is, L_λ is a "λ-homomorphism," $L_\lambda(\xi t) = \lambda(q)\, g\, L_\lambda(t)$.
Next, define a linear transformation $\mathcal{E}_v$ of P_T by

$$\mathcal{E}_v(\alpha t) = \alpha v L_\lambda(t)$$

for $\alpha \in kD$, $t \in T$; the bilinearity means this is well-defined. We claim that $\mathcal{E}_v$ is an endomorphism of the kG-module P_T. Indeed, if $\beta \in kD$ then

$$\begin{aligned}
\mathcal{E}_v(\beta . \alpha t) &= \mathcal{E}_v(\beta\alpha . t) \\
&= \beta\alpha . v . L_\lambda(t) \\
&= \beta . \alpha v L_\lambda(t) \\
&= \beta\, \mathcal{E}_v(\alpha t)
\end{aligned}$$

while if $q \in Q$ then

$$\begin{aligned}
\mathcal{E}_v(q . \alpha t) &= \mathcal{E}_v(\xi\alpha\xi^{-1} . \xi t) \\
&= \xi\alpha\xi^{-1} . v . L_\lambda(\xi t)
\end{aligned}$$

$$= g \alpha g^{-1} v \, \lambda(g) \, g \, L_\lambda(t)$$
$$= g\alpha \, . \, g^{-1} v g \, . \, \lambda(g) \, L_\lambda(t)$$
$$= g\alpha \, . \, \lambda(g)^{-1} v \, \lambda(g) \, L_\lambda(t)$$
$$= g \, . \, \alpha \, v \, L_\lambda(t)$$
$$= g \, . \, \varepsilon_v(\alpha t) \, .$$

This gives the endomorphism algebra, in the following sense. If $v_1, \ldots v_n$ are linearly independent eigenvectors then $\varepsilon_{v_1}, \ldots, \varepsilon_{v_n}$ are linearly independent endomorphisms. Indeed, suppose that there are scalars $\mu_1, \ldots, \mu_n$ such that

$$\mu_1 \varepsilon_{v_1} + \cdots + \mu_n \varepsilon_{v_n} = 0 .$$

Hence, if $t \in T$, then if v_i is an eigenvector for $\lambda_i \in Q^*$,

$$\mu_1 v_1 L_{\lambda_1}(t) + \cdots + \mu_n v_n L_{\lambda_n}(t) = 0 .$$

But the spaces $v_1 T, \ldots, v_n T$ are linearly independent, as $P_T \cong kD \otimes T$ as vector spaces and the v_i are linearly independent, so all $\mu_i = 0$ as $v_i \neq 0$ by choice and $L_{\lambda_i}(t) \neq 0$ if $t \neq 0$. It remains now to get the ring structure of the endomorphism algebra.

<u>Lemma</u>. If $\lambda \in Q^*$ then there is $x \in Q$ such that $\lambda(g) = \theta([x, g])$ for all $g \in Q$.

Of course this makes sense by taking pre-images in $\hat{Q}$ and calculating the commutator.

<u>Proof</u>. Let gZ be a cyclic direct factor of order a power of the prime r of $Q = \hat{Q}/Z$; it suffices to find $h \in \hat{Q}$ such that $[g, h]$ has order equal to the order of gZ

since θ is faithful on Z. However, let $q_1 \in \hat{Q}$ be such that $\langle q_1 Z\rangle = \Omega_1(\langle q Z\rangle)$. The central-type condition implies that $q_1 \notin Z(\hat{Q}) = Z$ so there is $h \in \hat{Q}$ such that $[q_1, h] \neq 1$. Hence, commutation with h defines a homomorphism of $\langle q Z\rangle$ into Z which is now proved to be one-to-one.

Next, let $\lambda \in Q^*$ and choose $q \in \hat{Q}$ so that we have $\lambda(\ ) = \theta([q_\lambda,\ ])$. Let $\tau(q_\lambda)$ be the linear transformation of T induced by q_λ.

<u>Lemma.</u> $\tau(q_\lambda)$ is a "λ-homomorphism."

Proof. We are using left modules so are commutators are "left" in that $[a, b] = aba^{-1}b^{-1}$. Hence,

$$\begin{aligned}\lambda(q)\, q\tau(q_\lambda)\, t &= \lambda(q)\, q\, q_\lambda\, t \\ &= \theta([q_\lambda, q])\, q\, q_\lambda\, t \\ &= [q_\lambda, q]\, q\, q_\lambda\, t \\ &= q_\lambda\, q\, t \\ &= \tau(q_\lambda)\, q\, t\end{aligned}$$

as desired.

Hence, in our previous construction we set $L_\lambda = \tau(q_\lambda)$ where for each λ we select an element q_λ once and for all.

Now let f be the cocycle of the extension $\hat{Q}$ (i.e. Z by Q) so f is a function of two variables and we shall also denote by $\hat{f}$ this function inflated to $\hat{Q}$.

We can choose coset representatives for Z in $\hat{Q}$ by choosing the elements q_λ. Hence $q_\lambda q_\mu = \hat{f}(q_\lambda, q_\mu)\, q_{\lambda\mu}$ as $q_{\lambda\mu}$ is the representative in the coset containing $q_\lambda q_\mu$ since $[q_\lambda q_\mu, \;] = [q_\lambda, \;][q_\mu, \;]$. Setting $\gamma_{\lambda,\mu}$ to be this cocycle viewed now on Q^* with values in k^*, we have

$$L_\lambda L_\mu = \gamma_{\lambda,\mu} L_{\lambda\mu}$$

since

$$\begin{aligned} L_\lambda L_\mu t &= q_\lambda q_\mu t \\ &= \gamma_{\lambda,\mu} q_{\lambda\mu} t \\ &= \gamma_{\lambda,\mu} L_{\lambda\mu} t \end{aligned}$$

It's easy to check that $\gamma_{\lambda\mu}$ is a cocycle on Q^*.

We can calculate now the multiplication in the endomorphism algebra.

<u>Lemma</u>. If v is an "eigenvector for λ and w is one for μ then

$$\varepsilon_v \varepsilon_w = \gamma_{\lambda,\mu} \varepsilon_{wv}$$

<u>Proof</u>. We calculate, for $\alpha \in kD$, $t \in T$,

$$\begin{aligned} \varepsilon_v \varepsilon_w (\alpha t) &= \varepsilon_v . \alpha w L_\mu(t) \\ &= \alpha w v L_\lambda L_\mu(t) \\ &= \gamma_{\lambda,\mu} \alpha w v L_{\lambda\mu}(t) \\ &= \gamma_{\lambda\mu} \varepsilon_{wv}(\alpha t). \end{aligned}$$

Now we can similarly "twist" the group algebra kD by the same cocycle $z_{\lambda,\mu}$ using the action of Q. We have the

Theorem. The block B is isomorphic with a matrix algebra over the group algebra kD twisted by γ.

Proof. We need only show the algebra opposite to the endomorphism algebra is this twist of kD. But

$$\varepsilon_w \varepsilon_v = \gamma_{\mu,\lambda}\, \varepsilon_{vw}$$

and $\gamma_{\mu,\lambda} = \gamma_{\lambda,\mu}$ as Q is abelian.

A different sort of result, in a more general situation was obtained by R. W. Sharpe, "The modular group algebra of certain finite groups with normal Sylow p-subgroups," J. Algebra 81 (1983), 403-19.

Let's do an example. Set $p = 2$, $D = Z_p \times Z_p = \langle x, y\rangle$, $\hat{Q} = Q_8 = \langle q, r\rangle$ so that q inverts x, centralizes y while r centralizes x and inverts y. Choose coset representatives q, r, qr & 1 so it follows that $\gamma_{\lambda,\mu} = -1$ where $\lambda(q) = -1$, $\lambda(r) = 1$, $\mu(q) = 1$, $\mu(r) = -1$. Indeed, $q_\lambda = r$, $q_\mu = q$ and $rq = z \cdot qr$ where $\langle z\rangle = Z(Q_8)$ and $\theta(z) = -1$. Now $x - x^{-1}$ is an eigenvector for λ and $y - y^{-1}$ is for μ. Set $X = x - x^{-1} = (x^2-1)x^{-1}$, $Y = y - y^{-1} = (y^2-1)y^{-1}$ so kD is a truncated polynomial algebra in $X + Y$. Hence, we get that B is isomorphic with 2×2 matrices over the algebra in non-commuting variables $\mathfrak{X}$ and $\mathfrak{Y}$, truncated as above, with $\mathfrak{X}\mathfrak{Y} = -\mathfrak{Y}\mathfrak{X}$. Indeed, the new product of X and Y is $-XY$ while the new product of Y and X is YX as qr is a coset representative.

Of course, one can check (as was the original argument) that the maps

$$\alpha t \to \alpha (x^{-1}-x) r\, t$$
$$\alpha t \to \alpha (y^{-1}-y) g\, t$$

are endomorphisms satisfying the right relations and so forth.

We just want to mention an example of a non-abelian central quotient for a group of central type even though this is only a triviality compared to what is known. Take the semi-direct product of a Sylow 2-subgroup of $GL(3,4)$ by a field automorphism of order two.

One final observation – originally made by DeMeyer + Janusz. If G is a group of central type then so is every Sylow subgroup. Indeed, let χ be the irreducible character defining G so it is faithful, vanishes off the center Z of G. Let P be a Sylow p-subgroup of G. Now $\chi(1)^2 = |G:Z|$ as a calculation of $(\chi,\chi)=1$ shows. Let φ be the restriction of $(1/\chi(1)_{p'})\chi$ to P. We can calculate that $(\varphi,\varphi)=1$:

$$(\varphi,\varphi) = \frac{1}{|P|} \sum_{z \in P \cap Z} \left| 1/\chi(1)_{p'}\, \chi(z) \right|^2$$

$$= \frac{1}{|P|} \frac{\chi(1)^2}{\chi(1)_{p'}^2} |P \cap Z|$$

$$= \frac{1}{|G|_p} \chi(1)_p^2 \, |P \cap Z|$$

$$= 1.$$

Hence, we need only show that φ is a generalized character as then P is of central type (if $P \cap Z = 1$ we get $P = 1$).

But since $\chi(1)_{p'}\varphi$ is a generalized character we need only prove that $|P|\varphi$ is a generalized character since $(\chi(1)_{p'}, |P|) = 1$. However φ is a generalized character on all cyclic subgroups of P. Hence, if we compute $|P|(\varphi, \varphi_i)$ for any irreducible character φ_i of P we get an integer as it is rational and an algebraic integer.

Now one easy remark on the general construction of twisting an algebra A, graded by G, by a 2-cocycle $\in Z^2(G, k^*)$. This depends only on the cohomology class. Indeed, say $a_g \in A_g$, $a_h \in A_h$ and $z_x \in C^1(G, k^*)$. Use $*$ for the new multiplication. Then

$$z_g a_g * z_h a_h = z_g z_h f_{g,h} a_g a_h$$
$$= z_g z_h z_{gh}^{-1} f_{g,h} (z_{gh} a_g a_h)$$

so the algebra is isomorphic with the twist by the product of f and the coboundary of z, as is easy to see.

Symmetric algebras and Hochschild cohomology

We fix a symmetric algebra A, with defining form $(\ ,\ )$. We note that A° and $A\otimes A^\circ$ are symmetric algebras also. Indeed, if the multiplication in A° is denoted by "$\circ$" then

$$(a\circ b, c) = (ba, c) = (c, ba) = (cb, a) = (a, cb) = (a, b\circ c)$$

and the other required properties are immediate. Likewise, if B is another symmetric algebra then so is $A\otimes B$ via, with the obvious notation

$$(a_1\otimes b_1, a_2\otimes b_2) = (a_1, a_2)(b_1, b_2)$$

as is easily verified.

We also have that $A\otimes A$ is an $A\otimes A^\circ$-module, in fact a free module via

$$a\otimes b.\ c\otimes d = ac\otimes db.$$

Moreover, as usual A is an $A\otimes A^\circ$-module via $a\otimes b.c = acb$. Let π be the linear transformation of $A\otimes A$ to A with $\pi(a\otimes b) = ba$. (This is not an $A\otimes A^\circ$-homomorphism as $\pi(a\otimes b.c\otimes d) = \pi(ac\otimes db) = dbac$ and we have that $a\otimes b.\pi(c\otimes d) = a\otimes b.dc = adcb$; the map $\rho : A\otimes A\to A$ via $\rho(a\otimes b) = ab$ is an $A^\circ\otimes A$-homomorphism since $\rho(a\otimes b.c\otimes d) = \rho(ac\otimes db) = acdb$ while we have that $a\otimes b.\rho(c\otimes d) = acdb$.) Let π^* be the adjoint linear transformation. Then π^* <u>is</u> an $A\otimes A^\circ$-homomorphism. Indeed,

$$\begin{aligned}(\pi^*(a\otimes b.c), x\otimes y) &= (\pi^*(acb), x\otimes y)\\ &= (acb, \pi(x\otimes y))\\ &= (acb, yx)\end{aligned}$$

while, by the associativity of the form,

$$\begin{aligned}(a\otimes b.\ \pi^*(c),\ x\otimes y) &= (\pi^*(c),\ x\otimes y.\ a\otimes b)\\ &= (\pi^*(c),\ xa\otimes by)\\ &= (c,\ \pi(xa\otimes by))\\ &= (c,\ byxa)\\ &= (acb,\ yx).\end{aligned}$$

(note that ρ^* is not a homomorphism as

$$\begin{aligned}(\rho^*(a\otimes b.\ c),\ x\otimes y) &= (\rho^*(acb),\ x\otimes y)\\ &= (acb,\ \rho(x\otimes y))\\ &= (acb,\ xy)\end{aligned}$$

while

$$\begin{aligned}(a\otimes b.\ \rho^*(c),\ x\otimes y) &= (\rho^*(c),\ x\otimes y.\ a\otimes b)\\ &= (\rho^*(c),\ xa\otimes by)\\ &= (c,\ \rho(xa\otimes by))\\ &= (c,\ xaby).)\end{aligned}$$

Now let's turn to Hochschild cohomology. Recall that if U, V are A-modules then $Hom_k(U,V)$ is an $A\otimes A^\circ$-module via

$$a\otimes b.\ \varphi(-) = a\varphi(b-)$$

(since $a\otimes b\,(c\otimes d\ \varphi(-)) = a\otimes b\,(c\varphi(d-)) = ac\,\varphi(db-)$ which is $(ac\otimes db)\varphi(-) = (a\otimes b.\ c\otimes d)\,\varphi(-)$.) The basic result relating Hochschild cohomology to A-modules is the observation (see Cartan-Eilenberg) that

$$Hom_{A\otimes A^\circ}(A,\ Hom_k(U,V)) \simeq Hom_A(U,V).$$

(Indeed, let $\varphi \in Hom_A(U,V)$. Define Φ, a map of A to $Hom_k(U,V)$ by $\Phi(a) = a\varphi$. This is certainly linear and we have

$$\Phi(a\otimes b . c) = \Phi(acb) = acb\varphi$$

while

$$a\otimes b . \Phi(c) = a\otimes b . c\varphi = ac\varphi(b-) = acb\varphi(-)$$

so Φ is in $\mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U, V))$. Also, it is clear that the map which sends φ to Φ is linear. Now $\Phi(1) = \varphi$ so the map sending φ to Φ is one-to-one. It remains only to see that it is surjective as well. Suppose that $\Phi \in \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U,V))$ and set $\varphi = \Phi(1)$ so $\varphi \in \mathrm{Hom}_k(U, V)$. We claim that φ is mapped to Φ. Let Φ_φ be the image of φ so

$$\Phi_\varphi(a) = a\varphi$$

while

$$\begin{aligned}\Phi(a) &= \Phi(a\otimes 1.1)\\ &= a\otimes 1 . \Phi(1)\\ &= a\otimes 1 . \varphi\\ &= a\varphi(1-)\\ &= a\varphi\end{aligned}$$

so all is proved.)

This basic remark on "Hom" leads to results for all Ext^n - see Cartan-Eilenberg again. We wish to add the

Proposition. $\overline{\mathrm{Hom}}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U, V)) \simeq \overline{\mathrm{Hom}}_A(U, V)$.

To get at this we first return to π^* and calculate $\pi^*(1)$. Let $e_1, \dots, e_n$ be a basis of A and $f_1, \dots, f_n$ the dual basis of A with respect to the defining form. Then

$$\pi^*(1) = \sum_{i=1}^{n} e_i \otimes f_i$$

It suffices to prove that $(\sum e_i \otimes f_i,\ b \otimes c) = (1,\ \pi(b \otimes c))$.
However, if $b = \sum_s \beta_s e_s$, $c = \sum_t \gamma_t f_t$ then

$$\begin{aligned}
(\sum e_i \otimes f_i,\ b \otimes c) &= (\sum e_i \otimes f_i,\ (\sum \beta_s e_s) \otimes (\sum \gamma_t f_t)) \\
&= \sum (e_i \otimes f_i,\ \sum \beta_s \gamma_t\, e_s \otimes f_t) \\
&= \sum \beta_i \gamma_i \\
&= (\sum \beta_s e_s,\ \sum \gamma_t f_t) = (b, c) = (c, b) = (1, cb) \\
&= (1,\ \pi(b \otimes c)),
\end{aligned}$$

as required.

Now preserving the notation of the Proposition, we have the

__Lemma.__ If $\varphi \in Hom_A(U,V)$ and $\Phi \in Hom_{A \otimes A^o}(A, Hom_k(U,V))$ correspond then φ factors through an A free module iff Φ factors through $A \otimes A$ via π^* (so Φ is the composite of π^* and a homomorphism of $A \otimes A$ to $Hom_k(U,V)$).

