LECTURES ON STRING TOPOLOGY OF CLASSIFYING SPACES

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These are notes for a series of three lectures to be given during the conference *Loop spaces in Geometry and Topology* in Nantes, September 1st-5th 2014. The subject is *string topology of classifying spaces*. String topology began in 1999 with the paper of Chas and Sullivan [1], which found new algebraic structure on $H_*(\Lambda M)$ when M is a closed oriented manifold. An excellent introduction can be found in Cohen and Voronov's notes [3], and the highpoint of the theory is perhaps Godin's paper [4] on higher string topology operations. String topology of classifying spaces was introduced by Chataur and Menichi [2], who studied the structure of $H_*(\Lambda BG)$ for compact Lie groups G. The present lectures are based on a paper of Anssi Lahtinen and myself [6], where we extend Chataur and Menichi's work. Any errors in the lectures or notes are mine!

Our aim in these lectures is to answer the following question.

> **Question:** Let G be a finite group, let BG denote the classifying space of G, and let ΛBG denote the space of all maps from S^1 to BG. What is the structure of $H_*(BG)$ and $H_*(\Lambda BG)$?

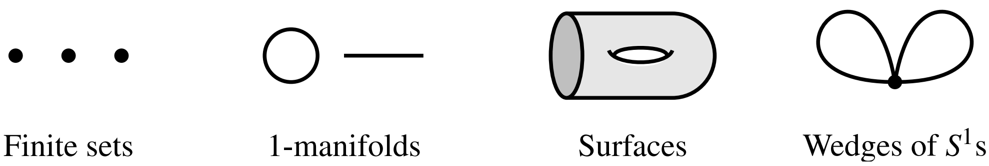
Here, and throughout what follows, homology is taken with coefficients in a field $\mathbb{F}$. The answer that Chataur and Menichi gave to the question above is that $H_*(BG)$ and $H_*(\Lambda BG)$ are part of a *homological conformal field theory*, which is an algebraic structure governed by surfaces and their diffeomorphisms. However, Anssi and I found that only the most basic homotopical properties of surfaces are important, and so our answer to the question is that $H_*(BG)$ and $H_*(\Lambda BG)$ are part of what we call a *homological h-graph field theory*, which are similar to homological conformal field theories, but with surfaces and diffeomorphisms replaced with with more general (and frequently bizarre) spaces and their homotopy equivalences.

1. H-GRAPHS AND H-GRAPH COBORDISMS

This section will introduce h-graphs and h-graph cobordisms, which are our homotopical versions of 1-manifolds and surfaces, respectively. I hope that these sections are self-contained, but if you've never encountered field theoretical structures before then you might like to look at section 1.1 of Lurie's expository article [7].

Definition 1 (H-graph)**.** An *h-graph* is a space with the homotopy type of a finite CW-complex of dimension at most 1.

Example 2. Here are some basic examples of h-graphs.



(The 1-manifolds must be compact, and the surfaces must have boundary in every component.)

Definition 3 (H-graph cobordism)**.** Let X and Y be h-graphs. An *h-graph cobordism* $S\colon X \nrightarrow Y$ consists of an h-graph S and a zig-zag of continuous maps

$$X \xrightarrow{i} S \xleftarrow{j} Y$$

satisfying the following conditions:

(i) $i \sqcup j\colon X \sqcup Y \to S$ is a closed cofibration.
(ii) $i(X)$ meets every path-component of S.
(iii) There is a homotopy cofibre square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow j \\ B & \longrightarrow & S \end{array}$$

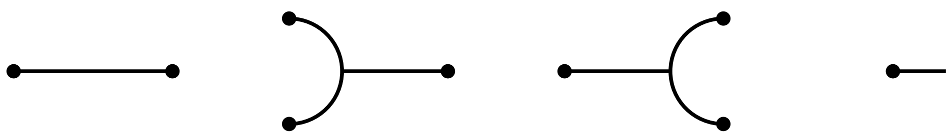
in which B is an h-graph and A has the homotopy type of a finite set.

Remark 4. Why do we impose conditions (i), (ii) and (iii) above? Condition (i) is necessary because later on we will want to consider homotopy automorphisms of S that respect X and Y, and without this restriction these automorphisms will behave in the wrong way. Condition (ii) is 'optional' in some sense, but without it we wouldn't be able to include string topology of classifying spaces as an example. And condition (iii) is the very heart of the definition: it is a homotopy-theoretical property of surfaces, and it is the *only* property of surfaces that we will need.

