RESEARCH NOTES

VOLUME III

Research Notes

Volume III

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<u>Strongly self-centralizing subgroups. A start.</u>

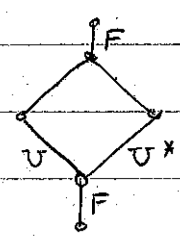
Let's look at the easiest non-cyclic case, $A \simeq Z_2 \times Z_2$ and strongly self-centralizing, $N(A) \simeq A_4$, $G \supset N(A)$.
Want: $B_0(2)$ contains at least three modular irreducibles.
Assume otherwise, $B_0(2)$ containing only F, V with F a splitting field of characteristic two. $\therefore V \cong V^*$.

Let $1, \omega, \bar{\omega}$ denote the linear characters in characteristic 2 for $N(A)$ and the corresponding modules as well. Under the Green correspondence

$$1 \longleftrightarrow F$$
$$\omega \longleftrightarrow U$$
$$\bar{\omega} \longleftrightarrow U^*$$

when U^* is the dual of U, as $\bar{\omega}$ and ω are duals.
(Green correspondence commutes with taking of duals - check out!)
This forces P_F the projective cover of F as $F[G]$-module to look as follows:

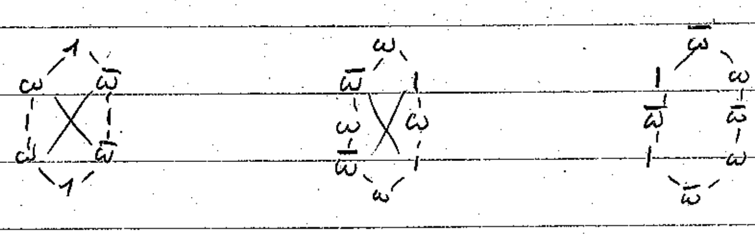
P_F :



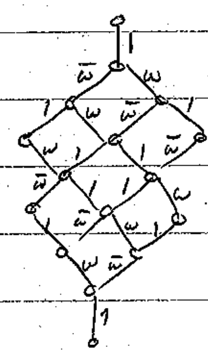
Of course as G is 2-perfect, $Ext^1(F, F) = 0$ so U has only V's "at the top". Seems hard to proceed.

<u>Projectives and Resolutions for SL(2,3) in characteristic 2.</u>

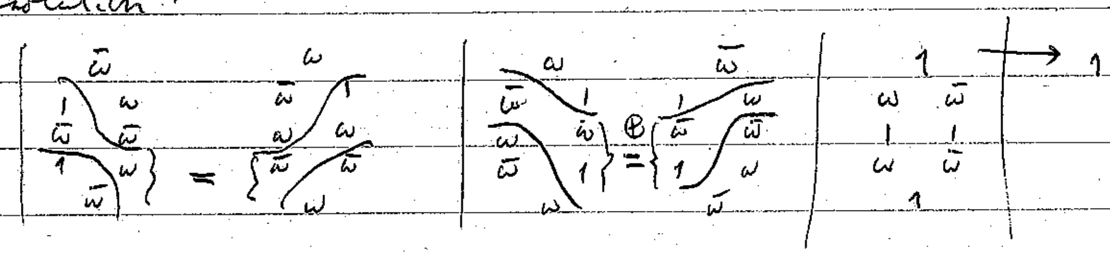
Projectives: First get proj cover of trivial module, then tensor:

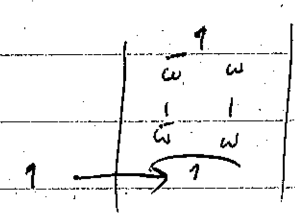


First of these is:



Resolution:





$\mathcal{L}_1(P_F)$, a conjecture

Conjecture: Let F be a splitting field of characteristic p for the group G. Let w be the number of irreducible summands in a decomposition into such of $\mathcal{L}_1(P_F)$ (= layer below F in the projective cover of F). Let d be the number of generators of a Sylow p-subgroup, P, of G. Then $w \le d$.

Remarks

1. If P is cyclic then we know that $w = 1$.
2. Say G is p-solvable. Hence, $P_F \mid P$ is free on one-generator. Thus, if M_F is maximal submodule of P_F then w is at most the dimension of $M_F \mid P$ mod e trivial, i.e. $\dim \mathcal{L}_1(P_F \mid P) = d$. $\therefore$ conjecture true here.
3. Can have $w < d$. Consider G: $\Sigma_3 \left\{ \begin{array}{l} Z_2 \\ Z_3 \end{array} \right.$ $Z_5 \times Z_5$ (obvious action)

Classes:	1;	5,	5,	5,	5;	5,	5;	2,	3;	10,	10,	10	10
# el'ts:	1	3	3	3	3	6	6	15	50	15	15	15	15

Degrees: $1, 1, 2, \overbrace{3 \cdots 3}^{a}\ \overbrace{6 \cdots 6}^{b}$

Hence
$$3 + a + b = 13$$
$$6 + 9a + 36b = 150$$
$$-21 \qquad + 27b = 33 \qquad b = 2,\ a = 8$$

Mod 3, the six dimensional ones are irreducible. If a six can be written over $GF(3)$, extension of it by G has $d = 3$ $w = 2$ as module is projective and by Maschke's theorem. If not writable over $GF(3)$ get a twelve then, getting a group with $d = 5$, $w = 3$.

Note: Must be "twelve" as minimal degree of $Z_5 / GF(3)$ is 4.

Loewy series for tensor products.

Definition. If M is an $F[G]$-module, F field, G a group, then $L_0(M) = M$, $L_{i+1}(M) = \text{Radical}(F[G]) \cdot L_i(M)$ recursively. Let $\mathcal{L}_i(M) = L_i(M) / L_{i+1}(M)$, the i-th (lower) Loewy layer.

Now let M be as above, let N be an $F[H]$-module for a group H and regard $M \otimes N$ as a $F[G\times H]$-module.

Proposition For $k \geq 0$

$$\mathcal{L}_k(M\otimes N) \simeq \bigoplus_{j=0}^{k} \mathcal{L}_j(M) \otimes \mathcal{L}_{k-j}(N)$$

Let's denote $L_i(M) = M_i$, $L_i(N) = N_i$, $i \geq 0$.

Picture:

$$M_0 \otimes N_0$$

$$M_0 \otimes N_1 \qquad M_1 \otimes N_0$$

$$M_0 \otimes N_2 \qquad M_1 \otimes N_1 \qquad M_2 \otimes N_0$$

$$M_0 \otimes N_3 \qquad M_1 \otimes N_2 \qquad M_2 \otimes N_1 \qquad M_3 \otimes N_0$$

$$\vdots$$

As a first step we establish that

$$L_k(M\otimes N) = \sum_{j=0}^{k} M_j \otimes N_{k-j},$$

that is, the row sums in the above diagram. It then remains to deal with the successive quotients.

Let R and S be the radicals of $F[G]$, $F[H]$, respectively.

We have $R, S \subseteq F[G] \otimes F[H]$ "=" $F[G]\, F[H]$. Now $M_i = R^i M$, $N_i = S^i N$. Also $R^i S^j \subseteq (\mathrm{Rad}\ F[G \times H])^{i+j}$ as R, S are nilpotent and commute with $F[H]$, $F[G]$ resp. Thus,

$$M_j \otimes N_{k-j} = R^j M \otimes S^{k-j} N$$
$$= R^j S^{k-j} (M \otimes N)$$
$$\subseteq L_k(M \otimes N)$$

And so certainly $L_k(M \otimes N) \supseteq \sum_{j=0}^{k} M_j \otimes N_{k-j}$.

To establish the first step need only know successive quotients of these sums are semi-simple. Thus, it suffices to show that

$$\sum_{j=0}^{k} M_j \otimes N_{k-j} \Big/ \sum_{l=0}^{k+1} M_l \otimes N_{k+1-l}$$

is isomorphic with

$$\bigoplus_{j=0}^{k} \mathcal{L}_j(M) \otimes \mathcal{L}_{k-j}(N)$$

and that each summand in this direct sum is completely reducible. But this will also prove the proposition as well!

However, say U is an irreducible $F[G]$-module and V is an irreducible $F[H]$-module. Let $\hat{F}$ be algebraic closure of F so going up to $\hat{F}$ get $U \otimes V$ completely reducible. But then $U \otimes V$ must be or else there would be an element of $\mathrm{Rad}\, F[G \times H]$ not annihilating it and this stays in the radical going up to $\hat{F}$. Hence, it remains only to examine the successive quotients.

~~First we require the~~

Now set $K_k = \sum_{j=0}^{k} M_j \otimes N_{k-j}$ and $V_{i,j} = M_i \otimes N_j + K_{i+j+1}$.
Hence, to conclude the proof we need only prove the following two assertions:

a) $V_{i,j} / K_{i+j+1} \simeq M_i/M_{i+1} \otimes N_j/N_{j+1}$ $(= \mathcal{L}_i(M) \otimes \mathcal{L}_j(N))$

b) K_k/K_{k+1} is the direct sum of all the $V_{i,j}/K_{k+1}$ over all i, j with $i+j = k$.

However,

$$V_{i,j} / K_{i+j+1} = M_i \otimes N_j + K_{i+j+1} / K_{i+j+1}$$
$$\simeq M_i \otimes N_j / (M_i \otimes N_j) \cap K_{i+j+1}$$

And $M_i \otimes N_j \cap K_{i+j+1} \supseteq M_i \otimes N_{j+1} + M_{i+1} \otimes N_j$ so $V_{i,j}/K_{i+j+1}$ is a homomorphic image of $M_i \otimes N_j / M_i \otimes N_{j+1} + M_{i+1} \otimes N_j$.
But there is an easy result

<u>Lemma</u> If $A' \subseteq A$ and $B' \subseteq B$ are $F[G]$ and $F[H]$ modules, respectively, Then

$$A \otimes B / A \otimes B' + A' \otimes B \simeq A/A' \otimes B/B'.$$

This is relevant as follows. We now have that $V_{i,j}/K_{i+j+1}$ is an image of $M_i/M_{i+1} \otimes N_j/N_{j+1} = \mathcal{L}_i(M) \otimes \mathcal{L}_j(N)$.
Moreover K_{k}/K_{k+1} is the sum of all these $V_{i,j}/K_{k+1}$.
Thus,

$$\dim_F K_k/K_{k+1} \leq \sum_{i+j=k} \dim V_{i,j}/K_{k+1}$$

$$\leq \sum_{i+j=k} \dim \mathcal{L}_i(M) \cdot \dim_F \mathcal{L}_j(N).$$

Summing over k we get $\dim M \cdot \dim N$ on both sides!
Hence, we must have equalities in our arguments everywhere!
Hence done! It remains to prove the lemma

Proof. It suffices to prove that $A\otimes B' \cap A'\otimes B = A'\otimes B'$. For then

$$\dim A\otimes B' + A'\otimes B = \dim A\otimes B' + \dim A'\otimes B - \dim A'\otimes B'$$

so

$$\dim A\otimes B / A\otimes B' + A'\otimes B = \dim A\otimes B - \dim A\otimes B' - \dim A'\otimes B + \dim A'\otimes B'$$
$$= (\dim A - \dim A')(\dim B - \dim B')$$
$$= \dim A/A' . \dim B/B'.$$

But $A/A' \otimes B/B'$ is clearly an image of $A\otimes B$ with kernel containing $A\otimes B' + A'\otimes B$. So this is the kernel!

But choose a basis - over F - for A intersecting A' in a basis and do the same for B and B'. Then we take the "product" of these bases getting one for $A\otimes B$. Now, if $A\otimes B' \cap A'\otimes B \supset A'\otimes B'$, the only case to be eliminated then would get a non-trivial relation on this basis, expressing the element in LHS not in RHS in terms of product basis for $A\otimes B'$ and for $A\otimes B'$ not all terms being element of product basis for $A'\otimes B'$.

Shifting modules - Invertible modules

Usual set up: G finite group, F field of characteristic p.
Convention: U indecomposable means U is a non-zero non-projective indecomposable FG-module
Notation: Congruence means modulo projectives.

Def. An $F[G]$-module M is a shifting module if

a) whenever U is indecomposable then $M \otimes U \equiv U_1$ with U_1 indec, and

b) whenever U_1 is indec, there is indec U with $M \otimes U = U_1$.

Example. Consider $\sigma(F)$: $0 \to \sigma(F) \to P_F \to F \to 0$, where σ is the shift.
Then if U is indec, $\sigma(F) \otimes U \equiv \sigma(U)$. For

$$\sigma(U) = \sigma(F \otimes U) \equiv \sigma(F) \otimes U.$$

Note: $M \otimes F = M$ so M is indec + proj. WLOG: M indec henceforth.

Prop For an indec module M the following are equivalent:

1) M is a shifting module;

2) $M \otimes M^* \equiv F$;

3) There is a module N with $M \otimes N \equiv F$;

4) $p \nmid \dim M$ and M satisfies a) in the Def.

Proof 1) $\Rightarrow$ 3) by property b). 4) $\Rightarrow$ 2) as $M \otimes M^* = F \oplus \ldots$ by $p \nmid \dim M$. 2) $\Rightarrow$ 4) as $(\dim M)^2 \equiv 1\ (p)$ by 2). 2) $\Rightarrow$ 3) clearly.
Hence: ETS 3) $\Rightarrow$ 1), 4). But 3) $\Rightarrow$ $p \nmid \dim M$. $\therefore$ ETS 3) $\Rightarrow$ 1)
as then M certainly satisfies a) of Def. As for b),

$$\begin{aligned} M \otimes (N \otimes U_1) &= (M \otimes N) \otimes U_1 \\ &\equiv F \otimes U_1 \\ &= U_1. \end{aligned}$$

As far as, say
$$M \otimes U \cong U_1 \oplus U_2 \oplus \dots \oplus U_r$$
each U_j indec. Hence
$$U \cong (M \otimes N) \otimes U \cong (N \otimes U_1) \oplus (N \otimes U_2) \oplus \dots \oplus (N \otimes U_r)$$
∴ if $r > 1$ then as $i \to N \otimes U_i$ is projective. Hence so is $M \otimes N \otimes U_1$.
But $M \otimes N \otimes U_i \cong F \otimes U_i = U_i$ ∴ $r = 1$.

Remarks

1) $N = U^*$ as $N \cong N \otimes (M \otimes M^*) = (N \otimes M) \otimes U^* \cong U^*$.

2) $\otimes M$ is 1-1 and onto the indec. For only need see the 1-1.
But U_1, U_2 indec, $M \otimes U_1 \cong M \otimes U_2 \Rightarrow M^* \otimes M \otimes U_1 \cong M^* \otimes M \otimes U_2$
or $U_1 \cong U_2$ so $U_1 = U_2$.

Examples 1) Let $G = GL(3,2)$, V be the standard 3-dim module so
$V \otimes V^* \cong F$, V is a shifting module. Note: $V, \sigma(F)$ generate a $\mathbb{Z} \times \mathbb{Z}$ (as is seen by restricting to non-conjugate $Z_2 \times Z_2$'s)

2) Let $G = A_5$. Let ω be linear char = 1 dim module for A_4.
Then $\omega^G = \begin{matrix} V_1 \\ F \\ V_2 \end{matrix}$ is a shifting module.

For ω^G has dim 5, by Green maps, $\omega^G \cong$ indec
so ω^G is indec as $4 \mid |A_5|$. $\mathrm{Hom}\left(\begin{matrix} F \\ V_1 \\ V_2 \end{matrix}, \omega^G\right) = \mathrm{Hom}\left(\begin{matrix} 1 \\ \omega \\ \bar\omega \end{matrix}, \omega\right) = \begin{matrix} 0 \\ F \\ 0 \end{matrix}$.
∴ $\omega^G = \begin{matrix} V_1 \\ F \\ V_2 \end{matrix}$ is shifting because ω is (as $\omega \bar\omega = 1$)
and by Mackey product theorem.

3) $G = A_5$, $\begin{matrix} F \\ V_1 \end{matrix} \otimes \begin{matrix} V_1 \\ F \end{matrix} = F \oplus P(V_1)$ as is easy to prove! $\left(\begin{matrix} F \\ V_1 \end{matrix} \cong \sigma(F) \otimes \begin{matrix} V_1 \\ F \\ V_2 \end{matrix}\right)$

Remark: In $0 \to M_F \to P(F) \to F \to 0$ can now give
another proof that M_F is a shift by showing that
$$M_F \otimes M_F^* \cong F.$$
Now also have, as $P_F^* \cong P_F$
$$0 \to F \to P_F \to M_F^* \to 0.$$

Hence, using projectivity, get

$$\begin{array}{ccccccc} 0 \to & M_F \otimes M_F^* & \to & P_F \otimes M_F^* & \to & M_F^* & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ & F & \to & P_F & \longrightarrow & M_F^* & \to 0 \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

Let $0 \to K \to M_F \otimes M_F^* \to F \to 0$ since

$0 \to K \to P_F \otimes M_F^* \to P_F$

get K is projective, hence injective so $M_F \otimes M_F^* = F \oplus K$, as required.

[Or put it this way: a little chasing gives the exact comm diag:

$$\begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ 0 \to & K & \to & K & \to & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & M_F \otimes M_F^* & \to & P_F \otimes M_F^* & \to & M_F^* & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & F & \to & P_F & \to & M_F^* & \to 0 \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

]

Next, we consider restriction to subgroups.

<u>Prop.</u> Let H be a subgroup of G with $p \mid |H|$. If M is a shifting module for G then there is a shifting module N for H with $M_H \equiv N$.

That is, M restricted to H is the direct sum of a shifting module and projectives.

<u>Pf</u> Let $M_H = U_1 \oplus \dots \oplus U_r$ where each U_i is indecomposable.
Now $M \otimes M^* \equiv F$ so $M_H \otimes M_H^* \equiv F$ as projectives and F

restrict to projectives and F, respectively. Hence,

$$\bigoplus_{i,j} U_i \otimes U_j^* \equiv F$$

so there are indices i_0 and j_0 with $U_{i_0} \otimes U_{j_0}^* \equiv F$ and all other $U_i \otimes U_j^*$ projectives. But $U_{j_0} \otimes U_{i_0}^* \simeq (U_{i_0} \otimes U_{j_0}^*)^*$ so we deduce that $i_0 = j_0$. If $i \neq i_0$ get $U_i \otimes U_i^*$ is projective so U_i is projective by D. G. Higman's averaging criterion, as $Hom(U_i, U_i) \simeq U_i \otimes U_i^*$. Hence, the result holds.

Suppose now U is a shifting module for a p-group P of order p^n so $U \otimes U^* \equiv F$ yields $\dim U$ is $kp^n \pm \varepsilon$, $\varepsilon = \pm 1$ if p is odd and $kp^n \pm \varepsilon$, $kp^{n-1} \pm \varepsilon$ if $p = 2$. Suppose that p is odd so $U \otimes U^*$ has dimension $k^2p^{2n} + 2\varepsilon kp^n + 1 = 1 + p^n(k^2p^n + 2\varepsilon k)$ Thus U of this dimension is shifting if and only if $Hom_{FG}(U, U)$ is of dimension $k^2p^n + 2\varepsilon k + 1$. For $U \otimes U^* = F \oplus \cdots$ and $Hom_{FG}(U, U)$ corresponds to fixed points in $U \otimes U^*$. Here we are using the fact that the free FP-module of dimension p^n is the unique indecomposable projective.

If M is a shifting module its vertex is the Sylow p-subgroup since its dimension is not divisible by p. By the above, M_L, where L is a Sylow p-subgroup normalizer is a shifting module N plus projectives. Does every N so arise? The answer is no!

<u>Lemma</u> Suppose that

a) G/G' is a p-group,

b) $N(P)/N(P)'$ is not a p-group,

c) $P \cap P^g \neq 1$ for every $g \in G$.

It follows that there is a shifting module N for L such that there is no shifting module M of G with $M_L \equiv N$ modulo projectives.

For example, $p=3$, $G = Z_3 \times Z_3 \cdot SL(2,3)$.

<u>Proof</u> Let N be a non-trivial one-dimensional module for L which exists by 2). It suffices to assume M exists and show M has dimension one, by a).

Now since M is a shifting module we have M_L is a shifting plus projectives so this shifting is an L-source, the unique one by the Green correspondence. Hence $M \mid N^G$, $M_L \equiv N$.

But M_L is a summand of $(N^G)_L$ as M is a summand of N^G. However, if $LTL = G$, T a double coset transversal, $1 \in T$,

$$N^G{}_L = \bigoplus_{t \in T} (N^t_{L \cap L^t})^L$$

But if $t \neq 1$ then $p \mid |L \cap L^t|$ so as $\dim N = 1$ get $(N^t_{L \cap L^t})^L$ has no projective summand. Indeed, this is so, but let's come back to this. Hence get $M_L = N$ as desired as one also have that $\dim M = 1$.

Left to show: If K is a subgroup of L, $p \mid |K|$ and U is a one-dimensional module for K then K^L has no projective summand. But K also has a normal Sylow p-subgroup Q and inducing the trivial Q-module F_Q to K gives a direct sum of all irreducibles for K with mult. Hence, it suffices to assume $K = Q$, U the trivial Q-module. $\therefore$ $\dim U^P < |P|$ so $\sum_{x \in P} x$ annihilates U^P and $\therefore$ annihilates $(U^P)^L$ so $(U^P)^L_P$ has no projective summand and so $(U^P)^L = U^L$ doesn't either.

