RESEARCH NOTES

VOLUME II

Research Notes

Volume II

Contents

Projectives for $L_2(2^n)$ in characteristic 2

The principal indecomposable projective for $L_2(2^n)$.

Theorem Let Z be a cyclic subgroup of order 2^n+1 in $L_2(2^n)$ and let F be a field of characteristic two. It follows that $F \otimes_{FZ} FL_2(2^n)$ is the principal indecomposable projective for $L_2(2^n)$.

Proof We may assume F is a splitting field. If V is an irreducible FG-module then the multiplicity of P_V — the corresponding projective — in the projective $F \otimes_{FZ} FL_2(2^n) = Q$ — the latter is projective as $|Z|$ is odd — equals $\dim_F \mathrm{Hom}_{FZ}(F, V/Z)$. Thus, this multiplicity is the number of eigenvalues equal to 1 for a generator z of Z in the representation corresponding to V.

Let λ be a primitive 2^n+1 st root of unity so the eigenvalues of z in V are the summands of the product

$$(\lambda^{2^{i_1}} + \lambda^{-2^{i_1}}) \cdots (\lambda^{2^{i_s}} + \lambda^{-2^{i_s}})$$

where $0 \le i_1 < i_2 < \dots < i_s < n-1$. Hence each term is

$$\lambda^{\pm 2^{i_1}} \lambda^{\pm 2^{i_2}} \cdots \lambda^{\pm 2^{i_s}} = \lambda^{\pm 2^{i_1} \pm 2^{i_2} \pm \dots \pm 2^{i_s}}$$

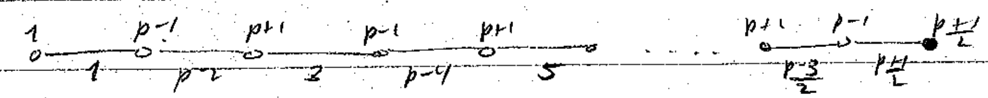
The maximum value for the exponent is

$$1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1 < 2^n + 1$$

and the minimum is the negative of this. Hence, if the term is 1 then the sum $\pm 2^{i_1} \pm \dots \pm 2^{i_s} = 0$. But it is congruent to $\pm 2^{i_1}$ modulo 2^{i_1+1}.

Picture for $L_2(p)$ is standard. Believe it is as follows:

$p \equiv -1 \ (4)$



Have projectives: $P_1, P_3, P_5, \ldots, P_{p-2}$ and P_p.
Part of modular characters: $\langle x \rangle = Z_{\frac{p+1}{2}}$, λ primitive $\underline{\underline{p+1\text{-st}}}$ root of 1.

	1	x
φ_1	1	1
φ_3	3	$\lambda^2+1+\lambda^{-2}$
φ_5	5	$\lambda^4+\lambda^2+1+\lambda^{-2}+\lambda^{-4}$
$\vdots$		$\vdots$
φ_{p-2}	$p-2$	$\lambda^{p-3}+\cdots+\lambda^{-(p-3)}$
φ_p	p	$\lambda^{p-1}+\cdots+\lambda^{-(p-1)}$

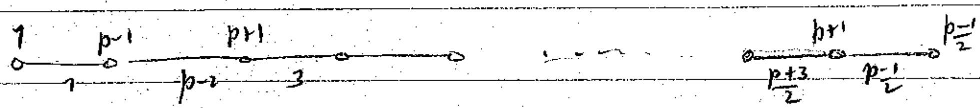
Now, using ideas of last page,

$$F \otimes_{FZ_{\frac{p+1}{2}}} FL_2(p) \simeq P_1 + \cdots + P_p$$

Dimensions check:

$$p(p-1) = \overbrace{2p + 2p + \cdots + 2p + p}^{\frac{p+1}{2}}$$

$p \equiv 1 \ (4)$



Next let $G = L_2(2^n)$, $L, M \subseteq G$ cyclic of orders 2^n+1, 2^n-1, respectively. Let $\mathcal{L}$ be a set of linear characters of L, one from each orbit in $N(L)$ and $\mathcal{M}$ be similarly defined for M with typical elements $\lambda \in \mathcal{L}$, $\mu \in \mathcal{M}$. Let V_λ, V_μ be corresponding FL, FM modules, F a large field of characteristic two. Let P_λ, P_μ be the corresponding induced modules so they are projective inasmuch as L and M are of odd order. Let λ_0, μ_0 be principal characters in $\mathcal{L}, \mathcal{M}$.

<u>Lemma 1</u> $\dim_F \mathrm{Hom}_{FG}(P_\lambda, P_\mu) = 2^n$

<u>Proof</u> $\mathrm{Hom}_{FG}(P_\lambda, P_\mu) = \mathrm{Hom}_{FG}(V_\lambda \otimes_{FL} FG, V_\mu \otimes_{FM} FG)$

$\simeq \mathrm{Hom}_{FL}(V_\lambda, V_\mu \otimes_{FM} FG \mid L)$

But the latter module is, by Mackey's theorem, a free FL-module. Since $\dim_{FG} V_\mu \otimes_{FM} FG = (2^n-1)\,2^n\,(2^n+1) / 2^n-1 = 2^n(2^n+1)$ the restriction is 2^n copies of a free module. Hence the result holds.

<u>Lemma 2</u> If $\lambda \neq \lambda_0$, $\dim_{FG}(P_\lambda, P_\lambda) = 2^n-1$, while $\dim_{FG}(P_{\lambda_0}, P_{\lambda_0}) = 2^n$

<u>Proof</u> We have as above, $P_\lambda / L \cong V_\lambda \oplus V_{\bar\lambda} \oplus$ free where if $\lambda \neq \lambda_0$ then $\bar\lambda \notin \mathcal{L}$ but is clear as to its meaning. The number of generators of the free module is $2^n(2^n-1) - 2 / 2^n+1 = 2^n - 2$.

<u>Lemma 3</u> If $\lambda \neq \lambda'$ then $\dim_{FG}(P_\lambda, P_{\lambda'}) = 2^n - 2$.

This is also now clear - details omitted.

Lemma 4 If $\mu, \mu' \in M$ and $\mu \neq \mu'$ then

a) $\dim_{FG}(P_\mu, P_{\mu'}) = 2^n + 2$;

b) $\dim_{FG}(P_\mu, P_\mu) = 2^n + 3$, if $\mu \neq \mu_0$;

c) $\dim_{FG}(P_{\mu_0}, P_{\mu_0}) = 2^n + 4$.

Proof. Have $P_\mu | M \cong V_\mu \oplus V_{\bar\mu} \oplus$ free, number of generators of the latter summand being $2^n(2^n+1) - 2 / 2^n - 1 = 2^n + 2$. Rest as before.

Let's use this to get Cartan matrix for $A_5 \cong L_2(4)$. Let λ be primitive 5-th root of unity, μ primitive cube root. We list restrictions of irreducibles of $L_2(4)$ to Z_5, Z_3 – at least their generators x, y

	1	x	y
φ_0	1	1	1
φ_1	2	$\lambda + \lambda^{-1}$	$\mu + \mu^{-1}$
φ_2	2	$\lambda^2 + \lambda^{-2}$	$\mu + \mu^{-1}$
φ_{12}	4	$\lambda^2 + \lambda + \lambda^{-1} + \lambda^{-2}$	$2 + \mu + \mu^{-1}$

Let P_a be projective for φ_a, P_{λ^i} projective for $(\lambda^i)^G$, λ linear character having value λ at x, etc – as above –

$P_{\lambda^0} \cong P_0$

$P_{\lambda^1} \cong P_1 \oplus P_{12}$

$P_{\lambda^2} \cong P_2 \oplus P_{12}$

$P_{\mu^0} \cong P_0 \oplus P_{12} \oplus P_{12}$

$P_{\mu^1} \cong P_1 \oplus P_2 \oplus P_{12}$

Matrix of dimensions of Hom(−, −):

	P_{λ^0}	P_{λ^1}	P_{λ^2}	P_{μ^0}	P_{μ^1}
P_{λ^0}	4	2	2	4	4
P_{λ^1}	2	3	2	4	4
P_{λ^2}	2	2	3	4	4
P_{μ^0}	4	4	4	8	6
P_{μ^1}	4	4	4	6	7

Now, with equations in the Green ring,

$$P_0 = P_{\lambda^0}$$

$$P_1 = P_{\mu^1} - P_{\lambda^2}$$

$$P_2 = P_{\mu^1} - P_{\lambda^1}$$

$$P_{12} = P_{\lambda} + P_{\lambda^2} - P_{\mu}$$

Now using the P_{λ^i}, P_{μ^j} except P_{μ^0} calculate Cartan matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 & 4 \\ 2 & 3 & 2 & 4 \\ 2 & 2 & 3 & 4 \\ 4 & 4 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 2 & 4 \\ 2 & 2 & 1 & 3 \\ 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's do $L_2(8)$ by this method. Now $\lambda^9 = \mu^7 = 1$, all notation similar to above notation

	1	x	y
φ_0	1	1	1
φ_1	2	$\lambda+\lambda^{-1}$	$\mu+\mu^{-1}$
φ_2	2	$\lambda^2+\lambda^{-2}$	$\mu^2+\mu^{-2}$
φ_3	2	$\lambda^4+\lambda^{-4}$	$\mu^3+\mu^{-3}$
φ_{12}	4	$\lambda^3+\lambda+\lambda^{-1}+\lambda^{-3}$	$\mu^3+\mu^1+\mu^{-1}+\mu^{-3}$
φ_{23}	4	$\lambda^3+\lambda^2+\lambda^{-2}+\lambda^{-3}$	$\mu^2+\mu+\mu^{-1}+\mu^{-2}$
φ_{13}	4	$\lambda^4+\lambda^3+\lambda^{-3}+\lambda^{-4}$	$\mu^3+\mu^2+\mu^{-2}+\mu^{-3}$
φ_{123}	8	$\lambda^4+\dots+\lambda+\lambda^{-1}+\dots+\lambda^{-4}$	$\mu^3+\mu^2+\mu+\mu^{-1}+\mu^{-2}+\mu^{-3}+2$

$P_{\lambda^0} \cong P_0$

$P_\lambda \cong P_1 \oplus P_{12} \oplus P_{123}$

$P_{\lambda^2} \cong P_2 \oplus P_{23} \oplus P_{123}$

$P_{\lambda^4} \cong P_3 \oplus P_{13} \oplus P_{123}$

$P_{\lambda^3} \cong P_{12} \oplus P_{23} \oplus P_{13} \oplus P_{123}$

$[P_{\mu^0} \cong P_0 \oplus P_{123} \oplus P_{123}]$

$P_{\mu^1} \cong P_1 \oplus P_{12} \oplus P_{23} \oplus P_{123}$

$P_{\mu^2} \cong P_2 \oplus P_{23} \oplus P_{13} \oplus P_{123}$

$P_{\mu^3} \cong P_3 \oplus P_{12} \oplus P_{13} \oplus P_{123}$

$P_0 = P_{\lambda^0}$

$P_{23} = P_\mu - P_\lambda$

$P_{13} = P_{\mu^2} - P_{\lambda^2}$

$P_{12} = P_{\mu^3} - P_{\lambda^4}$

$P_{123} = P_{\lambda^3} + P_\lambda + P_{\lambda^2} + P_{\lambda^4} - P_\mu - P_{\mu^2} - P_{\mu^3}$

$P_1 = -P_{\lambda^3} - P_{\lambda^2} + P_\mu + P_{\mu^2}$

$P_2 = -P_{\lambda^3} - P_{\lambda^4} + P_{\mu^2} + P_{\mu^3}$

$P_3 = -P_\lambda - P_{\lambda^3} + P_\mu + P_{\mu^3}$

Matrix of dim Hom(–,–) :

	P_{λ^0}	P_{λ}	P_{λ^2}	P_{λ^4}	P_{λ^3}	P_{μ^0}	P_{μ}	P_{μ^2}	P_{μ^3}
P_{λ^0}	8	6	6	6	6	8	8	8	8
P_{λ}	6	7	6	6	6	8	8	8	8
P_{λ^2}	6	6	7	6	6	8	8	8	8
P_{λ^4}	6	6	6	7	6	8	8	8	8
P_{λ^3}	6	6	6	6	7	8	8	8	8
P_{μ^0}	8	8	8	8	8	12	10	10	10
P_{μ^1}	8	8	8	8	8	10	11	10	10
P_{μ^2}	8	8	8	8	8	10	10	11	10
P_{μ^3}	8	8	8	8	8	10	10	10	11

$$C = \begin{array}{c} \\ P_0 \\ P_3 \\ P_1 \\ P_2 \\ P_{23} \\ P_{13} \\ P_{12} \\ P_{123} \end{array} \begin{array}{c} \lambda^0 \;\; \lambda^1 \;\; \lambda^2 \;\; \lambda^4 \;\; \lambda^3 \;\; \mu \;\; \mu^2 \;\; \mu^3 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \end{array} \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 8 & 8 & 8 \\ 6 & 7 & 6 & 6 & 6 & 8 & 8 & 8 \\ 6 & 6 & 7 & 6 & 6 & 8 & 8 & 8 \\ 6 & 6 & 6 & 7 & 6 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 & 7 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 & 11 & 10 & 10 \\ 8 & 8 & 8 & 8 & 8 & 10 & 11 & 10 \\ 8 & 8 & 8 & 8 & 8 & 10 & 10 & 11 \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix}^T$$

$$= \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 8 & 8 & 8 \\ 4 & 3 & 4 & 4 & 3 & 5 & 4 & 5 \\ 4 & 4 & 3 & 4 & 3 & 5 & 5 & 4 \\ 4 & 4 & 4 & 3 & 3 & 4 & 5 & 5 \\ 2 & 1 & 2 & 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 4 & 4 & 2 & 2 & 2 & 0 \\ 4 & 4 & 2 & 2 & 2 & 0 & 1 & 0 \\ 4 & 2 & 4 & 2 & 1 & 2 & 0 & 0 \\ 4 & 2 & 2 & 4 & 0 & 1 & 2 & 0 \\ 2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

as desired

We go on to $L_2(16)$, with the obvious notation

	1	x	y
φ_0	1	1	1
φ_1	2	$\lambda+\lambda^{-1}$	$\mu+\mu^{-1}$
φ_2	2	$\lambda^2+\lambda^{-2}$	$\mu^2+\mu^{-2}$
φ_3	2	$\lambda^4+\lambda^{-4}$	$\mu^4+\mu^{-4}$
φ_4	2	$\lambda^8+\lambda^{-8}$	$\mu^7+\mu^{-7}$
φ_{12}	4	$\lambda^3+\lambda+\lambda^{-1}+\lambda^{-3}$	$\mu^3+\mu+\mu^{-1}+\mu^{-3}$
φ_{23}	4	$\lambda^6+\lambda^2+\lambda^{-2}+\lambda^{-6}$	$\mu^6+\mu^2+\mu^{-2}+\mu^{-6}$
φ_{34}	4	$\lambda^5+\lambda^4+\lambda^{-4}+\lambda^{-5}$	$\mu^4+\mu^3+\mu^{-3}+\mu^{-4}$
φ_{14}	4	$\lambda^8+\lambda^7+\lambda^{-7}+\lambda^{-8}$	$\mu^7+\mu^6+\mu^{-6}+\mu^{-7}$
φ_{13}	4	$\lambda^5+\lambda^3+\lambda^{-3}+\lambda^{-5}$	$\mu^5+\mu^3+\mu^{-3}+\mu^{-5}$
φ_{24}	4	$\lambda^7+\lambda^6+\lambda^{-6}+\lambda^{-7}$	$\mu^6+\mu^5+\mu^{-5}+\mu^{-6}$
φ_{123}	8	$\lambda^7+\lambda^5+\lambda^3+\lambda^1+\dots$	$\mu^7+\mu^5+\mu^3+\mu+\dots$
φ_{234}	8	$\lambda^7+\lambda^6+\lambda^3+\lambda^2+\dots$	$\mu^6+\mu^5+\mu^2+\mu^1+\dots$
φ_{134}	8	$\lambda^6+\lambda^5+\lambda^4+\lambda^3+\dots$	$\mu^5+\mu^4+\mu^3+\mu^2+\dots$
φ_{124}	8	$\lambda^8+\lambda^7+\lambda^6+\lambda^5+\dots$	$\mu^7+\mu^6+\mu^5+\mu^4+\dots$
φ_{1234}	16	$\lambda^8+\dots+\lambda+\lambda^{-1}+\dots$	$\mu^7+\dots+\mu^1+1+1+\mu^{-1}+\dots+\mu^{-7}$

$$P_{\lambda^0} = P_0$$

$$P_{\lambda} = P_1 + P_{12} + P_{123} + P_{1234}$$

$$P_{\lambda^2} = P_2 + P_{23} + P_{234} + P_{1234}$$

$$P_{\lambda^4} = P_3 + P_{34} + P_{134} + P_{1234}$$

$$P_{\lambda^8} = P_4 + P_{14} + P_{124} + P_{1234}$$

$$P_{\lambda^3} = P_{12} + P_{13} + P_{123} + P_{234} + P_{134} + P_{1234}$$

$$P_{\lambda^6} = P_{23} + P_{24} + P_{234} + P_{134} + P_{124} + P_{1234}$$

$$P_{\lambda^5} = P_{34} + P_{13} + P_{123} + P_{134} + P_{124} + P_{1234}$$

$$P_{\lambda^7} = P_{14} + P_{24} + P_{123} + P_{234} + P_{124} + P_{1234}$$

$$P_{\mu} = P_1 + P_{12} + P_{123} + P_{234} + P_{1234}$$

$$P_{\mu^2} = P_2 + P_{23} + P_{234} + P_{134} + P_{1234}$$

$$P_{\mu^4} = P_3 + P_{34} + P_{134} + P_{124} + P_{1234}$$

$$P_{\mu^7} = P_4 + P_{14} + P_{123} + P_{124} + P_{1234}$$

$$P_{\mu^3} = P_{12} + P_{34} + P_{13} + P_{123} + P_{134} + P_{1234}$$

$$P_{\mu^6} = P_{23} + P_{14} + P_{24} + P_{234} + P_{124} + P_{1234}$$

$$P_{\mu^5} = P_{13} + P_{24} + P_{123} + P_{234} + P_{134} + P_{124} + P_{1234}$$

Solving: (Dropping the symbol "p")

$$\begin{cases} 234 = \mu - \lambda \\ 134 = \mu^2 - \lambda^2 \\ 124 = \mu^4 - \lambda^4 \\ 123 = \mu^8 - \lambda^8 \end{cases}$$ (use Galois automorphism or subtraction)

$34 - 234 = \mu^3 - \lambda^3$ so
$$\begin{cases} 34 = \mu + \mu^3 - \lambda - \lambda^3 \\ 14 = \mu^2 + \mu^6 - \lambda^2 - \lambda^6 \\ 12 = \mu^3 + \mu^4 - \lambda^4 - \lambda^5 \\ 23 = \mu^6 + \mu^7 - \lambda^7 - \lambda^8 \end{cases}$$