Example 5 (Graphs as h-graph cobordisms)**.** Let S be a finite graph, let X and Y be finite sets, and let $i\colon X \to S$ and $j\colon Y \to S$ be injections with disjoint images, and such that $i(X)$ meets every component of S. Then S, i and j determine an h-graph cobordism

$$S\colon X \nrightarrow Y.$$

Conditions (i), (ii) and (iii) are easily verified. Here are some specific examples. We let $\mathrm{pt} = \{p\}$ denote the space with a single point p.



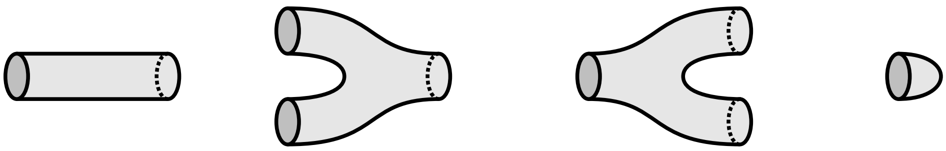
$i\colon \mathrm{pt} \nrightarrow \mathrm{pt}$ $\quad$ $m\colon \mathrm{pt} \sqcup \mathrm{pt} \nrightarrow \mathrm{pt}$ $\quad$ $w\colon \mathrm{pt} \nrightarrow \mathrm{pt} \sqcup \mathrm{pt}$ $\quad$ $c\colon \mathrm{pt} \nrightarrow \emptyset$

The pictures above just show the 'S' part of an h-graph cobordism $S\colon X \nrightarrow Y$. The maps i and j, which we have not drawn, are always the inclusion of the left and right 'ends' of S in the picture.

Example 6 (Surfaces as h-graph cobordisms)**.** Let X and Y be closed 1-manifolds and let S be a surface equipped with a diffeomorphism $i \sqcup j\colon X \sqcup Y \to \partial S$ between $X \sqcup Y$ and the boundary of S, such that $i(X)$ meets every component of S. Then S, i and j determine an h-graph cobordism

$$S\colon X \nrightarrow Y.$$

Here are some specific examples of h-graph cobordisms obtained in this way.



$I\colon S^1 \nrightarrow S^1$ $\quad M\colon S^1 \sqcup S^1 \nrightarrow S^1$ $\quad W\colon S^1 \nrightarrow S^1 \sqcup S^1$ $\quad C\colon S^1 \nrightarrow \emptyset$

Conditions (i) and (ii) are easily verified. Condition (iii) is less trivial, and relies on the assumption that $i(X)$ meets every component.

Exercise 7. Find h-graph cobordisms that are not of the form considered in Example 5 and 6. Include ones of the form $S^1 \nrightarrow \mathrm{pt}$, $\mathrm{pt} \nrightarrow S^1$ and $S^1 \to S^1$.

Given h-graph cobordisms $S\colon X \nrightarrow Y$ and $T\colon Y \nrightarrow Z$, the *composite* $T \circ S\colon X \nrightarrow Z$ is defined by taking $T \circ S = T \cup_Y S$. And given $S_1\colon X_1 \nrightarrow Y_1$ and $S_2\colon X_2 \nrightarrow Y_2$, the *disjoint union* $S_1 \sqcup S_2\colon X_1 \sqcup X_2 \nrightarrow Y_1 \sqcup Y_2$ is defined in the evidence way.

Definition 8 (2-cells)**.** Let $S\colon X \nrightarrow Y$ and $S'\colon X' \nrightarrow Y'$ be h-graph cobordisms. A *2-cell* $\varphi\colon S \Rightarrow S'$ consists of three homotopy equivalences

$$\varphi_X\colon X \xrightarrow{\simeq} X'$$
$$\varphi_Y\colon Y \xrightarrow{\simeq} Y'$$
$$\varphi_S\colon S \xrightarrow{\simeq} S'$$

compatible with the maps defining the h-graph cobordisms.

Example 9. There is a 2-cell

$$(c \sqcup i) \circ w \Rightarrow i.$$

For the domain and range are

$(c \sqcup i) \circ w$ $\qquad i$

and there is a homotopy equivalence between them that preserves the two copies of pt. If we regard w as a coproduct, then this 2-cell shows that we can regard c as a counit for w.