__Proof.__ First, suppose that we have a commutative diagram

$$\begin{tikzcd}
& A \arrow[dr, "\mu"] & \\
U \arrow[ur, "\lambda"] \arrow[rr, "\varphi"'] & & V
\end{tikzcd}$$

and we wish to fill in the diagram

$$\begin{tikzcd}
& A \otimes A \\
A \arrow[ur, "\pi^*"] \arrow[r, "\Phi"'] & Hom_k(U,V)
\end{tikzcd}$$

If $a, b \in A$ define $\Psi_{a,b} \in \mathrm{Hom}_k(U, V)$ by $\Psi_{a,b}(u) = (b, \lambda(u))\mu(a)$. This depends on a and b in a bilinear fashion so defines a linear transformation Ψ of $A \otimes A$ to $\mathrm{Hom}_k(U, V)$. We claim that it is an $A \otimes A^0$-module homomorphism. Indeed, if $x \otimes y \in A \otimes A^0$ then $(x \otimes y) a \otimes b = xa \otimes by$ is mapped to $\Psi_{xa, by}$ while

$$\begin{aligned}
((x \otimes y)\Psi_{a,b})(u) &= x\, \Psi_{a,b}(yu) \\
&= x(b, \lambda(yu))\mu(a) \\
&= (b, y\lambda(u))\mu(xa) \\
&= (by, \lambda(u))\mu(xa) \\
&= \Psi_{xa, by}(u).
\end{aligned}$$

We next assert that the following diagram commutes:

$$\begin{tikzcd}
& A \otimes A \arrow[dr, "\Psi"] & \\
A \arrow[ur, "\pi^*"] \arrow[rr, "\Phi"'] & & \mathrm{Hom}_k(U, V)
\end{tikzcd}$$

Since $1 \in A$ is a generator of A as an $A \otimes A^0$-module, it suffices to check that $\Psi \pi^*(1) = \Phi(1)$. However, say $u \in U$ as usual and $\lambda(u) = \sum l_j e_j$, $l_j \in k$. Then

$$\begin{aligned}
(\Psi \pi^*(1))(u) &= \Psi\left(\sum e_i \otimes f_i\right)(u) \\
&= \sum \Psi_{e_i, f_i}(u) \\
&= \sum (f_i, \lambda(u))\mu(e_i) \\
&= \sum \left(f_i, \sum l_j e_j\right)\mu(e_i) \\
&= \sum l_i\, \mu(e_i) \\
&= \mu\left(\sum l_i e_i\right) \\
&= \mu(\lambda(u)) \\
&= (\Phi(1))(u
\end{aligned}$$

On the other hand, suppose that we have a commutative diagram

$$\begin{tikzcd} & A \otimes A \arrow[dr, "\rho"] & \\ A \arrow[ur, "\pi^*"] \arrow[rr, "\Phi"'] & & \mathrm{Hom}_k(U, V) \end{tikzcd}$$

Let $\psi = \rho(1 \otimes 1) \in \mathrm{Hom}_k(U, V)$ so

$$\begin{aligned} \varphi = \Phi(1) &= \rho\pi^*(1) = \rho\left(\sum e_i \otimes f_i\right) = \sum \rho(e_i \otimes f_i) \\ &= \sum \rho(e_i \otimes f_i \,.\, 1 \otimes 1) = \sum (e_i \otimes f_i)\psi = \sum e_i \psi f_i . \end{aligned}$$

This looks familiar (see Curtis-Reiner for the special case $\varphi = 1$).

We shall deduce that φ factors through a free module, in fact the free module $A \otimes_k V$ where $a \,.\, b \otimes v = ab \otimes v$ and a basis of V leads to the freeness. There is an epimorphism of $A \otimes_k V$ to V which sends $a \otimes v$ to v, as is easily verified. Thus, we need only define an A-homomorphism of U to $A \otimes_k V$ whose composition with this surjection is $\sum e_i \psi f_i$.

Now there is a linear transformation of U to $A \otimes_k V$ which maps each $u \in U$ to $\sum e_i \otimes \psi(f_i u)$ and its composite with the surjection is correct. Hence, we need only prove that this linear transformation is an A-module homomorphism. However first observe that if $a \in A$ then $\sum a e_i \otimes f_i = \sum e_i \otimes f_i a$. Indeed,

$$\begin{aligned} \sum e_i \otimes f_i a &= (1 \otimes a) \sum e_i \otimes f_i \\ &= (1 \otimes a)\, \pi^*(1) \\ &= \pi^*(a) \\ &= (a \otimes 1)\, \pi^*(1) \\ &= (a \otimes 1) \sum e_i \otimes f_i \end{aligned}$$

$$= \sum a e_i \otimes f_i .$$

There is also a linear transformation of $A \otimes A$ to $A \otimes \mathrm{Hom}_k(U,V)$ sending each $a \otimes b$ to $a \otimes \rho(1 \otimes b)$ so

$$\sum e_i \otimes \rho(1 \otimes f_i a) = \sum a e_i \otimes \rho(1 \otimes f_i).$$

And there is a linear transformation of $A \otimes \mathrm{Hom}_k(U,V)$ which maps via evaluation at u, where u is a fixed element of U. Thus,

$$\sum e_i \otimes \rho(1 \otimes f_i a)(u) = \sum a e_i \otimes \rho(1 \otimes f_i)(u)$$

$$\sum e_i \otimes \big((1 \otimes f_i a)\,\rho(1 \otimes 1)\big)(u) = \sum a e_i \otimes \big((1 \otimes f_i)\,\rho(1 \otimes 1)\big)(u)$$

$$\sum e_i \otimes \big((1 \otimes f_i a)\psi\big)(u) = \sum a e_i \otimes \big((1 \otimes f_i)\,\psi\big)(u)$$

$$\sum e_i \otimes \psi(f_i a u) = \sum a e_i \otimes \psi(f_i u)$$

which, if T is the linear transformation in question, is

$$T(au) = a\,T(u)$$

as needed.

Let us now turn to the Proposition. Note that π^* is one-to-one. Indeed, if $a \in A$ choose $e_j \ni (a, e_j) \neq 0$ so

$$\pi^*(a) = \pi^*((1 \otimes a)1) = (1 \otimes a) \sum e_i \otimes f_i = \sum e_i a \otimes f_i$$

and

$$\left(\pi^*(a),\ 1 \otimes f_j\right) = \left(\sum e_i a \otimes f_i,\ 1 \otimes f_j\right) = (e_j a, 1) = (e_j, a) \neq 0$$

so $\pi^*(a) \neq 0$. Hence, a map of A to any $A \otimes A^0$-module M is projective if, and only if, it factors through π^* (the usual argument). The rest is now routine and the result is established.

Next, we shall use this to deduce a fact from Cartan-Eilenberg, namely

$$\mathcal{E}xt^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U,V)) \simeq \mathcal{E}xt^1_A(U,V).$$

We have

$$\mathcal{E}xt^1_A(U,V) \simeq \overline{\mathrm{Hom}}(\Omega^1 U, V)$$

so

$$\mathcal{E}xt^1_A(U,V) \simeq \overline{\mathrm{Hom}}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(\Omega^1 U, V)).$$

But if PU is the projective cover of U then we have an exact sequence of vector spaces

$$0 \to \mathrm{Hom}_k(U,V) \to \mathrm{Hom}_k(PU,V) \to \mathrm{Hom}_k(\Omega^1 U, V) \to 0$$

so we get an exact commuting diagram

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U,V)) & \to & \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU,V)) & \to & \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(\Omega^1 U, V)) \\
& \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
0 \to & \mathrm{Hom}_A(U,V) & \to & \mathrm{Hom}_A(PU,V) & \to & \mathrm{Hom}_A(\Omega^1 U, V)
\end{array}$$

$$\begin{array}{ccc}
\to \mathcal{E}xt^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U,V)) & \to & \mathcal{E}xt^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU,V)) \\
\downarrow \simeq & & \\
\to \mathcal{E}xt^1_A(U,V) & \to & 0
\end{array}$$

so we need only demonstrate that

$$\mathcal{E}xt^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU,V)) = 0.$$

However, let IV be an injective envelope for V so we have an exact sequence of vector spaces

$$0 \to \mathrm{Hom}_k(PU,V) \to \mathrm{Hom}_k(PU, IV) \to \mathrm{Hom}_k(PU, \mho^1 V) \to 0$$

so we get the long exact sequence (as the last sequence consists of $A\otimes A^\circ$-modules)

$$0 \to \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, V)) \to \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, IV))$$

$$\to \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, \mho^1 V)) \to \mathrm{Ext}^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, V))$$

$$\to \mathrm{Ext}^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, IV)) \to \cdots$$

now $\mathrm{Hom}_k(A, A)$ is a free $A\otimes A^\circ$-module so $\mathrm{Hom}_k(PU, IV)$ is a projective $A\otimes A^\circ$-module and $\mathrm{Ext}^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, IV)) = 0$ Therefore, the cokernel of the map

$$\mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, IV)) \to \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, \mho^1 V))$$

is isomorphic with $\overline{\mathrm{Hom}}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, \mho^1 V))$ and

$$\mathrm{Ext}^1_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, V)) \simeq \overline{\mathrm{Hom}}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(PU, \mho^1 V)).$$

But the right-hand side is, by the Proposition, isomorphic with

$$\overline{\mathrm{Hom}}_A(PU, \mho^1 V)$$

which is clearly zero.

We now turn to the relationship between what we have been doing and the standard Hochschild resolution. We have been using the embedding $A \to A\otimes A$ which looks like the standard resolution in the reverse order! In fact, this is the case.

Consider the standard resolution

$$\cdots \xrightarrow{\partial} A\otimes A\otimes A \xrightarrow{\partial} A\otimes A \xrightarrow{\partial} A \to 0$$

and let r be the "reversal" linear transformation on each $A^{\otimes n}$ so $r(a_1\otimes\ldots\otimes a_n) = a_n\otimes\ldots\otimes a_1$ while $r(a_1) = a_1$. Use the product inner product on each $A^{\otimes n}$ and use "$*$" to denote the taking of adjoints. We have

Theorem The sequence
$$\cdots \leftarrow A\otimes A\otimes A \xleftarrow{(r\partial r)^*} A\otimes A \xleftarrow{(r\partial r)^*} A \leftarrow 0$$
is exact, the maps are $A\otimes A^\circ$-module homomorphisms and the first of these is the map π^* defined above.

Proof. First, consider the sequence
$$\cdots \to A\otimes A\otimes A \xrightarrow{r\partial r} A\otimes A \xrightarrow{r\partial r} A \to 0.$$
It is exact. Indeed, if $a_n \in A^{\otimes n}$ then
$$r\partial r \cdot r\partial r\,(a_n) = r\,\partial^2(r a_n) = 0$$
while if $(r\partial r)a_n = 0$ then $\partial(r a_n) = 0$ so $r a_n = \partial a_{n+1}$ and $a_n = r\partial a_{n+1} = r\partial r\,(r a_{n+1})$ as required. Hence, the sequence in the theorem is also exact. Moreover,
$$(r\partial r)(a_1\otimes a_2) = r\partial(a_2\otimes a_1) = r(a_2a_1) = a_2a_1$$
so the first $r\partial r$ is π as asserted. It remains to see that the maps are module homomorphisms.

We start with a couple observations. If $u, v \in A^{\otimes n}$ then $(ru, v) = (u, rv)$ since
$$\begin{aligned}(r(a_1\otimes\cdots\otimes a_n), b_1\otimes\cdots\otimes b_n) &= (a_n, b_1)\cdots(a_1, b_n)\\ &= (a_1, b_n)\cdots(a_n, b_1)\\ &= (a_1\otimes\cdots\otimes a_n, b_n\otimes\cdots\otimes b_1)\\ &= (a_1\otimes\cdots\otimes a_n, r(b_1\otimes\cdots\otimes b_n)).\end{aligned}$$
Also, with the same notation, if $x, y \in A$ so $x\otimes y \in A\otimes A^\circ$,
$$\begin{aligned}&((x\otimes y)(a_1\otimes\cdots\otimes a_n), b_1\otimes\cdots\otimes b_n)\\ &\qquad = (a_1\otimes\cdots\otimes a_n, b_1x\otimes b_2\otimes\cdots\otimes yb_n).\end{aligned}$$
For the left-hand side is
$$(xa_1, b_1)(a_2, b_2)\cdots(a_ny, b_n)$$
while the right-hand side is $(a_1, b_1x)(a_2, b_2)\cdots(a_n, yb_n)$.

We can now finish with a calculation. If $x, y \in A$ so $x \otimes y \in A \otimes A^{\circ}$ while $a = a_1 \otimes \ldots \otimes a_{n+1} \in A^{\otimes n+1}$, $b = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$ then we want to prove that $(a, (x \otimes y)(r \partial r)^*(b)) = (a, (r\partial r)^*(x \otimes y . b))$. However,

$$
\begin{aligned}
&(a, (x \otimes y)(r\partial r)^* b) \\
&\quad = (a_1 \otimes \ldots \otimes a_{n+1}, (x \otimes y)(r \partial r)^*(b_1 \otimes \ldots \otimes b_n)) \\
&\quad = (a_1 x \otimes \ldots \otimes y a_{n+1}, (r \partial r)^* b_1 \otimes \ldots \otimes b_n) \\
&\quad = (r \partial r (a_1 x \otimes \ldots \otimes y a_{n+1}), b_1 \otimes \ldots \otimes b_n) \\
&\quad = (\partial (y a_{n+1} \otimes \cdots \otimes a_1 x), b_n \otimes \ldots \otimes b_1) \\
&\quad = (\partial ((y \otimes x)(a_{n+1} \otimes \ldots \otimes a_1)), b_n \otimes \ldots \otimes b_1) \\
&\quad = (y \otimes x . \partial (a_{n+1} \otimes \ldots \otimes a_1), b_n \otimes \ldots \otimes b_1) \\
&\quad = (\partial (a_{n+1} \otimes \ldots \otimes a_1), b_n y \otimes \ldots \otimes x b_1) \\
&\quad = (\partial r (a_1 \otimes \ldots \otimes a_{n+1}), r(x b_1 \otimes \ldots \otimes b_n y)) \\
&\quad = (r \partial r (a_1 \otimes \ldots \otimes a_{n+1}), (x \otimes y)(b_1 \otimes \ldots \otimes b_n)) \\
&\quad = (a_1 \otimes \ldots \otimes a_{n+1}, (r \partial r)^* (x \otimes y)(b_1 \otimes \ldots \otimes b_n)) \\
&\quad = (a, (r \partial r)^* (x \otimes y . b)).
\end{aligned}
$$

Finally, it's not hard to describe the maps $(r \partial r)^*$ on the elements $1 \otimes \ldots \otimes 1$ (+ hence on many elements). Using $(\sum e_i \otimes f_i, a \otimes b) = (1, ba) = (b, a) = (ab, 1)$ it's easy to see that

$$1 \otimes \ldots \otimes 1 \to \sum 1 \otimes \ldots \otimes 1 \otimes e_i \otimes f_i - \sum 1 \otimes \ldots \otimes 1 \otimes e_i \otimes f_i \otimes 1 + \ldots$$

Symmetric algebras and twisted group algebras

The idea is to relate the material of the previous two sections and to develop ideas that can be used for generalizing our argument for TI nilpotent blocks. We let D be a p-group, Q an abelian p'-group acting on D, $A = kD$, $B = A_m$ the twisted algebra for a 2-cocycle m, etc.— We know that $\mathrm{Ext}^*_{kD}(k, -) \cong \mathrm{Ext}^*_{kD\otimes kD}(kD, -)$ and we want a twisted version. We can make some progress.

As a start let M be an $A \otimes A^\circ$-module, written as an A, A bimodule. We shall construct a $B \otimes B^\circ$-module (i.e. A_m, A_m bimodule) M_m as follows. We set

$$a \circ m = m_{\alpha,\mu}\, am, \quad m \circ b = m_{\mu,\beta}\, mb$$

Let's check:

$$\left\{\begin{aligned} (a\circ m)\circ b &= m_{\alpha,\mu}\, am\circ b = m_{\alpha,\mu}\, m_{\alpha\mu,\beta}\,(am)b \\ a\circ(m\circ b) &= a\circ m_{\mu,\beta}\, mb = m_{\mu,\beta}\, m_{\alpha,\mu\beta}\, a(mb) \end{aligned}\right.$$

$$\left\{\begin{aligned} a\circ(b\circ m) &= a\circ m_{\beta,\mu}\, bm = m_{\alpha,\beta\mu}\, m_{\beta,\mu}\, a(bm) \\ (a\circ b)\circ m &= m_{\alpha,\beta}\, ab\circ m = m_{\alpha,\beta}\, m_{\alpha\beta,\mu}\,(ab)m \end{aligned}\right.$$

$$\left\{\begin{aligned} (m\circ a)\circ b &= m_{\mu,\alpha}\, ma\circ b = m_{\mu,\alpha}\, m_{\mu\alpha,\beta}\,(ma)b \\ m\circ(a\circ b) &= m\circ m_{\alpha,\beta}\, ab = m_{\mu,\alpha\beta}\, m_{\alpha,\beta}\, m(ab). \end{aligned}\right.$$

<u>Lemma</u>. The above procedure applied to a free $A \otimes A^\circ$-module yields a free $B \otimes B^\circ$-module.

<u>Proof</u>. The $B \otimes B^\circ$-module $(A \otimes A^\circ)_m$ is generated by $1 \otimes 1$ and has the correct dimension.