Now also $P_{\mu^0} - P_{\lambda^0} = P_{1234} + P_{1234}$. But
$\sum_{i=0}^{14} P_{\mu^i} = \sum_{j=0}^{16} P_{\lambda^j}$ so $P_{\mu^0} + 2\sum_{i=1}^{7} P_{\mu^i} = P_{\lambda^0} + 2\sum_{j=1}^{8} P_{\lambda^j}$ so

$$\left\{ P_{1234} = -\mu - \mu^2 - \mu^4 - \mu^7 - \mu^3 - \mu^6 - \mu^5 + \lambda + \lambda^2 + \lambda^4 + \lambda^8 + \lambda^3 + \lambda^6 + \lambda^5 + \lambda^7 \right.$$

Next, $24 = \lambda^6 - 23 - 234 - 134 - 124 - 1234$
$= \lambda^6 - \mu^6 - \mu^7 + \lambda^7 + \lambda^8 - \mu + \lambda - \mu^2 + \lambda^2 - \mu^4 + \lambda^4 - 1234$

$$\begin{cases} 24 = \mu^3 + \mu^5 - \lambda^3 - \lambda^5 \\ 13 = \mu^5 + \mu^6 - \lambda^6 - \lambda^7 \end{cases}$$

Hence $1 = \lambda - 12 - 123 - 1234 = \lambda - \mu^3 - \mu^4 + \lambda^4 + \lambda^5 - \mu^7 + \lambda^8 - 1234$

$$\begin{cases} 1 = \mu + \mu^2 + \mu^5 + \mu^6 - (\lambda^2 + \lambda^3 + \lambda^6 + \lambda^7) \\ 2 = \mu^2 + \mu^3 + \mu^4 + \mu^5 - (\lambda^3 + \lambda^4 + \lambda^5 + \lambda^6) \\ 3 = \mu^4 + \mu^5 + \mu^6 + \mu^7 - (\lambda^5 + \lambda^6 + \lambda^7 + \lambda^8) \\ 4 = \mu + \mu^3 + \mu^5 + \mu^7 - (\lambda + \lambda^3 + \lambda^5 + \lambda^7) \end{cases}$$

Now we calculate the Cartan matrix:

$C =$

	λ^0	λ	λ^2	λ^4	λ^8	λ^3	λ^6	λ^5	λ^7	μ	μ^2	μ^4	μ^7	μ^3	μ^6	μ^5
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	-1	0	0	0	-1	0	-1	-1	1	0	0	1	1	0	1
1	0	0	-1	0	0	-1	-1	0	-1	1	1	0	0	0	1	1
2	0	0	0	-1	0	-1	-1	-1	0	0	1	1	0	1	0	1
3	0	0	0	0	-1	0	-1	-1	-1	0	0	1	1	0	1	1
34	0	-1	0	0	0	-1	0	0	0	1	0	0	0	1	0	0
14	0	0	-1	0	0	0	-1	0	0	0	1	0	0	0	1	0
12	0	0	0	-1	0	0	0	-1	0	0	0	1	0	1	0	0
23	0	0	0	0	-1	0	0	0	-1	0	0	0	1	0	1	0
24	0	0	0	0	0	-1	0	-1	0	0	0	0	0	1	0	1
13	0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	1	1
234	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
134	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0	0
124	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0	0
123	0	0	0	0	-1	0	0	0	0	0	0	0	1	0	0	0
1234	0	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1

$\cdot$

16	14	14	14	14	14	14	14	14	16	16	16	16	16	16	16
14	15	14						14	16						
14	14	15	14					14	16						
14		14	15	14				14	16						
14			14	15	14			14	16						
14				14	15	14		14	16						
14					14	15	14	14	16						
14						14	15	14	16						
14	14	14	14	14	14	14	14	15	16	16	16	16	16	16	16
16	16	16	16	16	16	16	16	16	19	18	18	18	18	18	18
								16	18	19	18				18
								16	18	18	19	18			18
								16	18		18	19	18		18
								16	18			18	19	18	18
								16	18				18	19	18
16	16	16	16	16	16	16	16	16	18	18	18	18	18	18	19

$\cdot \, [\quad]^T$

=

16	14	14	14	14	14	14	14	14	16	16	16	16	16	16	16
8	7	8	8	8	7	8	7	7	9	8	8	9	9	8	9
8	8	7	8	8	7	7	8	7	9	9	8	8	8	9	9
8	8	8	7	8	7	7	7	8	8	9	9	8	9	8	9
8	8	8	8	7	8	7	7	7	8	8	9	9	8	9	9
4	3	4	4	4	3	4	4	4	5	4	4	4	5	4	4
4	4	3	4	4	4	3	4	4	4	5	4	4	4	5	4
4	4	4	3	4	4	4	3	4	4	4	5	4	5	4	4
4	4	4	4	3	4	4	4	3	4	4	4	5	4	5	4
4	4	4	4	4	3	4	3	4	4	4	4	4	5	4	5
4	4	4	4	4	4	3	4	3	4	4	4	4	4	5	5
2	1	2	2	2	2	2	2	2	3	2	2	2	2	2	2
2	2	1	2	2	2	2	2	2	2	3	2	2	2	2	2
2	2	2	1	2	2	2	2	2	2	2	3	2	2	2	2
2	2	2	2	1	2	2	2	2	2	2	2	3	2	2	2
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$\cdot$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	-1	0	0	0	0	0	-1	0	0	0	1
0	0	-1	0	0	0	-1	0	0	0	0	0	-1	0	0	1
0	0	0	-1	0	0	0	-1	0	0	0	0	0	-1	0	1
0	0	0	0	-1	0	0	0	-1	0	0	0	0	0	-1	1
0	-1	-1	-1	0	-1	0	0	0	-1	0	0	0	0	0	1
0	0	-1	-1	-1	0	-1	0	0	0	-1	0	0	0	0	1
0	-1	0	-1	-1	0	0	-1	0	-1	0	0	0	0	0	1
0	-1	-1	0	-1	0	0	0	-1	0	-1	0	0	0	0	1
0	1	1	0	0	1	0	0	0	0	0	1	0	0	0	-1
0	0	1	1	0	0	1	0	0	0	0	0	1	0	0	-1
0	0	0	1	1	0	0	1	0	0	0	0	0	1	0	-1
0	1	0	0	1	0	0	0	1	0	0	0	0	0	1	-1
0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	-1
0	0	1	0	1	0	1	0	1	0	1	0	0	0	0	-1
0	1	1	1	1	0	0	0	0	1	1	0	0	0	0	-1

	0	4	1	2	3	34	14	12	23	24	13	234	134	124	123	1234
0	16	8	8	8	8	4	4	4	4	4	4	2	2	2	2	0
4	8	8	4	4	4	4	0	2	2	4	2	2	0	0	1	0
1	8	4	8	4	4	2	4	0	2	2	4	1	2	0	0	0
2	8	4	4	8	4	2	2	4	0	4	2	0	1	2	0	0
3	8	4	4	4	8	0	2	2	4	2	4	0	0	1	2	0
34	4	4	2	2	0	4	0	1	0	2	0	2	0	0	0	0
14	4	0	4	2	2	0	4	0	1	0	2	0	2	0	0	0
12	4	2	0	4	2	1	0	4	0	2	0	0	0	2	0	0
23	4	2	2	0	4	0	1	0	4	0	2	0	0	0	2	0
24	4	4	2	4	2	2	0	2	0	4	1	0	0	0	0	0
13	4	2	4	2	4	0	2	0	2	1	4	0	0	0	0	0
234	2	2	1	0	0	2	0	0	0	0	0	2	0	0	0	0
134	2	0	2	1	0	0	2	0	0	0	0	0	2	0	0	0
124	2	0	0	2	1	0	0	2	0	0	0	0	0	2	0	0
123	2	1	0	0	2	0	0	0	2	0	0	0	0	0	2	0
1234	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

We return to the general case. Let $\mathbb{Z}_n$ be the integers modulo n with elements $1, 2, \ldots, n$. Let $N_n = 2^{\mathbb{Z}_n}$, the subsets of $\mathbb{Z}_n$. Let V_1 be the standard module for $SL(2, 2^n)$ and let $V_2, V_3, \ldots, V_n$ be obtained by successive applications of the Frobenius automorphism. Let V_S for $S \subseteq N_n$ be the tensor product of the V_s for $s \in S$ – where $V_\emptyset$ is the trivial one-dimensional module. These are the irreducible modules in characteristic 2 for $SL(2, 2^n)$. Let φ_S be the corresponding Brauer character.

If $S \subseteq N_n$ we set $\bar{S} = N_n - S$. If $S, T \subseteq N_n$ we let $C_{S,T}$ be the Cartan invariant for V_S and V_T. If $S, T \subseteq N_n$ we say S and T are <u>linked</u> if for all $i \in \mathbb{Z}_n$, whenever $i \notin S \cap T$ and $i+1 \in S \cup T$ then $i+1 \in S \cap T$.

We can now state an important result:

<u>Theorem</u> With the above notation

$$C_{S,T} = \begin{cases} 2^{|\bar{S} \cap \bar{T}|}, & \text{if } \bar{S} \text{ and } \bar{T} \text{ are linked, } + \bar{S} \cap \bar{T} \neq \emptyset \text{ or } \bar{S} = T \\ 0, & \text{if } \bar{S} \text{ and } \bar{T} \text{ are not linked. or otherwise} \end{cases}$$

(see page 121 for possible restatement)

In $G = SL(2, 2^n)$ we have subgroups Z_{2^n+1}, Z_{2^n-1} with generators x and y. Suppose $\varphi_1(x) = \lambda + \lambda^{-1}$, $\varphi_1(y) = \mu + \mu^{-1}$, where λ and μ are primitive 2^n+1-st and 2^n-1 st roots of unity. Moreover, note that any Brauer character has support on $(Z_{2^n+1})^G \cup (Z_{2^n-1})^G$. <u>Also</u> let λ and μ be the obvious linear characters of Z_{2^n+1}, Z_{2^n-1}, respectively.

Now note that if $S = N_n$ (or, $T = N_n$) then the theorem holds for $2^{|\bar{N}_n \cap \bar{T}|} = 2^0$ as $\phi \cap \bar{T} = \phi$ so the possible right hand sides are 0 or 1. If $T = N_n$ then $\bar{S}$ and $\bar{T}$ are linked – "vacuously" and $C_{N_n N_n} = 1$ since V_{N_n} is the projective irreducible – coming from characteristic zero. Otherwise, clearly $\bar{N}_n$ and $\bar{T}$ are not linked and again we have the right answer.

We proceed by a sequence of lemmas. Let $\varphi_0 = \varphi_\phi$.

<u>Lemma 1</u> $\varphi_i^2 = 2\varphi_0 + \varphi_{i+1}$

<u>Proof</u> We have $\varphi_i \mid Z_{2^n+1} = \lambda^{2^i} + \lambda^{-2^i}$, $\varphi_i \mid Z_{2^n-1} = \mu^{2^i} + \mu^{-2^i}$ so $\varphi_i^2 \mid Z_{2^n+1} = \lambda^{2^{i+1}} + \lambda^{-2^{i+1}} + 2$, $\varphi_i^2 \mid Z_{2^n-1} = \mu^{2^{i+1}} + \mu^{-2^{i+1}} + 2$ and the result is immediate. (note that to be precise we need the following restriction table:

	Z_{2^n+1}	Z_{2^n-1}
φ_1	$\lambda + \lambda^{-1}$	$\mu + \mu^{-1}$
φ_2	$\lambda^2 + \lambda^{-2}$	$\mu^2 + \mu^{-2}$
$\vdots$		
φ_n	$\lambda^{2^{n-1}} + \lambda^{-2^{n-1}}$	$\mu^{2^{n-1}} + \mu^{-2^{n-1}}$

Hence

φ_n^2	$\lambda^{2^n} + \lambda^{-2^n} + 2$	$\mu^{2^n} + \mu^{-2^n} + 2$
	$\shortparallel$	$\shortparallel$
	$\lambda^{-1} + \lambda + 2$	$\mu + \mu^{-1} + 2$)

<u>Lemma 2</u> If $S \subseteq N$ and $\varphi_S \mid Z_{2^n+1}$ has a principal constituent then $S = \phi$. If $S \subseteq N$ and $\varphi_S \mid Z_{2^n-1}$ has a principal constituent then $S = \phi$ or $S = N$, in the latter case the relevant multiplicity being two.

<u>Proof</u> Of course $\varphi_S \mid Z_{2^n+1} = \prod_{s \in S} (\lambda^{2^{s-1}} + \lambda^{-2^{s-1}})$. The argument of the proof on page 107 completes the first half of the proof of this lemma.

An easy adaptation to Z_{2^n-1} and μ shows we only get the principal character of Z_{2^n-1} as constituent for $S = N_n$ as then $\prod_{s \in N} \mu^{2^{s-1}} + \mu^{-2^{s-1}}$ involves $\mu^{\pm(1+2+\ldots+2^{n-1})} = \mu^{\pm(2^n-1)} = 1$.

<u>Lemma 3</u> If $S, T \subset N_n$ then φ_N is not a component of $\varphi_S \varphi_T$ unless $\bar{S} = T$ (ie. S, T partition N_n) in which case φ_N appears in the product with multiplicity one.

<u>Proof</u> In fact, in the latter case $\varphi_S \varphi_T = \varphi_N$ as is clear by inspection. In general, if $S \cap T = \phi$ the result is clear, so we proceed by induction on $|S| + |T|$. We may assume that $i \in S \cap T$ so

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Note for p119: Restatement of theorem. $C_{S,T}$ is zero, unless whenever $i \in S \cap T$ and $i+1 \notin S \cap T$, then $i+1 \notin S \cup T$, in which case $C_{S,T}$ is the number of subsets of N_n disjoint from $S \cup T$. Here, with the exception that $N_n = S \cup T$ ~~put~~ $S + T$ $c = 0$.

$$\varphi_S \varphi_T = \varphi_i^2 \varphi_{S-\{i\}} \varphi_{T-\{i\}}$$

But $\varphi_i^2 = 2\varphi_0 + \varphi_{i+1}$ so

$$\varphi_S \varphi_T = 2\varphi_{S-\{i\}} \varphi_{T-\{i\}} + \varphi_{i+1} \varphi_{S-\{i\}} \varphi_{T-\{i\}}$$

By induction the first term does not involve φ_N since $(S-\{i\}) \cup (T-\{i\}) \subsetneq N$. We can again apply induction, using $S-\{i\} \cup \{i+1\}$ and $T-\{i\}$, or the other way, provided $i+1 \notin S \cap T$. Hence, we may assume $i+1 \in S \cap T$.

In fact, we may assume, as neither S nor T is N_n, that $i, i+1, \ldots, i+j \in S \cap T$, $i+j+1 \notin S \cap T$. But

$$\varphi_i^2 \cdot \varphi_{i+1}^2 \cdots \varphi_j^2 = (2\varphi_0 + \varphi_{i+1}) \cdots (2\varphi_0 + \varphi_{i+j+1})$$

we can clearly clearly now apply the above type of argument.

Let $\mathcal{L}^*$ be the set of non principal linear characters of Z_{2^n+1}, $\mathcal{M}^*$ similarly for Z_{2^n-1} with typical elements λ, μ respectively, not the λ, μ above.

For $\lambda \in \mathcal{L}^*$ let (φ_S, λ) be the multiplicity of λ in the restriction of the irreducible φ_S, $S \subseteq N$, to Z_{2^n+1} and define (φ_S, μ) similarly. Hence, next by Nakayama's result (adjoint law) have, if $P_\lambda = \lambda \otimes_{FZ_{2^n+1}} FG$ and P_μ are the induced projective modules, P_S is the projective indecomposable corresponding to V_S,

$$P_\lambda = \sum_{\emptyset \subset S \subseteq N} (\varphi_S, \lambda) P_S$$

$$P_\mu = \sum_{\emptyset \subset S \subseteq N} (\varphi_S, \mu) P_S$$

while similarly by Lemma 2, if λ^0, μ^0 are the corresponding principal characters

$$P_{\lambda^0} = P_\emptyset$$
$$P_{\mu^0} = P_\emptyset + 2P_N$$

(just as on page 107)

<u>Lemma 4</u> We have that $\sum_{\lambda \in \mathcal{L}^*} (\varphi_S, \lambda)(\varphi_T, \lambda)$ is the multiplicity of φ_0 in $\varphi_S \varphi_T$ and that $\sum_{\mu \in \mathcal{M}^*} (\varphi_S, \mu)(\varphi_T, \mu)$ is the multiplicity of φ_0 in $\varphi_S \varphi_T$ plus twice the multiplicity of φ_N.

<u>Proof</u>. But $(\varphi_S, \lambda) = (\varphi_S, \lambda^{-1})$ so $\sum_{\lambda \in \mathcal{L}^*} (\varphi_S, \lambda)(\varphi_T, \lambda)$ is the multiplicity of λ^0 in $\varphi_S \varphi_T$. By Lemma 2 we have the first statement. The second similarly follows from Lemma 2.

<u>Lemma 5</u>. If $\emptyset \subseteq S \subset N$ then

$$2P_S + \sum_{\lambda \in \mathcal{L}^*} (\varphi_{\bar{S}}, \lambda) P_\lambda = \sum_{\mu \in \mathcal{M}^*} (\varphi_{\bar{S}}, \mu) P_\mu .$$

<u>Proof</u>. The left hand side is

$$2P_S + \sum_{\lambda \in \mathcal{L}^*} (\varphi_{\bar{S}}, \lambda) \sum_T (\varphi_T, \lambda) P_T$$

$$= 2P_S + \sum_T \Big(\sum_{\lambda \in \mathcal{L}^*} (\varphi_{\bar{S}}, \lambda)(\varphi_T, \lambda) \Big) P_T$$

$$= 2P_S + \sum_T (\text{mult of } \varphi_0 \text{ in } \varphi_{\bar{S}} \varphi_T) P_T$$

On the other hand

$$\sum_{\mu \in m^*} (\varphi_{\bar{S}}, \mu)\, P_\mu = \sum_{\mu \in m^*} (\varphi_{\bar{S}}, \mu) \sum_T (\varphi_T, \mu)\, P_T$$

$$= \sum_T \Big(\sum_{\mu \in m^*} (\varphi_{\bar{S}}, \mu)(\varphi_T, \mu) \Big) P_T$$

$$= \sum_T \big(\text{mult of } \varphi_0 \text{ in } \varphi_{\bar{S}} \varphi_T + 2 \cdot \text{mult of } \varphi_N \text{ in } \varphi_{\bar{S}} \varphi_T \big)\, P_T$$

$$= \sum_T \big(\text{mult of } \varphi_0 \text{ in } \varphi_{\bar{S}} \varphi_T \big) P_T + 2 P_S ,$$

since when $T \subset N$ we have, by Lemma 3, $\varphi_{\bar{S}} \varphi_T$ does not involve φ_N unless $\bar{S} = T$. It remains to show that if $0 \subset S \subset N$ then $\varphi_S \varphi_N$ does not involve φ_N. That is,

<u>Lemma 6</u> If $0 \subset S \subset N$ then $\varphi_S \varphi_N$ does not involve φ_N.

