UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICS

| | |
|---|---|
| Author | Hepworth |
| Date | January 2016 |
| Code | |

MX4540 KNOTS
QUESTION BOOKLET

## 1. AN INTRODUCTION TO KNOTS AND LINKS

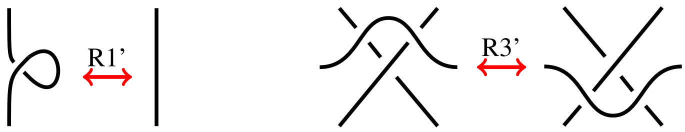
**1.** Draw diagrams of the unknot with 5, 10 and 15 crossings. Draw them such that it is not clear that they are unknots.

**2.** Look at the knots $3_1$, $5_1$ and $7_1$. Draw the knot $(2n+1)_1$ for a natural number $n$.

**3.** Look at the knots $4_1$, $6_1$ and $8_1$. Draw the knot $(2n+2)_1$ for a natural number $n$.

**4.** Given an integer $n$ construct a two component link with linking number equal to $n$.

**5.** Consider the following diagram moves R1' and R3'.




How do they differ from R1 and R3? Show that they can be obtained as a sequence of Reidemeister moves. List the sequence of Reidemeister moves that you use.

**6.** Determine whether the following are **true** or **false**. Prove the true ones and find a counterexample for each false one.

**(a)** A diagram with odd number of crossings represents a knot (and not a link with multiple components).

**(b)** The linking number does not depend on the choice of orientations of the components of a link.

**(c)** Every link with two crossings is trivial.

**(d)** Every knot with two crossings is trivial.

**7.** Show that the trefoil knot $3_1$, equipped with any orientation, is equivalent to its reverse.

**8.** Show that the figure-eight knot $4_1$ is equivalent to its mirror $m(4_1)$.

**9.** Consider the Hopf link. By choosing different orientations for the two components, and by taking mirrors, one can obtain eight different oriented links.

Draw them all. Which pairs are equivalent, and which are not? Justify your answer.

**10.** The *crossing number* of a knot $K$, denoted $c(K)$, is defined to be the minimum, taken over all diagrams of $K$, of the number of crossings in the diagram.

Let $K_1$ and $K_2$ be oriented knots. What is the relationship between $c(K_1)$, $c(K_2)$ and $c(K_1 \# K_2)$?

**11.** Show by example that the ‘sum of oriented links’ is ambiguous.

**12.** $(*)$ Let $L$ be a link of two components. Show that $\mathrm{lk}(L)$ is an integer.

**13.** $(*)$ Show that the two trefoil diagrams from Example 1.2 are equivalent. List the sequence of Reidemeister moves.

**14.** $(*)$ Show that any knot diagram can be turned into a diagram of the unknot by changing some of the crossings from *over* to *under* and vice versa.

## 2. The colouring of knots and links

**15.** Examine the colouring for all knots with up to 6 crossings, *including* all those with 6 crossings. (In other words, find the integers $n$ for which the knot can be coloured modulo $n$.)

**16.** Examine the colouring for the following links. (In other words, find the integers $n$ for which the link can be coloured modulo $n$.)

**(a)** The Hopf link.

**(b)** The Whitehead link.

**(c)** The Borromean rings.

**(d)** The link with two unknotted components and linking number $m \in \mathbb{N}$.

**17.** Determine which sentences are **true** and which are **false**. Prove the true ones and find a counterexample for each false one.

**(a)** There is a link that: is not splittable; has 4 components; and has linking number 0.

**(b)** There is a link that: is not splittable; has 2 components; and can be coloured modulo $n$ for some *odd* $n$.

**(c)** If $L$ is a link with linking number zero then it is not splittable.

**(d)** For any integer $n > 2$ there exists a $n$-component link with linking number zero which is not splittable.

**18.** Investigate the colouring of all knots with seven crossings.

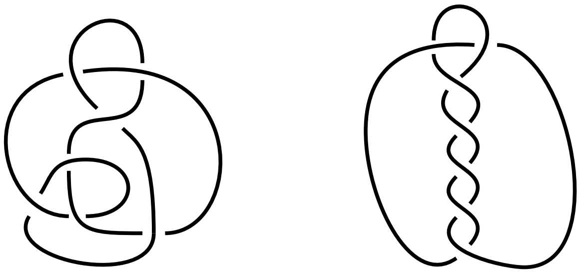
**19.** $(*)$ For which $n > 2$ is there a knot that can be coloured modulo $n$?