(We can make this explicit. We define a linear transformation from $B \otimes B^\circ$ to $(A \otimes A^\circ)_m$ by sending $c \otimes b$ to $m_{\alpha,\beta}\, a \otimes b$ (for the usual "eigenvectors" $a, b, \ldots$). Let's check that the element $c \otimes d$ of $B \otimes B^\circ$ with algebra action by $a \otimes b$ behaves:

$$\begin{tikzcd}
c \otimes d \arrow[r] \arrow[d, "a \otimes b -"] & m_{\gamma,\delta}\, c \otimes d \arrow[d, "a \otimes b -"] \\
m_{\alpha,\gamma}\, ac \otimes m_{\delta,\beta}\, db \arrow[dr, bend right] & m_{\gamma,\delta}\, m_{\alpha,\gamma\delta}\, m_{\alpha\gamma\delta,\beta}\, ac \otimes db \\
& m_{\alpha\gamma,\delta\beta}\, m_{\alpha,\beta}\, m_{\delta,\beta}\, ac \otimes db
\end{tikzcd}$$

The required identity should arise by polarising the usual identity, but we can do the calculation in a group extension:

$$\begin{aligned}
(g_\alpha g_\gamma)(g_\delta g_\beta) &= m_{\alpha,\gamma}\, g_{\alpha\gamma}\, m_{\delta,\beta}\, g_{\delta\beta} \\
&= m_{\alpha,\gamma}\, m_{\delta,\beta}\, m_{\alpha\gamma,\delta\beta}\, g_{\alpha\gamma\delta\beta} \\
(g_\alpha(g_\gamma g_\delta))\, g_\beta &= (g_\alpha(m_{\gamma,\delta}\, g_{\gamma\delta}))\, g_\beta \\
&= m_{\gamma,\delta}\, m_{\alpha,\gamma\delta}\, g_{\alpha\gamma\delta}\, g_\beta \\
&= m_{\gamma,\delta}\, m_{\alpha,\gamma\delta}\, m_{\alpha\gamma\delta,\beta}\, g_{\alpha\gamma\delta\beta} \, .)
\end{aligned}$$

This gives us the

<u>Lemma</u>. If I is a G-invariant submodule of the $A \otimes A^\circ$-module $A \otimes A^\circ$ then I is also a submodule of the $B \otimes B^\circ$-module $(A \otimes A^\circ)_m$.

<u>Proof.</u> Let c, d be "eigenvectors" so $c \otimes d$ is as well. Then, in $(A \otimes A^\circ)_m$ we have

$$a \otimes b \,.\, c \otimes d = (a \circ c \otimes d) \circ b$$
$$= m_{\alpha,\gamma\delta} (ac \otimes d) \cdot b$$
$$= m_{\alpha,\gamma\delta}\, m_{\alpha\gamma\delta,\beta}\; ac \otimes db.$$

The right-hand side is a multiple of the product of the element $a \otimes b$ of $A \otimes A^\circ$ and the module element $c \otimes d$. The multiple depends only on $\gamma\delta$, the product of γ and δ and not on γ or δ. Hence, if we have any eigenvector in I for eigenvalue $\gamma\delta$ — no matter how it arises — then its product with $a \otimes b \in B \otimes B^\circ$ as an element of $(A \otimes A^\circ)_m$ also lies in I.

We shall use these results to analyze the structure of the free $B \otimes B^\circ$-module. We begin with the (kD, kD) bimodule $kD \otimes kD$. If $v = \sum \alpha_d\, d \in kD$ then we set

$$\delta(v) = \sum \alpha_d\, d \otimes d^{-1} \in kD \otimes kD$$

so if V is a subspace of kD then $\delta(V)$ is a subspace of $kD \otimes kD$. If $e, f \in D$ then

$$e \otimes f \,.\, \delta(v) = e \otimes f \,.\, \sum \alpha_d\, d \otimes d^{-1}$$
$$= \sum \alpha_d\, ed \otimes d^{-1} f$$
$$= \sum \alpha_d\, ed \otimes d^{-1} e^{-1} . ef$$
$$= 1 \otimes ef \,.\, \sum \alpha_d\, ed \otimes (ed)^{-1} = 1 \otimes ef\, \delta(ev).$$

In particular, if V is a submodule of kD then

$$e \otimes f \,.\, \delta(V) = 1 \otimes ef \,.\, \delta(V)$$

Now let $\tau(V)$, in this case, be the two-sided submodule of $kD \otimes kD$ generated by $\delta(V)$; we therefore have that $\dim_k \tau(V) \leq |D| \dim_k V$ as $\tau(V)$ is spanned by all $1 \otimes e \,.\, \delta(V)$.

However, this forces $\dim_k \tau(V) = |D| \dim_k V$, or else, arguing as we just did, we would have

$$|D|^2 = \dim \tau(kD) \leq |D| \dim_k(kD/V) + \dim_k \tau(V) < |D|^2.$$

In particular, if v_i runs over a basis of V and e runs over D then the elements $1 \otimes e . \delta(v_i)$ are a basis of $\tau(V)$.

Next, suppose that $V \geq W$ are submodules of kD with $\dim(V/W) = 1$. We claim that $\tau(V)/\tau(W) \simeq kD$ as kD, kD bimodules. Indeed, let $v \in V - W$ so if $e \in D$ then $ev \equiv v$ (modulo W) so $\delta(ev) \equiv \delta(v)$ (modulo $\tau(W)$), as is easy to see. Hence, we proceed by defining a linear transformation that maps kD one-to-one and onto $\tau(V)/\tau(W)$ by demanding that $d \in D$ is sent to the coset containing $1 \otimes d . \delta(v)$. This is a bimodule map as the following commutative diagram shows:

$$\begin{array}{ccc} d & \longrightarrow & 1 \otimes d . \delta(v) \\ {\scriptstyle e\otimes f}\downarrow & & \downarrow{\scriptstyle e \otimes f} \\ edf & \longrightarrow & (e\otimes f)(1 \otimes d) . \delta(v) = e \otimes fd . \delta(v) \\ & & = 1 \otimes edf . \delta(v) \end{array}$$

We now turn to the action of Q, in preparation to passing to B from A. Suppose now that v is an eigenvector for eigenvalue λ – i.e. say V and W are also Q-invariant so that such a v exists. Now $v = \sum \alpha_d d$ so Q cycles the elements of D and the coefficients must satisfy a proportional relation so the same holds for $\delta(v) = \sum \alpha_d \, d \otimes d^{-1}$ and it is an eigenvector for λ.

The isomorphism between A and $\tau(V)/\tau(W)$

sends d to $1 \otimes d.\,\delta(V)$ (mod $\delta(W)$) so sends a to $1 \otimes a.\,\delta(V)$. If a has eigenvalue α then $1 \otimes a.\,\delta(V)$ has value $\lambda\alpha$. Thus, with the grading, we have that $\tau(V)/\tau(W)$ is like A but "shifted" by λ. Call this A, A bimodule A^λ.

The process of going from A to B using the coradical respects suitable submodules and quotients. Thus A^λ becomes the $B \otimes B^\circ$-module $(A^\lambda)_m$. Since there is certainly a G invariant sequence of submodules of $k D$ with all consecutive codimensions 1 we deduce that the B, B bimodule $B \otimes B$ is filtered by the (A^λ). Now we have the homomorphism of $B \otimes B$ to B that figured in the last section. We claim that the kernel is $\tau(\Delta\, k D)$. Indeed, the dimension is right so we need only see that all $\delta(d-1)$ get mapped to 0. But $\delta(d-1) = d \otimes d^{-1} - 1 \otimes 1$. Hence our filtration can be chosen "compatible" with the map of $B \otimes B$ to B and $B = (A^1)_m$ as bimodule.

We would like to be able to deduce that for B-modules U, V when $\overline{\mathrm{Hom}}_B(U,V) = 0$ then $\mathrm{Ext}^1_B(U,V) = 0$. That is, use $\mathrm{Hom}_{B \otimes B^\circ}(B, \mathrm{Hom}_k(U,V)) = 0$ to get $\mathrm{Ext}^1_{B \otimes B^\circ}(B, \mathrm{Hom}_k(U,V)) = 0$. By dimension shifting, using $B \otimes B \to B$, it's enough to see that the kernel of this map is filtered by copies of $B = (A^1)_m$ — which is not quite what we have shown.

The difficulty seems to be demonstrated by the following example, which needs proving as we have not verified it all, which is not though a case of a block with one modular irreducible.

We take $p=2$, $D = \mathbb{Z}_2 \times \mathbb{Z}_2$, $G = \mathbb{Z}_3$ with obvious action so have

$$\mathbb{Z}_3 \to \begin{cases} 1 \\ \omega \\ \bar{\omega} \end{cases}$$

$$A = kD: \quad \begin{array}{ccc} & 1 \; (1) & \\ X\,(\omega) & & Y\,(\bar{\omega}) \\ & XY\,(1) & \end{array}$$

Let U be the indecomposable 2-dimensional kD-module with

$$X \to \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \qquad Y \to \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

so U^ω is given by

$$X \to \begin{pmatrix} 0 & \alpha\omega \\ 0 & 0 \end{pmatrix} \qquad Y \to \begin{pmatrix} 0 & \beta\bar{\omega} \\ 0 & 0 \end{pmatrix}$$

so

$$\overline{\mathrm{Hom}}_A(U, U^\omega) = 0, \quad \overline{\mathrm{Hom}}_A(U^\omega, U^\omega) \neq 0$$

so

$$\overline{\mathrm{Hom}}_{A\otimes A^\circ}(A, \mathrm{Hom}_k(U, U^\omega)) = 0$$
$$\overline{\mathrm{Hom}}_{A\otimes A^\circ}(A^\omega, \mathrm{Hom}_k(U, U^\omega)) \neq 0.$$

Robinson reciprocity

We give a new proof of a result of G. Robinson, originally proved using idempotents and the Reynold's ideal. Let H be a subgroup of G, k algebraically closed field of characteristic p. Let S and T be simple modules for kG and kH, respectively.

Theorem. The multiplicity, as a direct summand, of the projective cover of S in T^G equals the multiplicity, as a direct summand, of the projective cover of T in S_H.

Proof. We first assert that the multiplicity of P_S (the projective cover of S) in the kG-module U (as a direct summand) equals

$$\dim_k \mathrm{Hom}_{kG}(S, U) - \dim_k \overline{\mathrm{Hom}}_{kG}(S, U)$$

and also equals

$$\dim_k \mathrm{Hom}_{kG}(U, S) - \dim_k \overline{\mathrm{Hom}}(U, S)$$

(where $\overline{\mathrm{Hom}}$ denotes, as usual, "Hom mod projectives = injectives"). It suffices to prove these hold when U is indecomposable. Let's first do the first equality. If $U \cong P_S$ then it is clear we get the value 1 from the difference, as desired. Suppose $U \not\cong P_S$. We assert that $\dim_k \mathrm{Hom}_{kG}(S, U) = \dim_k \overline{\mathrm{Hom}}_{kG}(S, U)$; indeed, there are no non-zero projective homomorphisms of S to U. If there were, then as projective = injective, the maps would factor as follows

$$\begin{tikzcd} & P_S \arrow[dr, dotted] & \\ S \arrow[ur] \arrow[rr] & & U \end{tikzcd}$$

and P_S would be embedded in U. A similar argument gives the second equality, reversing the use of projectives and injectives.

Now, to prove the theorem, the first multiplicity equals

$$\dim_k \mathrm{Hom}_{kG}(S, T^G) - \dim_k \overline{\mathrm{Hom}}_{kG}(S, T^G)$$

$$= \dim_k \mathrm{Hom}_{kH}(S_H, T) - \dim_k \overline{\mathrm{Hom}}_{kH}(S, T)$$

(by Frobenius reciprocity and by Frobenius reciprocity for Ext^1 and dimension shifting)

and this is the second multiplicity by the above paragraph applied to H instead of G.

On a question of Auslander and Reiten

Let A be a finite dimensional algebra over a field k. A subcategory of mod (A) is called an AR category if it is a full subcategory containing all projectives and injectives, closed under the taking of summands, containing the third term of any short exact sequence in which two terms already belong to the subcategory and satisfying, finally, the following condition: if M is an A-module with $\mathrm{Ext}^i(X, M) = 0$ for all $i > 0$ and all X in the subcategory then M is injective. The question is the existence of the existence of proper subcategories which are AR.

Fix a finite group G and an algebraically closed field k of prime characteristic p.

Theorem. If c is a positive integer then the subcategory of mod (kG) consisting of all modules of complexity at most c is AR.

Lemma 1. If M is a non-projective kG-module then there is a periodic kG-module U such that $U \otimes M$ is not projective.

Proof. By Chouinard's theorem there is an elementary abelian p-subgroup E such that M_E is not projective. Choose an indecomposable periodic kE-module V whose rank variety (a line) is contained in the rank variety of M_E. Hence, the intersection of these rank varieties, which is the variety for $V \otimes M_E$, is certainly non-empty so $V \otimes M_E$ is not projective. Hence, so is $\mathrm{Ind}_E^G(V \otimes M_E)$, as $V \otimes M_E$ is a summand of the restriction of the induced module, that is, $(\mathrm{Ind}_E^G V) \otimes M$ is not projective.

But $\mathrm{Ind}_E^G V$ has complexity (at most) one so there is a periodic summand U of $\mathrm{Ind}_E^G V$ with $U \otimes M$ not projective, as claimed.

Lemma 2. If M is a non-projective kG-module then there is a periodic module W such that $\mathrm{Ext}^1_{kG}(W, M) \neq 0$.

Proof. Choose U as in Lemma 1; hence, there is a simple kG-module S such that $\mathrm{Ext}^1_{kG}(S, U \otimes M) \neq 0$ so $\mathrm{Ext}^1_{kG}(S \otimes U^*, M) \neq 0$. Now U^* is also periodic so $S \otimes U^*$ also has a periodic resolution (by tensoring a resolution of U^*) and we're done.

Lemma 3. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of kG-modules and two of the M_i have complexity at most c then so does the third.

Proof. If M_j is the "third" module of undetermined complexity then we have to estimate the "rate of growth" of $\mathrm{Ext}^i(M_j, N)$ in i for any kG-module N. The long exact sequence for Ext and the short exact sequence, gives the needed estimate.

Homotopy type

We shall show that a standard result, in the case of simplicial complexes, is a consequence of Quillen's homotopy theorem for posets. Let K be a simplicial complex covered by subcomplexes $L_1, \ldots, L_r$ such that whenever I is a subset of $R = \{1, \ldots, r\}$ then $L_I = \bigcap_{i \in I} L_i$ is either empty or contractible.

<u>Lemma</u>. The complex K is of the same homotopy type as the nerve N of the covering of K given above.

<u>Proof</u>. We shall first construct an (order reversing) map of the poset $\mathcal{S}(K)$ of simplices of K to the poset of simplices $\mathcal{S}(N)$ of N, the latter consisting of the subsets I of R such that $L_I \neq \emptyset$. If σ is a simplex of K then $\sigma \subseteq L_i$ for some i, $1 \le i \le r$, by hypothesis so $I(\sigma) = \{ i \mid \sigma \subseteq L_i \}$ is non-empty and $I(\sigma) \in \mathcal{S}(N)$ since $L_{I(\sigma)} \neq \emptyset$ clearly. Moreover, we have $I(\sigma) \supseteq I(\tau)$ if $\sigma \subseteq \tau$ another simplex of K. Our map is the function I.

Next, let I be a simplex of N and consider the set of simplices σ of K such that $I(\sigma) \supseteq I$. But $I(\sigma) \supseteq I$ if, and only if, $\sigma \subseteq L_i$ for all $i \in I$. That is, the set of simplices in question is the set of simplices contained in L_I so the geometrical realization of this set is $|L_I'|$, the realization of the barycentric subdivision of L_I, and is contractible by hypothesis.

Reversing the order in $\mathcal{S}(K)$, which just makes no difference for the space, we get from Quillen that $|\mathcal{S}(K)| = |K'|$ and $|N'|$ are of the same homotopy type.

Remark: This is just a homotopy version of a homology theorem which is a special case of the Leray spectral sequence.

Remark: If the group G acts simplicially on K and permutes the L_i then the proof gives a G-map of K' to N' which is a homotopy equivalence. It would be interesting to know, in general, when this is a G-homotopy equivalence. Probably, Webb-Thévenaz answers it.

Let's turn to some applications to the Brown complex. Fix a finite group G, a prime p and let $\mathcal{S} = \mathcal{S}_p = \mathcal{S}_p(G)$ be the Brown complex of G. First, let $\mathcal{I} = \mathcal{I}_p = \mathcal{I}_p(G)$ be the complex whose vertices are the Sylow p-subgroups of G and where a set of t Sylow subgroups is a simplex if their intersection is non-empty.

Theorem $\mathcal{S}$ and $\mathcal{I}$ are of the same homotopy type.

Proof. Let $S_1, \dots, S_h$ be the Sylow p-subgroups of G and let $L_1, \dots, L_h$ be the subcomplexes of $\mathcal{S}$ consisting of the chains of subgroups of $S_1, \dots, S_h$, respectively. Hence, these subcomplexes cover $\mathcal{S}$ and each L_i is contractible. Moreover, $L_{i_1} \cap \dots \cap L_{i_s} \neq \emptyset$ if, and only if, $S_{i_1} \cap \dots \cap S_{i_s} \neq 1$ so the nerve of the covering is isomorphic, as simplicial complex, with $\mathcal{I}$ and is of the same homotopy type as $\mathcal{S}$ by the lemma.

Theorem If $\mathcal{A}$ is the Quillen complex then $\mathcal{A}$ and $\mathcal{I}$ are of the same homotopy type.

As usual, $\mathcal{A}$ comes from the poset of elementary abelian p-subgroups.