<u>Proof</u> It suffices to show $\mathrm{Hom}(V_N, V_S \otimes V_N) = 0$ that is, $\mathrm{Hom}(V_N \otimes V_N, V_S) = 0$. But claim: $V_N \otimes V_N \simeq P_\emptyset \oplus V_N$. The dimensions are right, as $P_\emptyset$ has dimension $2^n(2^n - 1)$, by p 107. Also $V_N \otimes V_N$ is projective and $\mathrm{Hom}(V_N \otimes V_N, V_\emptyset) \simeq \mathrm{Hom}(V_N, V_N) \simeq F$
$\therefore$ enough to see that V_N also appears in $V_N \otimes V_N$. But

$$(\varphi_1 \varphi_2 \cdots \varphi_n)^2 = (2\varphi_0 + \varphi_2) \cdots (2\varphi_0 + \varphi_{n-1})(2\varphi_0 + \varphi_1)$$
$$= 2^n \varphi_0 + \cdots + \varphi_1 \varphi_2 \cdots \varphi_n .$$

<u>Corollary</u> $C_{\emptyset S}$ is as claimed.

Proof This follows immediately from the expansion of $(\varphi_1 \cdots \varphi_n)^2$, as in the previous argument.

Lemma 7 If $\phi \subset S, T \subset N$ then

$$C_{S,T} = \begin{cases} 1, & \text{if } \bar{S} = T \\ \text{multiplicity of } \varphi_0 \text{ in } \varphi_{\bar{S}} \otimes \varphi_{\bar{T}}, & \text{otherwise.} \end{cases}$$

Proof Let $\mathcal{L}^+$, m^+ be subsets of $\mathcal{L}^*$, m^*, one element from each orbit in $\mathcal{L}^*$, m^* under $N(Z_{2^{n+1}})$, $N(Z_{2^{n-1}})$, respectively. Now $C_{S,T} = \dim_F \mathrm{Hom}_{FG}(P_S, P_T)$. Now can calculate this in the Green ring, using the distributive nature for Hom. For

$$P_S = \sum_{\mu \in m^+} (\varphi_{\bar{S}}, \mu) P_\mu - \sum_{\lambda \in \mathcal{L}^+} (\varphi_{\bar{S}}, \lambda) P_\lambda$$

$$P_T = \sum_{\tilde{\mu} \in m^+} (\varphi_{\bar{T}}, \tilde{\mu}) P_{\tilde{\mu}} - \sum_{\tilde{\lambda} \in \mathcal{L}^+} (\varphi_{\bar{T}}, \tilde{\lambda}) P_{\tilde{\lambda}}$$

so

$$C_{S,T} = \sum_{\mu, \tilde{\mu}} (\varphi_{\bar{S}}, \mu)(\varphi_{\bar{T}}, \tilde{\mu}) \dim_F \mathrm{Hom}_{FG}(P_\mu, P_{\tilde{\mu}})$$

$$- \sum_{\mu, \tilde{\lambda}} (\varphi_{\bar{S}}, \mu)(\varphi_{\bar{T}}, \tilde{\lambda}) \dim_F \mathrm{Hom}_{FG}(P_\mu, P_{\tilde{\lambda}})$$

$$- \sum_{\lambda, \tilde{\mu}} (\varphi_{\bar{S}}, \lambda)(\varphi_{\bar{T}}, \tilde{\mu}) \dim_F \mathrm{Hom}_{FG}(P_\lambda, P_{\tilde{\mu}})$$

$$+ \sum_{\lambda, \tilde{\lambda}} (\varphi_{\bar{S}}, \lambda)(\varphi_{\bar{T}}, \tilde{\lambda}) \dim_F \mathrm{Hom}_{FG}(P_\lambda, P_{\tilde{\lambda}})$$

so, by pages 109–10,

$$= 2^n \Big(\sum_{\mu \in \mathcal{M}^+} (\varphi_{\bar{S}}, \mu) - \sum_{\lambda \in \mathcal{L}^+} (\varphi_{\bar{S}}, \lambda) \Big) \Big(\sum_{\tilde{\mu} \in \mathcal{M}^+} (\varphi_{\bar{T}}, \tilde{\mu}) - \sum_{\tilde{\lambda} \in \mathcal{L}^+} (\varphi_{\bar{T}}, \tilde{\lambda}) \Big)$$

$$+ 2 \sum_{\mu, \tilde{\mu}} (\varphi_{\bar{S}}, \mu)(\varphi_{\bar{T}}, \tilde{\mu}) + 2 \sum_{\lambda, \tilde{\lambda}} (\varphi_{\bar{S}}, \lambda)(\varphi_{\bar{T}}, \tilde{\lambda})$$

$$+ \sum_{\mu} (\varphi_{\bar{S}}, \mu)(\varphi_{\bar{T}}, \mu) + \sum_{\lambda} (\varphi_{\bar{S}}, \lambda)(\varphi_{\bar{T}}, \lambda)$$

$$= 2^n \Big(\tfrac{1}{2} \varphi_{\bar{S}}(1) - \tfrac{1}{2} \varphi_{\bar{S}}(1) \Big) \Big(\tfrac{1}{2} \varphi_{\bar{T}}(1) - \tfrac{1}{2} \varphi_{\bar{T}}(1) \Big)$$

$$+ 2 \Big(\sum_{\mu} (\varphi_{\bar{S}}, \mu) \Big) \Big(\sum_{\tilde{\mu}} (\varphi_{\bar{T}}, \tilde{\mu}) \Big) - 2 \Big(\sum_{\lambda} (\varphi_{S}, \lambda) \Big) \Big(\sum_{\tilde{\lambda}} (\varphi_{\bar{T}}, \tilde{\lambda}) \Big)$$

$+ \frac{1}{2}$ mult of φ_0 in $\varphi_{\bar{S}} \varphi_{\bar{T}}$ $+ \frac{1}{2}$ mult of φ_0 in $\varphi_{\bar{S}} \varphi_{\bar{T}}$ + mult φ_N

$= 2^n \cdot 0 + 2 \cdot \frac{1}{2} \varphi_{\bar{S}}(1) \frac{1}{2} \varphi_{\bar{T}}(1) - 2 \frac{1}{2} \varphi_{\bar{S}}(1) \frac{1}{2} \varphi_{\bar{T}}(1)$ + mult of φ_0 in $\varphi_{\bar{S}} \varphi_{\bar{T}}$ + mult of φ_N

= mult of φ_0 in $\varphi_{\bar{S}} \varphi_{\bar{T}}$ + mult of φ_N in $\varphi_{\bar{S}} \varphi_{\bar{T}}$, as desired.

<u>Lemma 8</u> If $\emptyset \subset S, T \subset N$ then the multiplicity $m_{S,T}$ of φ_0 in $\varphi_S \varphi_T$ is given as follows:

$$m_{S,T} = \begin{cases} 2^{|S \cap T|}, & \text{if } i \notin S \cup T \text{ and } i+1 \in S \cup T, \text{ then } i+1 \in S \cap T \text{ or } \bar{S} = \bar{T} \\ 0, & \text{otherwise} \end{cases}$$

<u>Remark</u> This result proves the theorem. For already have it except for $c_{S,T}$, $\emptyset \subset S, T \subset N$ and $\bar{S} \neq T$. We know $c_{S,T} = m_{\bar{S},T}$ in this case. Hence

$$c_{S,T} = \begin{cases} 2^{|\bar{S} \cap T|}, & \text{under the right conditions} \\ 0, & \text{otherwise} \end{cases}$$

First, let's give a preliminary result. Assume that

$$X = \{ i, i+1, \dots, i+j \}$$
$$Y = \{ i+j+1, \dots, i+j+k \}$$

and set

$$Y_0 = \emptyset$$
$$Y_1 = Y = \{ i+j+1, \dots, i+j+k \}$$
$$Y_2 = \{ i+j+2, \dots, i+j+k \}$$
$$\ddots$$
$$Y_k = \{ i+j+k \}.$$

Then we have

Lemma 9 $$\varphi_X^2 \varphi_Y = \sum_{W \subseteq X - \{i\}} \left(2^{|X|-|W|-1} \varphi_{W \cup \{y_{i+j+k+1}\}} + 2^{|X|-|W|} \sum_{l=0}^{k} \varphi_{W \cup Y_l} \right)$$

Proof. First

$$\varphi_X^2 = \varphi_i^2 \varphi_{i+1}^2 \cdots \varphi_{i+j}^2 = (2\varphi_0 + \varphi_{i+1}) \cdots (2\varphi_0 + \varphi_{i+j+1})$$
$$= \sum_{W \subseteq X - \{i\}} 2^{|X|-|W|} \varphi_W + 2^{|X|-|W|-1} \varphi_W \varphi_{i+j+1}.$$

Hence

$$\varphi_X^2 \varphi_Y = \sum_{W \subseteq X - \{i\}} 2^{|X|-|W|} \varphi_{W \cup Y} + 2^{|X|-|W|-1} \varphi_W \varphi_{i+j+1} \varphi_Y$$

But

$$\varphi_{i+j+1} \varphi_Y = \varphi_{i+j+1}^2 \varphi_{i+j+2} \cdots \varphi_{i+j+k}$$
$$= 2\varphi_{i+j+2} \cdots \varphi_{i+j+k} + \varphi_{i+j+2}^2 \varphi_{i+j+3} \cdots \varphi_{i+j+k}$$
$$= 2\varphi_{Y_2} + 2\varphi_{Y_3} + \cdots + 2\varphi_{Y_k} + 2\varphi_0 + \varphi_{i+j+k+1}.$$

Thus,

$$\varphi_X^2 \varphi_Y = \sum_{W \subseteq X - \{i\}} 2^{|X|-|W|} \varphi_{W \cup Y_1} + 2^{|X|-|W|-1} \varphi_W \left(2\varphi_{Y_2} + \dots + 2\varphi_{Y_k} + 2\varphi_0 + \varphi_{\{i+j+k+1\}} \right)$$

$$= \sum_{W \in X - \{i\}} 2^{|X|-|W|-1} \varphi_{W \cup \{\varphi_{i+j+k+1}\}} + 2^{|X|-|W|} \left(\sum_{l=0}^{k} \varphi_{W \cup Y_l} \right)$$

<u>Corollary</u> $\varphi_X^2 \varphi_Y$ has φ_0 with multiplicity $2^{|X|}$ and all other φ_S involved are such that $S \subseteq \{i+1, \dots, i+j+k+1\}$.

<u>Proof</u> (of <u>Lemma 8</u>). Let $X = S \cap T$ and $Y = S \cup T - S \cap T$ so that $\varphi_S \varphi_T = \varphi_X^2 \varphi_Y$. Express $X = X_1 \cup \dots \cup X_n$ the union of disjoint segments (note $X \subset N$ so this makes sense; circular ordering as usual). Then can express $Y = Y_1 \cup \dots \cup Y_n \cup Z$ where Y_i is a segment following X_i and no element of Z can be added to Y_i, any i, to give a bigger segment. Thus,

$$\varphi_X^2 \varphi_Y = \left(\prod_i \varphi_{X_i}^2 \varphi_{Y_i} \right) \varphi_Z$$

$$= \left(\prod_i \left(2^{|X_i|} \varphi_0 + \dots \right) \right) \varphi_Z$$

where the dotted terms are φ_S with disjoint S from one sum to the other and Z disjoint from everything. The result is now clear.

Theorem If $\emptyset \subset S \subseteq N$ then $P_S \simeq V_{\bar{S}} \otimes V_N$.

(see note at bottom of page 131 for short proof)

We have seen before that $V_N \otimes V_N \simeq P_\emptyset \oplus V_N$ - at least the argument gives this (see page 124, lemma 6). If $S = N$ then $V_{\bar{S}} = V_\emptyset$ and $V_S \otimes V_N \simeq V_N$; hence we may assume $S \subset N$ so also $\emptyset \subset \bar{S} \subset N$. Now V_N is projective so $V_{\bar{S}} \otimes V_N$ is also. Moreover, $\mathrm{Hom}_{FG}(V_{\bar{S}} \otimes V_N, V_S) = \mathrm{Hom}_{FG}(V_N, V_{\bar{S}} \otimes V_S)$ $\simeq \mathrm{Hom}(V_N, V_N) \simeq F$ so P_S is a component of $V_{\bar{S}} \otimes V_N$; it suffices, therefore, to show that the multiplicity of V_T, $\emptyset \subseteq T \subseteq N$ as composition factor of $V_{\bar{S}} \otimes V_N$ equals $C_{S,T}$ as the latter also equals the multiplicity of V_T as composition factor of P_S.

But we have that $\sum_{\emptyset \subset S} 2^{|S|}(V_{\bar{S}} \otimes V_N)$ contains as a summand $\sum_{\emptyset \subset S} 2^{|S|} P_S$. Hence, to conclude the proof it suffices to show that $P_\emptyset \oplus \sum_{\emptyset \subset S} 2^{|S|}(V_{\bar{S}} \otimes V_N)$ and the regular representation have the same composition factors. But the composition factors of $P_\emptyset$ are $2^{|T|} V_{\bar{T}}$ for $\emptyset \subset T$ so it suffices to show that $P_\emptyset \oplus (P_\emptyset \otimes V_N)$ is the regular representation. But

$$P_\emptyset \otimes V_N \simeq (\lambda^0 \otimes_{FZ_{2^n+1}} FG) \otimes V_N$$

$$\simeq (\lambda^0 \otimes_{FZ_{2^n+1}} V_N | Z_{2^n+1}) \otimes_{FZ_{2^n+1}} FG$$

But $V_N | Z_{2^n+1} + \lambda^0$ is the regular representation of Z_{2^n+1} by inspection of values. Hence

$$\lambda^0 \otimes_{FZ_{2^n+1}} FG \oplus (P_\emptyset \otimes V_N) \simeq FZ_{2^n+1} \otimes_{FZ_{2^n+1}} FG$$

$$P_\emptyset \oplus (P_\emptyset \otimes V_N) \simeq FG,$$

as required.

Now we go on to the structure of the projectives

<u>Lemma</u> $V_i \otimes V_i$ is uniserial with composition factors V_0, V_{i+1}, V_0.

<u>Proof</u> We know that $V_i \otimes V_i \doteq V_0 \oplus V_0 \oplus V_{i+1}$ as $\varphi_i^2 = 2\varphi_0 + \varphi_{i+1}$.
Also $Hom(V_i \otimes V_i, V_0) \simeq Hom(V_i, V_i) \simeq F$ and $Hom(V_0, V_i \otimes V_i) \simeq F$.
Finally, $Hom(V_i \otimes V_i, V_{i+1}) \simeq Hom(V_i, V_i \otimes V_{i+1}) \simeq Hom(V_i, V_{\{i,i+1\}}) = 0$
and similarly $Hom(V_{i+1}, V_i \otimes V_i) = 0$.

<u>Proposition</u> The projective $P_{N-\{n\}}$ is <u>uniserial</u> with composition factors

$$V_{\{1,2,\dots,n-1\}}, V_{\{2,\dots,n-1\}}, \dots, V_{n-1}, V_0, V_n, V_0, V_{n-1}, \dots, V_{\{2,\dots,n-1\}}, V_{\{1,\dots,n-1\}}$$

<u>Proof</u> We know by the previous theorem that $P_{N-\{n\}} \simeq V_n \otimes V_N$.
But $V_n \otimes V_N \simeq V_n \otimes V_n \otimes V_{N-\{n\}}$ and $V_n \otimes V_n$ is uniserial with composition factors V_0, V_1, V_0. Since $P_{N-\{n\}}$ has an irreducible socle and only one irreducible quotient we are reduced to showing that $V_1 \otimes V_{N-\{n\}}$ is uniserial with composition factors

$$V_{\{2,\dots,n-1\}}, V_{\{3,\dots,n-1\}}, \dots, V_{n-1}, V_0, V_n, V_0, V_{n-1}, \dots, V_{\{2,\dots,n-1\}}.$$

Hence, we are done, by the lemma above and the following result, applying an obvious induction. (Will use $V_1 \otimes V_{N-\{n\}} \simeq V_1 \otimes V_1 \otimes V_{\{2,\dots,n-1\}}$ to get down to looking at $V_2 \otimes V_{\{2,\dots,n-1\}}$ and so forth down to $V_{n-1} \otimes V_{n-1}$ which is uniserial with composition factors V_0, V_n, V_0.)

<u>Lemma</u> If $1 \le i < j < n$ then $V_i \otimes V_{\{i, i+1, \dots, j\}}$ has an irreducible socle and a unique irreducible quotient, both isomorphic to $V_{\{i, \dots, j\}}$. Hence, the resulting "middle" is isomorphic with $V_{i+1} \otimes V_{\{i+1, \dots, j\}}$.

<u>Proof</u> We know that $V_i \otimes V_i$ is uniserial with composition factors $V_\emptyset, V_{i+1}, V_\emptyset$ so, as $V_i \otimes V_{\{i, \dots, j\}} \simeq V_i \otimes V_i \otimes V_{\{i+1, \dots, j\}}$ it follows that $V_i \otimes V_{\{i, \dots, j\}}$ has a series with factors $V_{\{i+1, \dots, j\}}, V_{i+1} \otimes V_{\{i+1, \dots, j\}}, V_{\{i+1, \dots, j\}}$. Thus we need only prove, that if $\emptyset \subseteq S \subseteq N$ then

$$\mathrm{Hom}(V_i \otimes V_{\{i, \dots, j\}}, V_S) \simeq \mathrm{Hom}(V_S, V_i \otimes V_{\{i, \dots, j\}}) \simeq \begin{cases} F, & \text{if } S = \{i, \dots, j\} \\ 0, & \text{if } S \neq \{i, \dots, j\}. \end{cases}$$

But $\mathrm{Hom}(V_i \otimes V_{\{i, \dots, j\}}, V_S) \simeq \mathrm{Hom}(V_{\{i, \dots, j\}}, V_i \otimes V_S)$.
Now, if $i \notin S$ then $V_i \otimes V_S = V_{\{i\} \cup S}$ so the result is true in this case. However, if $i \in S$, then as $j < n$, no composition factor V_T of $V_i \otimes V_S$ has $i \in T$. Again our claim holds. The argument for $\mathrm{Hom}(V_S, V_i \otimes V_{\{i, \dots, j\}})$ is entirely similar.

<u>Note for p129</u> V_N projective yields $V_{\bar{S}} \otimes V_N$ projective.
Now $\mathrm{Hom}(V_{\bar{S}} \otimes V_N, V_T) \simeq \mathrm{Hom}(V_N, V_{\bar{S}} \otimes V_T)$. But V_N is constituent of $V_{\bar{S}} \otimes V_T$ if and only if $\bar{S} \cup T = N$, $\bar{S} \cap T = \emptyset$, (see page 124, lemma 6). Thus, $V_{\bar{S}} \otimes V_N$ has unique irreducible quotient, namely V_S, and is projective. Done!