Exercise 10. Let $l\colon \mathrm{pt} \nrightarrow S^1$ and $d\colon S^1 \nrightarrow \mathrm{pt}$ be as follows.

$l\colon \mathrm{pt} \nrightarrow S^1$ $\qquad d\colon S^1 \nrightarrow \mathrm{pt}$

Find 2-cells

$d \circ l \Rightarrow i \qquad (d \sqcup d) \circ W \Rightarrow w \circ d \qquad c \circ d \Rightarrow C \qquad W \circ l \Rightarrow (l \sqcup l) \circ w \qquad C \circ l \Rightarrow c$

and interpret them algebraically.

2. Homotopy automorphisms

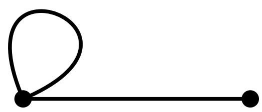
Definition 11 (Homotopy automorphisms)**.** Let $S\colon X \nrightarrow Y$ be an h-graph cobordism, determined by maps $i\colon X \to S$ and $j\colon Y \to S$. The *monoid of homotopy automorphisms of* S, denoted $\mathrm{hAut}(S)$, is defined to be the space of all homotopy equivalences

$$\alpha\colon S \longrightarrow S$$

that satisfy $\alpha \circ i = i$ and $\alpha \circ j = j$. It is a topological monoid under composition. Since $\mathrm{hAut}(S)$ is a monoid, so is $\pi_0(\mathrm{hAut}(S))$, and there is a homomorphism of monoids $\mathrm{hAut}(S) \to \pi_0(\mathrm{hAut}(S))$.

Proposition 12. *The morphism* $\mathrm{hAut}(S) \xrightarrow{\simeq} \pi_0(\mathrm{hAut}(S))$ *is a homotopy equivalence, and the monoid* $\pi_0(\mathrm{hAut}(S))$ *is a group.*

Example 13. Let $Q\colon \mathrm{pt} \nrightarrow \mathrm{pt}$ denote the following h-graph cobordism.



Then

$$\pi_0(\mathrm{hAut}(Q)) \cong \mathbb{Z} \ltimes \{\pm 1\}$$

where $\{\pm 1\}$ acts on $\mathbb{Z}$ by multiplication. The generator of $\{\pm 1\}$ corresponds to a homotopy automorphism that fixes the arc and flips the circle. The generator of $\mathbb{Z}$ corresponds to a homotopy automorphism that wraps the arc around the circle, and fixes the circle.

Exercise 14. Let $I\colon S^1 \to S^1$ denote the cylinder h-graph cobordism from Example 6. Show that

$$\pi_0(\mathrm{hAut}(I)) \cong \mathbb{Z}.$$

Hint: Choose a path in I that travels directly from the incoming S^1 to the outgoing S^1, and consider the effect of a homotopy automorphism on this path.

Example 15. If $S\colon X \nrightarrow Y$ is obtained from a surface in as in Example 6, then a variant of the Dehn-Nielsen-Baer Theorem shows that the map $\mathrm{Diff}(S) \to \mathrm{hAut}(S)$ is a homotopy equivalence, and so $\pi_0(\mathrm{hAut}(S))$ is the mapping class group of S.

Example 16. If we modify the cobordism $Q\colon \mathrm{pt} \nrightarrow \mathrm{pt}$ of Example 13 so that it features a wedge of n circles, then $\pi_0(\mathrm{hAut}(Q))$ is the *holomorph* $F_n \ltimes \mathrm{Aut}(F_n)$ of the free group F_n on n letters.

Example 17. The free groups with boundary $A^s_{n,k}$ of Hatcher and Wahl [5] occur as $\pi_0(\mathrm{hAut}(S))$ for h-graph cobordisms $S\colon X \nrightarrow Y$ where S is a graph and X and Y are unions of circles and points.

When we can form composites and disjoint unions, there are maps

$$\mathrm{hAut}(T) \times \mathrm{hAut}(S) \longrightarrow \mathrm{hAut}(T \circ S)$$
$$\mathrm{hAut}(S_1) \times \mathrm{hAut}(S_2) \longrightarrow \mathrm{hAut}(S_1 \sqcup S_2)$$

And if there is a 2-cell $\varphi\colon S \Rightarrow S'$, then there is a zig-zag of homotopy equivalences of monoids

$$\mathrm{hAut}(S) \xleftarrow{\simeq} \mathcal{H} \xrightarrow{\simeq} \mathrm{hAut}(S').$$

3. HOMOLOGICAL H-GRAPH FIELD THEORIES

Definition 18. A *homological h-graph field theory* or *HHGFT* ϕ consists of a (strong) symmetric monoidal functor ϕ_* from the category of h-graphs and homotopy equivalences into the category of graded $\mathbb{F}$-vector spaces, and for each h-graph cobordism $S\colon X \nrightarrow Y$ a map

$$\phi(S)\colon H_*(B\mathrm{hAut}(S))) \otimes \phi_*(X) \longrightarrow \phi_*(Y).$$

These data are required to be compatible with *2-cells*, *composition*, *identity*, and *disjoint unions*.