Now we start to investigate which shifting modules of L "come from" shifting modules of G.

Def a (shifting) module N of L is stable if for every $x \in G$

$$N^x_{L \cap L^x} \equiv N_{L \cap L^x} \quad \text{(mod projectives)}$$

Note that if N comes from a shifting module M of G then N is stable. For $M_L \simeq N \oplus X$, X projective, so $N^t_{L\cap L^t} \oplus X^t_{L \cap L^t} \cong N_{L \cap L^t} \oplus X_{L \cap L^t}$, and hence our claim holds.

Note that not every stable shifting module comes from G!! Let $G = Z_3 \times Z_3 \cdot SL(2,3)$ as above so $N(S_3) = Z_3 \times Z_3 \cdot Z_6$ and let N be non-trivial one-dimensional module. Easily seen to be stable as its restriction is trivial or non-trivial and uniquely determined as $L \cap L^x$ has order a power of three or order divisible by two.

Consider a faithful three dimensional module M for $Q_8 = \langle x, y \rangle$ in characteristic two. Can assume

$$x \to \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

so if $z = x^2 = y^2 = [x,y]$ then $z \to \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$. Can assume y is also represented by an upper uni-triangular matrix. Then $y^2 = z$ yields

$$y \to \begin{pmatrix} 1 & \omega & \zeta \\ & 1 & \omega^{-1} \\ & & 1 \end{pmatrix}$$

Then

$$[x,y] \to \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \omega & \zeta+1 \\ & 1 & \omega^{-1} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \omega & \zeta \\ & 1 & \omega^{-1} \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1+\omega & \omega^{-1}+\zeta \\ & 1 & 1+\omega^{-1} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1+\omega & \omega^{-1}+\zeta \\ & 1 & 1+\omega^{-1} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \omega+\omega^{-1} \\ & 1 & \\ & & 1 \end{pmatrix}$$

Hence, $\omega^3 = 1$, $\omega \neq 1$ a cube root of unity! Now $\dim_F \mathrm{Hom}_{FG}(M,M) \leq 2$ as centralizer of matrix representing x is three dimensional, being all

$$\begin{pmatrix} \alpha & \beta & \gamma \\ & \alpha & \beta \\ & & \alpha \end{pmatrix}$$

and $[y,x] \neq 1$. $\therefore \dim_F \mathrm{Hom}_{FG}(F, M\otimes M^*) \leq 2$ so as $\dim_F(M\otimes M^*) = 9$ get $M \otimes M^* = F \oplus$ free. M is a shifting module!

Note that transformation by $\begin{pmatrix} 1 & \zeta & \\ & 1 & \zeta \\ & & 1 \end{pmatrix}$ leaves matrix for x fixed and takes the one for y to $\begin{pmatrix} 1 & \omega & 0 \\ & 1 & \omega^{-1} \\ & & 1 \end{pmatrix}$ so can assume $\zeta = 0$. More easy thoughts - see what leaves matrix for x alone and does not centralize the matrix for y - shows there are exactly two such modules depending on choice of ω. They must be duals as a good guess! Or else $M \otimes M \cong M \otimes M^* = F \oplus$ free! Easy calculation involving taking of inverse transposes gives M^* is the other three-dimensional one!

Periodic modules

1. We let G be a finite group, F a field of prime characteristic p and we work with FG modules. Let's begin by recalling a few things.

If M is an FG-module then a "projective cover" P of M is a projective module P such that there is an exact sequence

$$0 \to R \to P \to M \to 0$$

satisfying one - and hence all - of the following equiv. conditions:

1) $R \subseteq Rad(P)$ (= smallest submodule with completely red quot);
2) $R \supseteq Socle(P)$;
3) R has no projective summand.

The P and the sequence is unique up to isomorphism. Moreover, if M has no projective summand then $M \to R$ is a 1-1 corr on iso classes of such modules, taking indec 1-1 and onto indec. We write $R = \Omega(M)$ (well def even if M has proj summand)

If Q is any projective FG-module and

$$0 \to S \to Q \to M \to 0$$

is exact then S is isomorphic to the direct sum of R and a projective (or 0).

Consider next a projective resolution of M:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

It is "minimal" provided one - and hence both - of the equiv cond's hold:

1) Image of P_{i+1} in P_i $\supseteq$ socle P_i, $i \geq 0$
2) Image of P_{i+1} in P_i $\subseteq$ rad P_i, $i \geq 0$

Hence, in a minimal resolution, if K_i is the kernel of P_{i-1} to P_{i-2} ($P_{-1} = M$) then $K_i = \Omega^i(M)$. The minimal resol is unique up to iso. Any resol is the direct sum of a resol of 0 and a minimal resol of M.

Remark: M periodic ⟺ each summand is (Easy using σ)

2. We say a module M is "periodic" if $\sigma^r(M) = M$ for some $r \geq 1$. In particular, M has no projective summand. This means, using a minimal resol. that there is an exact sequence

$$0 \to M \to P_{r-1} \to \cdots \to P_0 \to M \to 0$$

all the P_j projective. And conversely, such a sequence implies, minimality or not, that M is periodic, provided M has no proj summand. Such a sequence of course gives a resol which repeats. And a repeating resol, whatever that is, gives periodicity: Let M have no proj summand, consider a resol

$$\to P_j \to \cdots \to P_i \to \cdots \to P_0 \to M \to 0.$$

And say there exists

$$\begin{array}{ccc} P_{j+1} & \to & P_j \\ \wr\wr & & \wr\wr \\ P_{i+1} & \to & P_i \end{array}$$

Hence, $\sigma^{j+1}(M) \simeq \sigma^{i+1}(M)$ so $\sigma^{j-i}(M) = M$.

Our main results are as follows:

Theorem 1 If F is algebraic over the prime field then an FG-module is bounded if and only if it is the direct sum of a proj and a periodic.

Here, U is "bounded" if for all FG-module V there is an integer N_V such that for all $n \geq 1$

$$\dim_F \mathrm{Ext}^n_{FG}(U, V) \leq N_V.$$

Recall U is projective if $\mathrm{Ext}^n(U, V) = 0$ all V all $n \geq 1$.

Presumably the hypothesis on F is not needed but our proof uses this heavily.

One might think periodic modules are rare!

Theorem 2. If the Sylow p-subgroup of G is non-cyclic then FG has infinitely many non-isomorphic indecomposable periodic modules.

This result is old if the word "periodic" is dropped. We use Theorem 1 to prove Theorem 2. This is not necessary but avoids all calculations and is in the spirit of general results we seek.

Let's mention some examples. $G = A_5$ say, $p=2$ and F contains the field of four elements. But $A_5 \simeq SL(2,4)$ so FG has a two-dimensional module V. If $P(V)$ is a projective cover of V, $P(F)$ is of F, the trivial module, then there is an exact sequence:

$$0 \to P(V) \to P(F) \to P(V) \to 0.$$

More generally, let V be similarly defined for $G = SL(2, 2^n)$, $n \geq 2$. Let $V = V_1$, $V_2 = V_1^{(2)}$ the Frobenius applied, $V_3 = V_2^{(2)}$ etc. and let $W = V_1 \otimes V_2 \otimes \dots \otimes V_{n-1}$. Then W is irred of dimension 2^{n-1} and W is periodic.

Theorem 3. If $r_p(G) > 1$ and M is a periodic FG-module then $p \mid \dim_F M$.

3. Proof of Theorem 1.

First, we observe that if P is proj, M is periodic then $P+M$ is bounded. Since the functors Ext^n are additive and P is proj it suffices to show that M is bounded. Since M is periodic there is an exact sequence

$$0 \to M \to P_{r-1} \to P_{r-2} \to \dots \to P_0 \to M \to 0.$$

Hence, there is a resol. of M with $P_i \simeq P_j$ if $i \equiv j$ (modulo r). For any module V let $N_V = \max_{0 \leq i \leq r-1} \dim_F \mathrm{Hom}_{FG}(P_i, V)$ so then $N \geq \dim_F \mathrm{Hom}_{FG}(P_j, V)$ any j so $N_V \geq \dim_F \mathrm{Ext}^j_{FG}(M, V)$.

Thus, we henceforth assume M is bounded and prove it has the right structure.

We may also assume M is such that it has no projective summand. Let

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution of M. Let $V_1, \ldots V_r$ be the irred modules for FG, up to iso, let $Q_1, Q_2, \ldots Q_r$ be their proj covers, so every projective is a direct sum of modules iso. to the Q_i's.

Next, we may assume F is finite. For there is a finite field F_0 and an F_0G-module M_0 with $M \simeq F \otimes_{F_0} M_0$ as G is finite. For any F_0G module V_0 we have

$$F \otimes \mathrm{Ext}^n_{F_0G}(M_0, V_0) \simeq \mathrm{Ext}^n_{FG}(M, F \otimes_{F_0} V)$$

so M_0 is bounded. If we prove M_0 is periodic (it has no proj summand of course) the sequence exhibiting that tensored with F gives the result for M.

Since F is finite, it is therefore sufficient to show there is an integer N with $N \geq \dim_F P_i$ for all i. Indeed, if this is true then as $\sigma^{i+1}(M) \subseteq P_i$ we have $\sigma^j(M)$ has dimension at most N for all j. But σ maps the set of isomorphism classes of modules with no projective summands one-to-one onto itself and there are only finitely many such classes of dimension at most N. ~~(infinitely many)~~ Hence, $\sigma^j(M) = M$ for some $j \geq 1$ (as $\sigma^k(M) = \sigma^\ell(M)$ some k, ℓ unequal).

However, suppose $P_i \simeq \bigoplus_j a_{ij} Q_j$, a_{ij} a non-negative integer. Now $\mathrm{Ext}^i(M, V_j) \simeq \mathrm{Hom}_{FG}(P_i, V_j)$ as V_j is irreducible and the resol is minimal. Hence, $\mathrm{Ext}^i(M, V_j)$ has dimension $(\dim_F \mathrm{Hom}(V_j, V_j))a_{ij}$. Hence, $a_{ij} \leq N_j$ where this is the bounding constant for V_j. If Q_j has dimension d_j then we deduce that $\dim_F P_i \leq \sum_{j=1}^{r} d_j N_j$ and so the proof is complete.

4. Proof of Theorem 2

Let H be a subgroup of the group G. If U is an FG-module and V is a FH-module we say U and V are "related" if $U \mid V^G$ (U is isomorphic with a summand of the induced module $V^G = V \otimes_{FH} FG$) and if $V \mid U_H$ (U_H the restriction). We prove a couple easy results:

<u>Lemma 1</u> If V is any indec. FH-module then there is an indec. FG-module U related to V.

<u>Proof</u> $(V^G)_H$ contains V as a direct summand, $V \mid (V^G)_H$. Express $V^G = U_1 \oplus \cdots \oplus U_s$ as a sum of indecomposables. Then $(V^G)_H = (U_1)_H \oplus \cdots \oplus (U_s)_H$ so by the Krull Schmidt theorem $V \mid U_i$ some i. Hence done as $U_i \mid V^G$ by definition.

<u>Lemma 2</u> If U is an FG-module, V is an FH-module and U and V are related then U is bounded if and only if V is bounded.

<u>Proof</u> Suppose that V is bounded. Let U' be an FG-module. We must deal with all the $\mathrm{Ext}^n_{FG}(U, U')$. But $U \mid V^G$ so $\mathrm{Ext}^n_{FG}(U, U')$ is a summand of $\mathrm{Ext}^n_{FG}(V^G, U')$. Hence,

$$\dim_F \mathrm{Ext}^n_{FG}(U, U') \le \dim_F \mathrm{Ext}^n_{FG}(V^G, U') = \dim_F \mathrm{Ext}^n_{FH}(V, U'_H)$$

so U is bounded.

Conversely, say U is bounded, V' is an FH-module. Now $V \mid U_H$ so

$$\dim_F \mathrm{Ext}^n_{FH}(V, V') \le \dim_F \mathrm{Ext}^n_{FH}(U_H, V') = \dim_F \mathrm{Ext}^n_{FG}(U, V'^G)$$

and we are done.

In the case that F is algebraic over the prime field we deduce that U and V are periodic together. For they are certainly projective together. However, this is true in general and quite easy to prove.

Now we need a deeper result: ↓ Use $\mathrm{Ext}^p_{F(G/H)}(F, \mathrm{Ext}^q_{FH}(F,W)) \Rightarrow \mathrm{Ext}^n_{FG}(F,W)$
Same pf as in MacLane use F instead of $\mathbb{Z}$

<u>Lemma 3</u>. If H is normal in G and G/H has p-rank one and U is an FG-module with U_H projective then U is bounded.
(F algebraic over the prime field)

Presumably, U is also periodic. If H is a p-group this is true by a direct calculation shown to me by E. Dade.

<u>Proof</u>. Again we may assume F is finite. For say $U \simeq F \otimes_{F_0} U_0$, F_0 finite, U_0 a F_0G-module. It is easy to see U_0 satisfies the hypothesis so if U_0 is bounded then the dimensions of the projectives in a minimal resolution of U_0 are so the same holds for U by tensoring. Thus, we assume F is finite; hence we can assume F has p elements. For consider U as a F_pG-module, $F_p \subseteq F$, F_p having p elements. Then $U \mid F \otimes_{F_p} U$ and the rest is easy.

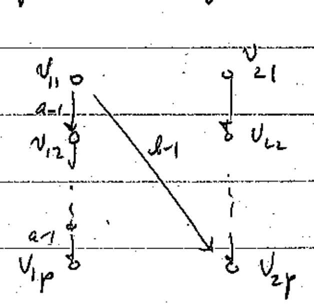
Now $\mathrm{Ext}^n_{FG}(U, V) = \mathrm{Ext}^n_{FG}(F, U^* \otimes V) \simeq H^n(G, U^* \otimes V)$.
But $W = U^* \otimes V$ has the same properties as U does. Hence $H^q(H, W) = 0$, if $q > 0$. Thus $H^p(G/H, H^q(H,W)^H)$ is zero unless $q = 0$ in which case this is $H^p(G/H, W^H)$.
But this is bounded independently of p so the Hochschild-Serre spectral sequence concludes the proof as it gives $H^n(G, W) \simeq H^n(G/H, W^H)$.

Now we construct some modules for $Z_p \times Z_p$. Let V_n be a vector space over F of dimension pn with basis $\{v_{ij}\}$, $1 \le i \le n$, $1 \le j \le p$. We let $Z_p \times Z_p = \langle a, b \rangle$ and let $\langle a, b \rangle$ act as follows:

$$v_{ij}\,a = \begin{cases} v_{ij} + v_{i\,j+1} & \text{if } j < p \\ v_{ip} & j = p \end{cases}$$

$$v_{ij}\,b = \begin{cases} v_{ij} & \text{if } j > 1 \text{ or } j = 1,\ i = n \\ v_{i1} + v_{i+1\,p} & \text{if } j = 1,\ i < n. \end{cases}$$

Picture of actions of $a-1$, $b-1$:



It is easy to see that we have a $F(Z_p \times Z_p)$-module which is free as a $F\langle a \rangle$-module. Hence V_n is bounded. D. Janusz has shown by a direct calculation that V_n is periodic of period 2 (or 1 if $p = 2$)

Lemma 4 V_n is indecomposable.

Besides our proof one can proceed as follows. In terms of matrices

$$a \to \begin{pmatrix} J_p & & & \\ & J_p & & \\ & & \ddots & \\ & & & J_p \end{pmatrix} \qquad b \to \begin{pmatrix} I & 1 & & \\ & I & 1 & \\ & & \ddots & 1 \\ & & & I \end{pmatrix}$$

and D. Janusz has calculated the centralizer:

$$\left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots \\ & \ddots & \alpha_2 \\ & & \alpha_1 \end{pmatrix} \right\}$$

Proof. Let U be a direct summand of V_n. Replacing U by a complement if necessary we may assume

$$v = v_{1p} + \alpha_2 v_{2p} + \dots + \alpha_n v_{np} \in U.$$

Since U is free as an $F\langle a\rangle$ there is $u \in U$ with $u(a-1)^{p-1} = v$. Then $u(b-1) \in U$ and it equals

$$v_{2p} + \dots$$

Keeping this up we finally get $v_{np} \in U$ and then working backwards $v_{n-1\,p} \in U$ and finally the whole socle of V_n in U. Done

Now we can prove the theorem. First assume F is finite (even with just p elements). Now since G has a non-cyclic Sylow p-subgroup it has subgroups T with $T \simeq Z_p \times Z_p$ or $T \simeq Q_8$. Let $V_1, V_2, \dots$ be the above modules for T when $T \simeq Z_p \times Z_p$ and also in the latter case for T/T'. In the latter case all modules for T are periodic. Hence, in both cases $V_1, V_2, \dots$ are indecomposable bounded modules for t of dimensions $p, 2p, \dots$. Let $U_1, U_2, \dots, U_n, \dots$ be FG modules related to the V_i, by Lemma 2. Hence, the U_i are indec; of growing dimensions as $V_i \mid (U_i)_T$, and bounded and hence periodic.

Now tensor with any field extension. The only problem is the indecomposability of the U_i. The V_i stay indec by the above argument. But we can always use a summand of the U_i also related to V_i since, as it easy to prove, a summand of a periodic module is periodic.

5. Questions

1) What is the period? In particular, if U is a periodic module for a p-group P does U have period 1, 2 or 4 and period 2 if $p > 2$?

2) Is every bounded module periodic over any field? Presumably the answer is "yes."

3) Do all periodic modules arise for a group G as follows: Choose a subgroup H, a normal subgroup K of H such that H/K has p-rank one, choose a module for H which is projective as a K-module induce it up and take a non-projective summand. This is plausible but there is no real evidence. Except the examples for $SL(2, 2^n)$ are of this type! ~~Ont, if $r_p(G) > 1$, M periodic then $p \mid \dim M$~~ ~~by Tanhen~~

4) If $0 \to U \to V \to W \to 0$ is exact, U, W are periodic is V? If the field is algebraic over the prime field then the long exact sequence in Ext shows V is bounded and hence periodic.

6. Proof of Theorem 3.

Say $r_p(G) > 1$ so $H = Z_p \times Z_p \subseteq G$. Hence, $(F_H)^G$ is unbounded. Hence (as $\operatorname{Ext}(U, V) \simeq \operatorname{Ext}(F, U^* \otimes V)$) F is an unbounded FG-module. But $p \nmid \dim M$ implies $F \mid M^* \otimes M$. Thus $M^* \otimes M$ is unbounded. Hence (again using $\operatorname{Ext}(M^* \otimes M, U) \simeq \operatorname{Ext}(M, M \otimes U)$) M is unbounded.

Complexity

Def. A sequence $\alpha = (a_0, a_1, a_2, \ldots)$ of non-negative integers has complexity $c = c(\alpha)$ if there is $f \in \mathbb{Z}[X]$, $\deg f = c-1$, $a_n \le f(n)$ for all but finitely many n and no such polynomial exists of smaller degree.

Remarks 1. Complexity zero means only finitely many non-zero terms as 0 is the only polynomial of degree -1.

2. If $\alpha = (a_0, a_1, \ldots)$, $\beta = (b_0, b_1, \ldots)$ have complexities $c(\alpha)$, $c(\beta)$ respectively (that's a hypothesis) then

$$\gamma = \Big(a_0 b_0, \ldots, \sum_{j=0}^{n} a_j b_{n-j}, \ldots\Big)$$

has complexity at most $c(\alpha) + c(\beta)$. For $a_n \le A n^{c(\alpha)-1}$, $b_n \le B n^{c(\beta)-1}$ for almost all n so "roughly"

$$\sum_{j=0}^{n} a_j b_{n-j} \le AB \sum_{j=0}^{n} j^{c(\alpha)-1}(n-j)^{c(\beta)-1}$$
$$\doteq AB \int_0^n x^{c(\alpha)-1}(n-x)^{c(\beta)-1}$$

so O.K. Need to deal with the cases of $c(\alpha)$ or $c(\beta)$ equal to 0 or 1 separately but that's easy.

(~~We next show that ? is in fact related~~ to Quillen's theorem.)

Prop If G is a finite group, F is a field of characteristic p and $a_n = \dim_F \mathrm{Ext}^n_{FG}(F, F)$ then $\alpha = (a_0, a_1, \ldots)$ has complexity $\ge$ to the p-rank of G.

Will prove it with a couple remarks.

Remarks 1. That α has a complexity is a consequence of the finite generation of the cohomology ring $Ext^*_{FG}(F,F)$. For, if it has generators $x_1, \ldots, x_d$ of degrees $n_1, \ldots, n_d$ then for $n > 0$ $\dim_F Ext^n_{FG}(F,F)$ is at most the number of sequences $(s_1, \ldots s_d)$ of non-negative integers such that $\Sigma s_i n_i = n$. This is clearly at most n^d.