Lemma If $1 \le i \le n$ then $V_i \otimes V_i \otimes V_i$ is semi-simple with composition factors V_i, $V_{\{i,i+1\}}$, V_i

Proof We know that $V_i \otimes V_i$ is uniserial with factors V_0, V_{i+1}, V_0 so $V_i \otimes V_i \otimes V_i$ has a series with factors V_i, $V_{\{i,i+1\}}$, V_i. Also $Hom(V_i \otimes V_i \otimes V_i, V_i) \simeq Hom(V_i \otimes V_i, V_i \otimes V_i) \simeq F \oplus F$ and also $Hom(V_i, V_i \otimes V_i \otimes V_i) \simeq F \oplus F$

Lemma If $j \neq i, i+1$ then $V_{j-1} \otimes V_{j-1} \otimes V_i$ is uniserial with composition factors V_i, $V_{\{i,j\}}$, V_i

Proof Now $V_{j-1} \otimes V_{j-1}$ has factors V_0, V_j, V_0 so $V_{j-1} \otimes V_{j-1} \otimes V_i$ has factors V_i, $V_{\{i,j\}}$, V_i. Moreover, $Hom(V_{j-1} \otimes V_{j-1} \otimes V_i, V_i) \simeq F$ as $j \neq i+1$. Similarly, $Hom(V_i, V_{j-1} \otimes V_{j-1} \otimes V_i) \simeq F$. Finally,

$$Hom(V_{j-1} \otimes V_{j-1} \otimes V_i, V_{\{i,j\}}) \simeq Hom(V_{\{j-1,i\}}, V_{\{j-1,j,i\}}) = 0$$

Remark: the last three results give us the structure of all $V_i \otimes V_j \otimes V_k$

Lemma $Ext(F, V_i) \simeq F$, $1 \le i \le n$.

Proof Since $V_{i-1} \otimes V_{i-1}$ is uniserial with composition factors $V_0 = F$, V_i, V_0 it suffices to prove that $\dim_F Ext(F, V_i) \le 1$. By automorphisms, it suffices to do this for $i = 1$. Suppose false and let M be an extension of $V_1 \oplus V_1$ by F with $Hom(M, V_1) = 0$. It suffices to prove that $Hom(M \otimes V_{\{2,\dots,n-1\}}, V_{\{1,\dots,n-1\}}) = 0$ as then we would have $\dim_F Ext(V_{\{2,\dots,n-1\}}, V_{\{1,\dots,n-1\}}) \ge 2$, contradicting the structure of $P_{\{1,\dots,n-1\}}$.

We shall prove that $\mathrm{Hom}(M \otimes V_{\{2,\ldots,i\}}, V_{\{1,\ldots,i\}}) = 0$ for $2 \le i \le n-1$, by induction on i. We are assuming $n \geq 3$ as otherwise the result is known. Now when $i=2$,

$$\mathrm{Hom}(M \otimes V_2, V_{12}) \simeq \mathrm{Hom}(M, V_{12} \otimes V_2)$$

but $V_{12} \otimes V_2$ has no composition factor isomorphic to F so we are done in this case as $M/\mathrm{Rad}\, M \simeq F$.

Similarly, if the result holds for i, then $M \otimes V_{\{2,\ldots,i\}}$ modulo its radical is $V_{\{2,\ldots,i\}}$. Then, if $i+1 < n$,

$$\mathrm{Hom}(M \otimes V_{\{2,\ldots,i+1\}}, V_{\{1,\ldots,i+1\}})$$
$$\simeq \mathrm{Hom}(M \otimes V_{\{2,\ldots,i\}}, V_{\{1,\ldots,i+1\}} \otimes V_{\{i+1\}})$$

while $V_{\{1,\ldots,i+1\}} \otimes V_{i+1}$ has no composition factor isomorphic with $V_{\{2,\ldots,i\}}$.

(Note: D.D. Higman states he knew this when he did his work on H^1; can do by direct calculation.)

Lemma $\mathrm{Ext}(V_{\{i,\ldots,j\}}, V_{\{i,\ldots,j+1\}}) \simeq F$ if $|\{i,\ldots,j\}| \le n-2$.

Proof Can assume $i=1$ by automorphism. Get $\dim \mathrm{Ext}\, F \geq 1$ by using above technique applied to a uniserial module with factors V_0, V_1. If $\dim_F \mathrm{Ext} \geq 2$ construct appropriate module and, as above, get contradiction to structure of $P_{\{1,\ldots,n-1\}}$.

<u>Proposition</u> If $X, Y \subseteq N$ then

$$Ext(V_X, V_Y) = 0,$$

unless $|X \cap Y| + 1 = |X \cup Y| < n$ and $i \in X \cup Y - X \cap Y$ implies $i-1 \notin X \cup Y$ in which case

$$Ext(V_X, V_Y) \simeq F.$$

<u>Proof</u> Let $T = X \cap Y$, $S = X \cup Y$ so $Ext(V_X, V_Y) = Ext(V_T, V_S)$, inasmuch as $Ext(V_X, V_Y) \cong Ext(V_{X \cap Y} \otimes V_{X - X\cap Y}, V_{X \cap Y} \otimes V_{Y - X \cap Y})$

$$= Ext(V_{X \cap Y}, V_{(X - X\cap Y) \cup (X \cap Y) \cup (Y - X \cap Y)}),$$

also $S \cap T = X \cap Y$, $S \cup T = X \cup Y$ so the result holds for X and Y if and only if it does for T and S. The point is that $T \subseteq S$. We may also assume that $S \subset N$, inasmuch as V_N is projective.

Now suppose T and S satisfy the conditions described in the proposition supposed to guarantee a non-zero $Ext(V_T, V_S)$. Let $i \in S$, $i \notin T$ so $S = \{i\} \cup T$ and, by hypothesis $i-1 \notin S \cup T$. Thus, $i-1 \notin S \cup T$, $i \in S \cup T$ and $i \notin S \cap T$, so by Lemma 8, page 126, $\mathbb{F}$ is <u>not</u> a composition factor of $V_T \otimes V_S$. Hence, if M is a non-split extension of V_i by F then

$$Hom\left(\tfrac{F}{V_i}, V_T \otimes V_S\right) = 0$$

so

$$Hom\left(\tfrac{F}{V_i} \otimes V_T, V_S\right) = 0$$

and $\frac{F}{V_i} \otimes V_T$ is a non-split extension of $V_{\{i\} \cup T} = V_S$ by V_T.

Next, suppose M is an extension of a sum of copies of V_S by V_T with $Hom(M, V_S) = 0$ and M violating the theorem. (Thus, one copy of V_S normally and two under the special circumstances) We shall use the technique done above. We also proceed by downward induction as the theorem holds when

$|S| = n-1$ by the structure of P_S.

Choose $i \notin S$ with, however, $i-1 \in S$. Since $|S| < n-1$ it follows that $T \cup \{i\}$, $S \cup \{i\}$ satisfy the same conditions as T and S do. Thus, it suffices to prove that

$$\mathrm{Hom}(M \otimes V_i, V_{S \cup \{i\}}) = 0$$

as then $M \otimes V_i$ will violate the induction assumption. But

$$\mathrm{Hom}(M \otimes V_i, V_{S \cup \{i\}}) \simeq \mathrm{Hom}(M, V_{S \cup \{i\}} \otimes V_i)$$

Hence, it suffices to show that $V_{S \cup \{i\}} \otimes V_i$ has no composition factor isomorphic with V_T or if it does then no section which is a non-split extension of S by T. Now if $i+1 \notin S$ then $V_{S \cup \{i\}} \otimes V_i$ has composition factors V_S, $V_{S \cup \{i+1\}}$, V_S and we are done.

Suppose that $i+1, \ldots, j \in S$, $j+1 \notin S$, so $j+1 \neq i$ as $|S| < n-1$. Let $I = \{i, i+1, \ldots, j\}$, so $S = I \cup (S - I)$. We know already the structure of $V_i \otimes V_I$ – it is uniserial with known factors, by the lemma on page 131.

Picture:

$V_i \otimes V_I$:

$V_{I-\{i\}}$
$V_{I-\{i,i+1\}}$
$\vdots$
V_j
F
V_{j+1}
F
V_j
$\vdots$
$V_{I-\{i,i+1\}}$
$V_{I-\{i\}}$

Hence, it suffices to prove that $V_i \otimes V_I \otimes V_{S-I}$ is uniserial. For then if V_T appears want it not "above" V_S. But then will have $T = S - \{i+1\}$ since $I - \{i\} \cup S - I = S$. In this case, we must go back to our choice of i and change it. We can do

this since $|S| \leq n-2$. For there is $j \notin S$, $j \neq i$.
If $j-1 \in S$ use j. If $j-1 \notin S$ and $j-1 \neq i$ can use $j-1$ unless $j-2 \notin S$. Hence can assume $N - S$ is a string $i, i+1, \ldots, k$. But then $i+1 \notin S$ so we would have been done before having to change i.

Now, in proving the uniseriality by induction on $|I|$ we need only prove less since $V_i \otimes V_I \otimes V_{S-I}$ has a series which is V_{S-I}, $V_{i+1} \otimes V_{I-\{i\}} \otimes V_{S-I}$, V_{S-I}, namely that V_S is the unique irreducible submodule and quotient module of the tensor product — i.e. the appropriate Hom's are one-dimensional. Say $U \subseteq N$ so

$$\mathrm{Hom}(V_i \otimes V_I \otimes V_{S-I}, V_U) \simeq \mathrm{Hom}(V_I \otimes V_{S-I}, V_i \otimes V_U)$$
$$\simeq \mathrm{Hom}(V_{S \cup \{i\}}, V_i \otimes V_U)$$

If $U = S$ this is one-dimensional as desired. If $U \neq S$ and $i \notin U$ then we also get the desired result so we can assume $U \neq S$, $i \in U$. Also can assume $U \subset N$ or we're done. Hence $V_i \otimes V_U$ has no composition factor involving "i" and the "Hom" is zero as desired. The argument for

$$\mathrm{Hom}(U, V_i \otimes V_I \otimes V_{S-I}, V_U)$$

is, of course, the same — or if you like a consequence of the preceding lines.

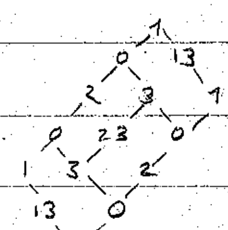
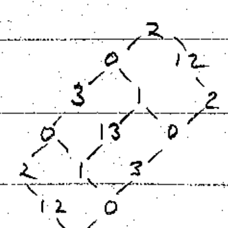
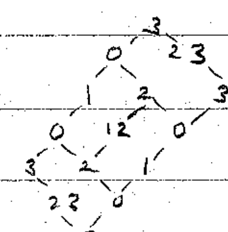
This completes the proof of the Proposition on page 134!!

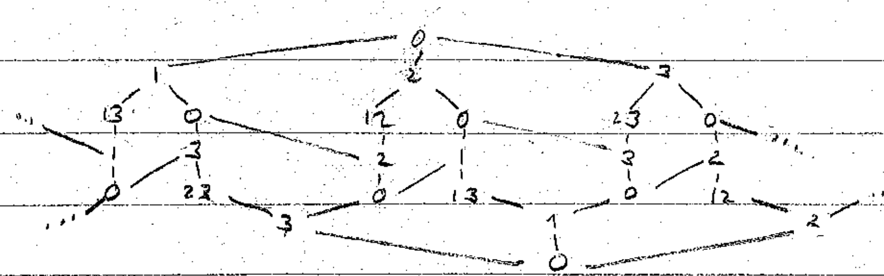
Now can make conjectures on structure of projectives for $SL(2,8)$!

Know the following:

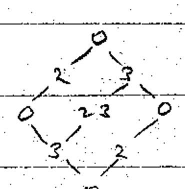
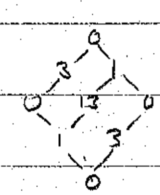
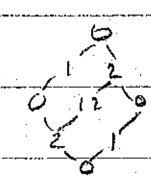
$$P_{12} \simeq \begin{matrix} 12 \\ 1 \\ 0 \\ 3 \\ 0 \\ 2 \\ 12 \end{matrix} \qquad P_{23} \simeq \begin{matrix} 23 \\ 3 \\ 0 \\ 1 \\ 0 \\ 3 \\ 23 \end{matrix} \qquad P_{13} \simeq \begin{matrix} 13 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 13 \end{matrix}$$

Guess:

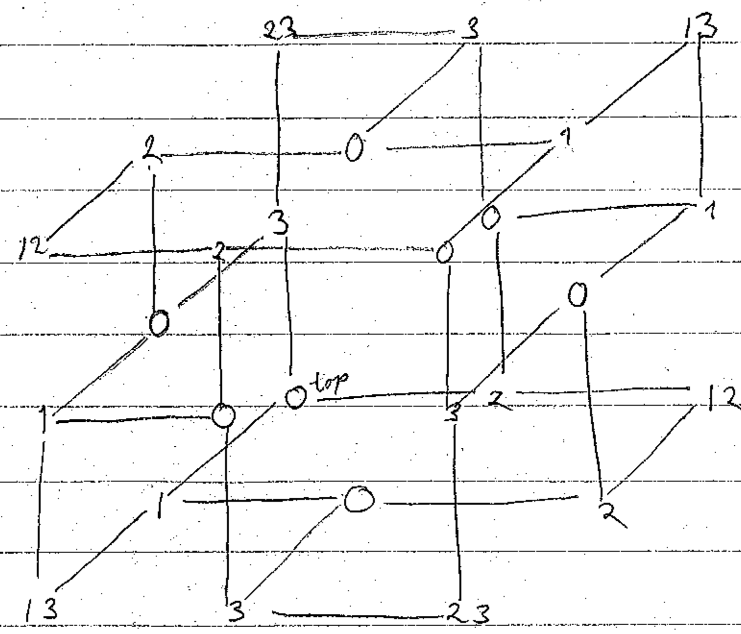
$P_1 \simeq$  ; $P_2 \simeq$  ; $P_3 \simeq$ 

$P_0 \simeq$ 

Probably need to prove following first - if they hold -

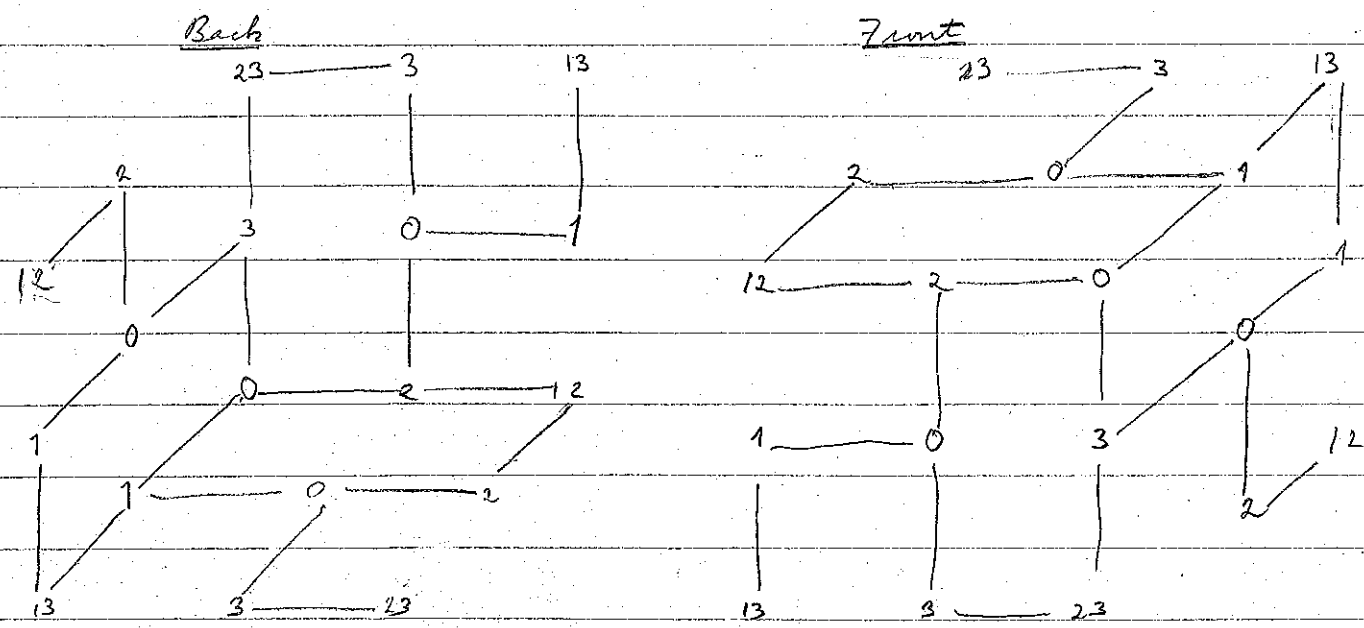
$1 \otimes 1 \otimes 2 \otimes 2 \simeq$  ; $2 \otimes 2 \otimes 3 \otimes 3 \simeq$  ; $1 \otimes 1 \otimes 3 \otimes 3 \simeq$  .

Let's picture our guess for P_0 in 3 dimensions, getting:

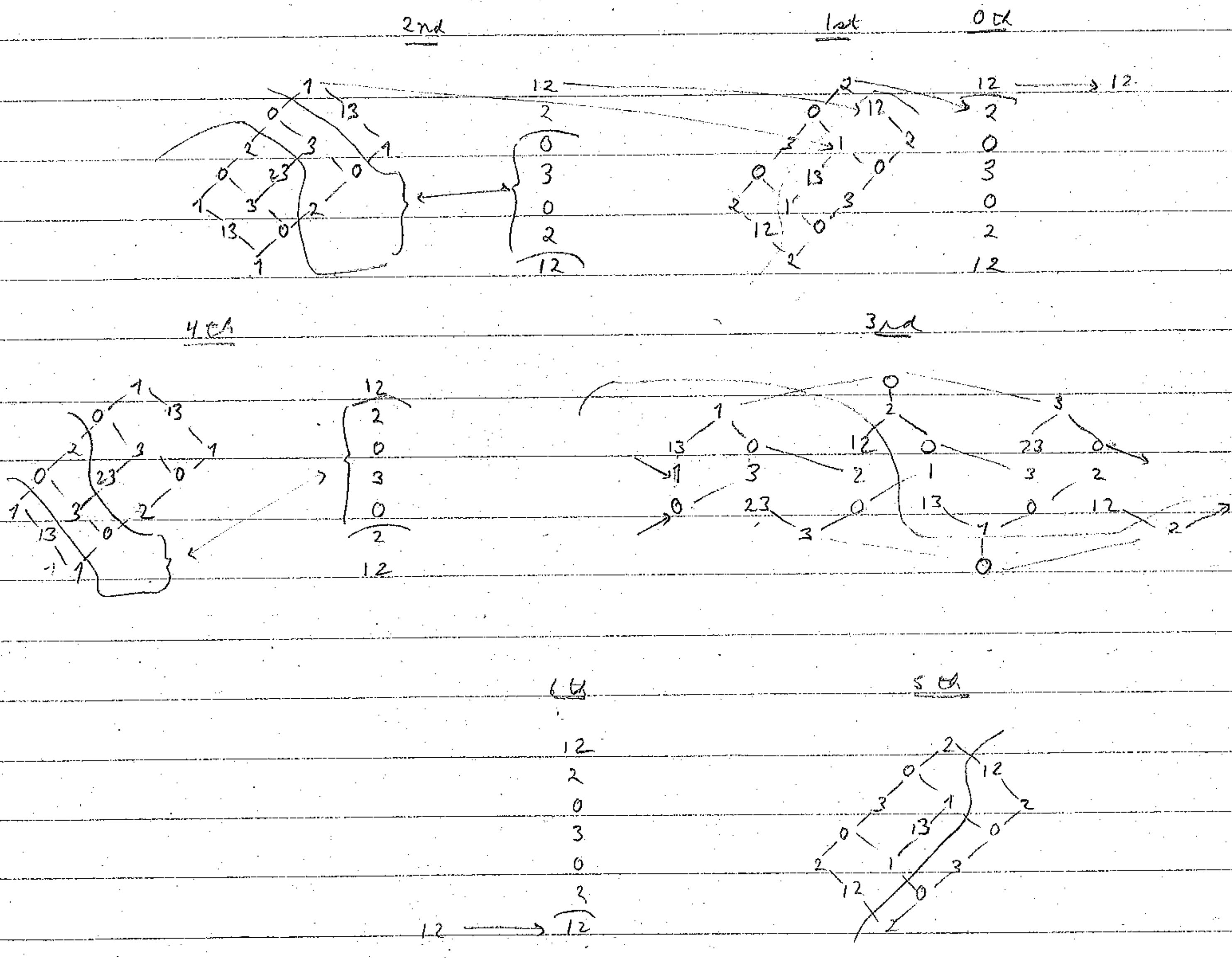


(Could really put 123 in center - disconnected - to get $123 \otimes 123$ and not just P_0)

Let's picture two halves



Next, let's calculate the resolution of V_{12}:



Periodic of period seven!