## 3. THE DETERMINANT OF KNOTS AND LINKS

**20.** Calculate the determinant of all knots up to seven crossings.

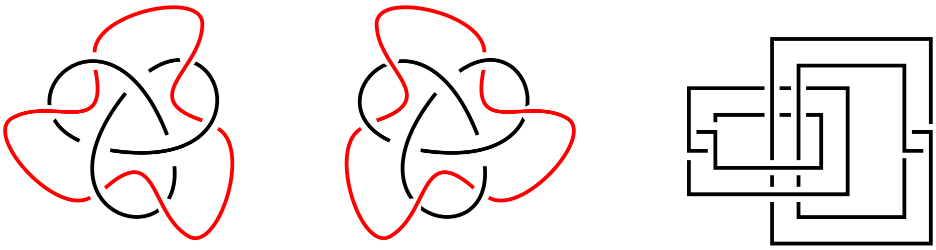
**21.** Calculate the determinant of the Hopf link, the Whitehead link and the Borromean rings.

**22.** Identify the following knots.



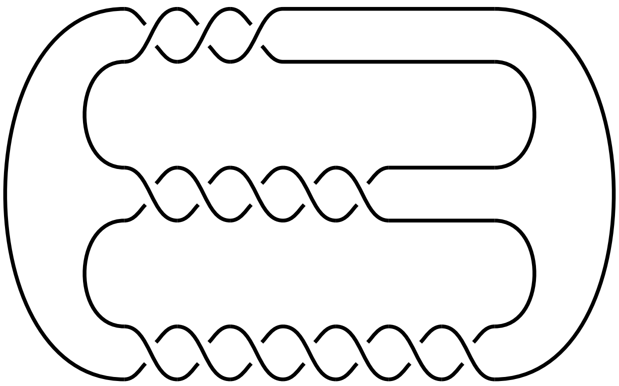
In other words, find knots in the `table of knots` to which they are equivalent.

**23.** Are the following links splittable?



('Yes', 'no' and 'maybe' are all acceptable answers — but give justification!)

**24.** $(*)$ Compute the determinant of the following knot.



What can you say about its colouring properties?

**25.** $(*)$ Calculate the determinant of the knots $10_{124}$ and $10_{153}$. What can you say about the colouring properties of these knots? *Remark: These are the simplest knots with the property in question.*

**26.** ($*$) Let $L$ be a link with diagram $D$. Let us write the crossing equations for $D$ in the form

$$a + b - 2c \equiv 0 \mod n.$$

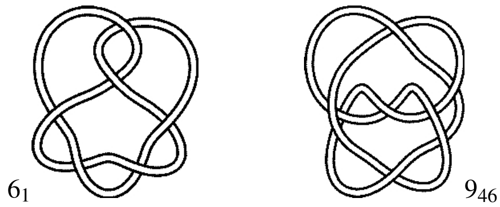
Under what conditions (on $D$) do these crossing equations sum to zero? Which is the first knot in the knot table for which the sum is not zero?

*Hint: A link is called* alternating *if it has a diagram with if the crossings alternate under, over, under, over, as you travel along each component of the link.*

## 4. The colouring group

**27.** Carefully read appendix A from the notes available online. You need the techniques presented there in order to answer the following questions.

**28.** For each of the knots $6_1$ and $9_{46}$




do the following.

**(a)** Write down the Goeritz matrix.

**(b)** Apply **A.1** to the Goeritz matrix to obtain a diagonal matrix.

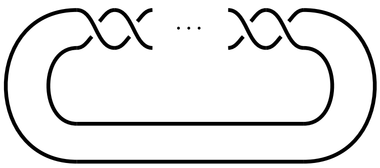
**(c)** Apply **A.2** to compute the determinant and the colouring group of the knot.

Are the knots equivalent? (The next question may help.)

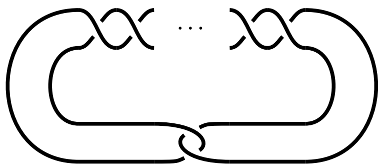
**29.** **(a)** Show that $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/ab\mathbb{Z}$ if $\gcd(a,b) = 1$.