Proof. For each of the Sylow p-subgroups S_i as above let A_i be the subcomplex of $\mathcal{A}$ given by the elementary abelian p-subgroups of S_i so A_i is contractible (even conically as $Z(S_i) \neq 1$). Moreover $A_{i_1} \cap \dots \cap A_{i_j} \neq \emptyset$ if, and only if, $S_{i_1} \cap \dots \cap S_{i_j} \neq 1$ – use subgroups of order p. Rest is easy.

Remark: We have a commuting diagram of maps

$$\begin{tikzcd} \mathcal{S} \arrow[r] & \mathcal{I} \\ \mathcal{A} \arrow[u] \arrow[ur] & \end{tikzcd}$$

where two have been shown to be homotopy equivalences so the third is too. Moreover all are G-maps as is easy to see.

Next, let $E_1, \dots, E_t$ be the subgroups of order p in G and let $\mathcal{E}$ be the simplicial complex with these vertices where a subset is a simplex if the subgroups generate a p-subgroup.

Theorem. $\mathcal{S}$ and $\mathcal{E}$ are of the same homotopy type.

Proof. Let M_i be the subcomplex formed from p subgroups containing E_i so M_i is contractible as it is a suspension (of the subcomplex of p-subgroups properly containing E_i). Moreover, the M_i cover $\mathcal{S}$ and $M_{i_1} \cap \dots \cap M_{i_j} \neq 0$ if, and only if, there is a p-subgroup containing $E_{i_1}, \dots, E_{i_j}$.

We can also get a direct relationship between $\mathcal{E}$ and $\mathcal{I}$:

Theorem $\mathcal{E}$ and $\mathcal{I}$ are of the same homotopy type

Proof. Let F_i be the subcomplex of $\mathcal{E}$ consisting of the subgroups of order p in S_i, $1 \le i \le l$, and all the faces, i.e. a simplex so certainly contractible. The F_i cover $\mathcal{E}$, the intersections are empty or similarly contractible. The nerve of this cover is isomorphic with $\mathcal{L}$.

Next, let C be the subcomplex of $\mathcal{E}$ with the same vertices but with vertices $E_{i_1}, \ldots, E_{i_p}$ forming a simplex if, and only if, they pairwise commute.

Theorem. $\mathcal{A}$ and C are of the same homotopy type.

Proof. Let H_i be the subcomplex of $\mathcal{A}$ formed by elementary abelian p-subgroups containing E_i so H_i is contractible, being a suspension, as is every non-empty intersection of such complexes. Moreover, $H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset$ if, and only if, $E_{i_1}, \ldots, E_{i_k}$ are contained in an elementary abelian p-subgroup.

We can also get the direct relation

Theorem. C and $\mathcal{L}$ are of the same homotopy type.

Proof. Let C_i be the subcomplex of C formed by all subgroups of order p in S_i and with all possible faces. The previous result gives us the desired contractibility properties for this cover of C and the nerve is as desired.

The maps we have so far (with either maps or containments):

$$\begin{tikzcd}
\mathcal{E} \arrow[dr] & & \mathcal{S} \arrow[ll] \arrow[dl] \\
& \ell & \\
\mathcal{C} \arrow[uu, dotted] \arrow[ur] & & a \arrow[ll] \arrow[ul] \arrow[uu, dotted]
\end{tikzcd}$$

Let's prove a technical version of some of this that will be quite useful. Let $\mathcal{X}$ be a collection of non-identity p-subgroups closed under the taking of subgroups. Let $\mathcal{I}$ be the corresponding simplicial complex. Let $\mathcal{J}$ be the "intersection" simplicial complex using the maximal members of $\mathcal{I}$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$ and let $\mathcal{F}$ be the complex whose vertices are the subgroups in $\mathcal{Y}$ and where the subgroups $Y_1, \dots, Y_t$ of $\mathcal{Y}$ form a simplex if the subgroup they generate is contained in $\mathcal{X}$.

Theorem The complexes $\mathcal{I}$ and $\mathcal{J}$ are of the same homotopy type. The same is also true for $\mathcal{F}$ and $\mathcal{J}$ provided the subgroups of $\mathcal{Y}$ contained in any non-identity intersection of maximal members of $\mathcal{X}$ form a non-empty contractible complex.

Proof. Let $T_1, \dots, T_r$ be the maximal members of $\mathcal{X}$. Let $L_1, \dots, L_r$ be the subcomplexes of $\mathcal{I}$ given by the subgroups of $T_1, \dots, T_r$ so each L_i is contractible, as is any non empty intersection, so we have the first assertion in the usual way.

Let M_i, $1 \le i \le r$, be the simplex of $\mathcal{F}$ whose vertices are the subgroups of $\mathcal{Y}$ contained in T_i. Thus, M_i is contractible, these simplices cover $\mathcal{F}$ and the intersections behave properly.

We want to record that the covering technique also gives a proof of Brown's theorem. Let P be a Sylow p-subgroup of G and let K be the subcomplex of $\mathcal{S}_p$ consisting of simplices formed by chains of p-subgroups all of which are contained in a Sylow p-subgroup which intersects P in a non-identity subgroup. We claim that K is contractible: we just use the obvious covering by Sylow subgroups and the nerve of this covering is a simplex! The quotient of the chain complex of $\mathcal{S}_p$ by the chain complex of K has a basis corresponding with simplices one of whose subgroups is not in a Sylow subgroup intersecting P non-trivially and so P acts regularly on the quotient! Hence, the Euler characteristic of this quotient is divisible by $|P|$ while that of K is one so the Euler characteristic of $\mathcal{S}_p$, the sum of the other two, is as desired.

Next, we shall prove part of Quillen's theorem on certain p-nilpotent groups using our simpler techniques. First, we need a "Hall-Higman result."

<u>Lemma</u> Suppose that $G = EK$ where E is an elementary abelian p-subgroup of order p^n and K is a normal solvable p'-group containing its centralizer. Let M be a subgroup of K minimal subject to being E-invariant and with E acting faithfully.

1) M is nilpotent, $\Phi(M)$ is central and equals $C_M(E)$.

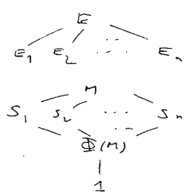
2) There exist n maximal subgroups $F_1, \dots, F_n$ of E such that $C_M(F_i)$ properly contains $\Phi(M)$ with the quotient a simple E-module and M is their central product.

That is, the subgroups $C_M(F_i)$ commute elementwise and $M/\Phi(M)$ is the direct product of the groups $C_M(F_i)/\Phi(M)$. Note that $\cap F_i = 1$ so $\cap F_i$ centralizes M.

<u>Proof</u>. Since $O_p(EM) = 1$, it follows that $F(M) = F(EM)$ so E acts faithfully on $F(M)$. Thus, M is nilpotent. In particular, $M/\Phi(M)$ is abelian of square-free exponent, that is, a direct product of elementary abelian groups for primes other than p, so it is a completely reducible module for E. Since E has order p^n there are n simple modules in $M/\Phi(M)$, $S_1, \dots, S_n$ such that the intersection of the kernels of the action of E is trivial. By minimality, $M/\Phi(M)$ is the direct product of the S_i. Set F_i to be the kernel of the action of E on S_i so F_i is a maximal subgroup of E.

Next, we assert that E centralizes $\Phi(M)$ (so it will equal the centralizer by what we have just shown). If this is not the case then there is a simple E-module "involved" in $\Phi(M)$ such that the kernel F in E is proper in E. Hence, there is j such that $F \cap (\bigcap_{i \neq j} F_i) = 1$, so M is not minimal, by the complete reducibility. We now assert that $\Phi(M)$ is central in M. Indeed, $[M, \Phi(M), E] \leq [\Phi(M), E] = 1$, $[\Phi(M), E, M] = [1, M] = 1$ so $[E, M, \Phi(M)] = 1$, that is, $[M, \Phi(M)] = 1$. This implies that F_i centralizes S_i, not just $S_i / \Phi(M)$, so $S_i = C_M(F_i)$ as $\bigcap F_i = 1$. It remains only to see that $[S_i, S_j] = 1$. However, $[S_i, S_j]$ is a homomorphic image of $S_i / \Phi(M) \otimes S_j / \Phi(M)$ and F_i acts trivially on the first factor but not the second so can only act trivially on $[S_i, S_j]$ if that is trivial, as required.

Continuing with this notation, let $E_i = \bigcap_{j \neq i} F_j$ so E_i is of order p. E is the direct product of the E_i and F_i is the direct sum of the E_j, $j \neq i$. The picture:



Thus, $S_i / \Phi(M)$ is an irreducible E_i-module on which all the E_j, $j \neq i$, act trivially.

We shall next produce a cycle in the complex built on the subgroups of order p in G. For each i, $1 \le i \le n$, let $s_i \in S_i - \Phi(M)$ so the elements $s_1, \dots, s_n$ commute elementwise but s_i is not centralized by E_i. In fact, E_i and $E_i^{s_i}$ do not commute. However, E_i and $E_i^{s_i}$ do commute with E_j and $E_j^{s_j}$ if $i \neq j$. Consider the subcomplex on the subgroups $E_1, E_1^{s_1}, \dots, E_n, E_n^{s_n}$. We have that $E_n, E_n^{s_n}$ is a 0-sphere, $E_{n-1}, E_{n-1}^{s_{n-1}}, E_n, E_n^{s_n}$ is the double suspension of a 0-sphere, that is, a 1-sphere and so on. Hence, we have an $n-1$ sphere, in particular an $n-1$ cycle. If this particular complex were $n-1$ dimensional, we would therefore have a non-zero element of homology. We must use this cycle to construct a cycle in the original Brown complex which is $n-1$ dimensional.

However the poset of simplices in the complex of subgroups of order p that we've constructed corresponds one-to-one with the poset of certain p-subgroups of G, namely the subgroups generated by the set of subgroups of order p. Hence, we have a non-trivial $(n-1)$-cycle in the Brown complex of G so its $(n-1)$st homology is non-zero.

<u>Remark</u>: A little thought and induction gives in other situation, without assuming that K is solvable, that the p-subgroups R of G with $R = O_p(N(R))$ are exactly the intersection of Sylow subgroups (thus exactly true by Brauer's student result). Moreover, if $R_1 < R_2$ are two such then there is another R_3 with $R_1 < R_3 < R_2$ provided $|R_1 : R_2| > p$

Next, we turn our attention to the symmetric groups.

<u>Theorem</u> The reduced homology $\hat{H}_{[n/p]-1}(\mathcal{S}_p(\Sigma_n)) \neq 0$ if $p > 3$.

This is the best possible in terms of highest dimension because of the Quillen complex. Let $C_p(\Sigma_n)$ be the subcomplex given by p-subgroups whose orbits on $\{1, \dots, n\}$ are all of size 1 or p, that is, direct products of groups of order p consisting of p-cycles. In the top dimension, this agrees with the Quillen complex so we need only prove the corresponding statement for this complex. However, any two such p-subgroups (with the prescribed orbits) intersect in such a subgroup and so contain a p-cycle. Hence, we need only look at the complex built from subgroups of order p (consisting of p-cycles) and the commuting relationship; call it $C_p'(\Sigma_n)$. Now p-cycles commute if, and only if, they are disjoint or generate the same subgroup. We can thus construct a cycle in the top dimension just as we did for the solvable groups in Quillen's theorem: Let E_1, E_1' be distinct subgroups of order p "on the first p letters," E_2, E_2' on the next p letters and so on!

<u>Remarks</u> 1. If n is not too large we can also count the dimensions of the chains on $C_p'(\Sigma_n)$ in the top two dimensions and a suitable inequality gives non-zero homology. The counts are – for $r = [\frac{n}{p}]$ –

$$\tfrac{1}{r!} \binom{n}{p}\binom{n-p}{p} \cdots \binom{n-rp}{p}((p-1)!)^r$$

and

$$\frac{1}{(r-1)!}\binom{n}{p}\binom{n-p}{p}\cdots\binom{n-rp+p}{p}\left((p-2)!\right)^{r-1}$$

so we have to compare

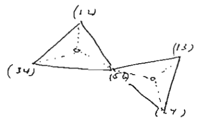
$$\frac{1}{r}\binom{n-rp}{p}(p-2)! \quad \text{and} \quad 1.$$

2. If $p=2$ and $n=2m$ is even then the top dimension of homology of $C_p'(\Sigma_n)$ is zero. Indeed, consider a typical $(m-1)$-simplex

$$\langle (12)\rangle, \ldots, \langle (n-1\ n)\rangle$$

It is retractible onto the subspace which is the join of the barycenter to the $(m-3)$-skeleton. Since any two such $(m-1)$-simplices intersect in such skeletons, as the $(m-1)$-simplex is the unique one containing any of its $(m-2)$-simplices (e.g. $\langle(12)\rangle, \ldots, \langle(n-3, n-2)\rangle$ determines $(n-1, n)$), we have that $C_p'(\Sigma_n)$ retracts onto a subspace of dimension $m-2$ (which is the geometric realization of a simplicial complex).

Here's a relevant picture for $n=6$:



3. Next, we claim that $H_1(\mathcal{S}_p(\Sigma_{3p}))\neq 0$ if $p>3$ showing that there are two non-zero homology groups as "soon" as is possible. Let $C_p' = C_p'(\Sigma_n)$ and let C_p'' be the simplicial complex whose vertices are the p-element subsets of $\{1,\ldots,3p\}$ and where disjoint unions define the simplices. Hence, there is a

simplicial map of C_p' onto C_p''. The argument in the previous comment (there for $p=2$) applies to C_p'' and so we get $H_2(C_p'')=0$. But the reduced Euler characteristic of C_p'' is

$$-1+\binom{3p}{p}-\frac{1}{2}\binom{3p}{p}\binom{2p}{p}+\frac{1}{6}\binom{3p}{p}\binom{2p}{p}\binom{p}{p}$$
$$=-1+\binom{3p}{p}\left\{1-\frac{1}{3}\binom{2p}{p}\right\}\neq 0$$

so as C_p'' is connected we have that $H_1(C_p'')\neq 0$. But then we have cycles in C_p'', pictured as follows,

which then clearly lift to C_p'. But if the lift is a boundary then so is the original using the map of C_p' onto C_p''. Hence, $H_1(C_p')\neq 0$ as claimed.

We have a general construction in play. If $k>1$ and X is a simplicial complex let $X^{(k)}$ be another mapping onto X with k vertices "over" each one of X and with t vertices of $X^{(k)}$, over distinct vertices of X, forming a simplex if, and only if, their images in X do. Question: Compare the homology of X and $X^{(k)}$.

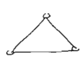
What we can say is that the homology of X is a summand of the homology of $X^{(k)}$. Indeed, there is a map of X to $X^{(k)}$, sending any vertex of X to one of the vertices of $X^{(k)}$ above it, so that the composition with the map of $X^{(k)}$ onto X is the identity on X. Hence, there is a "retract" of the homology of $X^{(k)}$ onto the homology of X, as claimed. (Taking the case of X a triangle, $k=2$, we can easily see that this decomposition is non-trivial.

4. We next assert that $H_{p-2}(\mathcal{S}_p(\Sigma_{p^2})) \neq 0$ if $p \geq 3$ (so the two top dimensions are non-vanishing in homology). By the end of the previous remark, and using the Quillen complex, it suffices to show that $H_{p-2}(C_p''(\Sigma_{p^2})) \neq 0$ (as the "other" maximal abelian p-subgroups have rank much smaller). But the $(p-2)$-skeleton of $C_p''(\Sigma_{p^2})$ is a subcomplex of a simplicial complex of dimension $p-2$ which is itself of the same homotopy type as $C_p''(\Sigma_{p^2})$, by the method of remark 2; hence, it suffices to exhibit a $(p-2)$-cycle. But $p^2 = (p+1)(p-1)$ so we can write a disjoint union

$$\{1, 2, \ldots, p^2\} = A_1 \cup \cdots \cup A_{p-1} \cup \{p^2\}$$

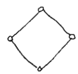
where $|A_i| = p+1$, for all i. Let E_i, F_i be subsets of size p in A_i, and distinct, so $E_1, F_1, E_2, F_2, \ldots$ give us a cycle in the "usual" way.

5. We now establish a vanishing result: $H_1(\mathcal{S}_p(\Sigma_n)) = 0$ if $5p-2 \leq n < p^2$. In fact, we prove that $H_1(C_p'(\Sigma_n)) = 0$ if $n \geq 5p-2$. (It should be easy to continue the argument and get a better lower bound). We have to see that any "geometric" 1-cycle is homologous to zero. But if we have

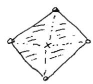


then, by the definition of C_p' this is the boundary of a triangle, the same one:

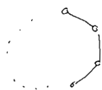
If we have



then it suffices to show that the sets of size p involved have a union of size at most $(5p-2)-p = 4p-2$ as then there is another vertex so that the quadrilateral is the boundary of the union of four triangles:



But opposite vertices must have overlapping corresponding sets so involve at most $2p-1$ letters, and $2(2p-1) = 4p-2$. Similarly, if we have



we have

so we can pass to a "shorter" cycle.

6. $H_1(\mathcal{S}_p(\Sigma_{p^2})) \neq 0$. For $\mathcal{A} = \mathcal{A}_p(\Sigma_{p^2}) = \mathcal{C} \cup \mathcal{D}$, $\mathcal{C} = \mathcal{C}_p(\Sigma_{p^2})$ and $\mathcal{D}$ the complex formed from regular subgroups. The Mayer-Vietoris gives

$$\leftarrow \underset{0=}{H_1(\mathcal{A})} \leftarrow \underset{\neq 0}{H_1(\mathcal{C}) \oplus H_1(\mathcal{D})} \leftarrow \underset{= 0}{H_1(\mathcal{C} \cap \mathcal{D})}$$

as is easy to see.