2. For p-groups it's easy to see that $c(\alpha) \geq r = r_p(G)$. For let G be such a p-group and choose E an elementary abelian p-subgroup. Then $Ext^n_{FG}(F, (F_E)^G) \cong Ext^n_{FE}(F,F)$. But $(F_E)^G$ has a composition series of length $|G:E|$ with each factor F. Using the long exact sequence for Ext iteratively, we get that $\dim_F Ext^n_{FG}(F,(F_E)^G) \leq |G:E| \dim_F Ext^n_{FG}(F,F)$. But $Ext^n_{FE}(F,F)$ has dimension of complexity r, by direct calculation. This gives the desired lower bound for $Ext^n_{FG}(F,F)$ by dividing by $|G:E|$.

3. But we can do better. Let M be any module, let $b_n = \dim_F Ext^n_{FG}(F,M)$, $\beta = (\ldots b_n \ldots)$, G not necessarily a p-gp. Claim $c(\beta) \leq c(\alpha)$. By Theorem 7.1 of L. Evens, The Cohomology Ring of a Finite Group, Trans. Amer. Math. Soc. 101 (1961) 224–239, $\mathcal{M} = Ext^*_{FG}(F,M)$ is a f.g. module over $Ext^*_{FG}(F,G)_\circ = R$. Say $\mathcal{M}$ has generators $m_1, \ldots m_r$ of degrees $k_1, \ldots k_r$. Then $\beta_n \leq \sum_{i=1}^{r} a_{n-k_i}$ so done. But now let $M = (F_E)^G$ as in 2. Get $c(\alpha) \geq r$.

Note that we've also proved in remark 3 the following (all modules f.g. dim/F)

Prop. If M is a FG-module then $(\dim_F \mathrm{Ext}^n_{FG}(F, M))$ has a complexity and it is at most $c(\alpha)$.

Def If M is an FG-module we let $q(M)$ be this complexity, the Quillen dimension of M.

Now let M be any module. Let U be any other. We can consider the sequence of dimensions $\dim_F \mathrm{Ext}^n_{FG}(M, U)$. Since $\mathrm{Ext}^n_{FG}(M, U) \simeq \mathrm{Ext}^n_{FG}(F, M^* \otimes U)$ this sequence has a complexity. Moreover, this is at most the maximum of the complexities we get by using U irreducible, by using the long exact sequence for Ext.

Def This maximum is the complexity $c(M)$ of the module M.

Hence $c(M) = 0$ if, and only if M is projective. $c(M) = 1$ if, and only if M is bounded and not projective.

Conjecture $q(M) = c(M_{B_0})$ where B_0 is the principal p-block. (This would imply $M \in B_0$ then $\mathrm{Ext}^n_{FG}(F, M) \neq 0$ for some n !!)

Here are a few easy results:

Prop If H is a subgroup of the group G, U, V are FG and FH modules, respectively, and U and V are related then $c(U) = c(V)$.

Proof Just as in bounded case.

Prop If M is an FG-module then $c(M)$ equals the complexity of the sequence $\dim_F P_n$ where

$$\to P_n \to \cdots \quad \to P_0 \to M \to 0$$

is a minimal projective resolution of M over FG.

Proof. Let $V_1, \ldots, V_s$ be the irreducible FG-modules, one from each isomorphism class. If $Q_1, \ldots Q_s$ are their projective covers then

$$P_i = \bigoplus_j a_{ij} Q_j$$

and

$$\dim_F \mathrm{Hom}_{FG}(P_i, V_k) = a_{ik} \dim_F \mathrm{Hom}_{FG}(V_k, V_k)$$

That is,

$$\dim_F \mathrm{Ext}^i_{FG}(M, V_k) = a_{ik} \dim_F \mathrm{Hom}_{FG}(V_k, V_k).$$

Thus,

$$a_{ik} \dim Q_k = \frac{\dim_F \mathrm{Ext}^i_{FG}(M, V_k) \cdot \dim_F \mathrm{Hom}_{FG}(V_k, V_k)}{\dim_F Q_k}$$

Hence

$$\dim P_i = \sum_j \left(\frac{\dim_F \mathrm{Hom}_{FG}(V_j, V_j)}{\dim_F Q_j} \right) \dim_F \mathrm{Ext}^i_{FG}(M, V_k)$$

so done.

A good guess? M has complexity c. Then there is a positive c-complex, each non-zero entry projective, all of dimensions below a bound such that the associated single complex has an augmentation onto M and the resulting augmented complex is the minimal projective resolution of M.

A program:

1. Minimality. If a resolution in the form of a complex exists show true for the minimal resolution.

2. Summands. If O.K for $A \oplus B$ then O.K for A.

3. Shrinking. Get dimension of complex down to c.

4. Wreaths. Do for $Z_p \wr Z_p \wr \ldots \wr Z_p$.

Then done as $G \subseteq \Sigma_{p^n}$ some n. Let P be a Sylow p-subgp for Σ_{p^n} so by 4. FP is O.K as FP-module. Then $F^{\Sigma_{p^n}} = F \oplus \ldots$ as index is not divisible by p. ∴ use summands and restriction to G.

Or perhaps skip steps 2 and actually calculate for Σ_n or $GL(n,p)$ or something.

Question: Is $\gamma(M)$ equal to the order of the pole at $t=1$ of

$$\sum \dim_F \mathrm{Ext}^n_{FG}(F, M)\, t^n$$

which is what Quillen gives as a dimension?

Answer: Yes, see Swan, J of Alg 17, 401-4

Our last result is a consequence of Quillen's Theorem:

<u>Prop</u> $c(\alpha) \leq r$

(Here $\alpha = (\alpha_0, \alpha_1, \ldots)$, $\alpha_i = \dim_F \mathrm{Ext}^n_{FG}(F,F)$, $r = r_p(G)$ as above.)

<u>Proof</u> Let R be the cohomology ring $\mathrm{Ext}^*_{FG}(F,F)$ and let K be the ideal which is the intersection of the kernels of the restriction maps of R into the elementary abelian subgroups of order p^r in G. Since there are a fixed number of such subgroups and their cohomology is well known, we deduce that if $\beta_n = \dim_F R_n/K_n$ (everything is graded) then $c(\beta) \leq r$, $\beta = (\beta_0, \beta_1, \ldots)$. But by Quillen-Venkov K is a nil ideal. But R is noetherian so K is nilpotent (remember R is skew-commutative). Say $K^s = 0$, $s > 0$. Hence K/K^2, $K^2/K^3, \ldots K^{s-1}/K^s$ are all R/K modules and each is finitely generated. Hence, if $\gamma_{in} = \dim_F K^i_n / K^{i+1}_n$, $i = 0, \ldots, s-1$ and $\gamma_i = (\gamma_{i0}, \gamma_{i1}, \ldots)$ then $c(\gamma_i) \leq r$. But

$$\alpha = \beta + \gamma_1 + \ldots + \gamma_{s-1}$$

so $c(\alpha) \leq r$ as desired.

Periodics for $Z_2 \times Z_2$

We shall give a direct proof, without using the classification of the following: (F a field of char. 2)

Prop If M is a periodic $F(Z_2 \times Z_2)$-module then $\text{Rad}\, M = \text{Soc}\, M$ and $\dim_F (M/\text{Rad}\, M) = \dim_F (\text{Soc}\, M)$.

Let $G = \langle x, y \rangle \cong Z_2 \times Z_2$

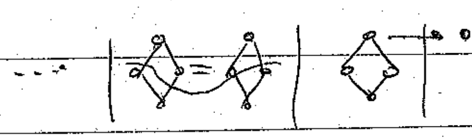
Lemma 1 If V is an FG-module then

$$\dim_F \text{Ext}^n_{FG}(F, V) \geq (2n+1) \dim_F C_V(G) - n \dim_F V$$

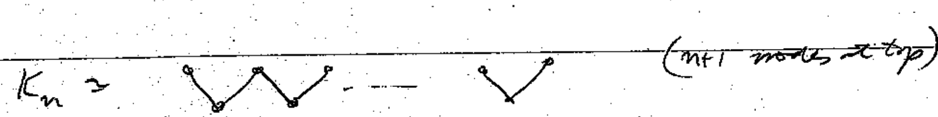
Proof Consider the minimal projective resolution of F:

$$\cdots \to P_2 \to P_1 \to P_0 \to F$$

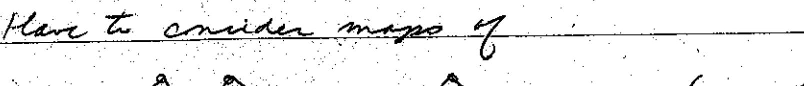
Or, diagrammatically,



Let K_n be kernel in P_{n-1}, $n \geq 1$. Hence the "cocycles" Z_n for $\text{Ext}^n(F,V)$ are $\text{Hom}(K_n, V)$. But



so $\dim_F Z_n \geq (n+1) \dim_F C_V(G)$. Next, let B_n be the coboundaries. Have to consider maps of



into V and B_n is restrictions to the socle of this module lifted to K_n

But map of L_n to V determines a map of L_n to $V/C_V(G)$ and also determines the restriction of the original map of L_n to V, to the socle of L_{n+1}. This is because $1+X$, $1+y$ annihilate $C_V(G)$.

Thus $\dim B_n \leq n \dim V/C_V(G) = n \dim V - n \dim C_V(G)$.

We deduce that

$$\dim \mathcal{E}xt^n(F,V) \geq (n+1) \dim C_V(G) - (n \dim V - n \dim C_V(G))$$

$$= (2n+1) \dim C_V(G) - n \dim V. \qquad Q.E.D.$$

<u>Lemma 2</u> If V is a periodic FG-module then

$$\dim_F C_V(G) \leq \tfrac{1}{2} \dim V$$

$$\dim_F V/[V,G] \leq \tfrac{1}{2} \dim V$$

<u>Proof</u> Say $\dim_F C_V(G) > \frac{1}{2} \dim V$ so $2 \dim_F C_V(G) > \dim V$

Hence $n(2 \dim_F C_V(G) - \dim V) \to \infty$ as $n \to \infty$ so

$\dim_F \mathcal{E}xt^n_{FG}(F,V) \geq n(2 \dim C_V(G) - \dim V) \to \infty$ as $n \to \infty$, a contradiction.

Now V periodic implies V^* the dual module is also, as

$$0 \to V \to Q_n \to \cdots \to Q_0 \to V \to 0$$

yields

$$0 \leftarrow V^* \leftarrow Q_n^* \leftarrow \cdots \leftarrow Q_0^* \leftarrow V^* \leftarrow 0$$

But $C_{V^*}(G) \cong V/[V,G]$ the "fixed pts at the top!"

<u>Proof of Prop.</u> $[V,G] = \operatorname{Rad} M$, $C_V(G) = \operatorname{Soc} M$. Now $\operatorname{Rad} M \subseteq \operatorname{Soc} M$ for $v(1+x)(1+y) = 0$ as no free summands. Hence Lemma 2 fulfills the proof.

McKay's Conjecture

McKay Conjecture: The number of irreducible characters of degree prime to p for a group G equals that of $N(P)$, P a Sylow p-subgroup.

Generalized Conjecture: If B is a p-block of G, D its defect group, b the corresponding block of $N(D)$ then the number of characters of height 0 in B is the same as the number in b.

Say G is p-nilpotent: $G = PK$, $K = O_{p'}(G)$, P a Sylow p-subgp. We apply Fong's thesis to get blocks and characters. Need one other fact. The covering groups Fong gets have p'-kernels and P is a p-group so these coverings are trivial.

Let $\mathcal{O}$ be an orbit of irreducible characters of K under P, let $\theta \in \mathcal{O}$, P_θ the stabilizer in P. Then θ lifts to a character $\tilde{\theta}$ of $P_\theta K$, one multiplies by each of the irreducible characters of $P_\theta K/K \simeq P_\theta$ induce to $PK = G$ and get a block $B_\mathcal{O}$ of G. This gives all blocks, a 1-1 correspondence $\mathcal{O} \leftrightarrow B_\mathcal{O}$ and all characters in $B_\mathcal{O}$. Hence, its easy to see that the generalized correspondence follows from Glauberman's one-to-one correspondence on characters. We shall give a new proof. Here's what we prove. (We need only a special case here.)

Theorem Let A be a solvable group of automorphisms of the group G, with $(|A|, |G|) = 1$ and set $F = C_G(A)$. There is a one-to-one correspondence between the A-invariant irreducible characters of G and the irreducible characters of F. Moreover, if

χ is an A-invariant irreducible character of G then there is a unique irreducible character φ of F such that (χ_F, φ) is relatively prime to $|A|$ and φ is the character corresponding to χ, if $|A|$ is a prime-power.

Presumably the result still holds if the word "solvable" is dropped.

Proof. Suppose the result holds in the case that A has prime-power order. We proceed by induction on $|A|$. Let Q be a normal q-subgroup, q a prime, $Q \neq 1$, of A. Let $E = C_G(Q)$. Let I_Q, (I_A) be the Q-invariant (A-invariant) irreducible characters of A. If $\chi \in I_Q$ let φ be the corresponding character of E. If $\alpha \in A$ then $(\chi^\alpha, \varphi^\alpha) = (\chi, \varphi)^q$ so χ^α and φ^α correspond. Hence, if $\chi \in I_A$ then $\varphi \in J_A$ the A-invariant characters of E and this is a one-to-one correspondence. Moreover,

$$\chi_E = (\chi_E, \varphi)\,\varphi + q\,\Phi$$

Applying induction to A/Q acting on E we are easily done.

Hence, we may assume A is a p-group, p a prime. We apply Brauer's first main theorem to blocks of AG with defect group A. (R. Dow informs me he knew this first part of the proof too.) Let O be an orbit such that B_O has defect group A, so $O = \{\chi\}$, χ a A-invariant character. And conversely these are the blocks with A as defect group. The blocks of $AF = N_{AG}(A)$ correspond to the characters of F, 1-1 and onto, a block being the product of all the characters of A and a single character of F (just a special case of the previous paragraph. Let $\tilde{\varphi} = 1 \times \varphi$ and let $\tilde{\chi}$ extend χ to AG. The corresponding $\tilde{\omega}$'s agree on F, the p-regular elements of F. That is, if $u \in F$

$$\frac{|f^{AG}| \ \tilde{\chi}(f)}{\tilde{\chi}(1)} \equiv \frac{|f^{AF}| \ \tilde{\varphi}(f)}{\tilde{\varphi}(1)} \quad (\mathfrak{p})$$

or

$$|f^{G}| \ \chi(f)/\chi(1) \equiv |f^{F}| \ \varphi(f)/\varphi(1)$$

and this formula gives the one-to-one correspondence between the χ's and the φ's. It remains to see that it has the desired properties.

This congruence can be rewritten as

$$\chi(f) \equiv \frac{|C_G(f)|}{|C_F(f)|} \frac{|F|}{|G|} \frac{\chi(1)}{\varphi(1)} \varphi(f)$$

Now the classes of F do not fuse in G, by Sylow's theorem, so if $\Pi = (1_F)^G$, the permutation character, $\Pi(f) = \frac{|C_G(f)|}{|C_F(f)|}$. So we have

$$\chi_F \equiv \left(\frac{|F|}{|G|} \frac{\chi(1)}{\varphi(1)}\right) \Pi_F \varphi$$

$$\chi_F \equiv r_\chi \Pi_F \varphi$$

where r_χ is the constant not divisible by p. Can assume $0 < r_\chi < p$.

Now under the correspondence the principal characters correspond as the principal blocks correspond.(*) We deduce from the previous formula, for all $f \in F$

$$\Pi_F(f) \equiv 1$$

Hence,

$$\chi_F \equiv r_\chi \varphi$$

this implies that

$$\chi_F = r_\chi \varphi + p \, \Phi$$

(As the determinant of the character table of F is not in $\mathfrak{J}$
only l.c. with all coeff divisible by p give values in $\mathfrak{J}$.)
This proves the theorem.

<u>Remark</u>: It seems we get the same correspondence as Glauberman.
For let η be "canonical" extension of χ to AG in case where A is
cyclic of prime-power order, in Glauberman's notation.
Then $\eta(\alpha f) = \pm \varphi(f)$ for fixed sign where φ and χ
correspond in his correspondence. But $\eta(\alpha f) \equiv \eta(f)$ as α is a
p-element, and $\eta(f) = \chi(f)$ so $\chi_F \equiv \pm \varphi$.
Glauberman seems to get more!

Algebraic Modules.

As usual, F a field, G a finite group, p = char (F) > 0.

Def An FG-module V is *algebraic* if there is $n > 0$ and integers $a_0, \dots, a_n$ not all zero such that

$$\sum a_i [V]^i = 0$$

in the Green ring.

Prop. If V is an FG-module the following are equivalent:

1) V is algebraic;

2) There are only finitely many isomorphism types of indecomposable summands of tensor powers of V;

3) Every indecomposable summand of V is algebraic

Proof. It is clear that 1) implies 2) as 1) yields that for all $k \geq k_0$ V^k can be expressed in terms of smaller tensor powers. More specifically, can assume in the definition that $a_n > 0$ so we get

$$a_n V^n + \dots \quad \simeq b_{n-1} V^{n-1} + \dots \quad \text{(all coeff non-neg)}$$

Hence, each summand of V^n is a summand of one of F, V, $V^2, \dots, V^{n-1}$.

next,

$$a_n V^{n+1} + \dots \quad \simeq b_{n-1} V^n + \dots$$

so if $b_{n-1} = 0$ still O.K. If not have

$$a_n^2 V^{n+1} + \dots \quad \simeq b_{n-1} a_n V^n + (\ ) V^{n-1} + \dots$$

so using the fact that summands of V^n are O.K. so also for V^{n+1}. Etc... quite easily.

(Better: Components of V^n in $F, V, \dots V^{n-1}$. Tensor this statement with V so components of V^{n+1} in $F \otimes V, \dots V^n$ so in $F, V, \dots, V^{n-1}$ etc.)

Next, suppose 2) holds, letting $W_1, \ldots, W_s$ be representatives of the isomorphism classes so appearing. Hence, for non-negative a_{ij},

$$V \cong a_{11} W_1 + \cdots + a_{1s} W_s$$
$$V^2 \cong a_{21} W_1 + \cdots + a_{2s} W_s$$
$$\vdots$$

and so $V, V^2, \ldots, V^{s+2}$ will satisfy a relation.

Now clearly 2) implies that a similar statement holds for summands of V so any summand of V is algebraic as 2) implies 1). Thus 2) yields 3). Now assume 3) and let $V = U_1 + \cdots + U_r$ a direct sum of indecomposables. As 1) yields 2) we know there is a finite number of iso classes of indec which contain all indec appearing in powers of any U_i. Therefore by decomposing a bounded number of products of these we easily establish that 2) holds for V.

Prop Let H be a subgroup of G and let W be an FH-module "related" to V (so $V | W^G$, $W | V_H$). Then V is algebraic if and only if W is algebraic.

Corollary If P is a Sylow p-subgroup of G then an FG-module V is algebraic if, and only if V_P is algebraic.

Pf (of Cor). If V is algebraic then V_P satisfies the same equation. If V_P is algebraic so is every summand by the previous proposition so V is by this proposition.

Proof. If V is algebraic then V_H satisfies the ~~same~~ polynomials V does so V_H is algebraic. Assume W is algebraic. Since $V \mid W^G$ we need "only" establish the following result:

Prop. If H is a subgroup of the group G and W is an algebraic FH-module then W^G is an algebraic FG-module.

Let $\mathcal{A}$ be the additive subgroup of the Green ring for G of all linear combinations of components of algebraic modules of proper subgroups of H induced to G. We may assume, as the proposition is trivial if $H = 1$, that $\mathcal{A}$ consists of algebraic modules. As usual we say $M \in \mathcal{A}$ if $[M] \in \mathcal{A}$.

Lemma 1. If $A \in \mathcal{A}$ then $A \otimes W^G \in \mathcal{A}$.

Proof. We may as well assume $A = U^G$, U an indec. algebraic FK-module, where K is a proper subgroup of G. It suffices to show the components of $A \otimes W^G$ are in $\mathcal{A}$. But

$$A \otimes W^G = U^G \otimes W^G \simeq \bigoplus_t \left(U^t_{K^t \cap H} \otimes W_{K^t \cap H}\right)^G.$$

And $U^t_{K^t\cap H}$ and $W_{K^t \cap H}$ are ~~proper~~ algebraic modules.

Next, let $W = W_1, \ldots, W_s$ be the distinct conjugates of W under $N(H)$. Consider all components of tensor products of these and all components of these induced to G. Let $\mathcal{U}$ be the latter set of indecomposables. Thus, $\mathcal{U}$ is finite. Now let $\mathcal{U}$ also denote the additive subgroup of the Green ring spanned by these.

Lemma 2 For all n, $(W^G)^n \in \bar{\mathcal{U}} + \mathcal{A}$.