Remark 19. The definition of what it means for ϕ_* to be symmetric monoidal can be read in chapters VII and XI of [8]. In particular ϕ_* gives us a graded vector space $\phi_*(X)$ for every h-graph X, and an isomorphism $\phi_*(X) \otimes \phi_*(Y) \cong \phi_*(X \sqcup Y)$ for any two h-graphs X and Y.

Remark 20. What does an HHGFT tell us? One point of view is that it is a rich algebraic structure on the spaces $\phi_*(X)$, which in our example will be the homology groups $H_*(BG^X)$. Another is that it gives us information about the spaces $H_*(B\mathrm{hAut}(S))$, which are interesting in their own right.

Definition 21 (Degree-0 operations)**.** Let ϕ be an HHGFT. Let $S\colon X \nrightarrow Y$ be an h-graph cobordism. The *degree-0 operation*

$$\phi_S\colon \phi_*(X) \longrightarrow \phi_*(Y)$$

is defined by $\phi_S(a) = \phi(S)(1 \otimes a)$ for $a \in \phi_*(X)$, where $1 \in H_0(B\mathrm{hAut}(S))$ denotes the canonical generator.

Proposition 22. *Degree-0 operations satisfy the following compatibilities.*

- *Given a 2-cell $\varphi\colon S \Rightarrow S'$, the square*

$$\begin{array}{ccc} \phi_*(X) & \xrightarrow{\phi_S} & \phi_*(Y) \\ \downarrow & & \downarrow \\ \phi_*(X') & \xrightarrow[\phi'_S]{} & \phi_*(Y') \end{array}$$

  commutes. (The unlabelled maps are obtained by applying the functor ϕ_* to the homotopy equivalences φ_X and φ_Y.)
- *Given h-graph cobordisms $X \overset{S}{\nrightarrow} Y \overset{T}{\nrightarrow} Z$, we have $\phi_{T\circ S} = \phi_T \circ \phi_S$.*
- *Given h-graph cobordisms $S_1\colon X_1 \nrightarrow Y_1$ and $S_2\colon X_2 \nrightarrow Y_2$, the following diagram commutes.*

$$\begin{array}{ccc} \phi_*(X_1) \otimes \phi_*(X_2) & \xrightarrow{\phi_{S_1} \otimes \phi_{S_2}} & \phi_*(Y_1) \otimes \phi_*(Y_2) \\ \downarrow & & \downarrow \\ \phi_*(X_1 \sqcup X_2) & \xrightarrow[\phi_{S_1 \sqcup S_2}]{} & \phi_*(Y_1 \sqcup Y_2) \end{array}$$

(The unlabelled maps come from monoidality of ϕ_*.)

- *Given an h-graph X, and $X\times[0,1]\colon X \nrightarrow X$ the 'cylinder' cobordism, then $\phi_{X\times[0,1]}$ is the identity map.*

Example 23 ($\phi_*(\mathrm{pt})$ is a Frobenius algebra)**.** Let ϕ be an HHGFT. Then $\phi_*(\mathrm{pt})$ admits the structure of a non-unital commutative Frobenius algebra. The operations, obtained using the h-graph cobordisms m, w and i of Example 5, are as follows.

Product: $\phi_*(\mathrm{pt})\otimes\phi_*(\mathrm{pt})\cong\phi_*(\mathrm{pt}\sqcup\mathrm{pt})\xrightarrow{\phi_m}\phi_*(\mathrm{pt})$

Coproduct: $\phi_*(\mathrm{pt})\xrightarrow{\phi_w}\phi_*(\mathrm{pt}\sqcup\mathrm{pt})\cong\phi_*(\mathrm{pt})\otimes\phi_*(\mathrm{pt})$

Counit: $\phi_*(\mathrm{pt})\xrightarrow{\phi_c}\phi_*(\emptyset)\cong\mathbb{F}$

The algebraic properties required to make $\phi_*(\mathrm{pt})$ into a Frobenius algebra now all follow from a combination of algebraic properties of the degree 0 operations and properties of the cobordisms m, w and c.