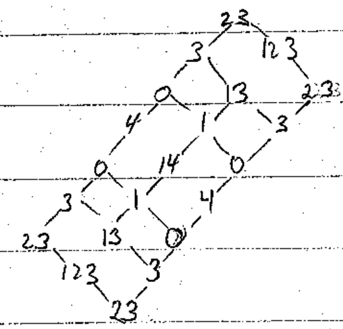
Guess for $L_2(2^n)$: V_{12} n-1 is periodic of period $2^n - 1$.

Let's go on to $L_2(16)$, first guessing some projectives:

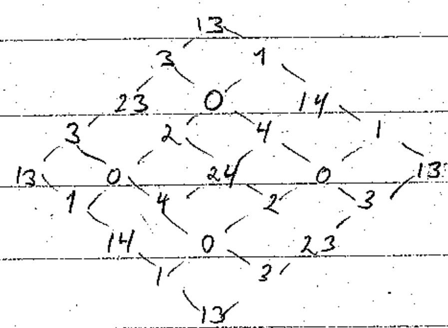
P_{123} :

123
23
3
0
4
0
3
23
123

P_{23} :

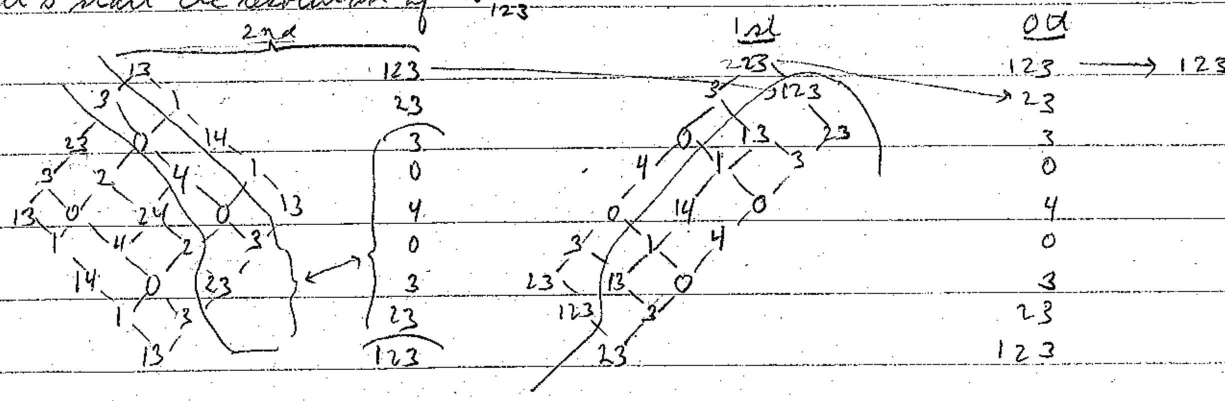


P_{13} :



(All by guessing and using tensor products in various orders.)

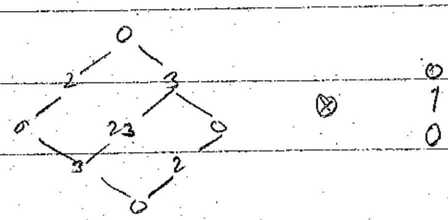
Let's start the resolution of V_{123} :



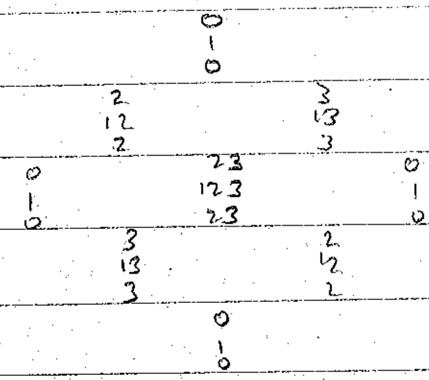
Thus, next term is P_3. Must now determine its structure.

Need some tensor products first.

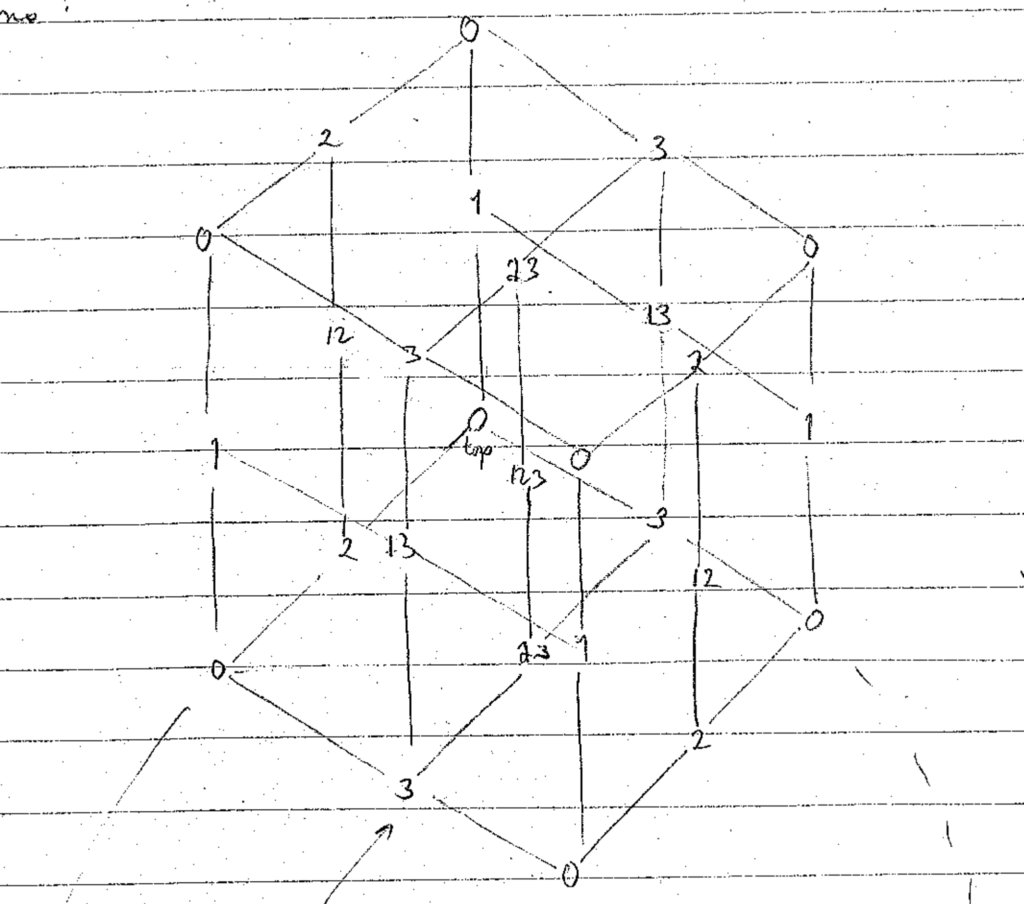
$1 \otimes 1 \otimes 2 \otimes 2 \otimes 4 \otimes 4 =$



$=$

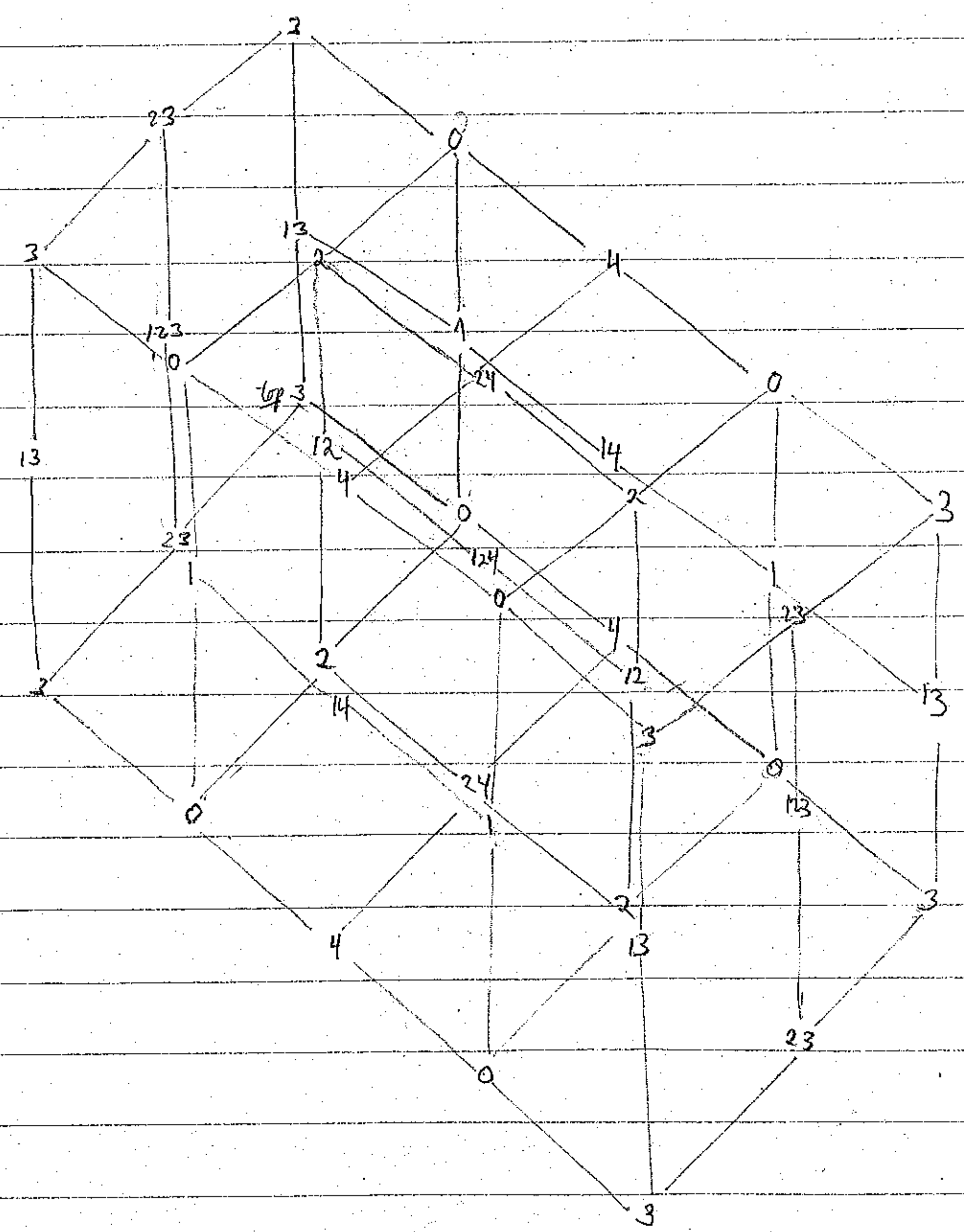


In three dimensions!



This slice $4 \otimes 4 \otimes 1 \otimes 1 \otimes 3$

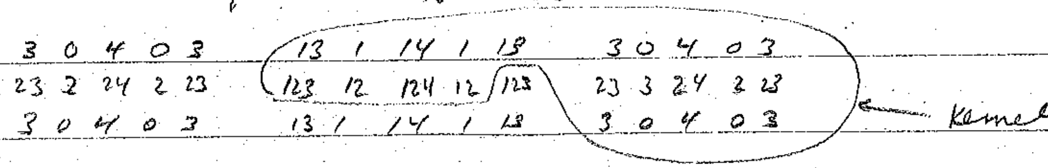
Hence, to get P_3 have to tensor with V_3 so "middle slice" goes to $4\otimes4\otimes1\otimes1\otimes3\otimes3$ which is just this cube with the indices "rotated." The two ends tensor easily with V_3. Only problem is fitting together; trial and error with comparison with other projections gives the following:



Better way to picture it is just write down three levels and then going down is moving right, down the page and to same point on next level:

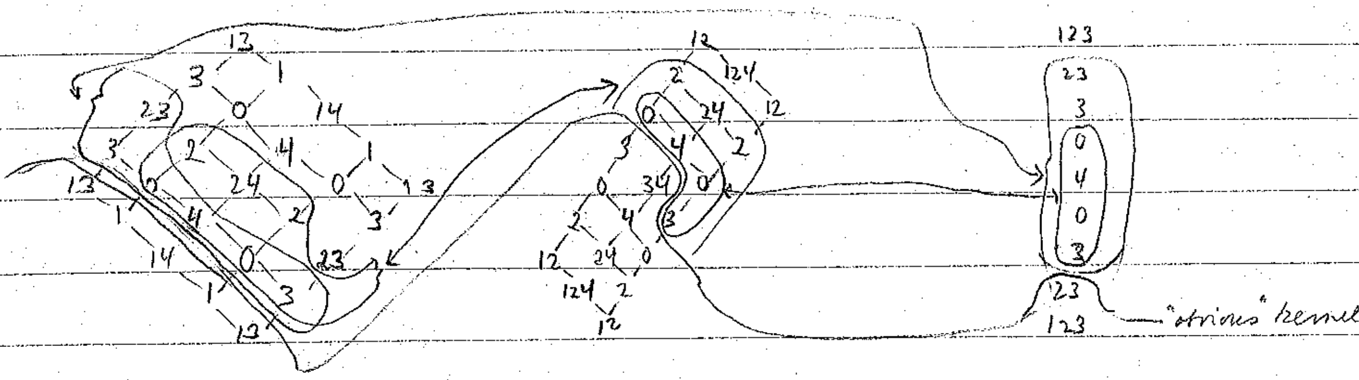
3	0	4	0	3	13	1	14	1	13	3	0	4	0	3
23	2	24	2	23	123	12	124	12	123	23	2	24	2	23
3	0	4	0	3	13	1	14	1	13	3	0	4	0	3

Next, let's compute kernel of $P_3 \to P_{13} \oplus P_{123}$ in resolution. Get:

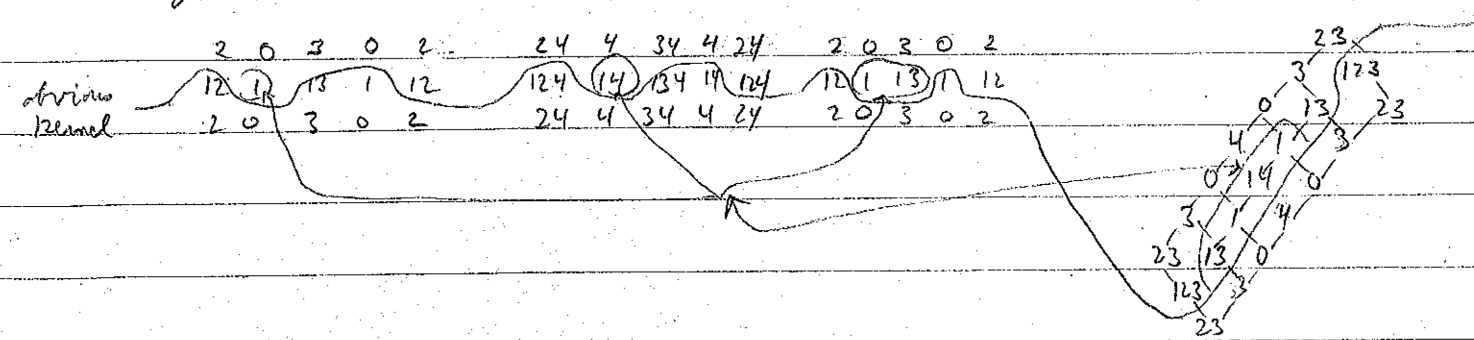


Hence, next term of resolution is $P_{13} \oplus P_{123} \oplus P_{12}$

Let's picture these and compute kernel:



Hence, by careful inspection, see next projective is $P_2 \oplus P_{23}$



Hence, next projective is $P_1 \oplus P_{12} \oplus P_{123}$.