Proof If $n=1$ then $W^G \in \bar{\mathcal{U}}$. Suppose $(W^G)^r \in \bar{\mathcal{U}} + \mathcal{A}$. Then, say $(W^G)^r = U \oplus A$, $U \in \bar{\mathcal{U}}$, $A \in \mathcal{A}$, so $(W^G)^{r+1} = (W^G)^r \otimes W^G = (U \otimes W^G) \oplus (A \otimes W^G)$. But $A \otimes W^G \in \mathcal{A}$ by the previous lemma, so we need only deal with $U \otimes W^G$. Hence, it suffices to let $W^{(1)}, \ldots W^{(m)}$ be conjugates of W and to study

$$\left(W^{(1)} \otimes \ldots \otimes W^{(m)}\right)^G \otimes W^G.$$

But this is isomorphic with

$$\bigoplus_t \left(W^{(1)} \otimes \ldots \otimes W^{(m)}\right)^t_{H^t \cap H} \otimes W_{H^t \cap H}$$

$$= \left(\bigoplus_{t \in N(H)} \left(W^{(1)} \otimes \ldots \otimes W^{(m)}\right)^t \otimes W\right) \oplus \left(\overbrace{\qquad\qquad}^{\in \mathcal{A}}\right)$$

so the lemma is established.

Proof of the Prop. Let $\mathcal{U} = \{U_1, \ldots, U_\ell\}$. Thus,

$$W^G = a_{11} U_1 + \ldots + a_{1\ell} U_\ell + A_1 \qquad A_1 \in \mathcal{A},\ a_{ij} \geq 0$$

$$(W^G)^2 = a_{21} U_1 + \ldots + a_{2\ell} U_\ell + A_2$$

$$\vdots$$

$$(W^G)^{\ell+1} = a_{\ell+1,1} U_1 + \ldots + a_{\ell+1,\ell} U_\ell + A_{\ell+1}$$

so an integral linear combination of the powers of W^G lie in $\mathcal{A}$ and is algebraic $\therefore$ W^G is algebraic!

Question If V is an irreducible FG-module, V in a block with abelian defect group, is V algebraic?

Addendum on invertible modules

Let V_ω be the 3-dimensional indecomposable for Q_8 – the quaternion group – which is faithful; the other one is $V_{\bar\omega}$. We shall see that $V_\omega^{4} \cong F$ so that V_ω is a invertible module of order 4 (not 2 as $V_\omega \otimes V_\omega \cong F$ yields $V_\omega \cong V_{\bar\omega}$ as $V_\omega^* \cong V_{\bar\omega}$.)

Consider the char 2 reps of $SU(3,2)$ = extra special order 27. Q_8. Have table

	1	z	z^{-1}	t
φ_1	1	1	1	1
φ_2	3	3ω	$3\bar\omega$	0
φ_3	3	$3\bar\omega$	3ω	0
φ_4	8	8	8	-1

where z, z^{-1} are central and t is of order 3 and not central. Now $\varphi_2 \otimes \varphi_3 = \varphi_1 \oplus \varphi_4$; as modules $V_2 \otimes V_3 = V_1 \oplus V_8$, with obvious notation, so V_2 is invertible. By Green correspondence get V_2 / Q_8 is indecomposable (as source is not trivial and vertex is Q_8) and certainly faithful so is V_ω or $V_{\bar\omega}$. (which?) Assume V_ω as argument is the same either way.

Now $\varphi_2^{3} = 3\varphi_3$ so get $V_2 \otimes V_2 \otimes V_2$ is projective plus a three-dimensional module with all φ_1 composition factors $\therefore$ by restricting to Q_8 see $V_\omega \otimes V_\omega \otimes V_\omega$ is a three-dim indec. Must be faithful or else would be invertible. Not V_ω by invertibility $\therefore$ by theorem we proved it is $V_{\bar\omega}$. $\therefore V_\omega^{4} \cong F$.

$L_3(4)$, characteristic 3   Order $2^6 \cdot 3^2 \cdot 5 \cdot 7$

Ord. characters

	1	2	5	5	4	4	4	7	7	3
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	20	4	0	0	0	0	0	−1	−1	2
χ_3	35	3	0	0	−1	−1	3	0	0	−1
χ_4	35	3	0	0	−1	3	−1	0	0	−1
χ_5	35	3	0	0	3	−1	−1	0	0	−1
χ_6	64	0	−1	−1	0	0	0	1	1	1
$\lambda^7=1$	45	−3	0	0	1	1	1	$\lambda^3+\lambda^5+\lambda^6$	$\lambda+\lambda^2+\lambda^4$	0
	45	−3	0	0	1	1	1	$\lambda+\lambda^2+\lambda^4$	$\lambda^3+\lambda^5+\lambda^6$	0
$\mu^5=1$	63	−1	$-\mu^2-\mu^3$	$-\mu-\mu^4$	−1	−1	−1	0	0	0
	63	−1	$-\mu-\mu^4$	$-\mu^2-\mu^3$	−1	−1	−1	0	0	0

Modulo 3

	1	2	5	5	4	4	4	7	7
φ_1	1	1	1	1	1	1	1	1	1
φ_2	19	3	−1	−1	−1	−1	−1	−2	−2
φ_3	15	−1	0	0	−1	−1	3	1	1
φ_4	15	−1	0	0	−1	3	−1	1	1
φ_5	15	−1	0	0	3	−1	−1	1	1

D_0

	φ_1	φ_2	φ_3	φ_4	φ_5
χ_1	1	0	0	0	0
χ_2	1	1	0	0	0
χ_3	1	1	1	0	0
χ_4	1	1	0	1	0
χ_5	1	1	0	0	1
χ_6	0	1	1	1	1

C_0

	φ_1	φ_2	φ_3	φ_4	φ_5
φ_1	5	4	1	1	1
φ_2	4	5	1	1	1
φ_3	1	1	2	1	1
φ_4	1	1	1	2	1
φ_5	1	1	1	1	2

Proceeding after the projectives, note that each φ_i is self-dual, being real-valued; so, for example, looking at $P(\varphi_5)$, the proj. cover, the "middle" (i.e. rad $P(\varphi_5)$ / soc $P(\varphi_5)$) is multiplicity free and self-dual so is semi-simple. Similarly, for $P(\varphi_4)$, $P(\varphi_3)$.

e.g.

$P(\varphi_5)$:

$$\begin{array}{c} \varphi_5 \\ \varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_4 \\ \varphi_5 \end{array}$$

Better notation:

$$\begin{array}{c} \varphi_{15}^{(3)} \\ \varphi_1 \oplus \varphi_{19} \oplus \varphi_{15}^{(1)} \oplus \varphi_{15}^{(2)} \\ \varphi_{15}^{(3)} \end{array} \quad \underline{\text{or}} \quad \begin{array}{c} 15^{(3)} \\ 1, 19, 15^{(1)}\ 15^{(2)} \\ 15^{(3)} \end{array}$$

Now let's study the action of $N(S_3) = Z_3 \times Z_3 \cdot Q_8$ on the 21 points of the projective space. A non-identity element of order 3 and determinant 1 in $SL(3,4)$ fixes exactly three one-dimensional spaces in the vector space. Hence, get each $x \in (Z_3 \times Z_3)^{\#}$ fixes exactly 3 pts of the 21. Usual character calculation

$$1/9\,(21 + 24)$$

gives that $Z_3 \times Z_3$ has five orbits ∴ one of length 9, four of length 3. Since Q_8 acts can get orbits for $Z_3 \times Z_3 \cdot Q_8$. For pt. stabilizer intersects $Z_3 \times Z_3$ in pt stabilizer, normalizes that intersection and has a relatively small bound on its index in $Z_3 \times Z_3 \cdot Q_8$. Get one orbit with Q_8 the stabilizer, one other with stabilizer being $Z_3 \cdot Z(Q_8) \simeq \Sigma_3$. ∴ degrees 9, 12.

Let $1, i, j, k$ be linear reps of Q_8. Since 12 dimensional module from this action is induced easily get $Soc(12) = 1 \oplus i \oplus j \oplus k$ ∴ restriction of $\begin{matrix}1\\19\\1\end{matrix}$ looks as follows:

$$\begin{matrix}1\\2\\i\ j\ k\\2\\1\end{matrix} \quad \oplus \quad \begin{matrix}\square\\1\ i\ j\ k\end{matrix}$$

first term is projective, from 9 dim action, second must be indec by Green correspondence. Get now that restriction of 19 has Green correspondent with a non-simple socle!

On the Green correspondence

The category form of the Green correspondence, as given by Green based on his work and the work of Feit (the "H^0") can be approached another way, at least as far as getting weaker results in a much more special case.

Let's assume we have a group G, subgroup H such that if M is a G-module (over alg. closed field of char. p) and m is an H-module, both indecomposable then

M_H = indec ⊕ proj

m^G = indec ⊕ proj

They correspond if $M_H = m \oplus$ proj and $m^G = M \oplus$ proj. We prove:

<u>Theorem</u>. Let S and s be simple G and H-modules, resp., corresponding to u and U respectively. Then

G:	U	S
H:	s	u

1) $\mathrm{Hom}(U, S) = 0 \iff \mathrm{Hom}(s, u) = 0$

2) $\mathrm{Hom}(S, U) = 0 \iff \mathrm{Hom}(u, s) = 0$

By duality and by symmetry between G and H, it suffices to show that if $\mathrm{Hom}(U, S) \neq 0$ then $\mathrm{Hom}(s, u) \neq 0$. Let's do that. Have submodule V of U such that $U/V \simeq S$. Express $U_H = s \oplus q$, q H-module which is projective. Where does V_H "sit" in this direct sum? $V_H \not\subseteq q$ or get $S_H \cong s \oplus q/V_H$, which is impossible as V_H not projective and so q/V_H will not be, unless $V = 0$ in which case all is clear anyway. Also $s \not\subseteq V_H$ for then V_H has s as summand so U is a summand of V!

Hence, there are submodules $q_1 \supseteq q_2$ of q and an isomorphism θ of s onto q_1/q_2 so that V_H is the corresponding "pull-back" inside $s \oplus q$. (a pull-back of s and q_1.) $\therefore S_H \simeq q/q_2$, $V_H \simeq q_1$

To conclude, as $S_H \simeq q/q_2$ and $q_1/q_2 \simeq s$ it suffices to see that q_1/q_2 is not contained in a projective submodule of q/q_2. So suppose otherwise: $q_0/q_2 \supseteq q_1/q_2$ and q_0/q_2 is projective $\therefore q_0 = q_2 \oplus \cdots$ so $q_1 = q_2 \oplus s_1^{\simeq s}$ so $s \oplus q_1 = s \oplus s_1 \oplus \cdots$ so $V_H = \overset{\simeq s}{s_2} \oplus q_2$ a contradiction.

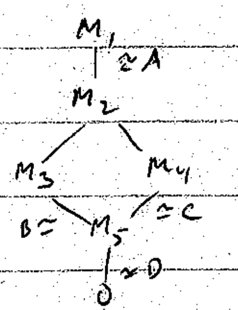
Diagrams

In many places we have diagrams to describe the structure of modules. This is an attempt to axiomatize some of this. Here's a list of examples of places where diagrams occurs:

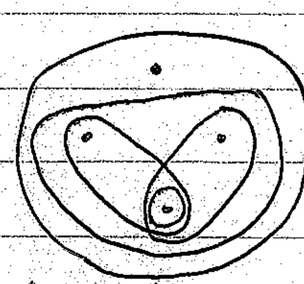
1. Conlon's description of $Z_2 \times Z_2$ modules;
2. Conlon's description of Szekeres' work;
3. Adams zig-zag diagrams;
4. Uniserial modules
5. Cyclic blocks
6. Modules for groups with dihedral Sylow 2-subgroups;
7. $SL(2, 2^n)$;
8. Ringel's classification for dihedral 2-groups (?);
9. Modules with no repeated composition factors cf. Carter-Cline on Weyl modules.

Besides axiomatics there are the questions of existence + uniqueness. What about existence for free modules in characteristic p for p-groups, extending Jennings' work on Loewy series and the p-dimension subgroups. Also, we should see what happens for tensor products.

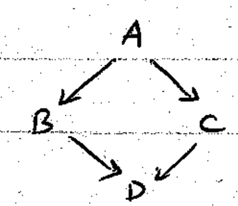
Here's a picture of what we're shooting for:



(the full lattice)



(a topological space)



(the diagram)

__Def 1.__ A __module space__ is a finite topological space with the property that each point is relatively closed in some open set containing it.

__Def. 2__ A __module diagram__ is a finite directed graph which is acyclic – in the directed sense – and completely intransitive, in that if there is an arc from a to b and one from b to c then there is no arc from a to c.

In particular, there are no arcs from a point to itself, i.e. no loops. Acyclic in the directed sense means we don't have: $a_1 \to a_2 \to \cdots \to a_1$, with the obvious notation.

If X is a module space and $x \in X$ let $O(x)$ be the smallest open set containing x, i.e. the intersection of all ($\because$ finitely many) open sets containing x. Form a directed graph $\mathcal{D}(X)$ on X by putting a (directed) arc from x to y if $O(x) \supset O(y)$ (i.e. proper inclusion) and if there is no z in X with $O(x) \supset O(z) \supset O(y)$.

__Prop 1.__ $\mathcal{D}(X)$ is a module diagram.

Clearly $\mathcal{D}(X)$ is a directed graph and completely intransitive. Also, it is acyclic in the directed sense because if x is arced to y then $|O(x)| > |O(y)|$.

Now let D be a module diagram. Define a family of open subsets of D by letting $O \subseteq D$ be open if, and only if, whenever $x \in O$ $y \in D$ and x is arced to y then $y \in O$. This is clearly a topology. Let $\mathfrak{X}(D)$ be this topological space.

__Prop 2.__ $\mathfrak{X}(D)$ is a module space.

Let $x \in D$. Let O be the set of all $y \in D$ such that there exist $z_1, \dots, z_k \in D$ with

$$x \to z_1 \to \dots \to z_k \to y$$

(with $k = 0$ allowed, of course). Clearly O is open, as is $O \cup \{x\}$. $\therefore$ x is closed in the relative topology on $O \cup \{x\}$.

__Prop 3.__ a) If D is a module diagram then $\mathcal{D}(\mathfrak{X}(D)) = D$.

b) If X is a module space then $\mathfrak{X}(\mathcal{D}(X)) = X$.

__Pf.__ a) By definition of the topology in $\mathfrak{X}(D)$ we have $O(x)$ to be $\{x\} \cup \{y \mid x \to z_1 \to \dots \to z_k \to y\}$. Hence, it suffices to show that $x \to y$ if, and only if $O(x) \supset O(y)$ and there is no $z \in D$ with $O(x) \supset O(z) \supset O(y)$. Of course, $x \to y$ implies $O(x) \supset O(y)$. Say there is such a z. Then $x \to \dots \to z$ and $z \to \dots \to y$ contradicting complete intransitivity. Now assume the conditions $O(x) \supset O(y)$ and the non-existence of the z. But $O(x) \supset O(y)$ and if we assume x is not arced to y then $x \to \dots \to z \to \dots \to y$, for some z, a contradiction.

b). Let $\mathcal{U}$ be a subset of X. It is required to show that $\mathcal{U}$ is open, as a subset of the topological space X if and only if it is open in $\mathfrak{X}(\mathcal{D}(X))$. Suppose $\mathcal{U}$ is open in X; we wish to see that if $x \in \mathcal{U}$, $y \in X$ and $x \to y$ in $\mathcal{D}(X)$ then $y \in \mathcal{U}$. But $x \to y$ means, in particular, that $\mathcal{O}(x) \supset \mathcal{O}(y)$ so $\mathcal{U} \supseteq \mathcal{O}(x) \supset \mathcal{O}(y)$ and $y \in \mathcal{U}$. Suppose, conversely, that $\mathcal{U}$ is open in $\mathfrak{X}(\mathcal{D}(X))$ so that if $x \in \mathcal{U}$, $x \to y$ in $\mathcal{D}(X)$ then $y \in \mathcal{U}$. We must prove that $\mathcal{U}$ is open in X. It suffices - as is easily seen - to establish the following

<u>Lemma</u> If $x \in X$ and $y \in \mathcal{O}(x)$, $y \neq x$ then there are $z_1, \ldots, z_k \in X$ (perhaps $k=0$) with $x \to z_1 \to \cdots \to z_k \to y$ in $\mathcal{D}(X)$.

<u>Pf</u> Since $y \in \mathcal{O}(x)$ we have $\mathcal{O}(x) \supset \mathcal{O}(y)$, with proper containment as $x \neq y$. Choose $z_1, \ldots, z_k \in X$ so that

$$\mathcal{O}(x) \supset \mathcal{O}(z_1) \supset \cdots \supset \mathcal{O}(z_k) \supset \mathcal{O}(y)$$

with no interpolations possible. Then $x \to z_1 \to \cdots \to z_k \to y$ in $\mathcal{D}(X)$.

Before proceeding, we make some observations.

<u>Lemma</u>. Let X be a module space and $\mathcal{O}$ an open subset.

a) The minimal open subsets of $\mathcal{O}$ (relatively = absolutely) are one-point sets.

b) The maximal open subsets of $\mathcal{O}$ have cardinality one less than that of $\mathcal{O}$.

<u>Pf</u> a) Let $\mathcal{U}$ be a minimal open subset of $\mathcal{O}$, $x \in \mathcal{U}$. Then $\mathcal{U} = \mathcal{O}(x)$. Since x is relatively closed in some open set it is in $\mathcal{O}(x)$. $\therefore \mathcal{O}(x) - \{x\}$ is

open so $\mathcal{O}(x) - \{x\} = \phi$ and $\mathcal{U} = \{x\}$. (Note: follows from b) as ϕ max $\mathcal{U}$) (*)

b) Let $\mathcal{V}$ be a maximal open subset of $\mathcal{O}$. Choose $x \in \mathcal{O}$, $x \notin \mathcal{V}$ minimizing $\mathcal{O}(x)$. Now if $y \in \mathcal{O}(x)$ then $\mathcal{O}(y) \subseteq \mathcal{O}(x)$ as $\mathcal{O}(x)$ is open. If $y \neq x$ then, as $\mathcal{O}(x) - \{x\}$ is open (as we saw in a)), $\mathcal{O}(y) \subset \mathcal{O}(x)$ so by minimality $y \in \mathcal{V}$. $\therefore$ $\mathcal{O}(x) = \{x\} \cup \mathcal{O}(x) \cap \mathcal{V}$ so $\mathcal{O}(x) \cup \mathcal{V} = \{x\} \cup \mathcal{V} \supset \mathcal{V}$ so $\{x\} \cup \mathcal{V} = \mathcal{O}$ by maximality of $\mathcal{V}$.

Def 3. If $\mathcal{O}$ is an open subset of the module space X then $soc(\mathcal{O})$ is the union of the minimal open subsets of $\mathcal{O}$ and $rad(\mathcal{O})$ is the intersection of the maximal open subsets of $\mathcal{O}$.

Remark: From the lemma, get clearly, $\phi \subseteq \mathcal{U} \subseteq soc(\mathcal{O})$, $rad(\mathcal{O}) \subseteq \mathcal{V} \subseteq \mathcal{O}$ imply $\mathcal{U}$ is open, $\mathcal{V}$ is open.

Actually, for nice symmetry really need another operation instead of soc.

Def 4. If $\mathcal{O}$ is an open subset of the module space X then $soc(X/\mathcal{O})$ is the union of all open subsets in which $\mathcal{O}$ is maximal & $soc(\overset{X}{\cancel{\mathcal{O}}}/\overset{X}{\cancel{\mathcal{O}}}) = X$.

From above, $\mathcal{O}$ max $\mathcal{O}_1$ implies $|\mathcal{O}_1| - 1 = |\mathcal{O}|$. Also $\mathcal{O} \subseteq \mathcal{U} \subseteq soc(X/\mathcal{O})$ implies $\mathcal{U}$ open. And $soc\, X = soc(X/\phi)$, $soc\, \mathcal{O} = soc\, X \cap \mathcal{O}$.

At last we can give the definition that we have been building up to!

(*) Proof: $\mathcal{O}_1$ max $\mathcal{O}_2 \Leftrightarrow \mathcal{O}_1 \subseteq \mathcal{O}_2$, $|\mathcal{O}_1| + 1 = |\mathcal{O}_2|$.

<u>Definition</u>. If D is a module diagram and M is a module (for some ring) then we say that D is a diagram for M provided there is a map μ from the topology of $\mathfrak{X}(D)$ to the submodules of M satisfying the following conditions:

1) $\mu(\emptyset) = 0$;

2) $\mu(D) = M$;

3) θ_1 is a maximal open subset of θ_2 if, and only if, $\mu\theta_1$ is a maximal submodule of $\mu\theta_2$;

4) $\mu(\theta_1 \cap \theta_2) = \mu(\theta_1) \cap \mu(\theta_2)$

5) $\mu(\theta_1 \cup \theta_2) = \mu(\theta_1) + \mu(\theta_2)$

6) $\mu(\text{rad}\ \theta) = \text{rad}(\mu\theta)$

7) $\mu(\text{soc}(D/\theta)) / \mu(\theta) = \text{soc}(M/\mu(\theta))$.

ie preserves sums, intersections, maximality + the rad + soc(D/) operators! Easy to get socles of modules in the image of μ and radicals of their cokernels from what we have using $\text{soc}(N) = \text{soc}(M) \cap N$ if $N \subseteq M$ and $\text{rad}(M/N) = (\text{rad}\ M) + N / N$. The map μ gives us an attachment of a simple module to each point of D, namely $\mu(\theta(x)) / \mu(\theta(x) - \{x\})$.