Definition 24 (Higher operations)**.** If we have an h-graph cobordism $S\colon X\nrightarrow Y$ and an element $\sigma\in H_i(B\mathrm{hAut}(S))$ with $i>0$, then the associated *higher operation*

$$\phi_*(X)\longrightarrow\phi_{*+i}(Y)$$

is defined by $a\mapsto\phi(S)(\sigma\otimes a)$.

Example 25. The *BV operator* $\Delta\colon\phi_*(S^1)\to\phi_{*+1}(S^1)$ is defined to be the higher associated to the generator $\sigma\in H_1(B\mathrm{hAut}(S^1\times[0,1]))\cong\mathbb{Z}$. It satisfies $\Delta\circ\Delta=0$.

4. String topology of BG

Theorem 26 (Hepworth-Lahtinen)**.** *Let G be a finite group. Then there is an HHGFT ϕ for which $\phi_*(X)=H_*(BG^X)$ for any h-graph X, where BG^X denotes the space of all maps from X to BG. In particular, $\phi_*(\mathrm{pt})=H_*(BG)$ and $\phi_*(S^1)=H_*(\Lambda BG)$.*

We will not prove the theorem. Instead, we will just sketch how to construct the *string topology operation*

$$\phi(S)\colon H_*(B\mathrm{hAut}(S))\otimes H_*(BG^X)\longrightarrow H_*(BG^Y).$$

The h-graph cobordism $S\colon X\nrightarrow Y$ is determined by maps

$$X\xrightarrow{i}S\xleftarrow{j}Y$$

which induce restriction maps

$$BG^X\longleftarrow BG^S\longrightarrow BG^Y.$$

There is a parameterised version

$$B\mathrm{hAut}(S)\times BG^X\xleftarrow{\alpha}B\mathrm{hAut}(S)\times_{\mathrm{twisted}}BG^S\xrightarrow{\beta}B\mathrm{hAut}(S)\times BG^Y$$

(the symbol in the middle is just a name) consisting of spaces and maps over $B\mathrm{hAut}(S)$ whose fibres are in some sense copies of the previous sequence, twisted according to the action of $\mathrm{hAut}(S)$. It turns out that the homotopy fibres of α have the homotopy type of finite sets, and so there is a *transfer map*

$$\alpha^*\colon H_*(B\mathrm{hAut}(S)\times BG^X)\longrightarrow H_*(B\mathrm{hAut}(S)\times_{\mathrm{twisted}}BG^S).$$

Then $\phi(S)$ is defined to be the composite

$$\begin{aligned} H_*(B\mathrm{hAut}(S)) \otimes H_*(BG^X) &\xrightarrow{\times} H_*(B\mathrm{hAut}(S) \times BG^X) \\ &\xrightarrow{\beta_* \circ \alpha^*} H_*(B\mathrm{hAut}(S) \times BG^Y) \\ &\xrightarrow{(\pi_2)_*} H_*(BG^Y). \end{aligned}$$

The rest of these lectures will tell you how to really compute these things.

5. Homotopy quotients

Definition 27 (Pairs)**.** We will consider pairs (X,G) consisting of a (discrete) group G and a G-set X, i.e. a set X with an action of G. A *map of pairs* $(X,G) \to (Y,H)$ consists of $f\colon X \to Y$ and $\varphi\colon G \to H$ satisfying $f(g \cdot x) = \varphi(g) \cdot f(x)$ for $x \in X$ and $g \in G$.

Definition 28 (Homotopy quotient)**.** Let G be a (discrete) group and let X be a G-set, or in other words a set with G-action. The *homotopy quotient* $X//G$ is defined by

$$X//G = (EG \times X)/G$$

where G acts diagonally on $EG \times X$. The assignment $(X,G) \mapsto X//G$ is functorial.

Example 29. $\mathrm{pt}//G = BG$

Example 30. Let $H \subset G$ and consider the G-set G/H. There is a map of pairs $(\mathrm{pt}, H) \to (G/H, G)$ determined the inclusion $H \hookrightarrow G$ and the map $\mathrm{pt} \to G/H$ that sends the point to eH. The induced map

$$BH = \mathrm{pt}//H \xrightarrow{\simeq} (G/H)//G$$

is a homotopy equivalence.