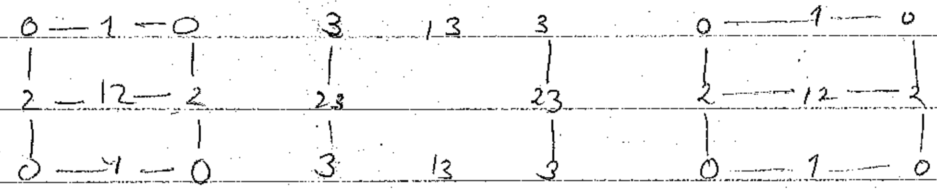
Sequence of projections in resolution:

123
23
13 + 123
3
12 + 13 + 123
2 + 23
1 + 12 + 123
0
↓ reversal

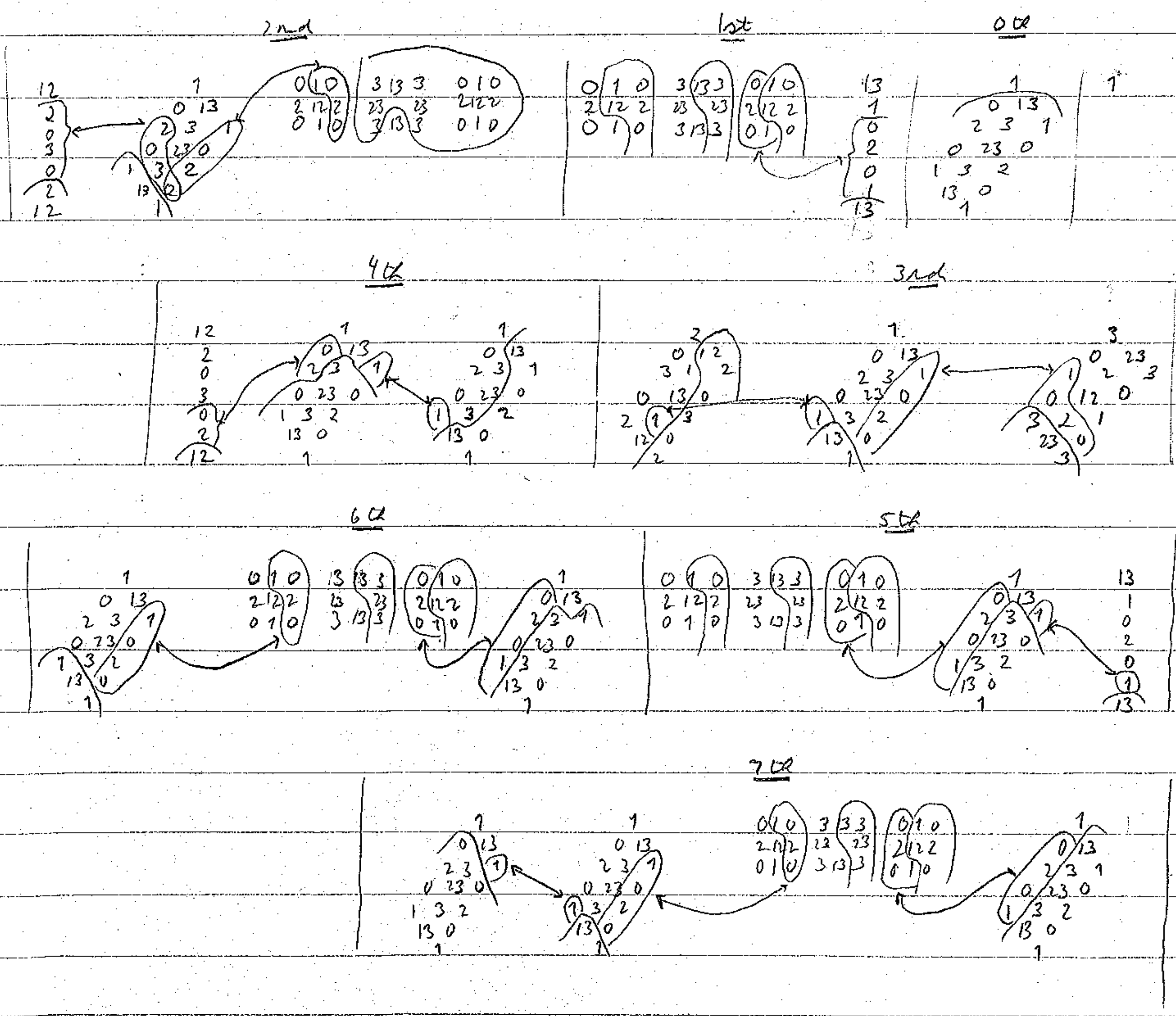
Pattern clear: First bit is prev seq with "3" tacked on and then "3". Then put prev seq plus reversal of part up to "3". Then "0" and reverse everything!! Thus, next case is

1234
234
134 + 1234
34
124 + 134 + 1234
24 + 234
14 + 124 + 1234
4
123 + 14 + 124 + 1234
23 + 24 + 234
13 + 123 + 124 + 134 + 1234
3 + 34
12 + 13 + 123 + 134 + 1234
2 + 23 + 234
1 + 12 + 123 + 1234
0
↓ reverse of above

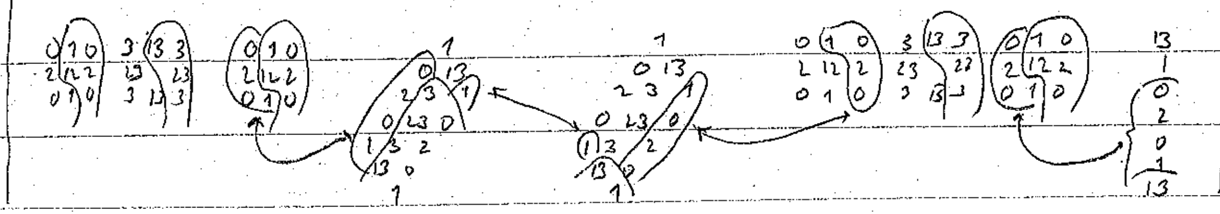
Now let's go back to $L_2(8)$. We have a better picture for P_0, looking at it differently and writing down the three levels, as above:



Using this let's determine next the resolution of V_1:



8th



9th

1 12 0 1 1 0

Let's picture this as a double complex!

1+12

0 0

1 1 1

1 1 1 1

12 0 0 0 0 0

2 1 1 1 1 13

1+12 1 1 13

0 0 3

1 13

Pattern is clear!

Let's return to supposed periodic case. Let's examine the sum of the projectives appearing in a given period. Suppose $n = 2$, so we are considering $L_2(4)$. The sum is

$$P_1 \oplus P_0 \oplus P_1$$

which is (after adding in the Steinberg irreducible)

$$(1 \otimes 2 \otimes 2) \oplus (1 \otimes 2 \otimes 1 \otimes 2) \oplus (1 \otimes 2 \otimes 2)$$

$$= (1 \otimes 2 \otimes 2) \otimes (0 \oplus 1 \oplus 0)$$

$$= (1 \otimes 2 \otimes 2) \otimes (2 \otimes 2)$$

$$= 1 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \quad [\text{"="}\ \underline{1\,2^4}]$$

Let's do $n = 3$. More informally:

Projectives	Tensor products (⊗ Steinberg)
12	3
2	13
1 + 12	23 + 3
0	123 (adding Steinberg)
1 + 12	23 + 3
2	13
12	3

That is,

$$3 \otimes (1 \otimes 2 \otimes 3) \otimes \{(0 \oplus 1 \oplus 0) + (2 + 12 + 2) + (0 + 1 + 0)\}$$

$$= (3 \otimes 1 \otimes 2 \otimes 3) \otimes \{(3 \otimes 3) \otimes (0 + 2 + 0)\}$$

$$= 3 \otimes 1 \otimes 2 \otimes 3 \otimes 3 \otimes 3 \otimes 1 \otimes 1$$

$$= 1 \otimes 1 \otimes 1 \otimes 2 \otimes 3 \otimes 3 \otimes 3 \otimes 3 \quad [\text{"="}\ \underline{1^3\,2\,3^4}]$$

Now we turn to $n=4$: Very informally, as above

123	4
23	14
13 + 123	24 + 4
3	124
12 + 13 + 123	34 + 24 + 4
2 + 23	134 + 14
1 + 12 + 123	234 + 34 + 4
0	1234
1 + 12 + 123	234 + 34 + 4
2 + 23	134 + 14
12 + 13 + 123	34 + 24 + 4
3	124
13 + 123	24 + 4
23	14
123	4

That is,

$$(1\otimes 2\otimes 3\otimes 4)\otimes\{4\otimes 4\otimes 4 + 2\otimes 4\otimes 4\otimes 4 + 4\otimes 4\otimes 4 + 3\otimes 4\otimes 4\otimes 4$$
$$+ 2\otimes 3\otimes 4\otimes 4\otimes 4 + 3\otimes 4\otimes 4\otimes 4 + 4\otimes 4\otimes 4$$
$$+ 2\otimes 4\otimes 4\otimes 4 + 4\otimes 4\otimes 4\}$$
$$= (1\otimes 2\otimes 3\otimes 4\otimes 4\otimes 4\otimes 4)\{0 + 2 + 0 + 3 + 23 + 3 + 0 + 2 + 0\}$$
$$= (1\otimes 1\otimes 1\otimes 2\otimes 3\otimes 4\otimes 4\otimes 4\otimes 4)\{0 + 3 + 0\}$$
$$= 1\otimes 1\otimes 1\otimes 2\otimes 2\otimes 2\otimes 3\otimes 4\otimes 4\otimes 4\otimes 4 = [1^3\ 2^3\ 3\ 4^4].$$

Hence guess for general pattern: $[1^3, 2^3, \ldots, (n-2)^3, n-1, n^4]$

Now, is it possible to describe the "boundary" directly using the tensor product structure? Let's examine the case $n=2$. The sum is $P_1 \oplus P_0 \oplus P_1$, that is

$$\begin{matrix}1\\0\\2\\0\\1\end{matrix} \oplus \begin{matrix} & 0 & \\ 1 & & 2\\ 0 & & 0\\ 2 & & 1\\ & 0 & \end{matrix} \oplus \begin{matrix}1\\0\\2\\0\\1\end{matrix} \oplus 12$$

The kernel is

$$1 \oplus \begin{matrix}1\\0\\2\\0\end{matrix} \oplus \begin{matrix}0\\2\\0\\1\end{matrix} \oplus 12$$

Is this anything nice? We claim it is

$$1 \otimes \left\{ 0 \oplus \begin{matrix}1\\0\end{matrix} \oplus \begin{matrix}2\\0\\1\end{matrix} \right\}.$$

That is kernel factors through the "1" in $1 \otimes 2 \otimes 2 \otimes 2 \otimes 2$.
To see this we must show the second factor is a submodule of $2 \otimes 2 \otimes 2 \otimes 2$ and that the product is the kernel.
But

$$2 \otimes 2 \otimes 2 \otimes 2 = \left(\begin{matrix}0\\1\\0\end{matrix} \otimes 2 \right) \otimes 2 = (2 \oplus 12 \oplus 2) \otimes 2$$
$$= \begin{matrix}0\\1\\0\end{matrix} \oplus \begin{matrix}1\\0\\2\\0\\1\end{matrix} \oplus \begin{matrix}0\\1\\0\end{matrix}$$

Hence, we need only establish the product is the kernel. We only require that

$$1 \otimes \begin{matrix}0\\1\end{matrix} = \begin{matrix}1\\0\\2\\0\end{matrix}, \quad 1 \otimes \begin{matrix}1\\0\end{matrix} = \begin{matrix}2\\0\\1\\0\end{matrix}$$

as 12 is projective.

But

$$1\otimes \begin{smallmatrix}1\\0\end{smallmatrix} \subseteq 1\otimes \begin{smallmatrix}0\\1\\0\end{smallmatrix} = 1\otimes 2\otimes 2 = P_1 = \begin{smallmatrix}1\\0\\2\\0\\1\end{smallmatrix}$$

And cokernel of $1\otimes \begin{smallmatrix}1\\0\end{smallmatrix}$ in $1\otimes \begin{smallmatrix}0\\1\\0\end{smallmatrix}$ is 1 so

$$1\otimes \begin{smallmatrix}1\\0\end{smallmatrix} \simeq \begin{smallmatrix}0\\2\\0\\1\end{smallmatrix}$$

as desired. Hence

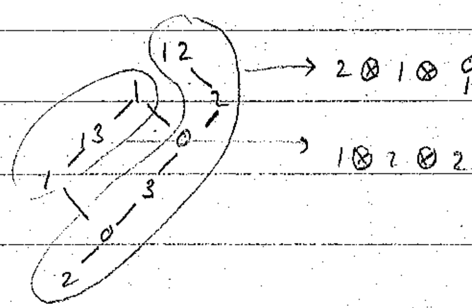
$$1\otimes \begin{smallmatrix}0\\1\end{smallmatrix} \simeq 1^*\otimes \begin{pmatrix}1\\0\end{pmatrix}^* \simeq \left(1\otimes \begin{smallmatrix}1\\0\end{smallmatrix}\right)^* \simeq \begin{pmatrix}0\\2\\0\\1\end{pmatrix}^* = \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$$

and our claim is valid.

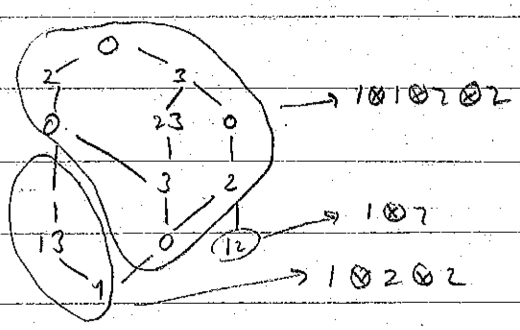
Therefore, we wish to find an endomorphism δ of $2\otimes 2\otimes 2\otimes 2$ with kernel $0 \oplus \begin{smallmatrix}0\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\0\\1\end{smallmatrix}$, or, as we want $\delta^2=0$, what is the same, an automorphism α of period 2 ($= 1+\delta$, $\delta = 1+\alpha$ as characteristic is two) with fixed point submodule being $0 \oplus \begin{smallmatrix}0\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\0\\1\end{smallmatrix}$.

Let's look at "kernel" in the next case, that is kernel of boundary operator on projective resolution. We proceed by <u>educated guesses</u> in tabular form:

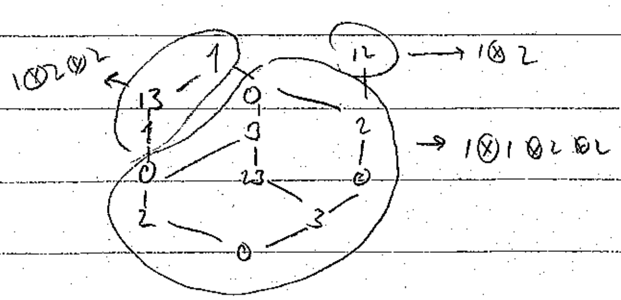
2 0 3 0 2 12	$2 \otimes$ 0 2 0 1	$2 \otimes 1 \otimes$ 1 0



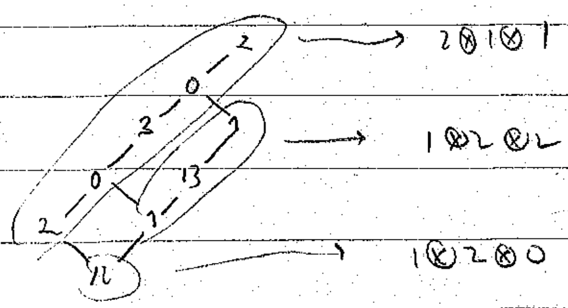
$1 \otimes 2 \otimes$ 2
0
1



$1 \otimes 2 \otimes$ 12
2
0



$1 \otimes 2 \otimes$ 0
2
12



$1 \otimes 2 \otimes$ 1
0
2

12
2
0
3
0
2

$2 \otimes 1 \otimes \substack{0\\1}$

12

$1 \otimes 2 \otimes 0$

Sum is

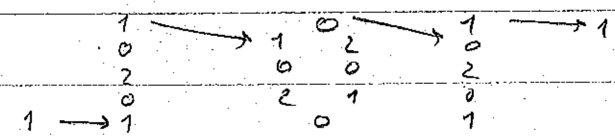
Putting in Steinberg

$$1 \otimes 2 \otimes \left\{ \substack{1\\0} \oplus \substack{2\\0\\1} \oplus \substack{12\\2\\0} \oplus \substack{0\\2\\12} \oplus \substack{1\\0\\2} \oplus \substack{0\\1} \oplus 0 \oplus 3 \right\}$$

But right-hand factor has dimension 33 so prospects are grim.
Of course, might also have kernel a product of $1 \otimes 2$ and a different right hand side, say with various summands glued together – these coming apart after being tensored with $1 \otimes 2$.

Next, let's try to factorize the whole periodic resolution!

Say $n=2$, we are considering



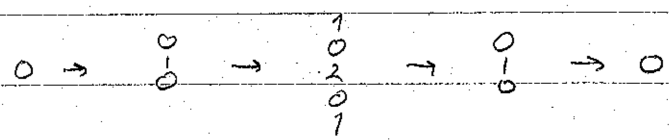
That is, after adding Steinberg module,

$$1 \to 1\otimes 2\otimes 2 \to 1\otimes 1\otimes 1\otimes 2 \to 1\otimes 2\otimes 2 \to 1$$

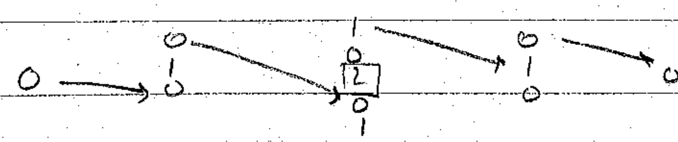
Factoring out 1 would like to have exact!

$$0 \to 2\otimes 2 \to 1\otimes 2\otimes 2 \to 2\otimes 2 \to 0$$

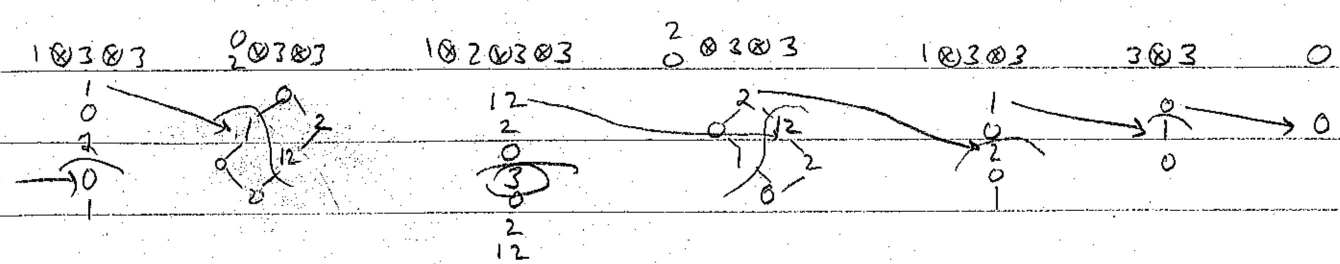
That is,

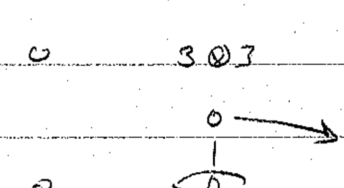


But can do!



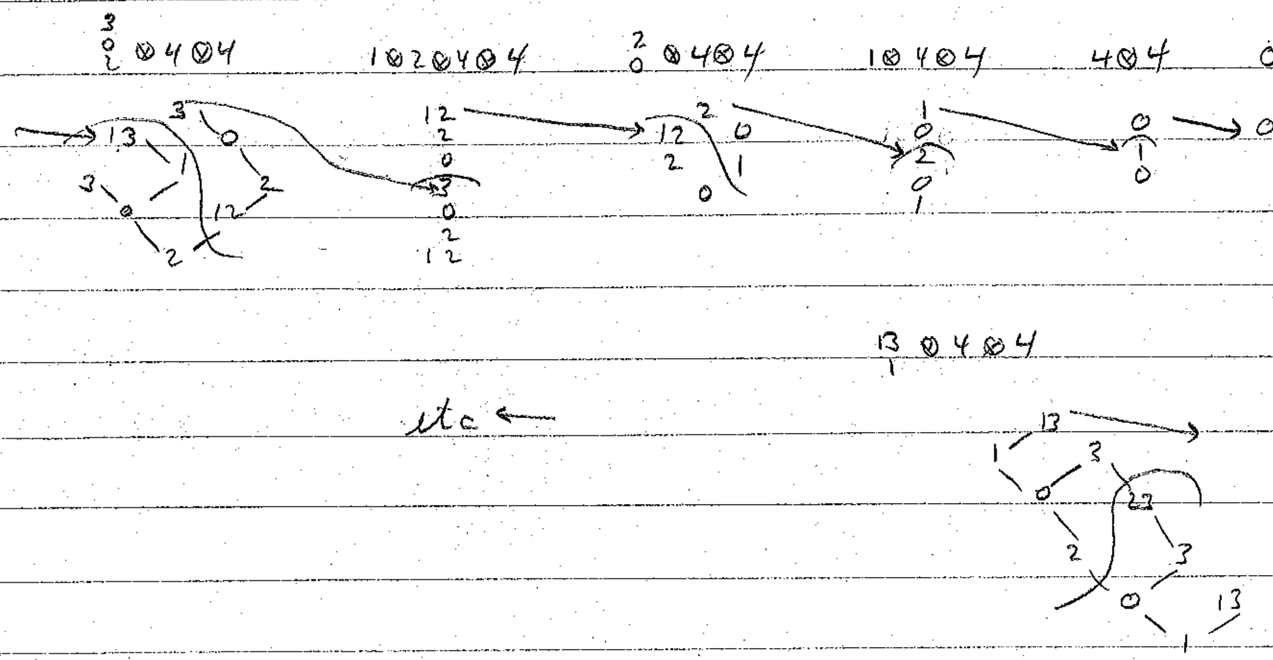
Let's try next the case $n=3$. Factoring out $1\otimes 2$ we have:





This does it! Seems to be workable into a proof!