<u>Remark</u>: Axioms should be of ~~three~~ two types

I $\subseteq$, $\cap$, $\cup$

II tied links, rad( ), soc(/)

Rest follows!

Heights and parabolics in the general linear groups

We are interested in finding formulas for the number of characters of a given height in a block which extend the generalized McKay conjecture on the number of height zero. We consider $GL(n, q)$ and disregard the blocks. If χ is a character of $GL(n,q)$, its degree is a polynomial in q divisible by q to a certain exponent – which we here call the height of the character.

There seems to be an easy rule to determine the number of characters of a given height; we haven't checked that is right. Let $\pi = a_1^{k_1} a_2^{k_2} \cdots a_r^{k_r}$ be a partition of n, $a_1 > \cdots > a_r$, $\sum k_i a_i = n$. To π corresponds

$$q^{(\sum k_i) - r} (q-1)^r$$

characters of height

$$\sum k_i \binom{a_i}{2}.$$

Running over all π we get all the characters. For example, $n = 5$

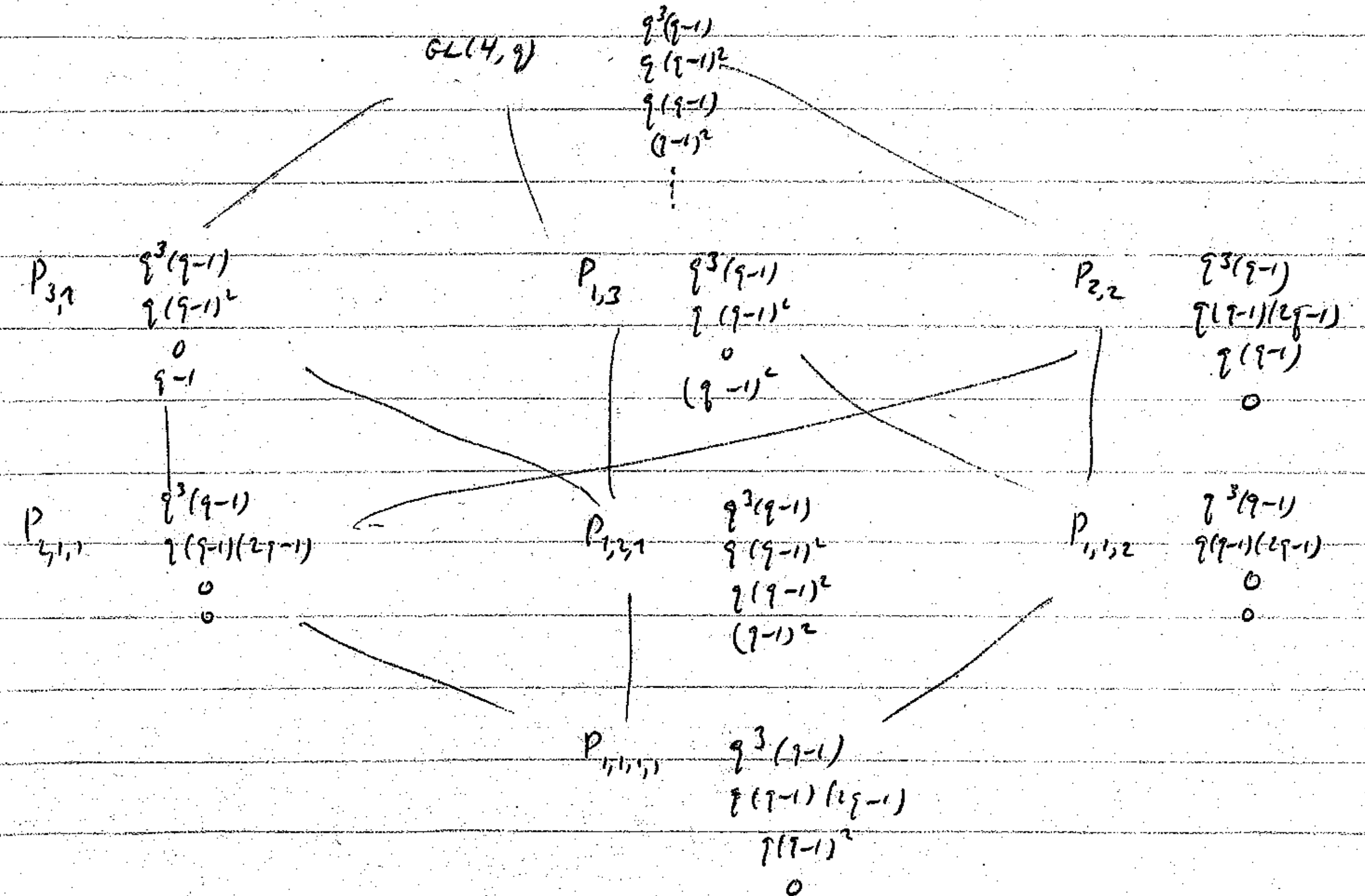
π	#	ht
1^5	$q^4(q-1)$	0
21^3	$q^2(q-1)^2$	1
$2^2 1$	$q(q-1)^2$	2
$3 1^2$	$(q-1)^2$	3
$3 2$	$(q-1)^2$	4
41	$(q-1)^2$	6
5	$(q-1)$	10

Next, we do some calculations for parabolics, $P_{s_1, s_2, \ldots s_t}$, $\Sigma s_i = n$, being the one with first an s_1 by s_1 block and so on. This is easy, especially using Clifford's Theorem in the case where the normal subgroup is abelian and the extension splits so projective, non-linear representations are not required. We simply tabulate the results:

n	Parabolic	Height	#
2	$P_{1,1}$	0	$q(q-1)$
	P_2	0	$q(q-1)$
		1	$q-1$
3	$P_{1,1,1}$	0	$q^2(q-1)$
		1	$(q-1)^2$
	$P_{2,1}$; $P_{1,2}$	0	$q^2(q-1)$
		1	$(q-1)^2$
	P_3	0	$q^2(q-1)$
		1	$(q-1)^2$
		2	0
		3	$(q-1)$
4	$P_{1,1,1,1}$	0	$q^3(q-1)$
		1	$q(q-1)(2q-1)$
		2	$q(q-1)^2$
	$P_{2,1,1}$; $P_{1,1,2}$	0	$q^3(q-1)$
		1	$q(q-1)(2q-1)$
		2	0
	$P_{1,2,1}$	0	$q^3(q-1)$
		1	$q(q-1)^2$

	2	$q(q-1)^2$
	3	$(q-1)^2$
$P_{2,2}$	0	$q^3(q-1)$
	1	$q(q-1)(2q-1)$
	2	$q(q-1)$
$P_{3,1}$; $P_{1,3}$	0	$q^3(q-1)$
	1	$q(q-1)^2$
	2	0
	3	$(q-1)^2$

Picture for $GL(4,q)$



Guess: Given a partition $\pi = a_1^{h_1} \cdots a_r^{h_r}$ the corresponding no. of characters of dt $\sum h_i \binom{a_i}{2}$ equals the no. for some parabolic of this type — or some such!

A new kind of projectivity

Definition. If $G \supseteq H \supseteq K$ are finite groups, F is a field (or more generally) and U is an FG-module then we say U is H/K-projective (H/K a coset space) provided U_H, the restriction, is relatively K-projective, that is, U_H is a summand of an FK-module induced to H.

Lemma. If $G \supseteq H \supseteq K \supseteq L$, U and V are FG-modules which are, respectively, H/K and K/L projective then $U \otimes V$ is H/L projective.

Proof. There is an FK-module W such that U_H is a summand of W^H. Hence $(U \otimes V)_H = U_H \otimes V_H$ is a summand of $W^H \otimes V_H$. Hence, it suffices to prove that $W^H \otimes V_H$ is relatively L-projective. But $W^H \otimes V_H \simeq (W \otimes V_K)^H$ so it suffices to prove that $W \otimes V_K$ is ~~relatively~~ L-projective. But V_K is a summand of X^K where X is an FL-module so $W \otimes V_K$ is a summand of $W \otimes X^K \cong (W_L \otimes X)^K$ and the proof is complete.

This perhaps can be used to construct resolutions. We would like to get a bounded complex giving *some* projective resolution of an FG-module M. $\therefore$ it suffices to deal with $M = F$. Perhaps the following is true: If $K \triangleleft H \leq G$ and H/K is a cyclic p-group then there is a complex

$$\to X_n \to \cdots \to X_1 \to X_0 \to F \to 0$$

where each X_i is an FG-module which is H/K-projective and there is a bound for all the dimensions of all the X_i.

If this is true, let S be a Sylow p-subgroup of G and let

$$S = S_0 \triangleright S_1 \triangleright S_2 \cdots \triangleright 1$$

be a subnormal series with cyclic factors. For each S_i/S_{i+1} apply the previous statement. Using tensor products and the lemma, we get a complex of S-projective and therefore projective FG-modules of the desired sort.

This complex will be multiperiodic provided the following holds: If $K \triangleleft H \leq G$, H/K a cyclic p-group then there is an exact sequence of FG-modules

$$0 \to F \to X_n \to \cdots \to X_1 \to F \to 0$$

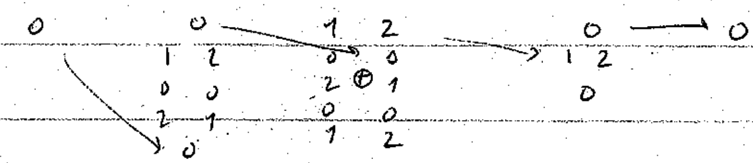
where each X_i is H/K-projective.

The arguments in Gene Lewis' thesis (see Trans. A.M.S. 132) show there is such a multiperiodic complex if G is poly-p-periodic, i.e. G is an iterated (subnormal) extension of groups with cyclic (or generalized quaternion) Sylow p-subgroups.

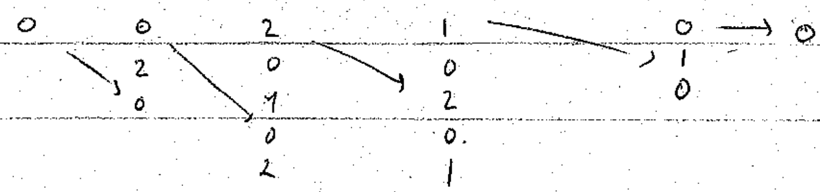
To get at these hypothetical statements perhaps need that the FG-modules which are H/K-projective, for all or suitable H/K, are a decent class of modules, so can be defined by a class of epimorphisms (see perhaps, the memoir, in A.M.S. memoirs, by Eilenberg and Moore). Want these epimorphisms to split on K and to "split on G/H" - whatever that should mean.

Let's give an example, for A_5 in characteristic 2, with the "standard" $SL(2, 2^n)$ - notation

Relatively Z_2-projectives:



Z_2-projective:



($\begin{smallmatrix}0\\1\\0\end{smallmatrix} = 2 \otimes 2$ is Z_2-projective as 2 is)

Minimal projective resolution for V_I for $SL(2,8)$. First $I = \{1\}$.

The pattern is clear:

	12 + 1				
	0	0			
	1	1	1		
	1	1	1	1	
12	0	0	0	0	0
2	1	1	1	1	13
12 + 1	1	1	13		
0	0	3			
1	13				

Can now determine the multiplicity of various indecomposable projectives in the minimal resolution of V_0, using above resolution and a cohomology calculation given below.

Answer:

dim \ comp type	0	1, 2, 3	12, 23, 31
0	1	0	0
1	0	1	0
2	0	1	0
3	1	0	1
4	3	0	0
5	3	1	0
6	4	1	0

Coho calc. Using eigenvalues of Z_7 on obvious poly ring

								dim
0								1
1	$x(\lambda)$	$y(\lambda^2)$	$z(\lambda^4)$					0
2	x^2	xy						0
3	x^3	x^2y	$xyz\ (\lambda^{1+2+4})$					1
4	x^4	x^3y; x^3z	x^2y^2	x^2yz				3
5	x^5	x^4y	x^3y^2	x^3yz	x^2y^2z			3
6	x^6	x^5y	x^4y^2	x^4yz	x^3y^3	x^3y^2z	$x^2y^2z^2$	4

Next, a partial calc of the resolution

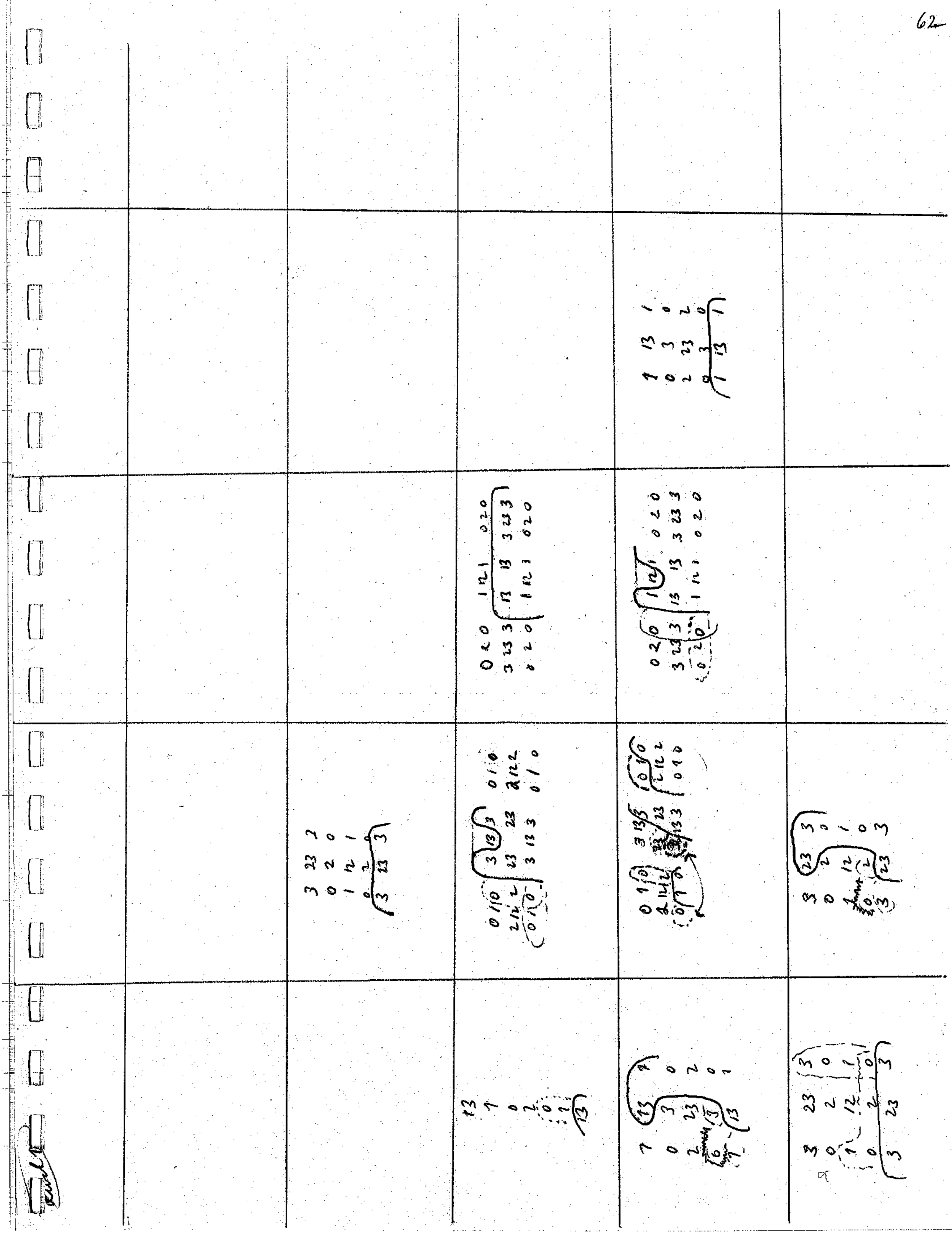
1 13 1
0 3 0
2 23 2
0 3 0
1 13 1

2 12 2
0 1 0
3 13 3
0 1 0
2 12 2

12
2
0
3
0
2
12

010 3133 010
2122 23 23 2122
010 3133 010

2 12 2
0 1 0
3 13 3
0 1 0
2 12 2



```
610   3 13 3   010
2 12 2 ⎡23   23   2 12 2
010   ⎣3 13 3   010
```

```
030 0 ⎡ 2 23 2   030
1 13 1 ⎢ 12  12   1 13 1
030 0 ⎣ 2 23 2   030
```

```
030   2 23 2   030
1 13 1 ⎡ 12  12   1 13 1
030   ⎣ 2 23 2   030
```

```
23
3
0
1
0
3
23
```

2	12	2
0	1	0
3	13	3
0	1	0
2	12	2

Nature of the pattern is clear: 2×2×2 block, 3×3×3 block, ... certain edges being two-dimensional modules, this pattern of edges rotating, with the four-dimensional modules periodically tacked in as extras!

```
        1  0  0   0  2
        0  0  0   0  2
    2   0  00  0   0  2
    0   0  0   0   0  2
3   0   0  0   0   0  2
0   0   0  0   0
2   2   2
.   .   .
```

```
    2  0  0  3
    0  0  0  3
3   0  0  0  3
0   0  0  0  3
0   0  0
.   .   .
```

```
  3  0  1
  0  0  1
1 0  0  1
3 3
```

```
1 2
0 2
```

<u>Remarks on blocks and the Brauer map</u>

G - a finite group

F - a field of characteristic p, "large enough" when necessary

<u>Lemma 1.</u> If K is a conjugacy class of elements of G which do not centralize $O_p(G)$ then their sum is in the radical of FG.

<u>Proof.</u> K is the disjoint union of subsets of the form $k^{O_p(G)}$. All the elements in $k^{O_p(G)}$ have the same image in $G/O_p(G)$ so the sum of all elements in $k^{O_p(G)}$ has zero image in $F\,G/O_p(G)$, by hypothesis. $\therefore$ the sum of all elements in K does also. But then this sum acts trivially on any simple FG-module as each such is an $F\,G/O_p(G)$-module.

More sophisticated argument: no element of K centralizes a defect group as each defect group contains $O_p(G)$. Hence, each central character (homomorphism of $Z(FG)$ to F) vanishes on the sum of elements in K so this sum is in the radical of $Z(FG)$.

We remark a consequence:

<u>Lemma 2.</u> If E is a central idempotent of FG then it is in $F\,C(O_p(G))$.

<u>Proof.</u> Express $E = e + r$, where $e \in F\,C(O_p(G))$ and r is a linear combination of elements not in $C(O_p(G))$. $\therefore r^{p^m} = 0$ some m, by Lemma 1. $\therefore E = E^{p^m} = e^{p^m} + r^{p^m} = e^{p^m} \in F\,C(O_p(G))$.

(So $E = e$, $r = 0$).

More sophisticated argument: Apply the Brauer homomorphism to G as $G = N(O_p(G))$. Let $E_1, \ldots, E_s$ be the primitive central idempotents of G and say E_i is mapped to e_i. ∴ the e_i's are orthogonal central idempotents so they must be primitive so there are s of them. But each $e_i \in F\,C(O_p(G))$. ∴ each E_i is and in fact thus have $E_i = e_i$.

Here's a consequence: If $g \in G$, $g \notin C(O_p(G))$ then

$$\sum_{\chi \in B} \chi(1)\,\overline{\chi(g)} = 0.$$

For

$$e_B = \sum_{\chi \in B} e_\chi$$

$$= \sum_{\chi \in B} \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)}\, g$$

$$= \frac{1}{|G|} \sum_{g \in G} \Big(\sum_{\chi \in B} \chi(1)\,\overline{\chi(g)} \Big)\, g.$$

But the coefficient of g is zero if $g \notin C(O_p(G))$.

Now, we also have from Lemma 1,

<u>Proposition</u>. If L is a p-local subgroup of G and b is a p-block of L which defines $B = b^G$ then b, as $L \times L$-module, is a summand of the restriction to $L \times L$ of the $G \times G$-module B.

<u>Proof</u>. This is just Srinivasan's argument. Say the image of E under the Brauer homomorphism, when E is the primitive

central idempotent corresponding to B, is just e. Then $E = e + r + s$, where r is a linear combination of elements from $L - C(Q)$, (where $L = N(Q)$) and s of elements from $G - L$. To see that

$$FLe \;\Big|\; B_{L\times L}$$

as $L\times L$-modules, Srinivasan's argument shows it suffices to see that $er \in Rad(FL)$. But $r \in Rad(FL)$ by Lemma 1. But b is a direct summand of FLe, as $b^G = B$.

With respect to understanding Srinivasan's argument the following result sheds some light:

<u>Proposition</u> If H is a subgroup of the group G, b and B are p-blocks of H and G while e and E are their primitive central idempotents and if b is a submodule of $B|_{H\times H}$ as $H\times H$-modules then the map, for $\beta \in b$,

$$\beta \to \beta E$$

of b into B is a one-to-one $H\times H$-homomorphism.

<u>Proof</u> It is a homomorphism as $E \in Z(FG)$. Suppose $\beta \in b$ and $\beta E = 0$. Express $\beta = \beta_B + \cdots$ with respect to $FG = B \oplus \cdots$. Hence $\beta E = \beta_B E$ so $\beta_B = 0$. Hence $\beta B = 0$. But $\beta b \neq 0$ now as $b \leq B$ as modules for $H\times H$ and so certainly for $Z(H)$. Thus, $\beta e = 0$ so $\beta = 0$ as desired.