Example 31. Let G be a group and let X be an arbitrary G-set. Given $x \in X$ we write Gx for the orbit and G_x for the stabiliser. Then

$$X = \bigsqcup Gx \cong \bigsqcup G/G_x,$$

so that

$$X//G \simeq \bigsqcup BG_x,$$

where x ranges over a set of orbit representatives.

6. How to understand BG^X

Definition 32 (Basepoints)**.** A set of *basepoints* for an h-graph X is a finite subset $P \subset X$ that contains at least one point in each path component and for which $P \hookrightarrow X$ is a cofibration. A *map of h-graphs with basepoints* $f\colon (X,P) \to (Y,Q)$ is a map $f\colon X \to Y$ sending P into Q.

Definition 33 (Fundamental groupoid etc.)**.** Let X be an h-graph with basepoints $P \subset X$.

- $\Pi_1(X,P)$ denotes the fundamental groupoid of X with basepoints in P. Its objects are the points of P and its morphisms are the path-homotopy classes of paths in X with basepoints in P.

- G^P is the set of functions $g\colon P \to G$. It is a group under pointwise multiplication. (It is the product of $\#P$ copies of G.)
- $G^{\Pi_1(X,P)}$ is the set of functions $f\colon \mathrm{Mor}(\Pi_1(X,P)) \to G$ satisfying $f(\delta\cdot\gamma) = f(\delta)f(\gamma)$ whenever δ,γ are composable morphisms in $\Pi_1(X,P)$.
- G^P acts on $G^{\Pi_1(X,P)}$ as follows. Given $g \in G^P$ and $f \in G^{\Pi_1(X,P)}$, we define $g\cdot f$ by the rule $(g\cdot f)(\gamma) = g(q)f(\gamma)g(p)^{-1}$ for $\gamma\colon p \to q$ in $\Pi_1(X,P)$.
- A map of h-graphs with basepoints $f\colon (X,P) \to (Y,Q)$ induces a map of pairs $(G^Q, G^{\Pi_1(Y,Q)}) \to (G^P, G^{\Pi_1(X,P)})$.

Theorem 34 (Model of the mapping space)**.** *Let X be an h-graph with basepoints P. Then there is a zig-zag of homotopy equivalences*

$$BG^X \longleftrightarrow G^P/\!/G^{\Pi_1(X,P)}.$$

Given $f\colon (X,P) \to (Y,Q)$, the "square"

$$\begin{array}{ccc} BG^Y & \longrightarrow & BG^X \\ \simeq\updownarrow & & \updownarrow\simeq \\ G^Q/\!/G^{\Pi_1(Y,Q)} & \longrightarrow & G^P/\!/G^{\Pi_1(X,P)} \end{array}$$

commutes.

Example 35. Let us study $\Lambda BG = \mathrm{Map}(S^1, BG)$. Let G^{ad} denote G with conjugation action. We apply Theorem 34 to $X = S^1$ and $P = \{p\}$ where p is any point of S^1. We let γ denote the path homotopy class of a loop that starts and ends at p and travels once around S^1. This gives us isomorphisms

$$G^P \xrightarrow{\cong} G,\ \ g \mapsto g(p) \qquad\qquad G^{\Pi_1(S^1,P)} \xrightarrow{\cong} G^{\mathrm{ad}},\ \ f \mapsto f(\gamma)$$

that form a map of pairs. So Theorem 34 gives a homotopy equivalence

$$\Lambda BG \simeq G^{\mathrm{ad}}/\!/G.$$

Next we study the right hand side in more detail. The orbits of G on G^{ad} are precisely the conjugacy classes of G, and the stabilizer of $h \in G^{\mathrm{ad}}$ is the centralizer $C(h)$. So $G^{\mathrm{ad}} \cong \bigsqcup G/C(h)$, and consequently

$$G^{\mathrm{ad}}/\!/G \cong \bigsqcup BC(h)$$

where h ranges over a set of representatives for the conjugacy classes of G.

Exercise 36. Describe $BG^{S^1\vee S^1}$ in a similar way.