**(b)** Show that $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \ncong \mathbb{Z}/a^2\mathbb{Z}$ if $a > 1$.

**30.** Compute the colouring group for the knots $(2n+1)_1$ and $(2n+2)_1$ drawn below.



$(2n+1)_1$



$(2n+2)_1$

The knots have a total of $(2n+1)$ and $(2n+2)$ crossings, respectively.

**31.** Prove that the third link from question 23 is not splittable by computing its colouring group.

## 5. The Alexander polynomial

**32.** Calculate the Alexander polynomial for $4_1$ and $5_1$.

**33.** Calculate the Alexander polynomial for the Hopf and Whitehead links.

**34.** Prove that $\Delta_L(t) \doteq \Delta_{rL}(t^{-1})$.

**35.** Let $K$ be an oriented knot, and suppose that there were a knot $J$ such that $J\#K$ were the unknot. What could you conclude about $\Delta_K(t)$?

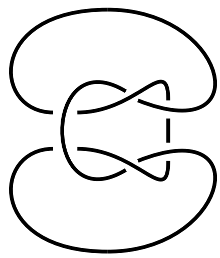
**36.** **(a)** Let $L$ be an oriented link. How are $\mathrm{col}(rL)$ and $\mathrm{col}(mL)$ related to $\mathrm{col}(L)$?

**(b)** Let $K$ and $J$ be oriented knots. Show that $\mathrm{col}(J\#K) \cong \mathrm{col}(J) \times \mathrm{col}(K)$.

**37.** Give an example of a knot or a link with the colouring group isomorphic to:

**(a)** $\mathbb{Z}$
**(b)** $\mathbb{Z}^2$
**(c)** $\mathbb{Z}^n$
**(d)** $\mathbb{Z}/3\mathbb{Z}$
**(e)** $\mathbb{Z}/5\mathbb{Z}$
**(f)** $\mathbb{Z}/7\mathbb{Z}$
**(g)** $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$
**(h)** $\mathbb{Z}/15\mathbb{Z}$
**(i)** $\mathbb{Z}/2\mathbb{Z}$

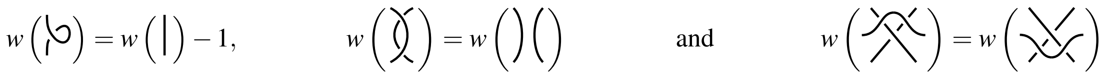
**38.** Express the following knot as a sum of nontrivial knots and determine its Alexander polynomial.



**39.** Find two knots with equal colouring group but with distinct Alexander polynomials.

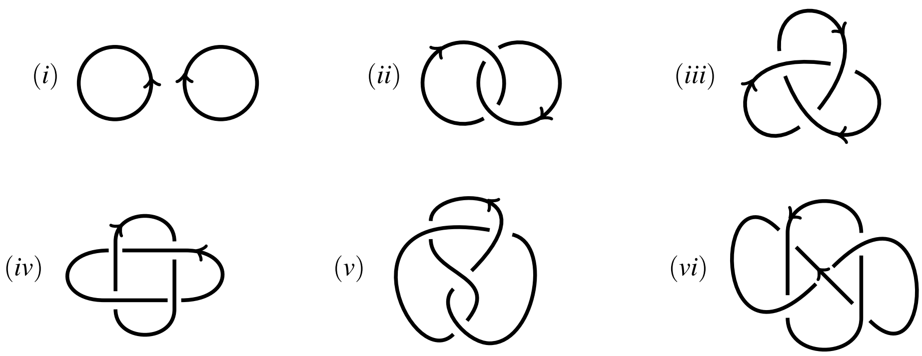
## 6. The Jones polynomial

**40.** Check that

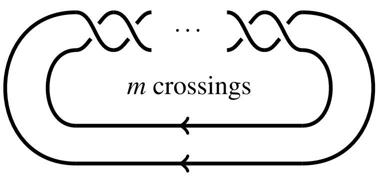



for every choice of orientations.

**41.** Using just the skein relation and the fact that $V(\bigcirc) = 1$, compute the Jones polynomial for all of the following knots and links.