We shall now prove the following result. Our field F as usual.

<u>Theorem</u> If Q is a p-subgroup of G and H is a subgroup of G with $QC(Q) \leq H \leq N(Q)$ then for blocks b and B of H and G, respectively, we have

$$b^G = B$$

if, and only if, b as an $F(H \times H)$-module is a direct summand of the $F(G \times G)$-module B restricted to $H \times H$.

Half has been proved above; the argument works just as well for H as for $N(Q)$. The other half will take a development. We begin with the

<u>Theorem</u> (Nagao, Nagoya 23) If Q is a p-subgroup of G and H is a subgroup of G with $Q\,C(Q) \leq H \leq G$ and U is an indecomposable FG-module in B then every indecomposable summand of U_H has one of the following properties:

1) It lies in a block of H corresponding to B;

2) Its vertex does not contain Q.

This is <u>implicit</u> in Nagao's arguments, but we repeat them here to be sure.

<u>Proof</u>. Let E be the idempotent corresponding to B; so let e be the image of E under the Brauer homomorphism of $Z(FG)$ to $Z(FH)$. Hence $E - e$ lies in the ideal $I_Q = \sum_{R \supset Q \neq R} Z_R(G:H)$ of $Z(G:H)$.

Recall $Z(G:H)$ is the algebra of H-classes & if R is a p-subgroup of H, $Z_R(G:H)$ consists of the class sums (i.e. H-classes) whose defect groups (i.e. Sylow p-subgroups of the centralizer in H) lies in R. (Up to conjugacy, as usual.)

Now set $f = E(E-e)$ so $E = E^2 = E(e + (E-e)) = Ee + f$. But $(Ee)^2 = Ee$, as $E \in Z(FG)$, $Ee.f = EeE(E-e) = Ee - Ee = 0$ and so Ee and f are orthogonal idempotents and $f \in I_Q$.

Hence, as FH-modules we have $U = U(Ee) \oplus Uf = Ue \oplus Uf$.

$\therefore$ it suffices to show that any indecomposable summand of Uf has vertex not containing Q.

But express $f = \sum_{i=1}^{n} f_i$ as orthogonal sum of primitive idempotents in I_Q. By Rosenberg's Lemma (see Green Math Zeit 79 L. 3.3 a)) have, as I_Q is a sum of ideals $Z_R(G:H)$, that for each $i = 1 \ldots n$ there is a p-subgroup Q_i of H with $Q \not\leq Q_i$ and $f_i \in Z_{Q_i}(G:H)$.

Now

$$Uf = Uf_1 \oplus \cdots \oplus Uf_n$$

an FH-decomposition, and by Lemma 4.1. a) of Green (above reference) Uf_i is relatively Q_i-projective as an FH-module. The proof is complete.

Our next step is to prove some results about blocks and direct products and to relate the Brauer correspondence of groups and their direct products. Once we have these formalities then we can prove our theorem.

<u>Lemma</u> Let G and H be groups and say

$$FG = B_1 + \dots + B_m$$

$$FH = C_1 + \dots + C_n$$

are the blocks and the idempotents are $e_1, \dots, e_m$, $f_1 \dots f_n$. Then

$$F(G\times H) = B_1 C_1 + \dots + B_i C_j + \dots + B_m C_n$$

is the block decomposition of $F(G\times H)$ and the corresponding idempotents are the $e_i f_j$.

<u>Note</u>: We just using $F(G\times H) = FG \otimes FH$ with classical notation.

<u>Proof</u> Certainly have such a decomposition and with corresponding idempotents $e_i f_j$. (+ products not zero as $B_i \neq 0$, $C_j \neq 0$ and we're in the tensor product). ∴ remains only to show $F(G\times H)$ has at most mn blocks.

But say χ_1, χ_2 be irreducible characters of G in same block, ψ_1, ψ_2 for H. Thus for all $g \in G$, $h \in H$

$$\frac{|G|}{|C(g)|}\,\frac{\chi_1(g)}{\chi_1(1)} \equiv \frac{|G|}{|C(g)|}\,\frac{\chi_2(g)}{\chi_2(1)}$$

$$\frac{|H|}{|C(h)|}\,\frac{\psi_1(h)}{\psi_1(h)} \equiv \frac{|H|}{|C(h)|}\,\frac{\psi_2(h)}{\psi_2(1)}$$

so multiplying get

$$\frac{|G\times H|}{|C_{G\times H}((g,h))|}\,\frac{\chi_1\otimes\psi_1(g,h)}{\chi_1\otimes\psi_1(1,1)} \equiv \frac{|G\times H|}{|C_{G\times H}((g,h))|}\,\frac{\chi_2\otimes\psi_2(g,h)}{\chi_2\otimes\psi_2(1,1)}$$

This proves the result.

Keeping the same notation

<u>Lemma</u> Suppose Q and R are p-subgroups of G and H, respectively, that X and Y are subgroups of G and H with $Q\,C(Q) \leq X \leq N_G(Q)$ and $R\,C(R) \leq Y \leq N_H(R)$ so that $(Q \times R)\, C_{G\times H}(Q \times R) \leq X \times Y \leq G \times H$ and let b and c be blocks of X and Y respectively. It follows that

$$b^G \otimes c^H = (b \otimes c)^{G \times H}.$$

(The equation is in Brauer's usual notation.)

<u>Proof</u> Let $b^G = B$, $c^H = C$ and let these have idempotents E, F, respectively. It suffices to show that

$$\tau_{C(Q)\times C(R)}(EF) = \tau_{C(Q)}(E) \cdot \tau_{C(R)}(F)$$

But this is clear.

<u>Lemma</u> If B is a block of the group G then B, as $F(G\times G)$-module lies in $B \otimes B^*$.

Here B^* is the dual block corresponding to E^*, the star map taking $g \to g^{-1}$. For $F(G\times G)$ acts on B by $(g,h)\,\beta = g\beta h^{-1}$ for $\beta \in B$.

<u>Proof</u> But $EE^*.\, B = EBE = B$ so $B \in B \otimes B^*$.

We now proceed with the proof. Suppose that b_1 is a block of H such that b_1 is a summand, as $F(H\times H)$-module, of B. Since b_1 is a block of H it has a defect group d_1 and $H \geq d_1 \geq Q$. We have, by a theorem of Green, that b_1 as $F(H\times H)$-module has vertex Δd_1, where Δ is the diagonal embedding operator.

Now we have

$$H\times H \geq N = N_{H\times H}(\Delta Q) \geq \Delta d_1 \geq \Delta Q$$

so, by Green (Math Zeit 70, Th 6) there is an indecomposable FN-module U with $U \mid (b_1)_N$ and b_1 a summand of $U^{H\times H}$ and U having vertex Δd_1. Hence, as $\Delta d_1 \geq \Delta Q$, Nagao's theorem says that the block of U corresponds to the block of b_1, namely $b_1 \otimes b_1^*$.

However, $H \geq C(Q)$ so $N_{H\times H}(\Delta Q) \geq C(Q)\times C(Q) = C_{G\times G}(\Delta Q)$ and so

$$N_{G\times G}(\Delta Q) \geq N \geq C_{G\times G}(\Delta Q)\Delta(Q).$$

Hence, as U is a summand of B_N we have that the block of U corresponds to the block of B, again by Nagao's theorem, since $\Delta d_1 \geq \Delta Q$. That is, the block of U corresponds to $B\otimes B^*$.

But the Brauer correspondence from N to $G\times G$ is the composition of the Brauer correspondences from N to $H\times H$ and from $H\times H$ to $G\times G$. Hence, $b_1\otimes b_1^*$ corresponds to $B\otimes B^*$. However, if b_1 as a block of H corresponds to the block B_1 of G then b_1^* corresponds to B_1 (easily seen) so by an above lemma, $b_1\otimes b_1^*$ corresponds to $B_1\otimes B_1^*$. Thus, $B = B_1$ by our above results.

A lemma of Hall-Higman type

Let F be an algebraically closed field of characteristic two and let $L = PSL(2,q)$ for q odd. Let D be a Sylow 2-subgroup of L so that D is dihedral, say of order 2^{n+1}. Let Y be a cyclic maximal subgroup of D and let E be a fours subgroup of D. In answer to a query of Gorenstein and Lyons we prove the

Theorem 1. If V is a non-trivial simple FL-module in the principal 2-block of L and $q > 9$ then V as FE-module has a free summand.

That is, $[V, E, E] \neq 1$ an equation in the semi-direct product $V.L$. This is done by proving a stronger result in each of the separate cases $q \equiv 1\ (4)$ and $q \equiv -1\ (4)$.

Theorem 2. If $q \equiv 1\ (4)$ then $V_D = S(q) \oplus P(q)$ where
1) $P(q)$ is projective,
2) $S(q)$ is indecomposable of dimension 2^n, $S(q)_Y$ is a free FY-module,
3) $S(q)$ has vertex a fours subgroup and source two-dimensional.

Let's see how this proves the first theorem in the case $q \equiv 1(4)$. We are done by 1) unless $q-1$ is a power of 2 so q is 9 or a Fermat prime. (For we're done if $\frac{q-1}{2} > 2^n$.) $\therefore$ remains only to deal with the case $n > 2$ so $|D| > 8$. Let E' be the vertex of $S(q)$ in D with W the source so $V = W^D$. (The dimensions imply this but we're really relying on F algebraically closed and Green's indecomposability criterion.) Say E and E' are not conjugate in D. Now $S(q)$ restricted to any involution is free as all involutions in L are conjugate and so $S(q)$ and $P(q)$ are free as FY-modules. Hence, Mackey's theorem immediately gives that $W^D{}_E$ is free. Similarly, if E and E' are conjugate and E is not normal in D we get a free summand in $W^D{}_E$.

In the other occurrence we shall prove

Theorem 3 If $q \equiv -1\ (4)$ then $V_D = S(q) \oplus P(q)$ where
1) $P(q)$ is projective,
2) $S(q)$ is indecomposable of dimension $2^n - 1$,
3) $S(q)_Y$ is cyclic.

To complete the proof of Theorem 1 from this we need the

Lemma 1 Suppose that $|D| > 4$ and that S is an FD-module of dimension 2^n-1 with S_Y cyclic. Then there are distinct dihedral maximal subgroups D_1 and D_2 of D such that

$$W_{D_1} = \Omega_1(F), \quad W_{D_2} = \Omega_{-1}(F).$$

(Here, as usual, $\Omega_1(F) = \text{Rad}\, P(F)$, $P(F)$ the projective cover of F so here $\Omega_1(F)$ is the augmentation ideal of FW_{D_1} as W_{D_1} module. And $\Omega_{-1}(F) = P(F)/\text{Soc}\, P(F) \simeq \Omega_1(F)^*$.)

Granting these last two results let's complete the proof of Theorem 1. We apply Lemma 1 to $S(q)$ in Theorem 3. Say $E \le D_1$. Then $S(q)_E = \Omega_1(F) \oplus$ projective, as a free module restricts to a sum of free modules and F restricts to F (i.e. $0 \to \Omega_1(F) \to FD_1 \to F \to 0$). By dimension comparisons, as $\Omega_1(F)$ for E has 3 dimensions, we're done. Similarly, if $E \le D_2$ the only other possibility.

Now let's prove Lemma 1 (which could be proved presumably by referring to Ringel's classification of all FD-modules in the Math. Annalen.) Say $D = \langle Y, x \rangle$, $x^2 = 1$, $Y = \langle y \rangle$. Now $W_{\langle y^2 \rangle} = J_{2^{n-1}} \oplus J_{2^{n-1}}$ as $W_Y = J_{2^n - 1}$ by assumption (Jordan block notation).

$\therefore$ if x does not centralize $\mathrm{Soc}(W_{\langle y^2 \rangle})$ then $W_{\langle x, y^2 \rangle}$ has a simple socle and so by dimensions can set $D_1 = \langle x, y^2 \rangle$.

If x does centralize this socle then xy, another involution, does not as y does not. Thus we can set $D_1 = \langle xy, y^2 \rangle$. In particular, $W_{D_1} / \mathrm{Rad}\, W_{D_1}$ is 2-dimensional so $\mathrm{soc}(W^*_{D_1})$ is two-dimensional. But our arguments apply to W^* so there is D_2 with $W^*_{D_2} = \Omega_1(F)$ so taking duals again $W_{D_2} = \Omega_1(F)^* = \Omega_{-1}(F)$.

Let's first prove Theorem 2 as that is easy. Now L has a cyclic subgroup of order $\frac{1}{2}(q+1)$ which we can take as $Y \times X$ where X has odd order. The ordinary irreducible characters of L of degree $\frac{1}{2}(q-1)$ restrict to $Y \times X$ as the regular representation less the sign character (the one linear character of $Y \times X$ of order two). $\therefore$ on V, which is just this character mod 2, X has each character with multiplicity 2^n except the principal character which appears with multiplicity 2^n-1. Now $N(Y\times X) = D.X$ $\therefore D$ acting on V decomposes the space into a space of dimension 2^n-1 (princ. char.) and spaces of dimension 2^{n+1} (two non-principal characters of X). Claim: Each of these is indecomposable, the first being $S(\xi)$ with all the required properties and the second ones are all free.

For V is an invertible FL-module, as is easily seen by multiplying Brauer characters. Thus V_D is an invertible plus projectives, by general results (see section on invertibles). $\therefore$ all is proved except that $S(\xi)_Y$ is cyclic. But $S(\xi)_Y$ is an invertible plus projectives and $\dim S(\xi) < |Y|$ so the proof is complete as $S(\xi)_Y \simeq J_{2^n-1}$.

We now turn to Theorem 2. We shall prove parts 1) and 2) right off. Let $B = H.U$ be the Borel subgroup so we can assume $H = Y\times X$, cyclic + X of odd order. Now B is a Frobenius group so V_H is free, in fact on one generator. $N(H) = D.X$ so D/Y is regular on the non-principal characters of X. Now $V_H = \bigoplus J_{2^n}\otimes\lambda$ over all the linear characters of X and parts 1) and 2) are now clear.

The rest is not so easy, though it might follow from Ringel's results. We shall prove several lemmas. Let G be a group, Q a p-subgroup, F a field of characteristic p for now. Let $H = N(Q)$.

<u>Lemma 2.</u> If U is an indecomposable FG-module (+ non projective) then U and ΩU have the same vertex.

Pf. Say R is the vertex of U so there is an FR-module $W \ni U \mid W^G$.
$\therefore \Omega V = \Omega(F)\otimes U \mid \Omega(F)\otimes W^G = (\Omega(F)|_R \otimes W)^G = (\Omega^{\text{for } R}(F)\otimes W)^G + \text{proj.}$

Hence, the vertex of ΩV is contained in R. A similar argument applies to Ω_{-1} so the vertex of V is contained in the vertex of ΩV and the lemma is proved.

Lemma 3 (cont.) Suppose W is an FH-module and the Green correspondent of U (so U, W have the same vertex contained in Q). Then ΩW and ΩU are also Green correspondents.

Pf. The vertices are right so it suffices to see that $\Omega W \mid (\Omega U)_H$. But $W \mid U_H$ so $\Omega W \mid \Omega U_H \mid (\Omega(F)|_H \otimes U_H) = \Omega U|_H$ and done.

Lemma 4 (cont.) U is periodic if, and only if W is periodic, in which case they have the same period.

Pf. Clear from Lemma 3.

Now let's go back to the proof of part 3) of Theorem 2.

Lemma 5. S is periodic of period dividing two.

This is because S_Y is free so an argument of Dade applies. It may even show the period is one as we're dealing with 2-groups. We can also use our spectral sequence argument showing S is bounded to give this result. For it seems to give the isomorphism of functors

$$Ext_D^i(S, -) \cong Ext_{D/Y}^0(F, Hom_{FY}(F, S^* \otimes -))$$

(assisted by Eisenbud here.) Hence the Hilton-Rees theorem applies.

Lemma 6 V is not periodic of period one or two.

(Actually we know that it is periodic of period three, but we give an easier argument here.)

Pf. Say V is of period one. Hence $0 \to V \to \text{proj} \to V \to 0$ exists so V is the only simple module in B_0, a contradiction. Suppose not, in the minimal resolution of V we have

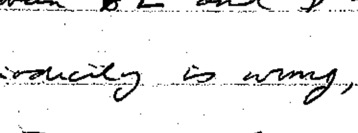
$$0 \to V \to P_1 \to P_0 \to V \to 0.$$

$\therefore P_1 = P(V)$ the projective cover of V (Also $P_0 = P(V)$) so $\mathrm{Ext}^1(V, F) = 0$. Applying an automorphism we deduce that $\mathrm{Ext}^1(F, -)$ is zero on the simple modules in B_0, as well as F, a contradiction.

Now we complete the proof. Suppose the vertex of $S(q)$ is not a fours group. It's not a $\mathbb{Z}_n$ cyclic as $S(q)_Y$ is free. Now the intersections of D and its conjugates are 1, D and fours groups and Y. Hence, V has a Green correspondent between L and $D = N(D)$ which of course is $S(q)$. But the periodicity is wrong, contradicting Lemma 4. Hence, the vertex is right. The source has dimension two by the fact that $\dim S(q) / (|D|/4) = 2$

Next, let's inspect A_6 a bit. Obvious notation. $x = (12)(34)$, $y = (34)(56)$

Action modulo $1+2+3+4+5+6$



$$x \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \qquad y \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

Let's compute endomorphisms to see indecomposable it is not!

Centralizer of t

$$\begin{pmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & l \end{pmatrix}$$

$$\begin{pmatrix} a & b & e & f \\ c & d & g & h \\ & & i & j \\ & & k & l \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} = \begin{pmatrix} a & b & b+e & a+f \\ c & d & d+g & c+h \\ 0 & 0 & i & f \\ 0 & 0 & k & e \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} a & b & e & f \\ c & d & g & h \\ & & i & j \\ & & k & l \end{pmatrix} = \begin{pmatrix} a & b & e+k & f+l \\ c & d & g+i & h+j \\ & & i & j \\ & & k & l \end{pmatrix}$$

$\therefore b = k,\ a = l,\ d = i,\ c = j$

$$\begin{pmatrix} a & b & e & f \\ c & d & g & h \\ & & d & c \\ & & b & a \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} = \begin{pmatrix} a & b & a+b+e & b+f \\ c & d & c+d+g & d+h \\ & & d & c \\ & & b & a \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} a & b & e & f \\ c & d & g & h \\ & & d & c \\ & & b & a \end{pmatrix} = \begin{pmatrix} a & b & e+d & f+c \\ c & d & g+d+b & h+c+a \\ & & d & c \\ & & b & a \end{pmatrix}$$

$\therefore a+b = d,\ b = c$

$$\begin{pmatrix} a & b & e & f \\ b & a+b & g & h \\ & & a+b & b \\ & & b & a \end{pmatrix}$$

$\left\{ \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \right\} \simeq F \oplus F$ if $F \supseteq GF(4)$. In fact

$e = \begin{pmatrix} \omega^2 & 1 \\ 1 & \omega \end{pmatrix}$ $f = \begin{pmatrix} \omega & 1 \\ 1 & \omega^2 \end{pmatrix}$ so $e^2 = e,\ f^2 = f,\ ef = 0$.

$\therefore$ source is unstable over $GF(4)$. Must be looking at $\left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$.

Same as for A_5 !!

<u>Let compute FD modules, $|D|=8$ which are free (of rank 1) for FY.</u>

Say $Y=\langle y\rangle$, $D=\langle x,y\rangle$, $x^2=1$.

$$x \to \begin{pmatrix} 1 & a & b & c \\ & 1 & d & e \\ & & 1 & f \\ & & & 1 \end{pmatrix} \qquad y \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

$$\therefore x^2 \to \begin{pmatrix} 1 & 0 & ad & ae+df \\ & 1 & 0 & df \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \qquad \therefore ad = df = ae+df = 0.$$

$$yx \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b & c \\ & 1 & d & e \\ & & 1 & f \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+1 & b+d & c+e \\ & 1 & d+1 & e+f \\ & & 1 & f+1 \\ & & & 1 \end{pmatrix}$$

$$yxy \to \begin{pmatrix} 1 & a+1 & b+d & c+e \\ & 1 & d+1 & e+f \\ & & 1 & f+1 \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & a+b+d+1 & b+c+d+e \\ & 1 & d & d+e+f+1 \\ & & 1 & f \\ & & & 1 \end{pmatrix}$$

$\therefore a+d=1,\ d+f=1,\ b+d+e=0.$

<u>Case 1</u> $a=0$. $\therefore d=1,\ f=0,\ 1+b=e$ $\quad x \to \begin{pmatrix} 1 & 0 & b & c \\ & 1 & 1 & b+1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad y^2 \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$

$\therefore \langle x, y^2\rangle$ not free action.

<u>Case 2</u> $a \neq 0$. $\therefore d=0,\ a=1$ $\therefore f=1,\ b=e$ so

$$yx \to \begin{pmatrix} 1 & 0 & & \\ & 1 & 1 & \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

$\therefore$ replacing x by yx get $a=0$ case. i.e. $a=0$ up to isomorphism.