7. Here's a quicker way to get the non-vanishing of the top homology of Σ_n. Let $n = tp + r$, $r < p$ and consider the Young subgroups

$$\overbrace{\Sigma_p \times \cdots \times \Sigma_p}^{t} \times \Sigma_r$$

which has the same p-rank as that of Σ_n. It's top homology is not zero as we're looking at a join of discrete points if $p > 3$.

We can also see that if $r = 0$ then all the top dimensional homology arises this way. Working with the "p-cycle complex" C_p' we need only see that every top cycle in C_p' is the sum of cycles each of which lies over a single simplex of C_p''. But this is easy as the boundary of a top-dimensional simplex of C_p' also determines the underlying partition: remember the simplex "comes from" t disjoint p-cycles.

In this last paragraph, we can also allow $p = 2, 3$ to get that the top homology is zero in these cases! This is easier to explain than the argument we have already seen before. Note that in the case $p = 2$ we mean only the top homology of the relevant subcomplex!

We will return in due course to consider the prime $p = 2$ further, in particular to compare our results with those of Bouc.

Next, let's return to Quillen's theorem on p nilpotent groups and give a simple proof of the other half. The set-up: $G = EK$, $K = O_{p'}(G)$ is solvable, E is elementary abelian of order p^n and we want to show that the reduced homology ($\mathbb{Z}$ coefficients) vanishes as claimed, that is, $\hat{H}_i(\mathcal{S}_p(G)) = 0$, $0 \le i < n-1$. We proceed by double induction on n and $|G|$. If $n = 1$ there is nothing to show. If $n = 2$ then we are only claiming that $\mathcal{S}_p(G)$ is connected, which is so as then G cannot contain a strongly p-embedded subgroup.

Let K/L be a chief factor of G so $|K/L| = r$, a prime power and K/L is a simple module for E. If K/L is a central chief factor then $\mathcal{S}_p(G) = \mathcal{S}_p(EL)$ so we're done, by induction. Thus, if $F = C_E(K/L)$ then $|E:F| = p$. Since $C_K(F)$ then covers K/L we can choose coset representatives $1 = x_1, \ldots, x_r$ for L in K from $C_K(F)$. Let $E_i = E^{x_i}$ so every Sylow p-subgroup of G lies in one of the groups $E_i L$, $1 \le i \le r$; that is, $\mathcal{S}_p(G) = \mathcal{S}_p(E_1 L) \cup \cdots \cup \mathcal{S}_p(E_r L)$.

Hence, to prove the assertion, we need only show that

$$\hat{H}_j(\mathcal{S}(E_1 L) \cup \cdots \cup \mathcal{S}(E_i L)) = 0$$

whenever $0 \le j < n-1$ and $1 \le i \le r$. If $i = 1$ this is true by induction. Hence, let $X = \mathcal{S}_p(E_1 L) \cup \cdots \cup \mathcal{S}_p(E_i L)$, $Y = \mathcal{S}_p(E_{i+1} L)$ and we have to deal with $X \cup Y$ using our inductive information on X and Y (so we're doing a triple induction).

First, suppose that $j > 1$ so $j - 1 > 0$. We have the Meyer-Vietoris sequence

$$H_{j-1}(X \cap Y) \leftarrow H_j(X \cup Y) \leftarrow H_j(X) \oplus H_j(Y)$$

where

$$H_j(X) = H_j(Y) = 0$$

as we have seen above. Moreover

$$X \cap Y = \mathcal{S}_p(FL).$$

Indeed, $\mathcal{S}_p(FL) \subseteq \mathcal{S}_p(E_i L)$ for all i as $\chi_i \in C(F)$. On the other hand, if P is a p-subgroup that is a vertex in X and Y, that is in $\mathcal{S}_p(E_t L)$ and in $\mathcal{S}_p(E_{i+1} L)$, $1 \leq t \leq i$ for some t, then P is a p-subgroup of FL: for $E_t L/L$ and $E_{i+1} L/L$ intersect in FL/L. Hence, since $j-1>0$, we have, by induction once again, that $H_{j-1}(X \cap Y) = 0$. Thus $H_j(X \cup Y) = 0$ as needed.

Finally, suppose that $j = 1$ (the case $j=0$ is trivial by connectedness). Here we have

$$0 \leftarrow \underset{\simeq \mathbb{Z}}{H_0(X \cup Y)} \leftarrow \underset{\simeq \mathbb{Z} \oplus \mathbb{Z}}{H_0(X) \oplus H_0(Y)} \leftarrow \underset{\simeq \mathbb{Z}}{H_0(X \cap Y)} \leftarrow H_1(X \cup Y) \leftarrow \underset{0}{H_1(X)} \oplus \underset{0}{H_1(Y)}$$

and the first three terms give an exact sequence by connectedness. Thus, $H_1(X \cup Y) = 0$ and everything is proved.

[We just need to record the genesis of the proof - though we didn't actually check this idea out line by line. The idea is to look at the Leray spectral sequence of the covering

$$\mathcal{S}_p(G) = \mathcal{S}_p(E_1 L) \cup \dots \cup \mathcal{S}_p(E_n L)$$

using inductive information to deal with all the homology of the intersections

$$\mathcal{S}_p(E_{i_1} L) \cap \dots \cap \mathcal{S}_p(E_{i_u} L). \quad]$$

Let N be a normal subgroup of the group G, set $\bar{G} = G/N$ and assume that N is a p'-group. Let $\mathcal{C}$ be a normal set of subgroups of order p of G, which generate G, and let $\bar{\mathcal{C}}$ be the image in $\bar{G}$. Let $|\mathcal{C}|$ and $|\bar{\mathcal{C}}|$ be the simplicial complexes given by the "commuting" relationship.

Theorem (Alperin + Blackburn) If $|\bar{\mathcal{C}}|$ is connected then the natural map

$$\pi : |\mathcal{C}| \to |\bar{\mathcal{C}}|$$

is a covering iff the following conditions hold:

1) whenever $X, Y \in \mathcal{C}$ then $C_N(X) = C_N(Y)$;

2) N admits no (proper) supplement in G.

This answers a conjecture of Bouc. Consider the non-trivial extension

$$1 \to N \to G \to \Sigma_7 \to 1$$

where N is cyclic of order three, $\bar{G} = \Sigma_7$, $\bar{\mathcal{C}}$ is the class of transpositions, $p = 2$, $\mathcal{C}$ the class of involutions "over" $\mathcal{C}$. We now have a triple cover of the complex $|\bar{\mathcal{C}}|$ studied by Bouc. He proved $H_1(|\bar{\mathcal{C}}|) \cong \mathbb{Z}/3\mathbb{Z}$ and conjectured the above group extension was relevant. (The covering shows $\pi_1(|\bar{\mathcal{C}}|)$ has a subgroup of index three - hence a normal one as there is no quotient of π_1 of order two.)

Let's turn to the proof of the result. We begin with an observation.

Lemma If $X \in \mathcal{C}$ then any simplex of $|\bar{\mathcal{C}}|$ containing $\bar{X}$ is the image under π of a simplex of $|\mathcal{C}|$ containing X.

Proof. Say $X = X_0$ and $X_1, \ldots, X_\alpha \in \mathcal{C}$ so that $\bar{X}_0, \bar{X}_1, \ldots, \bar{X}_\alpha$ are the distinct vertices of a simplex. Thus, $H = \langle N, X_0, \ldots, X_\alpha \rangle$ is a p-nilpotent group with abelian Sylow p-subgroups. Let P be a Sylow p-subgroup of H containing X_0. Since P covers H/N there are elements $m_1, \ldots, m_\alpha$ of N such that $X_1^{m_1}, \ldots, X_\alpha^{m_\alpha} \leq P$. Hence, $X_0, X_1^{m_1}, \ldots, X_\alpha^{m_\alpha}$ is the desired simplex.

Lemma (cont.) The simplex containing X with the desired property is unique iff

$$C_N(X_0) \subseteq \bigcap_{i=1}^{d} C_N(X_i^{m_i})$$

Proof. If $C_N(X_0) \nsubseteq C_N(X_j^{m_j})$, some j, $1 \leq j \leq d$, then there is $m \in C_N(X_0)$ with $m \notin C_N(X_j^{m_j})$. Thus,

$$X_0 = X_0^{m}, \quad X_1^{m_1 m}, \ldots, \quad X_d^{m_d m}$$

is another (as $X_j^{m_j m} \neq X_j^{m_j}$) such simplex.

Conversely, suppose that $X_0, Y_1, \ldots, Y_d$ is another simplex with $\bar{Y}_i = \overline{X_i^{m_i}} = \bar{X}_i$, $1 \leq i \leq d$. Since this is another simplex choose j, $1 \leq j \leq d$, with $Y_j \neq X_j^{m_j}$. (Hence there are edges $\{X_0, Y_j\}$ and $\{X_0, X_j^{m_j}\}$ over $\{\bar{X}_0, \bar{X}_j\}$.) Thus, $X_j^{m_j}$ and Y_j are Sylow p-subgroups of $\langle N, X_j^{m_j} \rangle = \langle N, Y_j \rangle$, in fact, Sylow p-subgroups of $C_{\langle N, Y_j \rangle}(X_0)$ so $Y_j = (X_j^{m_j})^c$ with $c \in C_N(X_0)$. Thus, $c \notin C_N(X_j^{m_j})$ as required.

Proof (of the theorem). Suppose that π is a covering map. To establish 1) it suffices to show that if $X, Y \in \mathcal{C}$ and commute then $C_N(X) = C_N(Y)$ since $|\mathcal{C}|$ is connected, by definition. However, if $C_N(X) \neq C_N(Y)$ then we can assume (by change of notation if necessary) that there is $m \in C_N(X)$, $m \notin C_N(Y)$. Then $\{X, Y\}$ and $\{X, Y^m\}$ are edges over $\{\bar{X}, \bar{Y}\}$ contradicting the path lifting properties of a covering.

Next, assume that H is a proper supplement to N in G; we shall derive a contradiction and so establish half of the theorem. Since N is a p'-group and H covers G/N it follows that H contains a Sylow p-subgroup of each subgroup of G containing N; in particular, $H \cap \mathcal{C}$ (the collection of subgroups in $\mathcal{C}$ which lie in H) has the property $\overline{H \cap \mathcal{C}} = \bar{\mathcal{C}}$. Since H is a proper subgroup and $\mathcal{C}$ generates G we have that $H \cap \mathcal{C}$ is properly contained in $\mathcal{C}$. Thus, it suffices to show that no vertex of $|H \cap \mathcal{C}|$, that is, subgroup of $\mathcal{C}$ in H commutes with such a subgroup of $\mathcal{C}$ not in H, as then we will have violated the connectedness of $|\mathcal{C}|$. But suppose $X \in \mathcal{C}$, $X \leq H$, $Y \in \mathcal{C}$, $Y \not\leq H$ and that X and Y commute. Then $\bar{X} \neq \bar{Y}$ as X does not commute with any of its conjugates in $\langle X, N \rangle$. We now apply the first lemma to the group H and $H \cap \mathcal{C}$ using the fact that $\overline{H \cap \mathcal{C}} = \bar{\mathcal{C}}$: there is $Y_1 \in H \cap \mathcal{C}$, $\bar{Y}_1 = \bar{Y}$ with X and Y_1 commuting. Again we have contradicted the path lifting.

We now prove the converse, assuming that 1) and 2) hold. In particular, the second lemma applies to give us a useful uniqueness statement. We next shall see [illegible] proper subset of $\mathcal{C}$, the vertices of a connected component of $|\mathcal{C}|$. Since $|\bar{\mathcal{C}}|$ is connected, the first lemma implies that $\bar{\mathcal{D}} = \bar{\mathcal{C}}$. Let $H = \langle \mathcal{D} \rangle$ so $HN = G$ as $G = \langle \mathcal{C} \rangle$ and $\bar{\mathcal{C}} = \bar{\mathcal{D}}$. To see that H is proper, it suffices to show that $\mathcal{D}$ is a normal subset of H since the equality $\bar{\mathcal{C}} = \bar{\mathcal{D}}$ forces $\mathcal{D}$ to contain subgroups from each conjugacy class making up $\mathcal{C}$. Let $D = \langle a \rangle \in \mathcal{D}$. Thus $\mathcal{D} \cap \mathcal{D}^a \supseteq \{D\}$ so by connectedness, $\mathcal{D}^a \subseteq \mathcal{D}$, $\mathcal{D}^a = \mathcal{D}$ and the connectedness is finally established.

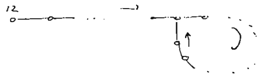
The fact that any simplex of $\mathcal{C}$ maps one-to-one to a simplex of $\bar{\mathcal{C}}$ and the lemmas imply that any simplex of $\bar{\mathcal{C}}$ has pre-image under π consisting of the disjoint union of r simplices. $r = |N : C_N(X)|$. This implies that the inverse image of the star neighborhood of a vertex of $\bar{\mathcal{C}}$ is similarly constituted and mapped. Since any point of $\bar{\mathcal{C}}$ has a neighborhood of this sort we have (see Hilton-Wylie, for example) that π is a covering.

<u>Remarks</u> 1. N is nilpotent when π is a covering by virtue of condition 2).

2. $C_N(X)$ is a normal subgroup of N: For $\mathcal{C}$ is normal and generates G and condition 1) applies when π is a covering. In particular, to construct examples, we will lose nothing by restricting ourselves to the situation that each X is fixed-point-free on N.

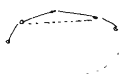
3. Consequently, the covering is normal and the corresponding quotient of the fundamental group is $N/C_N(X)$: this group is transitive on the "sheets."

Let's now show that the fundamental group of the commuting complex of transpositions in Σ_7 is cyclic. Fix a vertex, say (12). Consider a loop from that vertex, which may be assumed to be simplicial. Follow the loop to the first repetition of vertex, obtaining something of the form

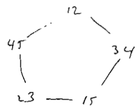


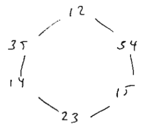
(a "handle" and a "spoon"). By returning to the vertex we started from and then back to the repetition and then completing the original loop we have a product of two loops, one a "handle and spoon." Hence, it suffices to show that each such handle and spoon is homotopic to a power of a fixed loop from (12).

Let's consider the spoon part of the loop. If two of its vertices commute, other than adjacent ones, then it is easy to see the handle and spoon is a product of two such with smaller spoons (the case

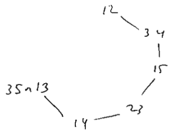


is reduced to one such loop with a smaller spoon). Hence we need to analyze polygons that are minimal in the sense of having only the given commuting relations between vertices. The types (up to Σ_7 conjugacy) are easy to enumerate:





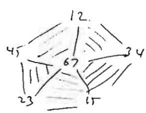
(and we can't go further as we get stuck at



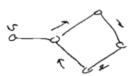
)

The triangle is "solid", i.e.  in the space.

The last two are contractible, filled with solid triangles using the additional vertex (67), e.g.

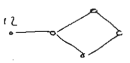


Hence, it suffices to show that a path (the last part of the handle and spoon)

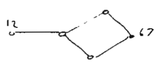


is homotopic to a quadrilateral, doing this with v fixed, and then deal with all quadrilaterals relevant to fixed vertex, say (12).

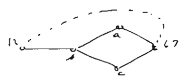
Now what are the Σ_7 orbits for short handle and spoons,



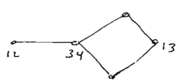
with the vertex (12) fixed (and we don't care about direction of the path because we're just after cyclicity of the fundamental groups). The types that have the "far" vertex not involving "1" or "2", e.g.



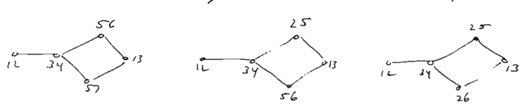
are easily seen to be a product of two quadrilaterals:



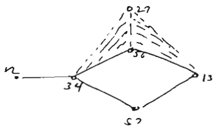
($12 \to b \to c \to 67 \to 12$ followed by $12 \to 67 \to c \to b \to 12$). Hence, we can assume we are dealing with



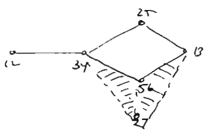
The remaining two vertices may not involve "2", one case, or one may and other does not, a second case, or they may both involve "2" the third case. Representatives of the orbits are drawn as follows:



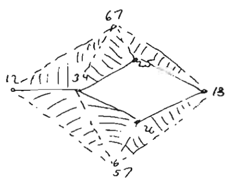
But the first type reduces to the second:



and the second to the third:

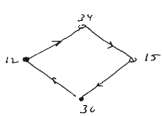


The third is easily dealt with:



Hence, we have to show all the quadrilaterals, with (12) remaining fixed, are all homotopic to each other or their inverses.

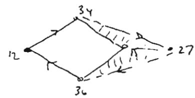
We begin with some notation for quadrilaterals:



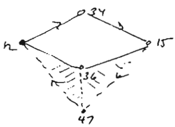
:

1 3
2 5 4 6
7

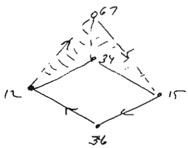
so the notation accounts for the initial point and the direction (so, if we were only interested in homology of quadrilaterals up to sign, we would need some identifications). Let's see how some homotopies are reflected in this notation.



2   3
1. 7   4 6
5

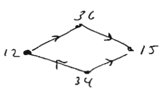


1   4
2 5   3 7
6



1   6
2 5   7 3
4

Change of direction gives



1   3
2 5   6 4
7

Let's work with these operations. What are the orbits of the diagrams

1   .
2 .   . .
.