Let's get a start on $n=4$, after factoring out $1\otimes2\otimes3$.



This seems to indicate though that we'll be getting into three dimensional diagrams when $n = 5$ even with the factorization.

$A_6 \cong PSL(2,9)$ in characteristic three:

Recall from pages 50, 51:

		1	2	4	5	5
ordinary		1	1	1	1	1
		9	1	1	−1	−1
		8	0	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
		8	0	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
		5	1	−1	0	0
		5	1	−1	0	0
		10	−2	0	0	0
modular		1	1	1	1	1
		3	−1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
		3	−1	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
		4	0	−2	−1	−1
		9	1	1	−1	−1

	1	3	3	4	9
1	1	0	0	0	0
9	0	0	0	0	1
8	1	1	0	1	0
8	1	0	1	1	0
5	1	0	0	1	0
5	1	0	0	1	0
10	0	1	1	1	0

	1	3	3	4	9
1	5	1	1	4	0
3	1	2	1	2	0
3	1	1	2	2	0
4	4	2	2	5	0
9	0	0	0	0	1

Easy calculations:

$$3\otimes 3 \doteq 1+1+3+4, \quad 3'\otimes 3' \doteq 1+1+3'+4, \quad 3\otimes 3' \doteq 9$$

$$3\otimes 4 \doteq 1+3'+4+4, \quad 3'\otimes 4 \doteq 1+3+4+4, \quad 4\otimes 4 \doteq 1+3+3'+9$$

Next, having composition factors, want actual structure. Clearly

$$3\otimes 3' = 9$$

Now $4\otimes 4$ is self-dual, $\mathrm{Hom}(1, 4\otimes 4) \cong \mathrm{Hom}(4,4) = \mathrm{Hom}(4\otimes 4, 1)$ so get easily

$$4\otimes 4 = 1 \oplus 3 \oplus 3' \oplus 9$$

Hence $\mathrm{Hom}(4\otimes 4, 3) \simeq F$ so $\mathrm{Hom}(4, 4\otimes 3) \simeq \mathrm{Hom}(4\otimes 3, 4) \simeq F$

Also $\mathrm{Hom}(4\otimes 3, 1) = 0$, $\mathrm{Hom}(4\otimes 3, 3') = 0$, $\mathrm{Hom}(3', 4\otimes 3) = 0$ etc.

so deduce - as again self-dual

$$3\otimes 4 = \begin{array}{c} 4 \\ 1\ \ 3' \\ 4 \end{array}$$

so

$$3'\otimes 4 = \begin{array}{c} 4 \\ 1\ \ 3 \\ 4 \end{array}$$

next $\mathrm{Hom}(3\otimes 3, 4) \simeq \mathrm{Hom}(3, 3\otimes 4) = 0$ and $\mathrm{Hom}(4, 3\otimes 3) = 0$

so two possibilities for $3\otimes 3$, namely $3 \oplus \begin{array}{c} 1 \\ 4 \\ 1 \end{array}$ or $\begin{array}{c} 1 \\ 3\ 4 \\ 1 \end{array}$.

But

$$\mathrm{Hom}(3\otimes 3, 4\otimes 4) \simeq \mathrm{Hom}(3\otimes 4, 3\otimes 4) \simeq F\oplus F$$

so $\mathrm{Hom}(3\otimes 3, 1\oplus 3\oplus 3'\oplus 9) \simeq F\oplus F$ so have

$$3\otimes 3 = 3 \oplus \begin{array}{c} 1 \\ 4 \\ 1 \end{array}$$

$$3'\otimes 3' = 3' \oplus \begin{array}{c} 1 \\ 4 \\ 1 \end{array}$$

Now we turn to the projectives. Now $(3\otimes 3)\otimes 3'$ $\simeq 3\otimes(3\otimes 3')$ is projective. Hence $\begin{array}{c} 1 \\ 4 \\ 1 \end{array} \otimes 3$ is projective. It has 3 in socle and comp factors are those of P_3. Hence $\begin{array}{c} 1 \\ 4 \\ 1 \end{array} \otimes 3 \simeq P_3$. Know $4\otimes 3$ and hence $4\otimes 3 \simeq M_3$, the middle. Hence

$$P_3 \simeq \begin{array}{c} 3 \\ 4 \\ 1\ \ 3' \\ 4 \\ 3 \end{array}, \quad P_{3'} \simeq \begin{array}{c} 3' \\ 4 \\ 1\ \ 3 \\ 4 \\ 3' \end{array}$$

Next, suppose we consider $3\otimes 3 \cong 3 \oplus \begin{smallmatrix}4\\1\end{smallmatrix}$, $3'\otimes 3' \cong 3' \oplus \begin{smallmatrix}4\\1\end{smallmatrix}$.
Now $\mathrm{Hom}(3\otimes 3, 3'\otimes 3') \cong \mathrm{Hom}(3\otimes 3', 3\otimes 3') \cong F$ so the two $\begin{smallmatrix}4\\1\end{smallmatrix}$ modules are non-isomorphic. Hence, P_1 has two 4's at the top two at the bottom and these are four different factors as we have the $\begin{smallmatrix}4\\1\end{smallmatrix}$ existing. But the corresponding Cartan invariant is 4 so $Ext^1(1,4) \cong F\oplus F$. Of course $Ext^1(1,1)=0$, $Ext^1(1,3)=Ext^1(1,3')=0$ by structure of P_3, $P_{3'}$. Hence,

$$P_1: \quad \begin{array}{c} 1 \\ \hline 4\ 4 \\ \\ \hline 4\ 4 \\ \hline 1 \end{array}$$

} ~~involves 1, 3, 3'~~

Hence, the middle of the middle is just a direct sum as all the relevant $Ext^1(-,-)$ groups are zero. Hence, Loewy series, upper and lower, of P_1 are

$$P_1: \quad \begin{array}{c} 1 \\ \hline 4,4 \\ \hline 1,1,1,3,3' \\ \hline 4,4 \\ \hline 1 \end{array}$$

Now we turn to P_4. Have $Ext(4,4) \cong Ext(1, 4\otimes 4) \cong Ext(1, 1\oplus 3\oplus 3'\oplus 9) = 0$.
$\therefore$ second Loewy factor of P_4 is $1\oplus 1\oplus 3\oplus 3'$. But also have

$$3\otimes 4\otimes 3' = \begin{smallmatrix} & 4 & \\ 1 & & 3' \\ & 4 & \end{smallmatrix} \otimes 3' = \begin{array}{c} \begin{smallmatrix} & 4 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \\ \hline 3'\oplus 3'\oplus \begin{smallmatrix}4\\1\end{smallmatrix} \\ \hline \begin{smallmatrix} & 4 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \end{array}$$

so by counting composition factors have $P_4 = 4\otimes 3\otimes 3'$.
Now $Ext(4,1) \cong F\oplus F$. And above partial structure shows the 1's at the top and the bottom of the middle

of P_4 are different. Also the 3's and by duality the 3''s. Thus get Loewy series of P_4, upper and lower

$$P_4 : \quad \begin{array}{c} 4 \\ \hline 1\ 1\ 3\ 3' \\ \hline 4\ 4\ 4 \\ \hline 1\ 1\ 3\ 3' \\ \hline 4 \end{array}$$

Next, wish to get at P_1. Now $3 \otimes 3 \otimes 3' \otimes 3'$ is projective and is

$$\left(3 \oplus \begin{smallmatrix}1\\4\\1\end{smallmatrix}\right) \otimes \left(3' \oplus \begin{smallmatrix}1\\4'\\1\end{smallmatrix}\right)$$

where $4' = 4$ but modules are different as $\mathrm{Hom}(3\otimes 3, 3' \otimes 3')$ is one-dimensional. From what we know already we get $P_9 \oplus P_3 \oplus P_{3'} \oplus \left(\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes \begin{smallmatrix}1\\4'\\1\end{smallmatrix}\right)$. But $\mathrm{Hom}(3 \otimes 3 \otimes 3' \otimes 3', 1)$ is one dimensional, so by counting $\dim P_1 + \dim P_9 + \dim P_3 + \dim P_{3'} = 72$ and since $4 \otimes 4$ involves P_9 have

$$\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes \begin{smallmatrix}1\\4'\\1\end{smallmatrix} = P_1 \oplus P_9 .$$

Hence, wish to get at $\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4$ for example.

Have
$$\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4 = \begin{array}{c} 4 \\ \hline 1,3,3' \\ \hline 4 \end{array} \oplus 9 .$$

<u>Claim</u>:

$$\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4 = \begin{smallmatrix}4\\1\ 3'\\4\end{smallmatrix} \oplus 3 \oplus 9$$

Indeed,

$\mathrm{Hom}\left(\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4, 4\right) \cong F$, $\mathrm{Hom}\left(\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4, 3'\right) \cong \mathrm{Hom}\left(\left(3 \oplus \begin{smallmatrix}1\\4\\1\end{smallmatrix}\right) \otimes 4, 3'\right)$ $\cong \mathrm{Hom}(3 \otimes 3 \otimes 4, 3') \cong \mathrm{Hom}(3 \otimes 4, 9) = 0$. Remains only to show, by duality, $\mathrm{Hom}\left(\begin{smallmatrix}1\\4\\1\end{smallmatrix} \otimes 4, 3\right) \neq 0$.

But:

$$\mathrm{Hom}\left(\begin{smallmatrix}1\\4\\1\end{smallmatrix}\otimes 4,\ 3\right) \simeq \mathrm{Hom}\left(\begin{smallmatrix}&4&\\1&&3'\\&4&\end{smallmatrix} \oplus \left(\begin{smallmatrix}1\\4\\1\end{smallmatrix}\otimes 4\right),\ 3\right)$$
$$\cong \mathrm{Hom}\left((3\otimes 4)\oplus\left(\begin{smallmatrix}1\\4\\1\end{smallmatrix}\otimes 4\right),\ 3\right)$$
$$\simeq \mathrm{Hom}(3\otimes 3\otimes 4,\ 3)$$
$$\simeq \mathrm{Hom}(4,\ 3\otimes 3\otimes 3).$$

But $3\otimes 3 = 3\oplus\begin{smallmatrix}1\\4\\1\end{smallmatrix}$ so $3\otimes 3\otimes 3 = 3\oplus\begin{smallmatrix}1\\4\\1\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\ \hline &4&\\1&&3'\\&4&\\ \hline &3&\end{smallmatrix}$

Hence, $\mathrm{Hom}(4, 3\otimes 3\otimes 3)=0$ implies, by dimension count that $3\otimes 3\otimes 3 = 3\oplus\begin{smallmatrix}1\\4\\1\end{smallmatrix}\oplus P_3 = (3\otimes 3)\oplus P_3$. Now $\mathrm{Hom}_{S_3}(1, 3\otimes 3)$ where S_3 is Sylow 3-subgroup is at most three-dimensional as $3/S_3$ is uniserial. Indeed, $S_3 = \left\{\begin{pmatrix}1&\alpha\\0&1\end{pmatrix} \mid \alpha\in GF(9)\right\}$, 3 is rep on symmetric tensors of degree 2,

$$\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\to\begin{pmatrix}1&2\alpha&\alpha^2\\0&1&\alpha\\0&0&1\end{pmatrix}$$

Hence, as P_3/S_3 is free, get $\mathrm{Hom}_{S_3}(1, 3\otimes 3\otimes 3)$ of dimension ≤ 5, as $5 = 3+2$. Thus, will be done as soon as we show that this dimension __exceeds__ five.

Let's look at representation $3\otimes 3$ on S_3.

$$\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\to\left(\begin{array}{ccc|ccc|ccc}
1&2\alpha&\alpha^2&2\alpha&4\alpha^2&2\alpha^2&\alpha^2&2\alpha^3&\alpha^4\\
&1&\alpha&0&2\alpha&2\alpha^2&0&\alpha^2&\alpha^3\\
&&1&0&0&2\alpha&0&0&\alpha^2\\ \hline
&&&1&2\alpha&\alpha^2&\alpha&2\alpha^2&\alpha^3\\
&&&&1&\alpha&0&\alpha&\alpha^2\\
&&&&&1&0&0&\alpha\\ \hline
&&&&&&1&2\alpha&\alpha^2\\
&&&&&&&1&\alpha\\
&&&&&&&&1
\end{array}\right)$$

Number vectors for canonical basis $v_1, v_2, \ldots v_9$ so v_7, v_8, v_9 give 3 back again. $v_3 - v_5$, $v_6 - v_8$ give submodule into

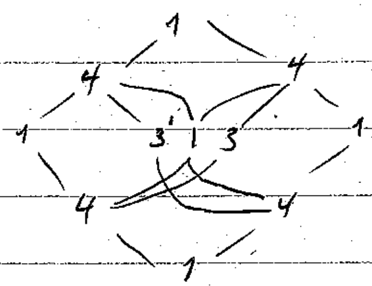
representation
$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$
Also $v_3 + v_5 + v_7$ is fixed. Hence, these six vectors give subrepresentation
$$\begin{pmatrix} 1 & & & & & \\ & 1 & \alpha & & & \\ & 0 & 1 & & & \\ & & & 1 & 2\alpha & \alpha^2 \\ & & & & 1 & \alpha \\ & & & & & 1 \end{pmatrix}$$
Symbolically, $3 \otimes 3 \cong 1 \oplus 2 \oplus 3$ as S_3 representation.

Hence, $3 \otimes 3 \otimes 3 \cong 3 \oplus (2 \otimes 3) \oplus (3 \otimes 3)$
$$\cong 3 \oplus (2 \otimes 3) \oplus 1 \oplus 2 \oplus 3$$
$\therefore$ suffice to show that $2 \otimes 3$ has 2 dimensions of fixed points.

But $Hom_{S_3}(1, 2 \otimes 3) = Hom_{S_3}(3, 2)$ as 3 is self dual for $L_2(q)$ so is for S_3. Now get one dim of homomorphisms from fixed pt of 2 and top of 3. For another need
$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -\alpha \\ & 1 \end{pmatrix}$$
to be the same representation. But in second, with regard to a basis x, y have
$$x \to x - \alpha y$$
$$y \to \quad y$$
so
$$x \to x + \alpha(-y)$$
$$(-y) \to \quad 1\,(-y)$$
and are done. The claim is proved.

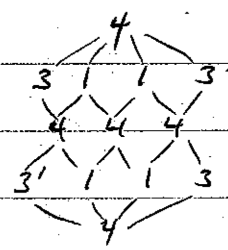
We can now *guess* P_1 pretty easily (but will not bother with proof)

P_1

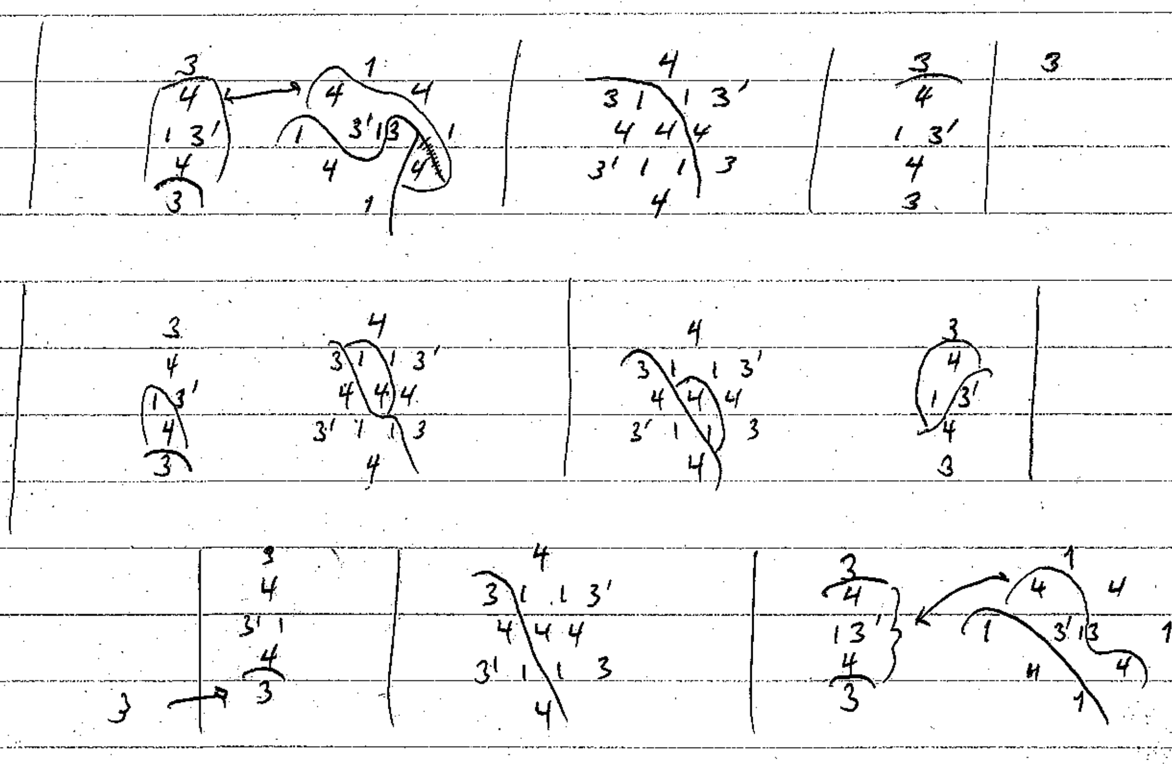


Next, let's make *stab* at P_4:

P_4 :



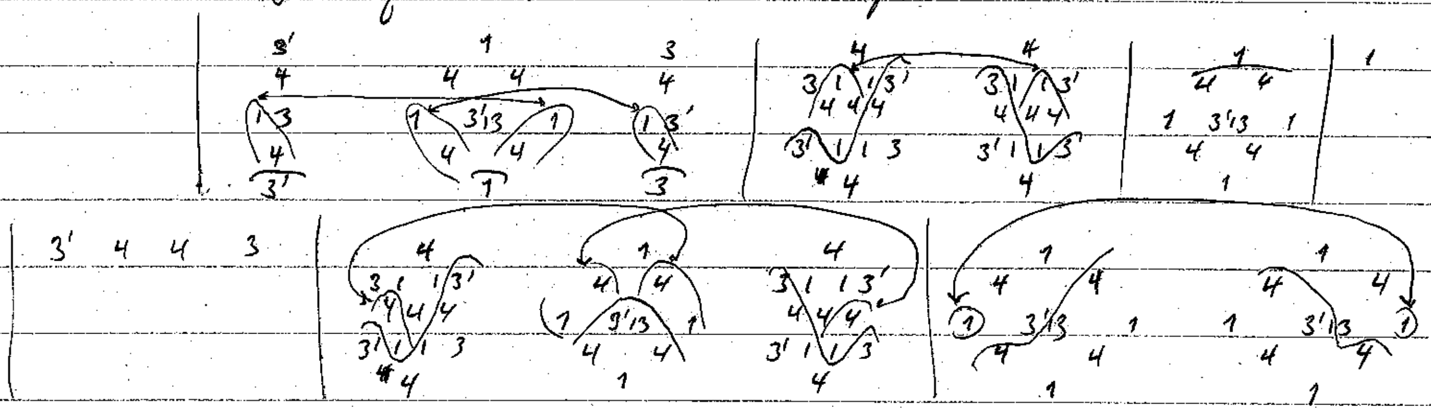
Using this let's try to get resolution for P_3:



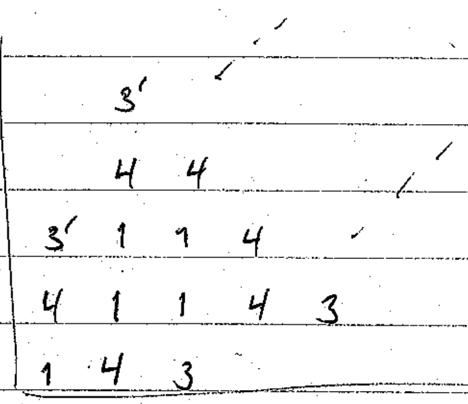
Periodic of period 8.