Periodicity for $SL(2,2^n)$

With the usual notation for $SL(2,2^n)$ we prove the

Theorem $V_{\{2,\dots,n\}}$ is periodic of period 2^n-1.

Proof. In view of the Green correspondence (see a lemma in the previous section!) it suffices to show that $V_{\{2,\dots,n\}}|_B$ (B the Borel subgroup) is periodic of period 2^n-1. Some notation first: $W_I = V_I|_B$, λ^i, $0 \le i < 2^n-1$, the one-dimensional FB-module, where $\lambda = W_1/\operatorname{soc} W_1$.

We have an exact sequence

$$0 \to \lambda^0 \to \lambda^1 \otimes W_1 \quad \cdots \to \lambda^{-2} \otimes W_1 \to \lambda^0 \otimes W_1 \to \quad \cdots \lambda^{-3} \otimes W_1 \to \lambda^{-1} \otimes W_1 \to \lambda^0 \to 0$$

For W_1 has composition factors λ, λ^{-1} and is uniserial. It is indecomposable on B' as W_N certainly is. Now tensor this whole sequence with $W_{N-\{1\}}$. Get a periodic resolution and since $W_{N-\{1\}}$ has a simple socle can see terms are different by inspection of socle.

Tensor products and projectives

Let F be a field, G a finite group and $V_1, \dots, V_s$ "the" irreducible FG-modules. Recall the Bryant-Kovacs theorem: If V is a faithful FG-module then there is $n \geq 0$ such that $F \oplus V \oplus (V \otimes V) \oplus \dots \oplus (\overbrace{V \otimes \dots \otimes V}^{n}) = \bigoplus_{i=0}^{n} V^{\otimes i}$ contains a free submodule. Immediately, we have, *henceforth* assuming F has characteristic p,

Corollary If $O_p(G) = 1$ then every indecomposable projective FG-module is irreducibly generated.

(is a direct summand of a tensor product of irreducible modules.)

Pf. Consider $V = V_1 \oplus \dots \oplus V_s$ and apply the Bryant-Kovacs theorem.

However, we can do better; it may be that there is a projective irreducible FG-module. For the moment assume $S \neq 0$ is just any projective FG-module. Let P_i be the projective cover of V_i so there are non-negative integers a_{ij} with

$$V_i \otimes S = a_{i1} P_1 \oplus \dots \oplus a_{is} P_s$$

Proposition 1. For any j, $1 \leq j \leq s$, there is i, $1 \leq i \leq s$ such that $a_{ij} \neq 0$.

In other words, all the P_i do occur. Lusztig observed this for S the

Steinberg module for a group of Lie type.

Pf. Choose ℓ such that $\mathrm{Hom}_{FG}(V_\ell, V_j^* \otimes S) \neq 0$, this exists as $V_j^* \otimes S$ has a socle! Hence, $\mathrm{Hom}_{FG}(V_j, V_\ell^* \otimes S) \neq 0$ so if $V_\ell^* \simeq V_i$ then P_j is a direct summand of $V_i \otimes S$.

(Here's another argument which works if $O_p(G) = 1$. Let $V = V_1 \oplus \cdots \oplus V_s$ and choose $n \ni \bigoplus_{i=0}^{n} W^{\otimes i} \supseteq U$ a free submodule. Hence, $S \otimes (\bigoplus_{i=0}^{n} W^{\otimes i}) \supseteq S \otimes U$ also free, and a summand! But $S \otimes X$ for any X is a direct sum of the $S \otimes V_i$ as S is injective and X has a composition series.) (~~Also, if S faithful can use~~

Now assume F is algebraically closed and that Φ is the Brauer character of S. Let supp Φ be the set of conjugacy classes where Φ does not vanish.

<u>Proposition 2</u> The rank of (a_{ij}) is the cardinality of supp Φ.

<u>Proof.</u> Let R be the Grothendieck type ring of projective FG-modules. (so it's isomorphic to the ring of Brauer characters of such modules.) (i.e. $K_0(FG)$). $\therefore R$ is a $\mathbb{Z}$-algebra, free as a $\mathbb{Z}$-module of rank s. For each $r \in R$ let Φ_r be the corresponding Brauer character.

Now $\mathbb{C} \otimes_{\mathbb{Z}} R = \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{s}$ as the s conjugacy classes of p'-elements give s independent homomorphisms to $\mathbb{C}$. (Each non-zero as for each p'-class there is a proj module with non-zero Brauer character there, as is easy to see using ~~Bra~~ induced characters.

What we want to compute is the rank of the submodule of R generated by S — or rather the class of S — under the Grothendieck type ring for FG-modules — the Brauer character ring. But if $\Phi_n(x) = 0$ for a p'-element x then $\Phi_n \varphi(x) = 0$ for any Brauer character φ of an irreducible. $\therefore$ in $\mathbb{C} \otimes R$ the dimension of the corresponding module is at most the given bound. But it must be that great by the ring structure. We have a direct sum of $\mathbb{C}$'s and an element with certain non-zero projections.

<u>Proposition 3</u>. If S is self-dual then (a_{ij}) is symmetric.

<u>Proof</u>. For

$$\begin{aligned} Hom(V_j, V_i \otimes S) &\cong Hom(V_i^* \otimes V_j, S) \cong Hom(S^*, V_i \otimes V_j^*) \\ &\cong Hom(S, V_i \otimes V_j^*) \cong Hom(S \otimes V_j, V_i) \\ &\cong Hom(V_i, S \otimes V_j) \end{aligned}$$

as FG is a symmetric algebra.

For example, now, $G = GL(n, q)$, $q = p^e$, S the Steinberg module. We now have (a_{ij}) non-singular, non-negative and symmetric. This is because certainly S does not vanish on a p'-element as it has a character with value $+1$ or -1 times p^f on such an element.

Now suppose we're in a Lie rank one situation. Let B be the Borel subgroup, $H = B \cap B^w$, w in the Weyl group. $S = F_B{}^G - F_G$, where F_G, F_B are the trivial FG and FB-modules. Have $(F_B{}^G)_B = F_B + (F_H)^B$, by Mackey's theorem. Let C be the Cartan matrix, $A = (a_{ij})$ as above and let $M = (m_{ij}) = (\dim_F \mathrm{Hom}_{FH}(V_i|_H, V_j|_H))$, $T = (\dim_F \mathrm{Hom}_{FB}(V_i|_B, V_j|_B))$

<u>Proposition 4.</u> $ACA + A = M$, $A + I = T$

<u>Proof.</u> $a_{ij} = \dim_F \mathrm{Hom}_{FG}(V_j, V_i \otimes S)$

$= \dim_F \mathrm{Hom}_{FG}(V_j, V_i \otimes F_B{}^G) - \dim_F \mathrm{Hom}_{FG}(V_j, V_i \otimes F_G)$

so

$$a_{ij} + \delta_{ij} = \dim_F \mathrm{Hom}_{FG}(V_j, (V_i|_B)^G)$$
$$= \dim_F \mathrm{Hom}_{FB}(V_j|_B, V_i|_B)$$
$$= t_{ji},$$

and the second equation holds, as T is symmetric as FH is semi-simple.

Next,

$\dim_F \mathrm{Hom}_{FG}(V_i \otimes S, V_j \otimes S)$

$= \dim_F \mathrm{Hom}_{FG}(V_i \otimes F_B{}^G, V_j \otimes F_B{}^G) - \dim_F \mathrm{Hom}_{FG}(V_i \otimes F_B{}^G, V_j)$

$- \dim_F \mathrm{Hom}_{FG}(V_i, V_j \otimes F_B{}^G) + \dim_F \mathrm{Hom}_{FG}(V_i, V_j)$

$= \dim_F \mathrm{Hom}_{FG}(V_i \otimes F_B{}^G, V_j \otimes F_B{}^G) - 2t_{ij} + \delta_{ij}$

and

$\dim_F \mathrm{Hom}_{FG}(V_i \otimes F_B{}^G, V_j \otimes F_B{}^G) = \dim_F \mathrm{Hom}_{FG}((V_i|_B)^G, (V_j|_B)^G)$

$= \dim_F \mathrm{Hom}_{FB}((V_i|_B)^G|_B, V_j|_B)$

$= \dim_F \mathrm{Hom}_{FB}(V_i|_B \oplus V_i|_H{}^B, V_j|_B)$

$$= \dim_F \mathrm{Hom}_{FB}(V_i|_B, V_j|_B) + \dim_F \mathrm{Hom}_{FH}(V_i|_H, V_j|_H)$$
$$= t_{ij} + m_{ij}.$$

Hence,

$$\dim_F \mathrm{Hom}_{FG}(V_i \otimes S, V_j \otimes S) = t_{ij} + m_{ij} - 2t_{ij} + \delta_{ij}$$
$$= m_{ij} + \delta_{ij} - t_{ij}.$$
$$= \delta_{ij} - t_{ij} + m_{ij} \qquad (G, B, H \text{ resp.})$$

But

$$\dim_F \mathrm{Hom}_{FG}(V_i \otimes S, V_j \otimes S) = \dim_F \mathrm{Hom}_{FG}\left(\bigoplus_k a_{ik} P_k, \bigoplus_l a_{jl} P_l\right)$$
$$= \sum_{k,l} \dim_F \mathrm{Hom}_{FG}(P_k, P_l)\, a_{ik} a_{jl}$$
$$= \sum_{k,l} a_{ik} C_{kl} a_{jl}$$
$$= \sum_{k,l} a_{ik} C_{kl} a_{lj} \quad \text{(by Prop. 3)}$$

Hence,

$$I - T + M = ACA$$

so

$$-A + M = ACA$$

and the proof is complete.

Remarks 1. Hence, if have T and M can get C. And easily if it is easy to invert A. Seems much better than computing C^{-1} by using inner products of Brauer characters.

2. Should generalize to arbitrary BN pairs — at least split ones.

Let's calculate the determinant of A now for general S, not in the rank one case. Let $x_1, \dots, x_s$ be representatives for the p'-classes of G and let σ be the Brauer character of S.

<u>Proposition 5</u> $\det A = \prod_{i=1}^{s} \sigma(x_i)/|C(x_i)|_p$.

<u>Proof</u>. Let φ_i and Φ_j be the obvious characters so

$$\varphi_i \sigma = \sum_j a_{ij} \Phi_j$$

and so

$$\varphi_i \sigma = \sum_{j,k} a_{ij} c_{jk} \varphi_k$$

and

$$\varphi_i(x_\ell)\sigma(x_\ell) = \sum_{j,k} a_{ij} c_{jk} \varphi_k(x_\ell)$$

and so

$$\varphi \begin{pmatrix} \sigma(x_1) & & \\ & \ddots & \\ & & \sigma(x_\ell) \end{pmatrix} = A C \varphi$$

where $\varphi = (\varphi_i(x_j))$. Now φ is invertible and $\det C = \prod_{i=1}^{s} |C(x_i)|_p$. This proves the result.

Now if we have $\sigma(x_i) = \pm |C(x_i)|_p$ for all i then we get

$$\det A = \pm 1.$$

This happens: Assume G simply connected alg gp / finite fld, σ endo $G \to G$ & G_σ is finite. Then there is such an irred. character. (R Steinberg, Endomorphisms of linear algebraic groups, Memoirs of the A.M.S # 80, 1968, p 99.

Let's get an application. Let $K_0(FG)$ be the abelian group of projectives - in the Grothendieck sense. Let $R_0(FG)$ be the ring of Brauer characters, also a Grothendieck type ring. Then, using tensor products we have that $K_0(FG)$ is an $R_0(FG)$-module.

<u>Theorem 6.</u> $K_0(RG)$ is a free $R_0(FG)$-module.

Proof. Let Φ be the complex-valued character (gen'd) on G given by

$$\Phi(g) = \begin{cases} 0 & \text{if } p \text{ divides the order of } g \\ |C(g)|_p & \text{otherwise.} \end{cases}$$

It is such by a result of Brauer (see Curtis-Reiner 84.16 - perhaps in Brauer's "A characterization of the characters of groups of finite order," Annals of Math. <u>57</u> ('53), 357-377.) Hence, it is a linear combination of indecomposable projectives. The element of $K_0(FG)$ corresponding is the free generator! Because the corresponding A is unimodular and the $\mathbb{Z}$-ranks are right.

We've seen above that the Steinberg module can serve as a free generator of the free module $K_0(FG)$! It would be interesting to have some other generators which are actually modules.

Let's return to the rank one case and in particular to $SL(2,2^n)=G$. Let V_I, $I \subseteq N = \{1,2,\dots n\}$ be the irreducible FG-modules as usual. We have done calculations before that carry out part of the program. To do the rest we need only establish the following:

__Proposition 7__ If $I, J \subseteq N$ then

$$\dim_F \operatorname{Hom}_{FB}(V_I/B, V_J/B) = \begin{cases} 2 & \text{if } I=J=N \\ 1 & \text{if } I=J\subset N \text{ or } I\cup J=N, I\cap J=\phi \\ 0 & \text{otherwise.} \end{cases}$$

__Proof__. Let $W_I = V_I/B$ and let λ be the faithful linear module for B/U so that W_1 is an extension of λ^{-1} by λ. Now $\lambda^{2^n-1}=1$ and no smaller power satisfies this. Now W_N has a simple socle and top as it is projective of dimension 2^n. As $\lambda^{1+2+\dots+2^{n-1}}=1$ this top and bottom is 1 and 1 does not appear elsewhere as composition factor of W_N. Hence $\dim_F \operatorname{Hom}(W_N, W_N)=2$.

Now W_I, $I=\{i_1,\dots i_k\}$ has composition factors $\lambda^{\pm 2^{i_1-1}\pm\dots\pm 2^{i_k-1}}$ so if $\phi\subset I\subset N$ it has no trivial factor so then $\operatorname{Hom}_{FB}(W_I, W_N)=0$. Hence the proposition is verified if $J=N$. The same sort of argument shows the top of W_I appears just once in W_I, $I\subset N$ so $\operatorname{Hom}_{FB}(W_I, W_I)\simeq F$. For the general case $\operatorname{Hom}_{FB}(V_I, V_J)\simeq \operatorname{Hom}_{FB}(V_{I\cap J}, V_{I\cup J})$. Hence, if $I\neq J$, $I\cap J\neq\phi$ then $\operatorname{Hom}_{FB}(V_I, V_J)\neq 0$ yields, $\operatorname{Hom}_{FB}(V_{I\cap J}, V_{I\cup J})\neq 0$, $\operatorname{Hom}(V_{I\cap J\cup(N-I\cup J)}, V_N)\neq 0$ a contradiction. The rest is quite easy now.

Diagonalizing AC

We return to the rank one Lie type situation, with the usual notation for such a group G and subgroups B, H, U element w and matrices A, C, T, M. In addition we assume B is a Frobenius group (from Lemma 2 onwards it's necessary).

Lemma 1 $F_H^{\,G} \simeq S \oplus (S \otimes S)$.

Proof. $F_B^{\,G} \simeq F \oplus S$ so $F_B^{\,G} \otimes F_B^{\,G} \simeq F \oplus S \oplus S \oplus S \otimes S$.
But, by Mackey's theorem (or could use tensor product one directly) as $G = B \cup BwB$

$$F_B^{\,G} \otimes F_B^{\,G} \simeq F_B^{\,G} \oplus F_H^{\,G}$$

so the lemma follows from the Krull Schmidt theorem.

Lemma 2 $S \otimes S \otimes S \cong S \oplus \overbrace{FG \oplus \dots \oplus FG}^{m}$, $m = \dfrac{|U|-1}{|H|}$.

Proof $F_H^{\,G} \otimes F_B^{\,G} \simeq [S \oplus (S \otimes S)] \otimes [F \oplus S]$

$$\simeq S \oplus (S \otimes S) \oplus (S \otimes S) \oplus S \otimes S \otimes S$$

But

$$F_H^{\,G} \otimes F_B^{\,G} = \left(F_H \otimes F_B^{\,G}{}_H\right)^G = \left(F_B^{\,G}{}_H\right)^G.$$

But

$$G = B \cup BwB = B1H \cup BwH \cup \bigcup_{\substack{u \in U^\# \\ \text{(over } H\text{ orbits)}}} BwuH.$$

Moreover,

$$B^{wu} \cap H = \left(B^w \cap H^{u^{-1}}\right)^u$$

$$B^w \cap H^{u^{-1}} \subseteq U^w H^w \cap H^{u^{-1}} \subseteq U^w H^w \cap B = H^w = H$$

so $\quad B^w \cap H^{u^{-1}} \subseteq H^w \cap H^{u^{-1}} = H \cap H^{u^{-1}} = 1$, by assumption.

Hence, by Mackey's theorem,

$$F_H^G \otimes F_B^G \cong (F_H \oplus F_H \oplus \overbrace{FH \oplus \cdots \oplus FH}^{m})^G$$

$$\cong S \oplus (S \otimes S) \oplus S \oplus (S \otimes S) \oplus \overbrace{FG \oplus \cdots \oplus FG}^{m}$$

∴ result is true by Krull-Schmidt.

∴ in the Green ring have

$$[S]^3 = [S] + m\,[FG]$$

so

$$[S]^4 = [S]^2 + m \cdot \dim_F S \; [FG]$$

But $\dim_F S = |U|$ so

$$[S]^4 - |U|\,[S]^3 = [S]^2 - |U|\,[S]$$

so $[S]$ satisfies

$$X^4 - |U|\,X^3 - X^2 + |U|\,X$$

which is

$$X(X-1)(X+1)(X-|U|).$$

Now in the ring of Brauer characters $[S]$ defines a linear transformation with matrix AC. ∴ this matrix is diagonalizable and has eigenvalues (with multiplicities ≥ 0 to be determined) $-1, 1, 0, |U|$. But AC is non-singular so the eigenvalues are $-1, 1, |U|$. Also $\det AC = \det A \det C = \pm \det C = \pm |U|$ by assumption & Brauer's theorem on $\det C$. (Also $\det A = \pm 1$ follows from values of the character of S, easily determined as $F_B{}^G \cong F \oplus S$). Hence, $|U|$ is an eigenvalue with multiplicity one.

To conclude the description of the canonical form of AC, we require the

<u>Lemma 3</u> M has rank equal to the number of G-classes in H.

We leave the proof for later. Now $ACA + A = M$ so $(AC + I)A = M$.
Hence, rank $AC + I$ = rank M. Thus, nullity $(AC + I)$ is
size of A − # of G classes in H. Hence, we have the

<u>Theorem</u> AC is diagonalizable and has eigenvalue $|U|$ with multiplicity one, eigenvalue 1 with multiplicity the number of G-classes in $H^{\#}$ and all remaining eigenvalues equal to -1

All done except for the following:

<u>Proof</u> (Lemma 3) Restricting the modular irreducible characters to H we get the same $\mathbb{Z}$-module as restricting all the ordinary characters so the rank of this $\mathbb{Z}$-module equals the number of G-classes in H. By taking suitable <u>integral</u> combinations get non-zero functions on H which have support on a fixed intersection of a G-class and H. Hence, the rank of M is at most as given. On the other hand, it is no more as the space of all complex valued functions on H constant on G-classes is a unitary space of the right dimension.

The Green ring for Z_p

Here's an alternative approach. Let l denote Loewy length.

Lemma. If $2 \le t \le k < p$ then

a) $l(\overbrace{J_2 \otimes \cdots \otimes J_2}^{k+1-t} \otimes J_t) = k+1$

b) $J_2 \otimes J_k = J_{k-1} \oplus J_{k+1}$.

Proof. Say $k=2$. Then there is an epimorphism of J_2^2 (obvious notation!) onto the symmetric square, which is J_3 (as in theory of incd reps of $GL(2,p)$) $\therefore l(J_2^2) \ge 3$. But $\mathrm{Hom}(J_1, J_2^2) \cong \mathrm{Hom}(J_2, J_2)$ of 2 dims so $J_2^2 = J_1 + J_3$ and a), b) hold for $k=2$.

$\therefore$ assume result holds for $k-1$. Now $l(J_2^k) \ge k+1$ as J_2^k maps onto the k-th symmetric product, which is J_{k+1}. This is $l(J_2^{k-1} \cdot J_2) \ge k+1$. But $J_2^{k-1} J_2 = J_2^{k-2}(J_1 + J_3) = J_2^{k-2} + J_2^{k-2} J_3$

But $l(J_2^{k-2}) = k-1$, by induction, so $k+1 \le l(J_2^{k-1} J_2) = l(J_2^{k-2} J_3)$.

Now $J_2^{k-2} J_3 = J_2^{k-3}(J_2 J_3) = J_2^{k-3}(J_2 + J_4)$ so now get

$$k+1 \le l(J_2^{k-1} J_2) = l(J_2^{k-2} J_3) = l(J_2^{k-3} J_4)$$

and continuing

$$k+1 \le l(J_2^{k-1} J_2) = \cdots = l(J_2 J_k)$$

But $\mathrm{Hom}(J_1, J_2 \otimes J_k) \simeq \mathrm{Hom}(J_2, J_k)$ of 2 dims so $J_2 J_k = J_a + J_b$, $a \ge b$, $a \ge k+1$. Also $\mathrm{Hom}(J_{k-1}, J_2 \otimes J_k) \simeq \mathrm{Hom}(J_2 \otimes J_{k-1}, J_k) \simeq \mathrm{Hom}(J_{k-2} + J_k, J_k)$ of dim $2k-2$.