We can assume that "3" is at the top right, i.e. in the "orbit" there is one such diagram, e.g.:

$$2^{1}3\;{}_{7}\;5^{4}6 \to 1^{2}7\;{}_{3}\;5^{4}6 \to 1^{2}7\;{}_{5}\;3^{6}4 \to 1^{2}7\;{}_{4}\;6^{3}5 \to 2^{1}4\;{}_{7}\;6^{3}5$$

Similarly, can get to either of the following:

$$2^{1}\cdot\;{}_{\cdot}\;4^{3}\cdot \quad \text{or} \quad 2^{1}\cdot\;{}_{4}\;\cdot^{3}\cdot$$

But can "eliminate" the second:

$$2^{1}5\;{}_{4}\;6^{3}7 \to 1^{2}4\;{}_{5}\;6^{3}7 \to 1^{2}4\;{}_{6}\;5^{7}3 \to 2^{1}6\;{}_{4}\;5^{7}3 \to 2^{1}6\;{}_{5}\;4^{3}7$$

However, have

$$2^{1}5\;{}_{7}\;4^{3}6 \to 1^{2}7\;{}_{5}\;4^{3}6 \to 1^{2}7\;{}_{6}\;3^{4}5 \to 2^{1}6\;{}_{7}\;3^{4}5 \to 2^{1}6\;{}_{5}\;4^{3}7$$

(i.e. rotation of the three "open" vertices which have to be 5, 6, 7)
However, using one change of direction - denoted $\Rightarrow$ - we have

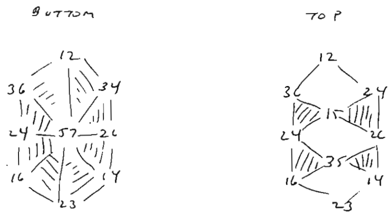
$$2^{1}5\;{}_{7}\;4^{3}6 \to 2^{1}5\;{}_{4}\;7^{6}3 \to 1^{2}4\;{}_{5}\;7^{6}3 \to 1^{2}4\;{}_{7}\;5^{3}6 \to 2^{1}7\;{}_{4}\;5^{3}6$$

$$\to 2^{1}7\;{}_{5}\;4^{6}3 \Rightarrow 2^{1}7\;{}_{5}\;3^{6}4 \to 2^{1}7\;{}_{4}\;6^{3}5 \to 1^{2}4\;{}_{7}\;6^{3}5$$

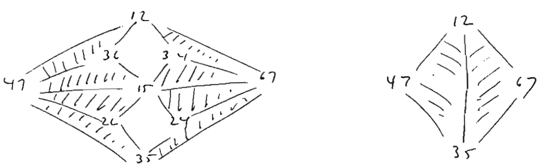
$$\to 1^{2}4\;{}_{6}\;7^{5}3 \to 2^{1}6\;{}_{4}\;7\,5 \to 2^{1}6\;{}_{7}\;4^{3}5$$

So we have also effected a transposition on the last two "5 and "6" and the proof is complete.

[Here's another way - not at all so nice as Bruce's - to see the quadrilaterals have order three. We construct a sphere with three quads cut out:

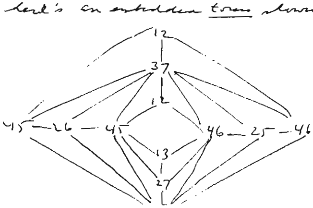


(and we can put on two more layers but no longer have a sphere (notice the switch of 26 and 24)

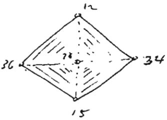


)]

Finally, here's an embedded torus showing $H_2 \neq 0$:



For Σ_n, $n \geq 8$, we can also consider the commuting complex for transpositions but here the fundamental group is trivial. Indeed, by an entirely similar analysis we need only show that quadrilaterals are suitably contractible. But we have



This has an amusing minor consequence: the fact that the multiplicator of Σ_n is a 2-group implies that the same holds for A_n. Indeed, if this were not the case we would have — by the usual arguments using free groups — an extension, for $p > 2$,

$$1 \to Z_p \to G \to \Sigma_n \to 1$$

whose restriction to A_n,

$$1 \to Z_p \to H \to A_n \to 1$$

is a covering but which is not a covering. Thus Σ_n/A_n inverts the subgroup Z_p and our result with Blackburn applies and we have the desired conclusion.

This last technique might well be useful for other groups with a normal subgroup of index two.

We now wish to give a converse to the Alperin-Glauberman result. Let $\bar{G}$ be a finite group, $\bar{C}$ a normal set of subgroups of order p in $\bar{G}$, Y the corresponding simplicial complex. Suppose that

$$X \to Y$$

is an n-fold normal covering of Y with N the group of covering transformations, so $|N| = n$. Assume that $p \nmid n$ and that there is an automorphism of the covering which induces on Y the action of any $g \in \bar{G}$. Finally, assume $\bar{G}$ acts faithfully on Y so there is a group G of automorphisms of the covering and an exact sequence

$$1 \to N \to G \to \bar{G} \to 1.$$

Let C be the corresponding normal set of subgroups of order p in G.

<u>Proposition</u> Under the above conditions, if for any $g \in G$ with $\langle g \rangle \in C$ we have g acting freely on N then the covering $X \to Y$ is isomorphic with the covering $|C| \to |\bar{C}|$.

Note that if $X \to Y$ is the largest abelian covering of such and such exponent then the fact that actions of $\bar{G}$ on Y come from automorphisms of X is automatic. Also note that by passing to a quotient of $\bar{G}$ and then pulling back to $\bar{G}$ we need not assume $\bar{G}$ faithful.

We have a one-to-one correspondence between subgroups of $\bar{C}$ and vertices of $Y = |\bar{C}|$. We shall define a one-to-one correspondence from C to X that is related. Let $\langle g \rangle \in C$ and let y be the vertex of Y corresponding with $\langle \bar{g} \rangle$. Since $p \nmid n$ there is a vertex of X in the fiber over y left invariant by g. It is unique since all the other vertices in the fiber are of the form nx, $n \in N - \{1\}$ and $gnx \neq nx$ or else $g^{-1}n^{-1}gnx = x$ and

$[g,n]=1$ as $[g,n]\in N$ and fixes a vertex. Hence, let $\langle g\rangle$ correspond to x. By a count this is one-to-one and onto.

Next, suppose that $\langle g'\rangle \in C$ with similar vertices x' and y'. We claim that there is an edge connecting x and x' if, and only if, g and g' commute. First, suppose that x and x' are connected by an edge. This implies that g fixes x': otherwise x and gx' are also joined by an edge, as $x=gx$, which contradicts the covering. Thus,

$$g g' x' = g x' = x' = g' x' = g' g x'.$$

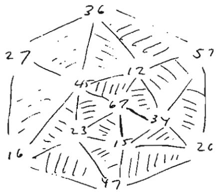
But $[g,g']\in N$ since x and x' joined implies that y and y' are too so $\bar g$ and $\bar g'$ commute. Hence, as N consists of covering transformations we get $[g,g']=1$. Second, suppose that g and g' commute. It suffices to show that $gx'=x'$ (as x is joined to the fixed-point over y' as y' is fixed by g as $\bar g\bar g'=\bar g'\bar g$), that is, gx' is fixed by g' (the unique fixed-point over y' for g' is x'). But $g'(gx') = g(g'x') = gx'$ as desired.

Finally, the covering property and the definition of Y in terms of commuting show that the simplices of X and $|C|$ coincide under the identification.

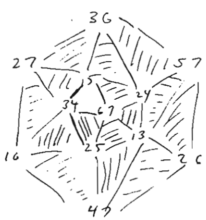
Question: In general, coverings give rise to group extensions. Which extensions arise in this way? How can the above considerations be generalized. Just assume $p \nmid n$ but non-free action, for example.

We wish to insert a correction here. The torus we exhibited to show $H_2 \neq 0$ on page 70 was in error. Here's a correct construction:

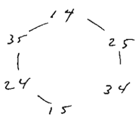
TOP



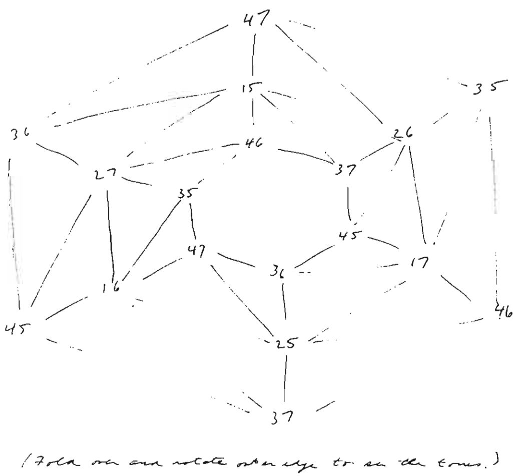
BOTTOM



But we can do much better. The idea is to partition the set $\{1, \ldots, 7\}$ into a set of size 4 and one of size 3 and use this (just as for Σ_5, the hexagons come this way, e.g.



).



(Fold over and rotate about edge to see the torus.)

(Compare with Cameron's picture in Kantor's "GAB paper", Europ. J. Comb 1981)

Let's see directly that this subcomplex, which turns out to be a torus, is orientable. The group $\Sigma_4 \times \Sigma_3$ is regular on ordered triangles such as

$$15/26/37.$$

Hence, $A_4 \times \Sigma_3$ acts so as to have two orbits on these ordered triangles. Using the lexicographic-numerical ordering of the vertices, it is clear that $A_4 \times \Sigma_3$ does not change the orientation of an ordered triangle so the two orbits are two orientations, that is an assignment of an orientation to each triangle. To see that the complex is oriented, we must see that an edge common to two triangles (and an edge is on exactly two triangles) receives two different orientations in this way. But, by transitivity on triangles, the case of the triangles

$$15/26/37 \quad \text{and} \quad 15/26/47$$

and common edge $15/26$ is general. However, the odd element, the transposition (34) carries the one ordered triangle to the other. Hence, if we have oriented $15/26/37$ one way then the other triangle meets the edge $15/26$ the other way.

Of course, this generalizes immediately, e.g. to

$$16/27/38/49 \quad \rightleftarrows \quad 16/27/38/59$$

Hence, we get an orientable subcomplex so $H_{n-1} \neq 0$.
It is easy to see that in higher dimensions this subcomplex is not a manifold (nbhd of a pt, e.g.)

We wish to develop the equivariant version of the contractible covering lemma, to deduce it from the Thévenaz-Webb equivariant poset result. Let X be a simplicial complex acted on by the group G and let $Y_1, \dots, Y_n$ be subcomplexes permuted by G which cover X so $|X| = |Y_1| \cup \dots \cup |Y_n|$. For each $I \subseteq \{1, \dots, n\}$ let $Y_I = \bigcap_{i \in I} Y_i$ and let G_I be the stabilizer in G of Y_I. Let N be the nerve of the cover so its simplices are the sets I for which $Y_I \neq \emptyset$.

__Proposition__. If Y_I is G_I-contractible for each simplex I of N then $|X|$ and $|N|$ are of the same G-homotopy type.

__Proof__. Consider the usual map from simplices σ of X to simplices I_σ of N: $I_\sigma = \{ i \mid \sigma \subseteq Y_i \}$. This is an order reversing equivariant map of posets. To use the result of Thévenaz and Webb, to prove that $|Sd X|$ and $|Sd N|$ are of the same G-homotopy type, it suffices to prove that if I is a simplex of N then the subposet of simplices of X consisting of those simplices τ for which $I_\tau \supseteq I$ is G_I-contractible. However, this is simply the set of simplices τ for which $\tau \subseteq Y_i$ for each $i \in I$. Its geometric realization is homeomorphic with $|Y_I|$ so we're done.

We next wish to apply this to a number of complexes that we have already constructed and shown to be of the same homotopy type as the Brown complex. (It may well be also possible to use the Bredon criterion.)

Let's consider the Sylow intersection complex. In this case we need to see that if $P_1, \dots, P_t$ are Sylow p-subgroups of G then $\mathcal{S}(P_1 \cap \dots \cap P_t)$ is $N_G(P_1 \cap \dots \cap P_t)$-contractible if $P_1 \cap \dots \cap P_t \neq 1$. But this poset is a cone on $P_1 \cap \dots \cap P_t$ so this is clear.

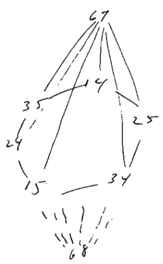
The same applies equally well to the complex of intersections of maximal elementary abelian subgroups - it is of the same homotopy type as the Quillen complex and this is equivariant now. Hence, the same holds for the Brown complex and it.

Let's go on to the commuting complex. Let $E_1, \dots, E_n$ be the maximal elementary abelian p-subgroups of G so these are the vertices of an "intersection" complex that we wish to relate to the commuting complex. Let $Q_1, \dots, Q_r$ be the subgroups of order p in G. To each Q_j associate the subcomplex of the intersection complex consisting of the E_i's which contain Q_j and their intersections which contain Q_j; that is $\{E_{i_1}, \dots, E_{i_u}\}$ is a simplex of this subcomplex iff $\cap E_{i_t}$ contains Q_j. Each of these subcomplexes is a simplex so is the intersection of any of them which is non-empty. But a simplex is a cone on its barycenter so is contractible equivariantly for any group acting on it. We therefore have the desired equivariant homotopy equivalence.

Continuing the discussion on page 76 about the geometric realization of homology, let's just briefly look at some other possibilities. For example, taking $p=2$, $n=8$ we can consider a "configuration"

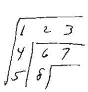
$$123/45 \,//\, 67/8$$

where we mean the join of the constructions on each side of the double bar — and then the symmetric groups applied to get more subcomplexes as before. Hence, we're looking at



which is a sphere.

Perhaps the way to look at this is in terms of main diagonal hooks:



Perhaps we're getting some of the homology that Bruce calculated.

Here is a stab at $p=3$, $n=10$. Consider

$$1234/567/8910.$$

We mean to choose two of the first four letters, one from each of the other groups of letters so part of the subcomplex we're considering is

We wish to observe how Quillen's theorem on p-nilpotent groups implies a much stronger result, namely

THEOREM. If G is a p-nilpotent solvable group with $O_p(G)=1$ and E is a maximal elementary abelian p-subgroup of G of order p^{r+1} then $\tilde{H}_r(\mathcal{S}_p(G)) \neq 0$.

PROOF. We deal with $\mathcal{A}_p(G)$ instead, as we may. Let $K=O_{p'}(G)$ so Quillen's theorem applies to $\mathcal{A}_p(EK)$ $(=\mathcal{S}_p(EK))$ and gives $\tilde{H}_r(\mathcal{A}_p(EK)) \neq 0$ as $O_p(EK)=1$. Let $z \in Z^r(\mathcal{A}_p(EK))$ a non-zero cycle so it represents a non-zero homology class as $C^{r+1}(\mathcal{A}_p(EK))=0$. It suffices to see that z is not a boundary in $\mathcal{A}_p(G)$!

But z is a linear combination of r-simplices in EK, that is chains $E_0 < E_1 < \dots < E_r$ of elementary abelian p-subgroups, $|E_i|=p^{i+1}$. This forces E_r to be a conjugate of E, by Sylow's theorem. If z is a boundary then this simplex must appear in the expression of a boundary of an $(r+1)$-simplex of $\mathcal{A}_p(G)$, something of the form $F_0 < F_1 < \dots < F_{r+1}$, F_i elementary abelian of order p^{i+1}. Hence, $F_r = E_r$ so E_r is not maximal as claimed. This proves the theorem.

Remark. We are actually looking at an edge homomorphism from a spectral sequence.

A doubly transitive action modulo p

Let G act doubly transitively on the set X with H a one-point stabilizer. Assume $p \mid |X|$ and k is algebraically closed of characteristic p. Let F be the one-dimensional submodule of kX spanned by $\sum_{x\in X} x$, so $F \cong k$ as kG-modules. Let Z be the "trace zero" submodule of kX, the set of all $\sum_{x\in X} \alpha_x x$, $\alpha_x \in k$, with $\sum \alpha_x = 0$ so $kX/Z \cong k$ and $kX \supseteq Z \supseteq F \supseteq 0$. Let $M = Z/F$.

Hypothesis. Assume $G = O^{p}(G)$, $H = O^{p}(G)$, the p part of the Schur multiplier of G is a non identity cyclic group and the p-part of the Schur multiplier of H is the identity.

This situation arose: $G = PSL(3, q)$, $q \equiv 1 \pmod{3}$, $p = 3$.

Proposition. Under the hypothesis, $\mathrm{End}_{kG}(M) \cong k$.

Corollary. If, in addition, every character in the principal p-block of G is real-valued, then M is a simple module.

The additional hypothesis holds if $q = 4$ in the example and presumably more generally. Let's derive the corollary from the proposition as our first step. The endomorphism algebra $\mathrm{End}_{kG}(kX)$ is two-dimensional, by the double transitivity, possesses an element of square zero (the map of kX onto F with kernel Z) so $\mathrm{End}_{kG}(kX) = k[X]/(X^2)$ and so kX is indecomposable. Hence, every composition factor of kX lies in the principal block character. Moreover, kX is self-dual: $kX \simeq \mathrm{Ind}_H^G k$ and k is self-dual. Furthermore, there is an isomorphism $kX \simeq (kX)^*$ mapping F to $\hat{Z}$ (the annihilator of Z) and Z to $\hat{F}$, so we have $M \simeq M^*$: the "point" x in X is mapped to the linear functional which has value 1 on x and vanishes elsewhere on X. Now, if M were not simple then it would now have a simple composition factor S of $M/\mathrm{rad}(M)$ which would also appear in $\mathrm{soc}(M)$ and we would contradict the proposition.