7. How to compute higher operations

Now we will explain how to compute the higher operation

$$\phi(S)\colon H_*(B\mathrm{hAut}(S)) \otimes H_*(BG^X) \longrightarrow H_*(BG^Y)$$

associated to an h-graph cobordism $S\colon X \nrightarrow Y$. Choosing basepoints $P \subset X$ and $Q \subset Y$, and writing $A_S = \pi_0(\mathrm{hAut}(S))$, Theorem 34 and Proposition 12 allow us to replace all the terms in the domain and range of $\phi(S)$ to obtain an operation

$$\overline{\phi}(S)\colon H_*(BA_S) \otimes H_*(G^{\Pi_1(X,P)}/\!/G^P) \longrightarrow H_*(G^{\Pi_1(Y,Q)}/\!/G^Q),$$

and this is what we will describe.

Proposition 37 (Higher operations)**.** *Let $S\colon X \twoheadrightarrow Y$ be an h-graph cobordism. Then $\overline{\phi(S)}$ is given by the composition*

$$\begin{aligned} H_*(BA_S)\otimes H_*(G^{\Pi_1(X,P)}/\!/G^P) &\xrightarrow{\ \times\ } H_*(G^{\Pi_1(X,P)}/\!/A_S\times G^P)\\ &\xrightarrow{\ \alpha^*\ } H_*(G^{\Pi_1(S,P)}/\!/A_S\times G^P)\\ &\xrightarrow{\ \beta_*^{-1}\ } H_*(G^{\Pi_1(S,P\sqcup Q)}/\!/A_S\times G^{P\sqcup Q})\\ &\xrightarrow{\ \gamma_*\ } H_*(G^{\Pi_1(Y,Q)}/\!/G^Q)\end{aligned}$$

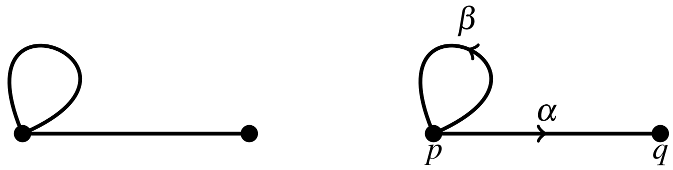
where α, β and γ are the following maps

$$\begin{array}{ccccc} G^{\Pi_1(X,P)}/\!/A_S\times G^P & \xleftarrow{\ \alpha\ } & G^{\Pi_1(S,P)}/\!/A_S\times G^P & & \\ & & \beta\uparrow\simeq & & \\ & & G^{\Pi_1(S,P\sqcup Q)}/\!/A_S\times G^{P\sqcup Q} & \xrightarrow{\ \gamma\ } & G^{\Pi_1(Y,Q)}/\!/G^Q \end{array}$$

and α^* is the transfer map associated to α, which is a finite covering space.

By choosing representatives for the orbits of each of the actions appearing in this diagram, we can rewrite the terms as disjoint unions of classifying spaces of groups. In particular, the representatives can be chosen so that β is replaced with an isomorphism, and so can be explicitly inverted.

Example 38. Let $Q\colon \mathrm{pt} \twoheadrightarrow \mathrm{pt}$ denote the h-graph cobordism on the left.



We will study the operation

$$\overline{\phi(S)}\colon H_*(BA_Q)\otimes H_*(BG) \longrightarrow H_*(BG),$$

or rather the simpler

$$\psi\colon H_*(B\{\pm 1\})\otimes H_*(BG) \longrightarrow H_*(BG)$$

obtained using the inclusion $\{\pm 1\}\to A_S$ that sends -1 to $[h]$. Here $h\colon S\to S$ is a homotopy automorphism whose effect on α and β is given by $\alpha\mapsto\alpha\circ\beta$, $\beta\mapsto\beta^{-1}$. To describe ψ we need some new notation.

- The *extended conjugacy class* $\mathrm{cl}^e(h)$ of an element $h\in G$ is the subset of G consisting of the conjugates of h and h^{-1}.
- The *extended centralizer* $C^e(h)$ of an element $h\in G$ is the subgroup of $\{\pm 1\}\times G$ consisting of pairs (ε,g) such that $h = gh^{\varepsilon}g^{-1}$.
- $\xi_h\colon C^e(h)\to\{\pm 1\}\times G$ is defined by $(\varepsilon,g)\mapsto g$.
- $\eta_h\colon C^e(h)\to G$ is defined by $(1,g)\mapsto g$ and $(-1,g)\mapsto gh^{-1}$.