$\therefore J_2 J_k = J_{k-1} + J_{k+1}$ as must have $b \ge k-1$ and $a+b = 2k$.

$\therefore$ b) holds. And $k+1 \le l(J_2^{k-1} J_2) = \cdots = l(J_2 J_k) = k+1$

so a) holds and the lemma is proved

Hence, the Drein ring has generators $J_1, J_2, \ldots, J_p$ with defining relations – apart from J_1 being the identity –

$$J_2 J_k = J_{k-1} + J_{k+1}, \quad J_2 J_p = 2 J_p$$

$2 \leq k < p$; for these force the $\mathbb{Z}$-rank to be at most p.
Thus, to see that this ring is semi-simple we need p "solutions" of these equations in $\mathbb{C}$.

Let λ be a primitive (i.e. $\neq 1$) p-th root of unity. Send

$$J_1 \to 1$$
$$J_2 \to \lambda + \bar{\lambda}$$
$$J_3 \to 1 + (\lambda^2 + \bar{\lambda}^2)$$
$$J_4 \to (\lambda + \bar{\lambda}) + (\lambda^3 + \bar{\lambda}^3)$$
$$\vdots$$
$$J_p \to 1 + (\lambda^2 + \bar{\lambda}^2) + (\lambda^4 + \bar{\lambda}^4) + \cdots + (\lambda^{p-1} + \bar{\lambda}^{p-1}) \underline{= 0}.$$

This works as is easily seen using, for $i \geq 1$,

$$(\lambda + \bar{\lambda})(\lambda^i + \bar{\lambda}^i) = (\lambda^{i-1} + \bar{\lambda}^{i-1}) + (\lambda^{i+1} + \bar{\lambda}^{i+1})$$

This gives $\frac{p-1}{2}$ solutions. The nature of the defining relations allows us to send J_2 to the negative of the above, J_3 to the same, J_4 negative and so on giving $\frac{p-1}{2}$ more solutions. Finally, send $J_i \to i$, $1 \leq i \leq p$.
We have $\frac{p-1}{2} + \frac{p-1}{2} + 1 = p$ solutions.

A criterion for projectivity

In discussing n-complexes and related problems with Eisenbud, the following seemingly unlikely idea arose:

Conjecture. If G is a finite group, F is a field of characteristic p and M is an FG-module which is projective for every elementary abelian p-subgroup of G then M is projective.

Proposition 1 If G is a minimal counterexample to the conjecture and M is a module displaying this then G is an extra-special p-group or the central product of an extra-special p-group and a cyclic group of order p^2. Moreover, M is projective for every maximal subgroup of G.

Proof. The property of M with respect to maximal subgroups is immediate from the minimality of G. Also, G is clearly a p-group. It suffices to show the following: If Z is any normal subgroup of order p in G then G/Z is elementary abelian.

Suppose not; thus if E/Z is an elementary abelian subgroup of G/Z then $E < G$. Hence M is a free (=projective) FE-module. Thus $C_M(Z)$ is a free $F[E/Z]$-module. Hence, by the minimality of G, $C_M(Z)$ is a free $F[G/Z]$-module. Thus, $\dim_F C_M(Z) = |G/Z| \dim C_{C_M(Z)}(G)$. But $\dim_F C_M(Z) = \frac{1}{p}\dim_F M$, as Z acts freely, and $C_{C_M(Z)}(G) = C_M(G)$ so $\dim_F M = p.\ |G/Z| . \dim_F C_M(G)$ or $\dim_F M = |G| . \dim_F C_M(G)$ and M is a free FG-module, a contradiction.

Now we turn to the case $p=2$ and some 2-groups to test the conjecture.

<u>Proposition 2</u>. If Q is a quaternion group of order eight, F is a field of characteristic two and M is an FQ-module which is free as $F[Z(Q)]$-module then M is a free FQ-module.

<u>Proof</u>. Let $Q = \langle x, y \rangle$ and let $Y = y+1 \in FQ$. Then the Loewy series of M as $F\langle y\rangle$-module is

$$M \supset MY \supset MY^2 \supset MY^3 \supset MY^4 = 0.$$

Now x centralizes M/MY and MY^3, or else M has a free summand and we can argue by induction on $\dim_F M$. Hence, if $X = x+1$, $MY = MX$ as M/MX has dimension one fourth of the dimension of M.

$\therefore MYX = MX^2 = MY^2$; for $MX^2 = MY^2$ since Q is quaternion.

Thus, with respect to a suitable basis, M affords the following representation of Q:

$$x \to \begin{pmatrix} I & A & D & F \\ & I & B & E \\ & & I & C \\ & & & I \end{pmatrix}, \quad y \to \begin{pmatrix} I & I & 0 & 0 \\ & I & I & 0 \\ & & I & I \\ & & & I \end{pmatrix}$$

(with A, B, C non-singular)

Hence $y^2 \to \begin{pmatrix} I & 0 & I & 0 \\ & I & 0 & I \\ & & I & 0 \\ & & & I \end{pmatrix}$ and

$$x^2 \to \begin{pmatrix} I & 0 & AB & AE+DC \\ & I & 0 & BC \\ & & I & 0 \\ & & & I \end{pmatrix}$$

so $AB = BC = I$, $AE + DC = 0$. $\therefore A = C$, $B = A^{-1}$, $AE + DA = 0$.

Now $yxy = x$ so this implies

$$\begin{pmatrix} I & A & D & F \\ & I & B & E \\ & & I & C \\ & & & F \end{pmatrix} = \begin{pmatrix} I & I & 0 & 0 \\ & I & I & 0 \\ & & I & I \\ & & & I \end{pmatrix} \begin{pmatrix} I & A & D & F \\ & I & B & E \\ & & I & C \\ & & & I \end{pmatrix} \begin{pmatrix} I & I & 0 & 0 \\ & I & I & 0 \\ & & I & I \\ & & & I \end{pmatrix}$$

$$= \begin{pmatrix} I & A+I & D+B & F+E \\ & I & B+I & E+C \\ & & I & C+I \\ & & & I \end{pmatrix} \begin{pmatrix} I & I & 0 & 0 \\ & I & I & 0 \\ & & I & I \\ & & & I \end{pmatrix}$$

$$= \begin{pmatrix} I & A & I+A+B+D & B+D+E+F \\ & I & B & I+B+C+E \\ & & I & C \\ & & & I \end{pmatrix}$$

so $A + B = I$, that is, $A + A^{-1} = I$. Also, we get $B + D + E + F = F$ so $D + E = B = A^{-1}$ (from above have $B = A^{-1}$) so $DA + EA = I$. But from above, $AE + DA = 0$, so adding these last two gives $AE + EA = I$.

Hence, we have $A^2 + A + I = 0$, $AE + EA = I$

Diagonalizing A we get similar equations for the diag A + the result of applying this similarly to E. Have

$$A = \begin{pmatrix} \omega & & \\ & \ddots & \\ & & \end{pmatrix}, \quad E = \begin{pmatrix} \cdot & \cdot & \cdot \\ & & \\ \cdot & & \_\_ \end{pmatrix}$$

so 1,1 entry of AE is ωe_{11} and 1,1 entry of EA is $e_{11}\omega = \omega e_{11}$ so $AE + EA$ has zero as 1,1 entry, contradicting $EA + AE = I$.

The result is proved.

Now let's prove a result (which Ringel says does follow from his classification):

<u>Proposition 3</u> If P is a dihedral group of order eight, F is a field of characteristic two and M is an FP-module which is free as a module for each fours subgroup of P then M is a free FP-module.

<u>Proof</u> Let $P = \langle x, y \rangle$ where $x^2 = y^4 = 1$, $y^2 \neq 1$. Let $Y = y+1$. Have Loewy series for $\langle y \rangle$: $M \supset MY \supset MY^2 \supset MY^3 \supset MY^4 = 0$ (as M is free as $F\langle y\rangle$-module since is as $F\langle y^2 \rangle$ module. Again, x centralizes M/MY and MY^3 or M has a free summand, since $\langle x, y^2 \rangle$ acts freely on M we get that $\langle x \rangle$ has free action on M/MY^2 and MY^2. Hence, $MX = MY$, $X = x+1$, $MY^2X = MY^3$. It follows, as x has order two that $MYX = MY^3$, that is, x is trivial on the middle. Hence, with respect to a suitable basis

$$x \longrightarrow \begin{pmatrix} I & A & C & D \\ & I & O & E \\ & & I & B \\ & & & I \end{pmatrix}, \quad y \longrightarrow \begin{pmatrix} I & I & O & O \\ & I & I & O \\ & & I & I \\ & & & I \end{pmatrix}.$$

[Thus

$$1 = x^2 \rightarrow I \text{ so } \begin{pmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{pmatrix} = \begin{pmatrix} I & O & O & AE+CB \\ & I & O & O \\ & & I & O \\ & & & I \end{pmatrix}$$

and $AE + CB = 0$.] Also

$$xy \to \begin{pmatrix} I & A & C & D \\ & I & 0 & E \\ & & I & B \\ & & & I \end{pmatrix} \begin{pmatrix} I & I & 0 & 0 \\ & I & I & 0 \\ & & I & I \\ & & & I \end{pmatrix}$$

$$= \begin{pmatrix} I & I+A & A+C & C+D \\ & I & I & E \\ & & I & I+B \\ & & & I \end{pmatrix}$$

and

$$(xy)^2 \to \begin{pmatrix} I & 0 & I+A & \leftarrow \\ & I & 0 & I+B \\ & & I & 0 \\ & & & I \end{pmatrix} \qquad \leftarrow E+AE+A+AB+C+CB$$

so $A = I$. Hence,

$$xy \to \begin{pmatrix} I & 0 & \cdot & \cdot \\ & I & \cdot & \cdot \\ & & \cdot & \cdot \\ & & & \cdot \end{pmatrix}$$

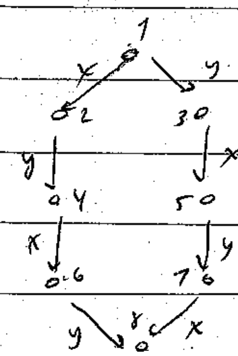
so xy centralizes M/MY^2. But the same argument that applies to xy equally well and x does not centralize M/MY^2. The proof is complete.

<u>Prop 4</u> Conjecture is false for $D_8 \circ Z_4$.

We do this by a detailed analysis, following the lines of the above arguments.

Now let $P = D_8 \circ Z_4$, $D_8 = \langle x, y \mid x^2 = y^2 = 1 \rangle$, $Z_4 = \langle t \rangle$.

A free module for D_8:



Matrices:

$$x \to \begin{pmatrix} I & I & 0 & 0 & 0 & 0 & 0 & 0 \\ & I & 0 & 0 & 0 & 0 & 0 & 0 \\ & & I & 0 & I & 0 & 0 & 0 \\ & & & I & 0 & I & 0 & 0 \\ & & & & I & 0 & 0 & 0 \\ & & & & & I & 0 & 0 \\ & & & & & & I & I \\ & & & & & & & I \end{pmatrix}$$

$$y \to \begin{pmatrix} I & 0 & I & 0 & 0 & 0 & 0 & 0 \\ & I & 0 & I & 0 & 0 & 0 & 0 \\ & & I & 0 & 0 & 0 & 0 & 0 \\ & & & I & 0 & 0 & 0 & 0 \\ & & & & I & 0 & I & 0 \\ & & & & & I & 0 & I \\ & & & & & & I & 0 \\ & & & & & & & I \end{pmatrix}$$

$$t \to \begin{pmatrix} I & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ & I & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ & & I & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ & & & I & A_{45} & A_{46} & A_{47} & A_{48} \\ & & & & I & A_{56} & A_{57} & A_{58} \\ & & & & & I & A_{67} & A_{68} \\ & & & & & & I & A_{78} \\ & & & & & & & I \end{pmatrix}$$

$$yt \rightarrow \begin{pmatrix} I & A_{12} & I+A_{13} & A_{14}+A_{34} & A_{15}+A_{35} & A_{16}+A_{36} & A_{17}+A_{37} & A_{18}+A_{38} \\ & I & A_{23} & I+A_{24} & A_{25}+A_{45} & A_{26}+A_{46} & A_{27}+A_{47} & A_{28}+A_{48} \\ & & I & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ & & & I & A_{45} & A_{46} & A_{47} & A_{48} \\ & & & & I & A_{56} & I+A_{57} & A_{58}+A_{78} \\ & & & & & I & A_{67} & I+A_{68} \\ & & & & & & I & A_{78} \\ & & & & & & & I \end{pmatrix}$$

$$ty \rightarrow \begin{pmatrix} I & A_{12} & I+A_{13} & A_{12}+A_{14} & A_{15} & A_{16} & A_{15}+A_{17} & A_{16}+A_{18} \\ & I & A_{23} & I+A_{24} & A_{25} & A_{26} & A_{25}+A_{27} & A_{26}+A_{28} \\ & & I & A_{34} & A_{35} & A_{36} & A_{35}+A_{37} & A_{36}+A_{38} \\ & & & I & A_{45} & A_{46} & A_{45}+A_{47} & A_{46}+A_{48} \\ & & & & I & A_{56} & I+A_{57} & A_{56}+A_{58} \\ & & & & & I & A_{67} & I+A_{68} \\ & & & & & & I & A_{78} \\ & & & & & & & I \end{pmatrix}$$

Hence,

$$t \rightarrow \begin{pmatrix} I & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ & I & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ & & I & A_{12} & 0 & 0 & A_{15} & A_{16} \\ & & & I & 0 & 0 & A_{45} & A_{26} \\ & & & & I & A_{56} & A_{57} & A_{58} \\ & & & & & I & A_{67} & A_{68} \\ & & & & & & I & A_{56} \\ & & & & & & & I \end{pmatrix}$$

$$xt \rightarrow \begin{pmatrix} I & I+A_{12} & A_{13}+A_{23} & A_{14}+A_{24} & A_{15}+A_{25} & A_{16}+A_{26} & A_{17}+A_{27} & A_{18}+A_{28} \\ & I & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ & & I & A_{12} & I & A_{56} & A_{15}+A_{57} & A_{16}+A_{58} \\ & & & I & O & I & A_{25}+A_{67} & A_{26}+A_{68} \\ & & & & I & A_{56} & A_{57} & A_{58} \\ & & & & & I & A_{67} & A_{68} \\ & & & & & & I & I+A_{78} \\ & & & & & & & I \end{pmatrix}$$

$$tx \rightarrow \begin{pmatrix} I & I+A_{12} & A_{13} & A_{14} & A_{13}+A_{15} & A_{14}+A_{16} & A_{17} & A_{17}+A_{18} \\ & I & A_{23} & A_{24} & A_{23}+A_{25} & A_{24}+A_{26} & A_{27} & A_{27}+A_{28} \\ & & I & A_{12} & I & A_{12} & A_{15} & A_{15}+A_{16} \\ & & & I & O & I & A_{25} & A_{25}+A_{26} \\ & & & & I & A_{56} & A_{57} & A_{57}+A_{58} \\ & & & & & I & A_{67} & A_{67}+A_{68} \\ & & & & & & I & I+A_{56} \\ & & & & & & & I \end{pmatrix}$$

Hence,

$$t \rightarrow \begin{pmatrix} I & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ & I & O & O & A_{13} & A_{14} & O & A_{17} \\ & & I & A_{12} & O & O & A_{15} & A_{16} \\ & & & I & O & O & A_{13} & A_{14} \\ & & & & I & A_{12} & O & A_{15} \\ & & & & & I & O & A_{13} \\ & & & & & & I & A_{12} \\ & & & & & & & I \end{pmatrix}$$

$$xy \rightarrow \begin{pmatrix} I & I & I & I & 0 & 0 & 0 & 0 \\ & I & 0 & I & 0 & 0 & 0 & 0 \\ & & I & 0 & I & 0 & I & 0 \\ & & & I & 0 & I & 0 & I \\ & & & & I & 0 & I & 0 \\ & & & & & I & 0 & I \\ & & & & & & I & I \\ & & & & & & & I \end{pmatrix}$$

$$(xy)^2 \rightarrow \begin{pmatrix} I & 0 & 0 & I & I & I & I & I \\ & I & 0 & 0 & 0 & I & 0 & I \\ & & I & 0 & 0 & 0 & I & I \\ & & & I & 0 & 0 & 0 & I \\ & & & & I & 0 & 0 & I \\ & & & & & I & 0 & 0 \\ & & & & & & I & 0 \\ & & & & & & & I \end{pmatrix}$$

$$t^2 \longrightarrow \begin{pmatrix} I & 0 & 0 & A_{13}A_{12} & A_{12}A_{13} & A_{12}A_{14}+A_{15}A_{12} & A_{13}A_{15}+A_{14}A_{13} & * \\ & I & 0 & 0 & 0 & A_{13}A_{12} & 0 & A_{13}A_{15}+A_{14}A_{13} \\ & & I & 0 & 0 & 0 & A_{12}A_{13} & A_{12}A_{14}+A_{15}A_{12} \\ & & & I & 0 & 0 & 0 & A_{13}A_{12} \\ & & & & I & 0 & 0 & A_{12}A_{13} \\ & & & & & I & 0 & 0 \\ & & & & & & I & 0 \\ & & & & & & & I \end{pmatrix}$$

$$* = (A_{12}A_{17} + A_{17}A_{12}) + (A_{13}A_{16} + A_{16}A_{13}) + A_{14}^2 + A_{15}^2 = I$$

Hence

$$A_{12}A_{13} = I$$

$$A_{12}A_{14} + A_{15}A_{12} = I$$

$$(A_{13}A_{15} + A_{14}A_{13} = I)$$

Let's examine $\langle x, t \rangle$. On fixed pts of x (blocks 2, 5, 6, 8) t has following action

$$\begin{pmatrix} I & A_{13} & A_{14} & A_{17} \\ & I & A_{12} & A_{15} \\ & & I & A_{13} \\ & & & I \end{pmatrix}$$

which has square — in view of above relations —

$$\begin{pmatrix} I & 0 & I & I \\ & I & 0 & I \\ & & I & 0 \\ & & & I \end{pmatrix}$$

so $\langle x, t \rangle$ acts freely.

Let's examine $\langle y, t \rangle$. On fixed pts of y (blocks 3, 4, 7, 8) t has following action

$$\begin{pmatrix} I & A_{12} & A_{15} & A_{18} \\ & I & A_{13} & A_{14} \\ & & I & A_{12} \\ & & & I \end{pmatrix}$$

whose square is

$$\begin{pmatrix} I & 0 & I & I \\ & I & 0 & I \\ & & I & 0 \\ & & & I \end{pmatrix}$$

so $\langle y, t \rangle$ acts freely.

next, $\langle xy, t\rangle$. Fixed pts of xy : 6+7, 8

Matrix there for t: $\begin{pmatrix} I & A_{12}+A_{13} \\ 0 & I \end{pmatrix}$

Hence need : $A_{12}+A_{13}$ non-singular.

next, $\langle x, z=(xy)^2, yt\rangle$. Fixed pts of $\langle x, z\rangle$: 6, 8

(by calc fixed pts of x + looking at $(xy)^2$ there) matrix there for yt:

$$\begin{pmatrix} I & I+A_{13} \\ & I \end{pmatrix}$$

Hence, need, $I+A_{13}$ non-singular.

next, $\langle y, z, xt\rangle$. Fixed pts of $\langle y, z\rangle$: 7, 8

$$\begin{pmatrix} I & I+A_{12} \\ & I \end{pmatrix}$$

so need $I+A_{12}$ non-singular.

The last subgroup : $\langle xy, yt\rangle \cong Q_8$. But xy acts freely so Q_8 does also by our theorem. Hence, conditions for a counterexample:

$$t \to \begin{pmatrix} I & A & A^{-1} & B & C & D & E & F \\ & I & 0 & 0 & A^{-1} & B & 0 & E \\ & & I & A & 0 & 0 & C & D \\ & & & I & 0 & 0 & A^{-1} & B \\ & & & & I & A & 0 & C \\ & & & & & I & 0 & A^{-1} \\ & & & & & & I & A \\ & & & & & & & I \end{pmatrix}$$

$AB + CA = I$

$AE + EA + A^{-1}D + DA^{-1} + B^2 + C^2 = I$

$\det(A + A^{-1}) \neq 0$

$\det(I + A^{-1}) \neq 0$

$\det(I + A) \neq 0$

These cannot be solved with 1×1 matrices. For then - using small letters - $a(b+c)=1$, $(b+c)^2=1$, $1+a\neq0$ so $a(b+c)=1$, $b+c=1$, $a\neq1$, a contradiction.

But set

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$B = \begin{pmatrix} \lambda^{-1} & \lambda^{-2} \\ 0 & \lambda^{-1} \end{pmatrix}$$

$$C = D = 0$$

$$E = \begin{pmatrix} 0 & 0 \\ 1+\lambda^{-2} & 0 \end{pmatrix}$$

with $\lambda + \lambda^{-1} \neq 0$, $1 + \lambda^{-1} \neq 0$, $1 + \lambda \neq 0$. Have the counterexample!