**42.** Let $A_m$ denote the following oriented link.




**(a)** Identify $A_0, A_1$ and $A_2$. Identify $A_{2n+1}$.

**(b)** Use the skein relations to find a recursive formula for $V(A_m)$ when $m \geqslant 2$.

**(c)** Compute $V(A_m)$ for $m = 0, 1, 2, 3, 4$.

**43.** Determine the Jones polynomial for the knots of five crossings and their mirrors.

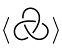
**44.** Show that the Jones polynomial of a link $L$ takes the value $(-2)^{\#L-1}$ when $t^{1/2} = 1$. Here $\#L$ denotes the number of components of $L$.

*Recall that one can convert a link diagram into a diagram of a trivial link by repeatedly changing crossings from 'over' to 'under'.*

**45.** Draw an oriented knot whose Jones polynomial is $-t^{-3} + t^{-2} - t^{-1} + 3 - t^{1} + t^{2} - t^{3}$.

**46.** Look at the `table of knots`. Each knot is drawn as if it consisted of two "parallel" thin strings. Think of this as of the diagram of a link with two components. Compute the linking number of this link in several examples. In general, what is the linking number of such a link?

## 7. The Jones polynomial and alternating knots

**47.** Use the state-sum formula to show that $\langle$  $\rangle = -A^5 - A^{-3} + A^{-7}$.

**48.** Let $f$ and $g$ be Laurent polynomials in some variable $X$.

**(a)** How are $M(f)$, $M(g)$ and $M(fg)$ related?

**(b)** How is $M(X^i f)$ related to $M(f)$?

**(c)** Let $F$ be the Laurent polynomial in a variable $Y$ obtained by setting $F = f|_{X=Y^k}$. Here $k$ is some positive number. How are $M(F)$ and $M(f)$ related?

**(d)** In the last part, suppose that $k$ is negative. What can you say now?

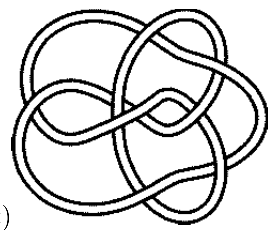
**49.** Let $L$ be a link and let $D$ be a connected diagram of $L$ with $n$ crossings. Show that

$$M(V(L)) \leqslant \frac{1}{4}(3w(D) + 2|s_-D| + n - 2) \qquad \text{and} \qquad m(V(L)) \geqslant \frac{1}{4}(3w(D) - 2|s_+D| + 2 - n)$$

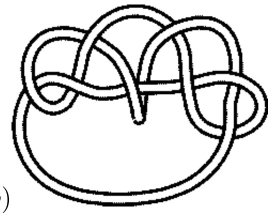
with equality if $D$ is reduced and alternating.

**50.** Let $D$ be a link diagram. Choose a chessboarding of $D$ and compute the sign of each crossing. Under what circumstances are all of the signs equal?

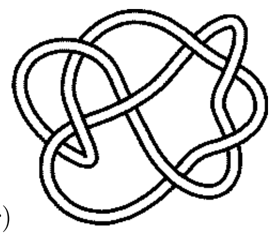
**51.** Try to match the knots to their Jones polynomials.



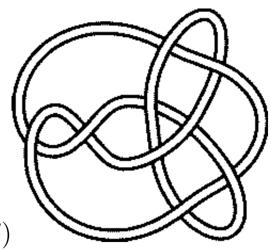
$(a)$



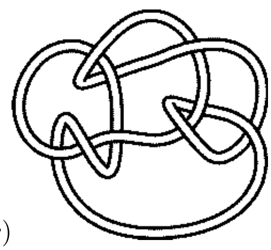
$(b)$



$(c)$



$(d)$



$(e)$

$(i)$ $t^{-1} - 3 + 4t - 4t^2 + 6t^3 - 4t^4 + 3t^5 - 2t^6$

$(ii)$ $-t^{-2} + 2t^{-1} - 2 + 4t - 4t^2 + 4t^3 - 3t^4 + 2t^5 - t^6$

$(iii)$ $t^2 - 2t^3 + 4t^4 - 4t^5 + 5t^6 - 4t^7 + 3t^8 - 2t^9$

$(iv)$ $-t^{-5} + 2t^{-4} - 4t^{-3} + 7t^{-2} - 7t^{-1} + 8 - 7t + 5t^2 - 3t^3 + t^4$

$(v)$ $-t^{-5} + 3t^{-4} - 5t^{-3} + 7t^{-2} - 8t^{-1} + 9 - 7t + 5t^2 - 3t^3 + t^4$

**52.** Let $K$ be a knot that admits a reduced, alternating diagram with an odd number of crossings. Show that $K$ is not equivalent to its mirror $m(K)$.