Let's turn to the proposition. First, note that our hypothesis can be restated in module-theoretic terms:

$$\mathrm{Ext}^1_{kG}(k,k) = \mathrm{Ext}^1_{kH}(k,k) = 0$$
$$\mathrm{Ext}^2_{kG}(k,k) = k, \quad \mathrm{Ext}^2_{kH}(k,k) = 0$$

Lemma 1 $\mathrm{Ext}^1_{kG}(k,M) \simeq k \oplus k$

Proof. We have the exact sequence $0 \to M \to kX/F \to k \to 0$ so

$$\mathrm{Hom}_{kG}(k, kX/F) \to \mathrm{Hom}_{kG}(k,k) \to \mathrm{Ext}^1_{kG}(k,M) \to \mathrm{Ext}^1_{kG}(k,kX/F) \to \mathrm{Ext}^1_{kG}(k,k)$$

is exact, that is,

$$\mathrm{Hom}_{kG}(k,kX/F) \to k \to \mathrm{Ext}^1_{kG}(k,M) \to \mathrm{Ext}^1_{kG}(k,kX/F) \to 0.$$

But $\mathrm{Hom}_{kG}(k,kX/F) = 0$ or else kX would have a two-dimensional submodule containing F with two composition factors isomorphic with k yielding either $\mathrm{Ext}^1_{kG}(k,k) \neq 0$ or $\mathrm{Hom}_{kG}(k,kX)$ of dimension greater than one, both contradicting (by hypothesis and as $\mathrm{Hom}_{kG}(k,kX) \simeq \mathrm{Hom}_{kH}(k,k) \simeq k$). Hence, to prove the lemma we need only prove that $\mathrm{Ext}^1_{kG}(k,kX/F) \simeq k$. But we also have the exact sequence $0 \to k \to kX \to kX/F \to 0$ so

$$\mathrm{Ext}^1_{kG}(k,k) \to \mathrm{Ext}^1_{kG}(k,kX) \to \mathrm{Ext}^1(k,kX/F) \to \mathrm{Ext}^2_{kG}(k,k) \to \mathrm{Ext}^2_{kG}(k,kX)$$

But $\mathrm{Ext}^1_{kG}(k,kX) \cong \mathrm{Ext}^1_{kH}(k,k) = 0$ and $\mathrm{Ext}^2_{kG}(k,k) \cong k$ while $\mathrm{Ext}^2_{kG}(k,kX) \cong \mathrm{Ext}^2_{kH}(k,k)$ is also zero. Hence, the lemma is proved.

<u>Lemma 2</u>. $\mathrm{Hom}_{kG}(kX/F, M) = 0$.

<u>Proof</u>. This vector space is a subspace of $\mathrm{Hom}_{kG}(kX, M)$ so we need only show the latter space is zero, that is, $\mathrm{Hom}_{kH}(k, M_H) = 0$. But $\mathrm{Hom}_{kH}(k, (kX)_H) \cong k \oplus k$ as H has exactly two orbits on X. Moreover, $(kX)_H \cong k \oplus k \oplus M_H$ (as the lemma is proved): let x be the point fixed by H, $Y = X - \{x\}$, $s = \sum_{y \in Y} s$, Z' the augmentation zero submodule of kY so $(kX)_H$ is the direct sum of kx, ks, Z' while $Z' + k(x+s) = Z$, $k(x+s) = F$.

To prove the proposition, we consider the exact sequence

$$\mathrm{Hom}_{kG}(kX/F, M) \to \mathrm{Hom}_{kG}(M,M) \to \mathrm{Ext}^1_{kG}(k,M) \to \mathrm{Ext}^1_{kG}(kX/F, M)$$

Since $\mathrm{Hom}_{kG}(kX/F, M) = 0$, by Lemma 2, and $\mathrm{Ext}^1_{kG}(k,M) \cong k \oplus k$ it suffices to prove that the map

$$\mathrm{Ext}^1_{kG}(k,M) \to \mathrm{Ext}^1_{kG}(kX/F, M)$$

has a non-zero kernel. But we have a commutative diagram, with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & kX/F \arrow[r] & k \arrow[r] & 0 \\ 0 \arrow[r] & Z \arrow[r] \arrow[u] & kX \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

so we get a commutative diagram with exact rows as follows:

$$\begin{tikzcd} \mathrm{Ext}^1_{kG}(k,M) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{kG}(kX/F, M) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{kG}(M,M) \arrow[d] \\ \mathrm{Ext}^1_{kG}(k,M) \arrow[r] & \mathrm{Ext}^1_{kG}(kX, M) \arrow[r] & \mathrm{Ext}^1_{kG}(Z,M) \end{tikzcd}$$

The first vertical column is an isomorphism while the second is a monomorphism since $\mathrm{Hom}_{kG}(k, M) = 0$ (using the exact sequence $0 \to k \to kX \to kX/F \to 0$). Hence, it suffices to prove that the map

$$\mathrm{Ext}^1_{kG}(k, M) \to \mathrm{Ext}^1_{kG}(kX, M)$$

has a non-zero kernel. But we assert there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{kG}(k, M) \arrow[rr] \arrow[dr, "\mathrm{Res}"'] & & \mathrm{Ext}^1_{kG}(kX, M) \\
& \mathrm{Ext}^1_{kH}(k, M) \arrow[ur, "\mathcal{E}"'] &
\end{tikzcd}$$

where $\mathcal{E}$ is the Eckmann (Shapiro) isomorphism. Once we show this we are done, as the extension kX/F of M by k splits on restriction to H, by the argument of Lemma 2 while the original extension is not split by an argument from Lemma 1. But suppose we have an injective resolution of M,

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$

so we have to look at the situation

$$\begin{tikzcd}
k \arrow[rr] \arrow[dr] & & \mathrm{coind}_H^G k \arrow[dl] \\
& I_1 &
\end{tikzcd}$$

which readily gives us what we desire.

Let's conclude with some remarks about the case $G = PSL(3, q)$, $q \equiv 1$ (modulo 3), $p = 3$. First, the modules given by the doubly transitive actions on points and lines of the projective plane are isomorphic. Indeed, letting p be a point now, the maps

$$p \to \sum_{\substack{\ell \\ p\in\ell}} \ell$$

$$\ell \to \sum_{\substack{p \\ p\in\ell}} p$$

are almost inverses: $p \to \sum_{\substack{\ell \\ p\in\ell}} \ell \to (q+1)p + \sum_{\substack{t\,\text{a}\,p\text{t} \\ t\neq p}} t = q\cdot p + \sum_{\substack{t \\ t\,\text{a}\,p\text{t}}} t = -p + \sum t.$

next, when $q \equiv 1 \pmod 3$ but $q \not\equiv 1 \pmod 9$ then the Sylow 3-subgroup is $Z_3 \times Z_3$ and we guess the following decomposition matrix:

	1	q^2+q-1	$(q^3-q^2-q+1)/3$	$(q^3-q^2-q+1)/3$	$(q^3-q^2-q+1)/3$
1	1	0	0	0	0
q^2+q	1	1	0	0	0
$(q+1)(q^2+q+1)/3$	1	1	1	0	0
$(q+1)(q^2+q+1)/3$	1	1	0	1	0
$(q+1)(q^2+q+1)/3$	1	1	0	0	1
q^3	0	1	1	1	1

Traces and relative projectivity.

Let H be a subgroup of the group G and let U, V be kG-modules. Let $f \in \mathrm{Hom}_{kG}(U, V)$.

Proposition The following are equivalent

1) f is a trace from H $\left(f \in \mathrm{Tr}_H^G \mathrm{Hom}_{kH}(U, V)\right)$

2) f factors through a relatively H-projective module;

3) f factors through $\mathrm{Ind}_H^G V$ via the natural map of $\mathrm{Ind}_H^G V$ to V;

4) f factors through $\mathrm{Ind}_H^G U$ via the natural map of U to $\mathrm{Ind}_H^G U$.

Proof. First, suppose that f factors as in 2), say

$$\begin{tikzcd} & \mathrm{Ind}_H^G W \arrow[dr, "\beta"] & \\ U \arrow[ur, "\alpha"] \arrow[rr, dashed, "f"'] & & V \end{tikzcd}$$

where W is an kH-module. Hence there is $\varphi \in \mathrm{Hom}_{kH}(U, W)$ such that $\alpha(u) = \sum t \otimes \varphi(t^{-1}u)$ for any $u \in U$ so that $f(u) = \beta(\alpha(u)) = \beta\left(\sum t \otimes \varphi(t^{-1}u)\right) = \sum t\psi\varphi(t^{-1}u)$ where $\psi \in \mathrm{Hom}_{kH}(W, V)$ so that $\beta(t \otimes w) = t\psi(w)$ for all t, w. Hence 1) holds.

Next, suppose $f = \mathrm{Tr}_H^G \gamma$, $\gamma \in \mathrm{Hom}_{kH}(U, V)$. Consider the maps

$$U \xrightarrow{\alpha} \mathrm{Ind}_H^G V \xrightarrow{\beta} V$$

where $\alpha(u) = \sum t \otimes \gamma(t^{-1}u)$ and $\beta(t \otimes v) = tv$. We have $f = \beta\circ\alpha$ so 3) holds. Similarly consider the maps

$$U \xrightarrow{u \to \sum t \otimes t^{-1}u} \mathrm{Ind}_H^G U \xrightarrow{t \otimes u \to t\gamma(u)} V$$

so again the composition is f and so 4) holds. The rest is

Semi-simple modular Hecke algebras.

Let the finite group G acts transitively on the set X with point stabilizer H. Let k be a field of characteristic p. Aloys Krieg (in "Maschke's theorem for Hecke-algebras" J. Indian Math. Soc. (N.S.) 50 (1986), no 1-4, 139-147 (1988)) (90b:16016) has shown that the corresponding Hecke algebra is semi-simple if the degree and the sub-degrees are relatively prime to p. Let us begin by sketching a recast of his proof. Let H be the stabilizer of $x_0 \in X$ and let $X_1, \dots, X_t$ be the sub-orbits of H, setting $X_0 = \{x_0\}$. Since $M = kX = \mathrm{Ind}_H^G(k)$, there is a basis $\varphi_0, \dots, \varphi_t$ of the Hecke algebra $E = \mathrm{End}_{kG}(M, M)$ where $\varphi_i(x)$ is the sum $\sum_{x \in X_i} x$ so $\varphi_0 = 1$. It is easy to see that $\mathrm{tr}_M(\varphi_i) = 0$ if $i > 0$ and $\mathrm{tr}_M(\varphi_0) = |X|$. (For $\varphi_i(x_0)$ does not involve x_0 so $\varphi(x)$ does not involve x for any $x \in X$). Hence, since $p \nmid |X|$, if $e = \sum a_i \varphi_i$ is in E and nilpotent then $a_0 = 0$. However, suppose that $e \neq 0$ and e is in the radical of E; we shall derive a contradiction. Suppose that $a_i \neq 0$. Using the usual business of $X \times X$ and paired orbits there is $j \ni \varphi_j(x_i) = x_0 + \dots$ for any $x_i \in X_i$ and $\varphi_j(x_\ell)$ does not involve x_0. Since the subdegrees are not divisible by p, the element $\varphi_j e$ is in the radical and gives a contradiction.

Two questions arise: Is this condition necessary? When are the modules and matrix algebras. Krieg shows the condition is not necessary, but does show E is not semi-simple if $p \mid |X|$ (the element $\sum \varphi_i$ is in the radical). We shall discuss the second question and also ask about algebras of finite type.

Let Q be a Sylow p-subgroup of H and let X^Q be the fixed-points of Q on X so X^Q is a transitive set for $N(Q)$, a "classical" observation. Let E_Q be the corresponding Hecke algebra.

<u>Theorem</u>. If the subdegrees are relatively prime to p then $E \cong E_Q$.

Now, if the degree $|X|$ is also prime to p, then Q is a Sylow p-subgroup of G so $N(Q)/Q$ is a p'-group so König's result follows. It also follows from deep results of Scott (Theorem 6 in "Modular permutation representations," Trans. Amer. Math. Soc. 175 (1973), 101–21). Our theorem seems to be a consequence of proofs (not results) of Scott's paper - see pp 109-10. We will give a direct proof however. First we note one new consequence.

<u>Corollary</u>. If the subdegrees are prime to p while $p \mid |X|$ and $p^2 \nmid |X|$ then E is of finite type.

<u>Proof</u>. Here we have $p \mid |N(Q)/Q|$, $p^2 \nmid |N(Q)/Q|$. Moreover, X^Q will be a free module for the Sylow p-subgroup of $N(Q)/Q$ so E_Q is the endomorphism algebra of a projective module for a group algebra of finite type so is also of finite type.

To prove the theorem first we note that each indecomposable summand of M has vertex Q: for M is induced from H which has Sylow p-subgroup Q so M is relatively Q-projective and, on the other hand, vertices of summands of M are Sylow subgroups of the intersection of H with conjugates of itself (as the hypothesis on subdegrees applies).

Next, consider the restriction $M_{N(Q)}$. The indecomposable summands are of two types. The first are the Green correspondents of the indecomposable summands of M and they are projective $N(Q)/Q$ modules so their restriction to Q is a direct sum of trivial modules. The second are indecomposable summands whose vertices are G-conjugates of proper subgroups of Q lying in $N(Q)$; hence, none of these has the trivial module k as a summand on restriction to Q. We write $M = N + W$ corresponding to the two kinds of summands. Now $\mathrm{End}_{kN(Q)}(N) = E_Q$ and we also have, with the usual notation from our work, that $\mathrm{Hom}^{\mathfrak{X}}_{kG}(M,M) \simeq \mathrm{Hom}^{\mathfrak{X}}_{kN(Q)}(N,N)$. Hence, it remains only to prove that $\mathrm{Hom}_{kG,\mathfrak{X}}(M,M) = \mathrm{Hom}_{kN(Q),\mathfrak{X}}(N,N) = 0$.

Let us first deal with $\mathrm{Hom}_{kG,\mathfrak{X}}(M,M)$. It suffices to see that $\mathrm{Hom}_{kG,R}(M,M) = 0$ where $R < Q$. However, suppose this is not zero so we have a diagram

$$\begin{tikzcd} & \mathrm{Ind}_R^G W \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & M \end{tikzcd}$$

which commutes with the bottom row non-zero. Mackey's theorem and the fact that $M = \mathrm{Ind}_H^G k$ then give us that $\mathrm{Hom}_{kH,S}(k, M_H) \neq 0$ where S is a p-subgroup of H, $|S| < |Q|$. But

$\dim_k \mathrm{Hom}_{kH}(k, M_H) = t+1$ (with our previous notation) and $M_H = \underbrace{k \oplus \dots \oplus k}_{t+1} \oplus \dots$ by our hypotheses on subdegrees. We deduce that $\mathrm{Hom}_{kH,S}(k,k) \neq 0$ which is a contradiction as k is not relatively S-projective.

Next, let's deal with $\mathrm{Hom}_{kN(Q),\mathfrak{X}}(N,N)$. It suffices to show that $\mathrm{Hom}_{kN(Q),R}(N,N) = 0$ if $R < Q$. However, from the previous section of this notebook we have that if $\alpha \in \mathrm{Hom}_{kN(Q),R}(N,N)$ then there is a commutative diagram

$$\begin{tikzcd} & \mathrm{Ind}_R^{N(Q)}(N_R) \arrow[dr, "\gamma"] & \\ N \arrow[ur, "\beta"] \arrow[rr, "\alpha"] & & N \end{tikzcd}$$

But $N_R = k \oplus \dots \oplus k$ so $\mathrm{Ind}_R^{N(Q)}(N_R)$ restricted to Q is the direct sum of copies of conjugates of $\mathrm{Ind}_R^Q(k)$. If I is the radical of kQ then $\mathrm{soc}(\mathrm{Ind}_R^Q(k)) \subseteq \mathrm{rad}(\mathrm{Ind}_R^Q(k))$ as kQ modules, that is, the submodule of $\mathrm{Ind}_R^Q(k)$ annihilated by I is in $\mathrm{rad}(\mathrm{Ind}_R^Q(k))$. But $IN = 0$, as N_Q is trivial, so $I.\beta N = 0$, that is, $I.\beta N \subseteq \mathrm{rad}(\mathrm{Ind}_R^{N(Q)}(N_R))$ as kQ-module. But the kernel of γ contains $\mathrm{rad}(\mathrm{Ind}_R^{N(Q)}(N_R))$ as its image in N is a module annihilated by I. (That is, $\ker\gamma \supseteq I.\mathrm{Ind}_R^{N(Q)}(N_R) \supseteq \mathrm{soc}\,\mathrm{Ind}_R^{N(Q)}(N_R) \supseteq \beta N$). Hence $\gamma \circ \beta = 0$ as desired.

Remark. We have not used transitivity so the results are even more general?? (just used transitivity to deduce that it holds on X^Q for $N(Q)$.) (Also used M was induced!)

Let's continue with the general analysis.

Proposition. If K is a subgroup of G with $\mathrm{Hom}_{kG,K}(M,M) \neq 0$ then K contains a Scott defect group (a Sylow p-subgroup of $H \cap xHx^{-1}$ for some x).