Then ψ is given by the composite

$$\begin{aligned} H_*(B\{\pm 1\}) \otimes H_*(BG) &\xrightarrow{\times} H_*(B\{\pm 1\} \otimes BG) \\ &\xrightarrow{\oplus \xi_h^*} \bigoplus H_*(BC^e(h)) \\ &\xrightarrow{\oplus \eta_{h*}} H_*(BG) \end{aligned}$$

where h ranges over a set of extended conjugacy class representatives. Now let's prove the claim. The composite from Proposition 37 takes the form:

$$\begin{array}{ccc} \mathrm{pt}//\{\pm 1\} \times G^p & \xrightarrow{\alpha} & G^\beta//\{\pm 1\} \times G^p & & \\ & & \beta \uparrow \simeq & & \\ & & G^\alpha \times G^\beta//\{\pm 1\} \times G^p \times G^q & \xrightarrow{\gamma} & \mathrm{pt}//G^q \end{array}$$

The action of $\{\pm 1\} \times G^p$ on G^β is given by $(\varepsilon, g_p) \cdot h_\beta = g_p h_\beta^\varepsilon g_p^{-1}$. So the orbit of an element $h \in G^\beta$ is exactly $\mathrm{cl}^e(h)$, and the stabilizer is exactly $C^e(h)$. So we may replace $G^\beta//\{\pm 1\} \times G^p$ with $\bigoplus BC^e(h)$, and we may replace α with $\bigsqcup \xi_h$. The action of $\{\pm 1\} \times G^p \times G^q$ on $G^\alpha \times G^\beta$ is given by

$$\begin{aligned} (1, g_p, g_q) \cdot (h_\alpha, h_\beta) &= (g_q h_\alpha g_p^{-1}, g_p h_\beta g_p^{-1}), \\ (-1, g_p, g_q) \cdot (h_\alpha, h_\beta) &= (g_q h_\alpha h_\beta g_p^{-1}, g_p h_\beta g_p^{-1}). \end{aligned}$$

If $h_1, \ldots, h_n$ are representatives for the extended conjugacy classes, then $(1, h_1), \ldots, (1, h_n)$ are representatives for the orbits of this action. And the stabilizer of a representative $(1, h)$ is given by

$$\overline{C^e(h)} = \{(1, g, g) \mid g \in C^e(h)\} \cup \{(-1, g, gh^{-1}) \mid g \in C^e(h)\}.$$

So we can replace the composite above with the following.

$$\begin{array}{ccc} B\{\pm 1\} \times BG^p & \xrightarrow{\bigsqcup \xi_h} & \bigsqcup BC^e(h) & & \\ & & \beta' \uparrow \cong & & \\ & & \bigsqcup B\overline{C^e(h)} & \xrightarrow{\gamma'} & BG^q \end{array}$$

The sums are all taken over a sequence h of representatives for the extended conjugacy classes. And β' and γ' are the maps induced by restricting the projections $\{\pm 1\} \times G^p \times G^q \to \{\pm 1\} \times G^p$ and $\{\pm 1\} \times G^p \times G^q \to G^q$. The former is the evident isomorphism $\overline{C^e(h)} \to C^e(h)$. And the latter amounts to the union of the η_h.

Example 39 (A nonzero higher operation)**.** In the last example, let us take $G = \{\pm 1\}$ Then there are two extended conjugacy classes $\{1\}$ and $\{-1\}$, and $C^e(1) = C^e(-1) = \{\pm 1\} \times \{\pm 1\}$. The maps ξ_1 and ξ_{-1} are both the identity. The map η_1 is projection onto the second factor, and η_{-1} is the addition map. So in this case ψ is

$$H_*(B\{\pm 1\}) \otimes H_*(B\{\pm 1\}) \xrightarrow{\times} H_*(B\{\pm 1\} \times \{\pm 1\}) \xrightarrow{\mathrm{id}_* \oplus \mathrm{add}_*} H_*(B\{\pm 1\})$$

Choosing $\mathbb{F} = \mathbb{F}_2$ to be the field with two elements, we find that

$$H_*(B\{\pm 1\}) = \mathbb{F}_2\{a_0, a_1, \ldots\}$$

where a_i is a generator in degree i, and the operation is given by

$$a_i \otimes a_j \mapsto \begin{cases} a_{i+j} & \text{if } i > 0 \text{ and } i, j \text{ share no 1s in their binary expansions} \\ 0 & \text{otherwise} \end{cases}$$

In particular, for each $i > 0$ there are infinitely many j such that ψ sends $a_i \otimes a_j$ to a nonzero element.

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