**53.** In each of the following situations, give an example of a link $L$ and a diagram $D$ of $L$ with the following properties:

**(a)** $D$ is reduced and alternating with $n$ crossings, but $span(V(L)) \neq n$.

**(b)** $D$ is connected and alternating with $n$ crossings, but $span(V(L)) \neq n$.

**(c)** $D$ is reduced and connected with $n$ crossings, but $span(V(L)) \neq n$.

**54.** Let $c(K)$ denote the *crossing number* of a knot $K$ — the smallest number of crossings in a diagram of $K$. Show that if $K_1$ and $K_2$ admit reduced alternating diagrams, then

$$c(K_1 \# K_2) = c(K_1) + c(K_2).$$

**55.** Let $L_+$, $L_-$ and $L_0$ be links that are related as in the skein relation. Define

$$\begin{aligned} m_+ &= \text{number of components of } L_+, \\ m_- &= \text{number of components of } L_-, \\ m_0 &= \text{number of components of } L_0. \end{aligned}$$

What is the relationship between $m_+$, $m_-$ and $m_0$?

**56.** Let $L$ be an oriented link, and let $D$ be an oriented diagram of $L$.

**(a)** Let $s$ be a state of $D$. Prove that $|s| - w(D)$ is *even*.

**(b)** Use the state-sum formula to prove that $(-A)^{-3w(D)}\langle D\rangle$ is a Laurent polynomial in $A^2$. (In other words, prove that it contains no terms of the form $A^m$ with $m$ odd.)

**(c)** Show that $V(L)$ is a Laurent polynomial in $t^{1/2}$. (In other words, prove that it contains no terms of the form $t^{m/4}$ with $m$ odd.)

**(d)** Why were you asked this question?

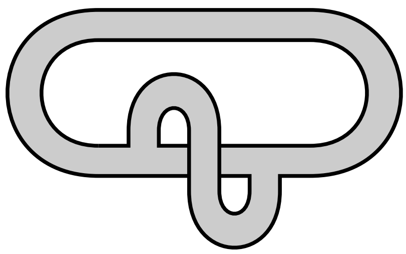
**57.** Let $D$ be a connected link diagram. Let $D_+$ and $D_-$ be the diagrams obtained from $D$ by smoothing some crossing in the positive and negative manner. Show that either $D_+$ or $D_-$ is connected.

## 8. The genus of a knot

**58.** Apply Seifert's algorithm to all knots with five crossings or fewer. In each case give the genus of the resulting surface.

**59.** In the previous question you applied Seifert's algorithm to the knots with 5 crossings or fewer. Estimate the genus of these knots. If there are any you cannot compute for certain, think about what information would help you to complete the answer.

**60.** Which knot is the boundary of the surface pictured below?



**61.** True or false?

**(a)** If $D_1$ and $D_2$ are diagrams representing the same knot then $g(\Sigma_{D_1}) = g(\Sigma_{D_2})$.

**(b)** Two knots that appear as the boundaries of homeomorphic surfaces are equivalent.

**(c)** The Seifert cycles of a diagram $D$ are just the boundaries of the shaded regions in some chessboarding of $D$.

**(d)** When applied to a knot diagram, Seifert's algorithm always results in at least two Seifert cycles.

**(e)** The surface produced by Seifert's algorithm depends on the choice of orientation of the knot.

**62.** Let $D$ be a knot diagram. Describe a state $s$ of $D$ such that $sD$ consists of the Seifert cycles of $D$.

**63.** In the lectures we proved that there are infinitely many prime knots by considering the knots $(2n+2)_1$. Does the same process work if we use the knots $(2n+1)_1$ instead?

**64.** **True** or **false**?

**(a)** There is a knot with genus 5912357.

**(b)** If $J$ and $K$ are knots, and $J$ is not the unknot, then $J\#K$ is not the unknot.

**(c)** There is a prime knot with genus greater than 1.

**65.** Estimate the genus of the following knot.