Proof. If $\mathrm{Hom}_{kG,K}(M,M) \neq 0$ then $M = \mathrm{Ind}_H^G k$ yields that there is a conjugate K^x of K so that $\mathrm{Hom}_{kH, K^x \cap H}(k, M_H) \neq 0$. Now, by Mackey's theorem, M_H is the direct sum of modules $\mathrm{Ind}_{H \cap xHx^{-1}}^H(k)$ and the summand in such a module with k in its socle (the Scott module) is the same as in $\mathrm{Ind}_Q^H(k)$ where Q is a Sylow p-subgroup of $H \cap xHx^{-1}$. Hence, we deduce that $\mathrm{Hom}_{kH,R}(k, \mathrm{Ind}_Q^H(k)) \neq 0$ where R is a Sylow p-subgroup of $K^x \cap H$; we need to deduce that R contains a Scott defect group such as Q.

However, the previous section tells us this occurs if, and only if, we have a commuting diagram

$$\begin{tikzcd} & \mathrm{Ind}_R^H(k) \arrow[dr] & \\ k \arrow[ur] \arrow[rr] & & \mathrm{Ind}_Q^H k \end{tikzcd}$$

where the map of k to $\mathrm{Ind}_R^H(k)$ is the natural one and the horizontal arrow is the given non-zero element of $\mathrm{Hom}_{kH,R}(k, \mathrm{Ind}_Q^H(k))$.

Now $\mathrm{Ind}_R^H(k) \cong kY$, where $Y = H/R$, and $\mathrm{Ind}_Q^H(k) \cong kZ$, where $Z = H/Q$. Suppose $y \in Y$ is stabilized by R and $\varphi \in \mathrm{Hom}_{kH}(kY, kZ)$ so $\varphi(y)$ is a linear combination of sums of orbits of R on Z. Let O be one such orbit and let now ψ be the element of $\mathrm{Hom}_{kH}(kY, kZ)$ which maps y to the sum of the elements of O (so ψ is one of an obvious basis of $\mathrm{Hom}_{kH}(kY, kZ)$). We ask: For how many elements of Y does the image under ψ involve a fixed element of Z? Well, the number of pairs (y', z)

where $y' \in Y$, $z \in Z$ and $f(y')$ involves z, is $|Y|\,|O|$. Hence the correct number is $|Y|\,|O|/|Z|$, that is $|Q|/|R|.|O|$. Now $|O| = |R : Q \cap lRl^{-1}|$ so the number we have been calculating is $|Q : Q \cap lRl^{-1}|$. Our hypothesis on $Hom_{kH,R}(k, Ind_Q^H k) \neq 0$ now forces that there is f, as above, such that f does not vanish on the image of $1 \in k$ in $Ind_R^H(k)$, that is $f(\sum_{y' \in Y} y') \neq 0$. But the coefficient of z in this non-zero element is the number we have just calculated, namely, $|Q : Q \cap lRl^{-1}|$ so $Q = lRl^{-1}$ and we are done.

Notice that the above can replace the argument in the proof on page 71. We now turn to the minimal Scott defect groups. For each p-subgroup Q let $M_{(Q)}$ be the sum of all indecomposable summands of M, in a decomposition of M, with vertex Q. Let $m_{(Q)}$ be the Green correspondent (sum of correspondents), the $N(Q)$ module.

<u>Proposition</u> If Q is a minimal Scott defect group then

$$End_{kG}(M_{(Q)}) \simeq End_{kN(Q)}(m_{(Q)}).$$

<u>Proof.</u> With the usual notation for the Green correspondence we have

$$Hom^{\mathfrak{X}}_{kG}(M_{(Q)}, M_{(Q)}) \simeq Hom^{\mathfrak{X}}_{kN(Q)}(m_{(Q)}, m_{(Q)})$$

(and the proof shows this is an algebra isomorphism as well). Moreover, the previous result shows $Hom^{\mathfrak{X}}_{kG}(M_{(Q)}, M_{(Q)}) = Hom_{kG}(M_{(Q)}, M_{(Q)})$ as $Hom_{kG,\mathfrak{X}}(M_{(Q)}, M_{(Q)}) = 0$. The argument on page 71 shows that $Hom_{kN(Q),\mathfrak{X}}(m_{(Q)}, m_{(Q)}) = 0$ (or the proof of the previous result) so the result is proved.

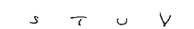
<u>Proposition</u> If Q and R are non-conjugate minimal Scott defect groups then $\mathrm{Hom}_{kG}(M_{(Q)}, M_{(R)}) = 0$.

<u>Proof</u> Suppose $\varphi \in \mathrm{Hom}_{kG}(M_{(Q)}, M_{(R)})$ so φ factors through a relatively Q-projective and an R-projective module. Hence, Lemma 4 (p 87 of my book), our hypothesis on the non-conjugacy of Q and R and the proposition on page 92 imply that $\varphi = 0$, as required.

We are interested in the question as to whether the Hecke algebra is also of finite type in the case that the degree is prime to p but the subdegrees can be divisible by p but not p^2. It seems the answer should be no. (In Scott's notation, the case at hand and the case of the degree divisible by p and not p^2 while the subdegrees are prime to p, together constitute the case $e = 1$).

We won't have a counterexample but the situation is not so difficult. Let's just consider the case $|G|_p = p$, $|G:H|_p = 1$. Thus $M \mid \mathrm{Ind}_P^G k$ where P is a Sylow p-subgroup of G contained in H. But if $\mathrm{Ind}_P^G k$ is of finite type then so is M (i.e. endomorphism algebras are of finite type). This can be seen in terms of the description in terms of short exact sequences of the indecomposable modules of the endomorphism algebra

But $\mathrm{Ind}_P^G k$ may well have as a summand the direct sum of all the projectives in the principal block plus all the Green correspondents of the simple modules for $N(P)/C(P)$. Let's look at a hypothetical example. Say the tree is



and $\mathrm{Ind}_P^G k$ has the following summands:

$$S \oplus T \oplus U \oplus V \oplus \begin{matrix} S \\ T \\ S \end{matrix} \oplus \begin{matrix} & T & \\ S & & U \\ & T & \end{matrix} \oplus \begin{matrix} & U & \\ T & & V \\ & U & \end{matrix} \oplus \begin{matrix} V \\ U \\ V \end{matrix}$$

The endomorphism algebra of this has the following structure:

$$\begin{matrix} A \\ E \end{matrix} \quad \begin{matrix} B \\ F \end{matrix} \quad \begin{matrix} C \\ G \end{matrix} \quad \begin{matrix} D \\ H \end{matrix} \quad \begin{matrix} & E & \\ A & & F \\ & E & \end{matrix} \quad \begin{matrix} & F & \\ B & E & G \\ & F & \end{matrix} \quad \begin{matrix} & G & \\ C & F & H \\ & G & \end{matrix} \quad \begin{matrix} & H & \\ D & & G \\ & H & \end{matrix}$$

so the bipartite graph for this algebra modulo the square of its radical involves

$$\begin{array}{ccccccc} A & & B' & & C & & \\ | & & | & & | & & \\ E' & — & F & — & G' & — & H \end{array}$$

so we're not in a finite type situation.

Let's try and do the same for a Hecke algebra. Take $G = \Sigma_7$, $p = 7$ (and we'll just look at the principal block part). The tree is

$$\begin{array}{ccccccccccccc} & 1 & & 5 & & 10 & & 10' & & 5' & & 1' & \\ 1 & — & 6 & — & 15 & — & 20 & — & 15' & — & 6' & — & 1' \end{array}$$

Let's now calculate restrictions first and then induction (within the principal block) by the usual games. We tabulate the results:

Restrictions		Inductions	
$G=\Sigma_7$	$N(Z_7)$	$N(Z_7)$	$G=\Sigma_7$
1	1	1	$1 \oplus \begin{array}{c} 10' \\ 10 \quad 5' \\ 10' \end{array}$
5	$\begin{array}{c} \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ \lambda^5 \end{array}$	λ	$\begin{array}{c} 5 \\ 10 \end{array}$
10	$\begin{array}{c} \lambda^5 \\ 1 \\ \lambda \end{array} \oplus \begin{array}{c} \lambda^3 \\ \lambda^4 \\ \lambda^5 \\ 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{array}$	λ^2	$\begin{array}{c} 10' \\ 5' \end{array}$
$10'$	$\begin{array}{c} \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{array} \oplus \begin{array}{c} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ \lambda^5 \\ 1 \end{array}$	λ^3	$1' \oplus \begin{array}{c} 10 \\ 5 \quad 10' \\ 10 \end{array}$
$5'$	$\begin{array}{c} \lambda^4 \\ \lambda^5 \\ 1 \\ \lambda \\ \lambda^2 \end{array}$	λ^4	$\begin{array}{c} 5' \\ 10' \end{array}$
$1'$	λ^3	λ^5	$\begin{array}{c} 10 \\ 5 \end{array}$

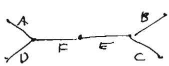
Hence, principal part of $\mathrm{Ind}_{Z_7}^{\Sigma_7}(1)$ is

$$\begin{array}{c} 5 \\ 10 \end{array} \oplus \begin{array}{c} 10' \\ 5 \end{array} \oplus \begin{array}{c} 5' \\ 10' \end{array} \oplus \begin{array}{c} 10 \\ 5 \end{array} \oplus \begin{array}{c} 10' \\ 10 \quad 5' \\ 10' \end{array} \oplus \begin{array}{c} 10 \\ 5 \quad 10' \\ 10 \end{array}$$

Structure of endomorphism algebra:

$$\begin{array}{c} A \\ D \\ F \end{array} \quad \begin{array}{c} B \\ E \\ C \end{array} \quad \begin{array}{c} C \\ B \\ E \end{array} \quad \begin{array}{c} D \\ F \\ A \end{array} \quad \begin{array}{c} E \\ \begin{smallmatrix} C \\ B \end{smallmatrix} \quad F \\ E \end{array} \quad \begin{array}{c} F \\ \begin{smallmatrix} A \\ D \end{smallmatrix} \quad E \\ F \end{array}$$

a quotient of the Brauer tree algebra



One final example, answering a question of Auslander-Reiten (about End$(A \oplus M)$ of infinite type when M is an A-module, $\mathrm{Ext}^1(A\oplus M, A\oplus M)=0$ so $\underline{\mathrm{End}}(A\oplus M)$ is of finite type).

$$\circ \overset{R}{\text{———}} \circ \overset{S}{\text{———}} \circ \overset{T}{\text{———}} \circ \overset{U}{\text{———}} \circ \overset{V}{\text{———}} \circ$$

Take $M = S \oplus U$. We have

$$S \oplus U \oplus \begin{smallmatrix} R \\ S \\ R \end{smallmatrix} \oplus \begin{smallmatrix} & S & \\ R & & T \\ & S & \end{smallmatrix} \oplus \begin{smallmatrix} & T & \\ S & & U \\ & T & \end{smallmatrix} \oplus \begin{smallmatrix} & U & \\ T & & V \\ & U & \end{smallmatrix} \oplus \begin{smallmatrix} V \\ U \\ V \end{smallmatrix}$$

Endomorphism algebra:

$$\begin{smallmatrix} A \\ D \end{smallmatrix} \qquad \begin{smallmatrix} B \\ F \end{smallmatrix} \qquad \begin{smallmatrix} C \\ D \\ C \end{smallmatrix} \qquad \begin{smallmatrix} & D & \\ A & C & E \\ & D & \end{smallmatrix} \qquad \begin{smallmatrix} & E & \\ D & & F \\ & E & \end{smallmatrix} \qquad \begin{smallmatrix} & F & \\ B & E & G \\ & F & \end{smallmatrix} \qquad \begin{smallmatrix} G \\ F \\ G \end{smallmatrix}$$

so get for separated graph, for modulo radical squared, in part

$$\begin{array}{ccccccccc} B' & — & F & — & E' & — & D & — & A' \\ & & | & & & & | & & \\ & & G' & & & & C' & & \end{array}$$

so type is infinite.

Let's return to the study of matrix summands of the Hecke algebra. Choose K, R so that (K, R, k) is a p-modular system. Let p^e (Scott's notation) be the largest power of p dividing the size of an orbit of G on $X \times X$ and let p^a be the order of a Sylow p-subgroup of G. If U is an indecomposable summand of M then we say U is of matrix type if it corresponds, in the obvious way, with a matrix summand of the Hecke algebra E.

<u>Theorem</u> If U is an indecomposable summand of M then the following are equivalent:

1) $K \otimes \hat{U}$ is irreducible of degree divisible by p^e;
2) U is of matrix type and has vertex of order p^{a-e};
3) U has vertex of order p^{a-e} and simple Green correspondent.

Here, $\hat{U}$ is the indecomposable summand of RX corresponding to the summand U of $kX = M$.

<u>Proof</u>. That 1) implies 2) is immediate from Scott's Theorem 6. Next, suppose that 2) holds. Then the proposition on page 93 forces the Green correspondent of U to have a one-dimensional endomorphism algebra so it must be simple (as it is projective modulo the vertex). Finally, suppose that 3) holds. Then again, by the same proposition, $\mathrm{Hom}_{kG}(U, U) \cong k$ so the endomorphism ring of $K \otimes \hat{U}$ is also one dimensional so $K \otimes \hat{U}$ is irreducible. The dimension of $\hat{U}$ has the right divisibility properties because of the vertex, by usual arguments (restrict each to a Sylow subgroup and use the Green indecomposability theorem).

Our next goal is to try and establish the equivalence of 1) and 3), in the previous theorem, in a direct module-theoretic way. Let's begin with a lemma that is, more or less, implicit in the literature.

Fix a subgroup L of the group G and let S be another subgroup of G. Let $\mathcal{S} = \{ gSg^{-1} \cap L \mid g \in G \}$, let M be a kG-module and V a kL-module.

<u>Lemma</u> The natural isomorphism
$$\mathrm{Hom}_{kL}(V, M_L) \cong \mathrm{Hom}_{kG}(V^G, M)$$
induces an isomorphism
$$\mathrm{Hom}_{kL,\mathcal{S}}(V, M_L) = \mathrm{Hom}_{kG,S}(V^G, M).$$

<u>Proof.</u> Suppose $\alpha \in \mathrm{Hom}_{kL}(V, M_L)$ and α' is the corresponding map in $\mathrm{Hom}_{kG}(V^G, M)$ so we have a commuting diagram

$$\begin{tikzcd} & V^G \arrow[dr, "\alpha'"] & \\ V \arrow[ur, "i"] \arrow[rr, "\alpha"'] & & M \end{tikzcd}$$

Hence, by Mackey's theorem, if α' factors through a relatively S-projective module then α factors through a relatively $\mathcal{S}$-projective module. Conversely, suppose α factors through the kL-module T which is relatively $\mathcal{S}$-projective, that is, we have a commuting diagram

$$\begin{tikzcd} & T \arrow[dr, "\gamma"] & \\ V \arrow[ur, "\beta"] \arrow[rr, "\alpha"'] & & M \end{tikzcd}$$

Then we have a commuting diagram

$$\begin{tikzcd} & T^G \arrow[dr, "\gamma'"] & \\ V^G \arrow[ur, "\beta^G"] \arrow[rr, "\alpha'"'] & & M \end{tikzcd}$$

which is commutating as can be checked on generators, i.e. on $V \subseteq V^G$. But then T^G is certainly relatively projective with respect to the set of conjugates of S, so it is relatively S-projective. Hence, the lemma is proved.

Now let's go back to the set-up of the result previous to the lemma. Hence, $M = kX$, Q is minimal among Sylow p-subgroups of intersections of two conjugates of H. Let $L = N_G(Q)$. We have an indecomposable summand U of M with Green correspondent V, a kL-module. We assume that V is simple and we want to deduce that U is of matrix type.

By the proposition on page 92 we know that if $\underline{Q}$ consists of the collection of proper subgroups of Q then

$$\mathrm{Hom}_{kG,\underline{Q}}(M,U) = 0$$

We also know from the proposition on page 93 that the endomorphism algebra of U is one-dimensional so we have only to prove that $\mathrm{Hom}^{\underline{Q}}_{kG}(U,M)$ is of dimension r, where r is the multiplicity of U as a summand of M (because with perm. modules the dimension of $\mathrm{Hom}(\ ,\ )$ is symmetric: use characters). By the lemma, and the structure of V^G in Green's theory, it suffices to show that $\mathrm{Hom}^{\mathcal{W}}_{kL}(V,M)$ is r-dimensional where $\mathcal{W}$ consists of intersections with L of conjugates of proper subgroups of Q. But V is relatively Q-projective so $\mathrm{Hom}^{\mathcal{W}}_{kL}(V,M_L) = \mathrm{Hom}^{\underline{Q}}_{kL}(V,M_L)$, by the usual argument - see my book eg..

Now $M_L = \bigoplus \mathrm{Ind}^{L}_{xHx^{-1}\cap L} k$, over the usual double coset representatives. Now, if $xHx^{-1}\cap L \not\geq Q$ then, again using the fact that V is relatively Q-projective, we can neglect that term in $\mathrm{Hom}^{\underline{Q}}_{kL}(V,M_L)$, that is

$$\mathrm{Hom}^{\underline{Q}}_{kL}(V,M_L) = \mathrm{Hom}^{\underline{Q}}_{kL}(V, \bigoplus \mathrm{Ind}^{H}_{yHy^{-1}\cap L})$$

where the g's run over the double coset representatives with $gHg^{-1} \cap L \geq Q$. Hence, $\mathrm{Ind}_{gHg^{-1}\cap L}^{H} k$ is a L/Q-module so has a map from V if, and only if, V is a summand. This happens at most r times, by Burry-Carlson-Puig, so $\dim \mathrm{Hom}_{kL}^{Q}(V, M_L) \leq r$ so $\dim \mathrm{Hom}_{kG}(U, M) \leq r$ and we are done. We have shown that "2) and 3)", of the theorem before the lemma, are equivalent, without recourse to Scott's work on characters.