<u>Good Guess</u>: The irreducible modules of dimension p^{r-1} for $SL(2, p^r)$ have periodic projective resolutions of period $q-1$. Indeed, the restriction of this module to a maximal subgroup of the Sylow p-subgroup of $SL(2, p^r)$ seems to be free so this subgroup "shouldn't count" homologically, leaving only its cyclic quotient as "counting."

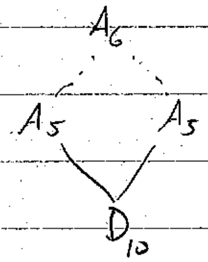
Some of the other resolutions for a bit:



Picture of double complex:



Remark: The two different modules of shape $\begin{smallmatrix}1\\4\\1\end{smallmatrix}$ are easily explained. After all A_6 has two different doubly transitive actions on a set of six elements corresponding to the two classes of A_5's. The classes of three-elements act differently in these two actions, exactly one having no fixed points. (*)
Picture:



Actually, $(1_{D_{10}})^{A_6} = P_1 \oplus P_9$. For $1_{D_{10}}^{A_6}$ is projective contains P_1 so by degree count it suffices to show P_9 is there — but can check this in characteristic zero:

D_{10}:

	#	1	5	2	2
		1	2	5	5
	φ_1	1	1	1	1
	φ_2	1	−1	1	1
	φ_3	2	0	$\frac{-1+\sqrt5}{2}$	$\frac{-1-\sqrt5}{2}$
	φ_4	2	0	$\frac{-1-\sqrt5}{2}$	$\frac{1+\sqrt5}{2}$
$\chi(1)=9$,	$\chi\vert D_{10}$	9	1	−1	−1

Hence, $\chi | D_{10} = \varphi_1 + 2\varphi_3 + 2\varphi_4$ so $P_9 \mid (1_{D_{10}})^{A_6}$.

(*) Is this right – have to reduce mod 3 after all!

Tensor powers for SL(2,4).

In SL(2, 2^n) notation V_1 = "1", $U = V_1 \otimes V_1 = \begin{smallmatrix}0\\2\\0\end{smallmatrix}$, V_{12} = "12", n = 3.

Consider the matrix

(projective: V_{12}, P_2, P_1, P_0)

	V_1	U	V_{12}	P_2	P_1	P_0
V_1	0	1	0	0	0	0
U	2	0	1	0	0	0
V_{12}	0	0	0	1	0	0
P_2	0	0	2	0	1	0
P_1	0	0	1	0	0	1
P_0	0	0	2	0	2	0

where the row corresponding to M gives the direct decomposition of $M \otimes V_1$. Hence, the n-th power has as its first row the structure of V_1^{n+1} $(= \overbrace{V_1 \otimes \dots \otimes V_1}^{n+1})$.

The block form of this matrix easily gives:

$$V_1^{2n+1} = \overbrace{V_1 \oplus \dots \oplus V_1}^{2^n} \oplus \text{projectives}$$
$$V_1^{2n} = \overbrace{U \oplus \dots \oplus U}^{2^{n-1}} \oplus \text{projectives.}$$

By diagonalizing the matrix should be able to calculate all its powers. Let's start by determining the characteristic equation. It's

$$\begin{vmatrix} -x & 1 \\ 2 & -x \end{vmatrix} \cdot \begin{vmatrix} -x & 1 & 0 & 0 \\ 2 & -x & 1 & 0 \\ 1 & 0 & -x & 1 \\ 2 & 0 & 2 & -x \end{vmatrix}$$

$$= (x^2-2) \cdot \begin{vmatrix} 0 & 1 & 0 & 0 \\ 2-x^2 & -x & 1 & 0 \\ 1 & 0 & -x & 1 \\ 2 & 0 & 2 & -x \end{vmatrix}$$

$$= -(x^2-2) \begin{vmatrix} 2-x^2 & 1 & 0 \\ 1 & -x & 1 \\ 2 & 2 & -x \end{vmatrix} = -(x^2-2)\left[(2-x^2)(x^2-2) - (-x-2)\right]$$

$$= -(x^2-2)(-x^4+4x^2-4+x+2) = (x^2-2)(x^4-4x^2-x+2)$$

$$= (x^2-2)(x+1)(x^3-x^2-3x+2)$$

$$= (x^2-2)(x+1)(x-2)(x^2+x-1)$$

$$= (x-\sqrt{2})(x+\sqrt{2})(x+1)(x-2)\left(x-\tfrac{-1+\sqrt{5}}{2},\ x-\tfrac{-1-\sqrt{5}}{2}\right)$$

Matrix is similar to: $\begin{pmatrix} \sqrt{2} &&&&& \\ & -\sqrt{2} &&&& \\ && -1 &&& \\ &&& 2 && \\ &&&& \frac{-1+\sqrt{5}}{2} & \\ &&&&& \frac{-1-\sqrt{5}}{2} \end{pmatrix}$.

Eigenvector	Eigenvalue
(0, 0, 4, 2, 3, 1)	+2
(0, 0, -1, 1, 1, -1)	-1

(Just $V_1 \otimes$ regular = 2regular)

Doesn't seem worth the effort.

Tensor powers for $SL(2, 2^n)$

Remark: It is obvious (!) that the irreducible modules generate a subring of rank $\leq 3^n$ of the Green ring.

Indeed, $V_i \otimes V_i \otimes V_i \simeq V_i \oplus V_i \oplus V_{i+1}$.
Hence irreducibles, their products and powers, are linear combinations of the modules

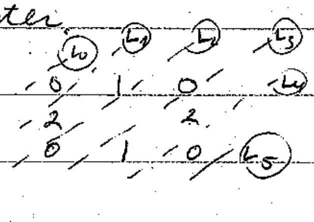
$$V_1^{e_1} V_2^{e_2} \cdots V_n^{e_n}$$

where $0 \leq e_i \leq 2$, $i = 1, \ldots, n$. It should be easy to see that these 3^n "monomials" are independent. Note: As summands get all possible projectives, by Bryant-Kovacs.

Loewy series of projectives for $SL(2, 2^n)$

We'll make some educated guesses for projective corresponding to principal character.

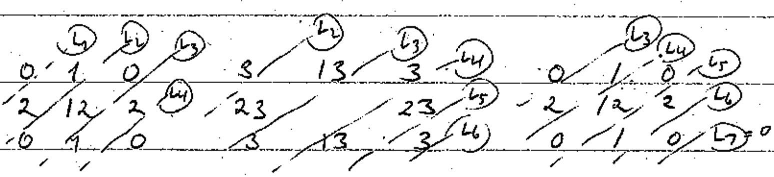
$n=2$



0
$1,2$
$0,0$
$1,2$
0

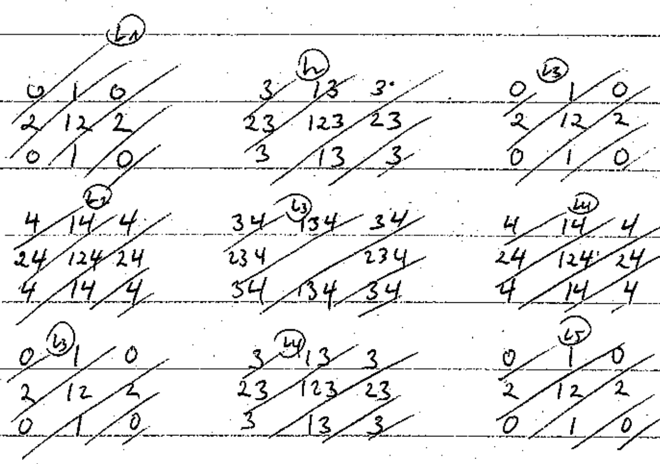
Here $L_i = (\text{Rad } FG)^i$. module, the i-th term of the lower Loewy series.

$n=3$



0
$1,2,3,4$
$0,0,0,12,13,23$
$1,1,2,2,3,3$
$0,0,0,12,13,23$
$1,2,3,4$
0

$n=4$



0
$1\ 2\ 3\ 4$
$0,0,0,0,12,13,14,23,24,34$
$1,1,1,2,2,2,3,3,3,4,4,4,123,124,134,234$
$0,0,0,0,0,0,12,12,13,13,14,14,23,23,24,24,34,34$
$1,1,1,2,2,2,3,3,3,4,4,4,123,124,134,234$
$0,0,0,0,12,13,14,23,24,34$
$1,2,3,4$
0

Hence, guess that the multiplicity of V_I, $0 \le |I| = r < n$ in $\mathcal{L}_k = L_k / L_{k+1}$ is $\binom{n-r}{\frac{k-r}{2}}$, when this makes sense, i.e. $0 \le \frac{k-r}{2} \le n-r$, $2 \mid k-r$ and zero otherwise.

$$\mathcal{L}_k(P_\phi) \simeq \bigoplus_{I \subset N} \binom{n-|I|}{\frac{k-|I|}{2}} V_I$$

Let's see how to guess this in terms of the "coordinates". Let us imagine the last module, for example, situated in 4-space with an irreducible constituent at each lattice point. As follows

first coord →   → third

second ↓

fourth ↓

so points are

(0,0,0,0)   (1,0,0,0)   (2,0,0,0)   (0,0,1,0) . -

(0,1,0,0)   (1,1,0,0)   (2,1,0,0)

(0,2,0,0   (1,2,0,0)   (2,2,0,0)

(0,0,0,1) . –

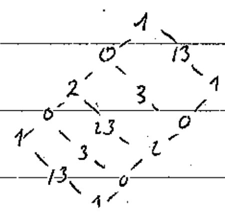
The module at lattice point (x_1, x_2, x_3, x_4) "belongs" to $L_{x_1+x_2+x_3+x_4}$. Here is rule determining the $I \ni V_I$ is isomorphic with $V_{(x_1,x_2,x_3,x_4)}$ the module at that point. We have $i \in I \iff x_i = 1$. Hence, the multiplicity of V_I in L_k is the number of ways to associate to each $j \notin I$ a number "0 or 2", to each $i \in I$, the number "1" so that the resulting coordinates have sum k. This is $\binom{n-|I|}{\frac{k-|I|}{2}}$.

Next, let's examine the projective P_1

$\underline{n=2}$

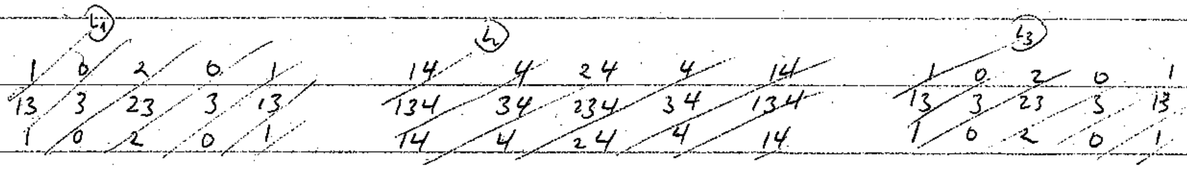
1
0
2
0
1

$\underline{n=3}$



1
0, 13
1, 2, 3
0, 0, 23
1, 2, 3
0, 13
1

$\underline{n=4}$



1
0, 13, 14
1, 1, 2, 3, 4, 134
0, 0, 0, 13, 14, 23, 24, 34
1, 1, 2, 2, 3, 3, 4, 4, 234
0, 0, 0, 13, 14, 23, 24, 34
1, 1, 2, 3, 4, 134
0, 13, 14
1

$V_{(x_1, x_2, x_3)} \to \mathcal{L}_{x_1+x_2+x_3}$ $\quad 0 \le x_1 \le 4, \quad 0 \le x_2, x_3 \le 2$

$V_{(x_1, x_2, x_3)} \sim I$ according to the rule:

$x_1 = 0, 2, 4$ put $1, 2, 1$ in I

$x_2 = 1$ put 3 in I

$x_3 = 1$ put 4 in I

Semi-simplicity of the "irreducible ring" for $SL(2, 2^n)$

We know that $V_i \otimes V_i \otimes V_i = V_i \oplus V_i \oplus V_{i,i+1}$ so that the subring of the Green ring generated by all the V_i consist of all $V_{i_1}^{e_1} \cdots V_{i_k}^{e_k}$, $i_1 < \dots < i_k$, each e_j is zero or one. Each of these modules is indecomposable, having a simple socle — or else tensor ups to get a projective — with the exception of $V_1 \otimes V_1 \otimes V_2 \otimes V_2 \cdots \otimes V_n \otimes V_n = P_F \oplus (V_1 \otimes \cdots \otimes V_n)$. Hence, are dealing with 3^n indecomposable, as these are pairwise non-isomorphic. Reason: Otherwise tensor up the two isomorphic ones keeping one side indecomposable and let other side split up!

Hence, this ring is given by generators $x_1, \dots, x_n$ with relations

$$x_1^3 = 2x_2 + x_1 x_2$$

$$\vdots$$

$$x_n^3 = 2x_1 + x_n x_1$$

(commuting variables). Thus, to complete the proof we must find 3^n distinct complex solutions of these equations.

Now no Brauer character of an irreducible vanishes on any element of odd order so get 2^n solutions — using the columns corresponding — of these equations which have the property that no x_i is zero.

Claim: If $I \subseteq \{1, \dots, n\}$ then there are ~~of~~ exactly $2^{n-|I|}$ solutions which have $x_i = 0$ if, and only if $i \in I$. Must only verify for $I \neq \emptyset$. Then, number of solutions is

$$\sum_I 2^{n-|I|} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = (1+2)^n = 3^n.$$

Before proving our claim, and thus the desired semi-simplicity, we assert that no x_i is ever -2 in a solution. For, without loss, say $x_1 = -2$. Hence $(-2)^3 = 2(-2) - 2x_2$ so $x_2 = 2$. Hence, $x_3 = 2$, $x_4 = 2$, ..., $x_n = 2$, $x_1 = 2$ a contradiction.

Without loss we may assume $1 \in I$, $n \notin I$ – as $I = \{1, \ldots n\}$ clearly gives exactly one solution: $x_1 = \ldots = x_n = 0$. Thus, last equation is $x_n^3 = 2x_n$ so there are two possibilities for x_n. If $x_{n-1} \notin I$ then have $x_{n-1}^3 = 2x_{n-1} + x_{n-1}x_n$ so $x_{n-1}^2 = 2 + x_n$ and have two possibilities as $x_n \neq -2$. Keep going down till we get to another element of I. And then start again!

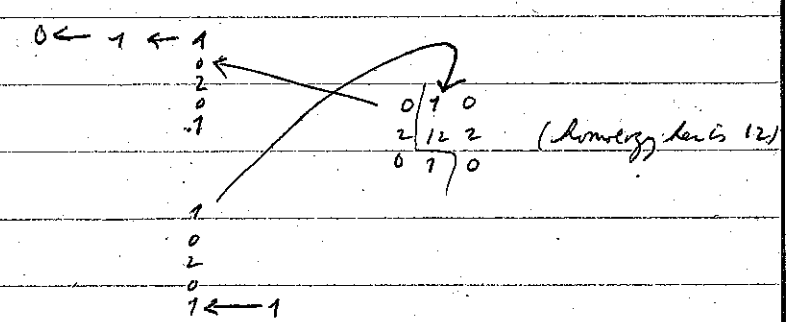
To look at the minimal resolution for $V_{12\cdots n-1}$ and hopefully prove its obvious properties we extend our conjectures to a family of exact sequences — or nearly exact. (Will allow the Steinberg to appear in the middle of the period for $V_{1,2,\ldots,n-1}$.)

Here are the sequences, the first module in the first line, etc..

nn1

nn11

nn1



Next case:

nn12

nn112

nn12 nn122

nn1122

nn12 nn122

nn112

nn12

Leaving out the nn have

1

11

1

12

112

12 122

1122

12 122

112

12

123

1123

123 1223

[illegible]

123 1223 1233

1123 [illegible] 11233

123 1233 12233

[illegible] [illegible] 112233

123 1233 12233

1123 11233

123 1223 1233

11223

123 1223

1123

To get at rectangular structure of projectives

Consider $V_{i_1}^{e_1} \cdots V_{i_k}^{e_k}$, each e_i zero or one. It is a tensor product of strings : $V_i^2 \cdots -$ where dots are square free – or it is irreducible. Each string is uniserial and using direct product Loewy series results should use these uniserials to get at rectangular structure.

For example, $V_1^2 V_2 \; V_3^2 V_4 V_5$

In short notation

$$\begin{matrix} 2 \\ 0 \\ 3 \\ 0 \\ 2 \end{matrix} \quad \otimes \quad \begin{matrix} 45 \\ 5 \\ 0 \\ 6 \\ 0 \\ 5 \\ 45 \end{matrix}$$

getting, hopefully, a 5 × 7 rectangle structure.

Structure of $V_{12\cdots n-1}$

We assert that there is an elementary abelian subgroup of order 2^{n-1} such that $V_{1,2,\cdots,n-1}$ restricted to it is a free module. We prove more, by induction, that there is an elementary abelian subgroup of order 2^k which acts freely on $V_{1,2,\cdots,k}$, $1 \le k \le n$. This is clear for $k=1$. Assume $E = \langle e_1, \ldots e_k \rangle$ works for $V_{1,\ldots,k}$, $1 \le k < n$. Hence, E is free on $V = V_{1,\cdots k,k+1} = V_{1,\cdots k} \otimes V_{k+1}$, and on two generators by the dimension. Hence, E has two dimensions of fixed points on V. Let T be a Sylow 2-subgroup of G containing E. If $t \in T-E$ and t acts faithfully on $C_V(E)$ then $\langle E, t \rangle$ is free on V as $\dim V = \log_2 |\langle E,t\rangle|$ and $C_V(\langle E,t\rangle)$ is of dimension 1. Hence, can assume false, so $t \in C(C_V(E))$ all $t \in T-E$. $\therefore C_V(T)$ is of two dimensions so $C_{V \otimes V_{k+2,\ldots,n}}(T)$ has dimension at least two, a contradiction, as T is free on one generator, by dimensions, on $V \otimes V_{k+2,\ldots,n} = V_{1,\ldots,n}$.