STRING TOPOLOGY OF CLASSIFYING SPACES

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These are the notes from a talk given at the meeting 'Algebraic Topology: Applications and New Directions' held in honour of Gunnar Carlsson, Ralph Cohen and Ib Madsen at Stanford, July 2012. The new results presented here are all joint work with Anssi Lahtinen. Mistakes, omissions and general chattiness are entirely my own.

Throughout the talk I will use $\mathbb{Z}/2$ coefficients. This allows me to ignore orientation issues and simplify the presentation. Be aware that the results stated here do *extend to general coefficients.*

Part I. String topology is the study of algebraic structures on objects like $H_*(LX)$, where X is an appropriate topological space and $LX = \mathrm{Map}(S^1, X)$ is the free loop space of X. The first such result was given by Chas-Sullivan in '99. They took $X = M$ a closed manifold, and showed that $H_*(LM)$ is a Batalin-Vilkovisky algebra. Godin was able to extend this to the following field-theoretic result.

Godin '07. *Let M be a closed manifold. Then $H_*(LM)$ is the value on S^1 of a homological conformal field theory of degree* $\dim(M)$.

Let us explain what this means. A *homological conformal field theory* of degree d, or HCFT for short, is an assignment that sends each 1-manifold X to a graded vector space $\mathcal{F}_*(X)$, and that sends each cobordism Σ from X to Y to a linear map

$$\mathcal{F}_*(\Sigma)\colon H_*(B\mathrm{Diff}(\Sigma)) \otimes \mathcal{F}_*(X) \longrightarrow \mathcal{F}_{*+d\chi(\Sigma,X)}(Y).$$

These must satisfy disjoint union and gluing conditions. Here $\mathrm{Diff}(\Sigma)$ is the group of orientation-preserving diffeomorphisms of Σ that fix every point in the incoming and outgoing boundaries, and $\chi(\Sigma, X)$ is the relative Euler characteristic.

The following parallel result of Chataur and Menichi was obtained at roughly the same time as Godin's.

Chataur-Menichi '07. *Let G be a compact Lie group. Then $H_*(L(BG))$ is the value on S^1 of a homological conformal field theory of degree* $-\dim(G)$.

As presented, the results of Godin and Chataur-Menichi are parallel. However there are some real qualitative differences between the two.

The first difference is in the type of surfaces that are permitted. For Godin the theory is *open-closed*, meaning that the 1-manifolds may have boundary and the cobordisms may have corners. (We call such cobordisms *open-closed cobordisms*. Boundary components that are neither incoming nor outgoing are called *free*.) For Chataur-Menichi the theory is *closed*, meaning that 1-manifolds must be closed, and that surfaces cannot have free boundary.

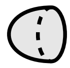
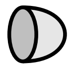
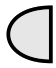
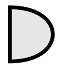
The second difference is that the cobordisms are also subject to *boundary conditions*. For Godin this boundary condition states that each component of Σ must have boundary that is either outgoing or free. For Chataur-Menichi the boundary condition states that each component of Σ must have both incoming and outgoing boundary.

So string topology of manifolds is in fact significantly more general than that of classifying spaces. The first new result of this talk closes the gap between the two, so far as it is possible.

H-Lahtinen '12. *Let G be a compact Lie group. Then $H_*(L(BG))$ is the value on S^1 of a homological conformal field theory of degree $-\dim(G)$.*

To explain the difference between this result and Chataur-Menichi's I must specify the surface type and boundary condition. Indeed, the theory is now open-closed, and the boundary condition states that each component of Σ must have incoming boundary. So in terms of generality, string topology of classifying spaces and of manifolds are now almost identical, but not quite.

The following table demonstrates which of the standard unit and counit cobordisms are admitted by the various theories discussed so far.

	 U_{S^1}	 C_{S^1}	 U_I	 C_I
Godin	✓	✗	✓	✓
Chataur-Menichi	✗	✗	✗	✗
H-Lahtinen	✗	✓	✗	✓

One might have hoped for a full row of checkmarks in each case, but that is not possible. For if an HCFT admits U_{S^1} and C_{S^1} then $\mathcal{F}_*(S^1)$ is finite-dimensional. But for us $\mathcal{F}_*(S^1)$ is either $H_*(LM)$ or $H_*(L(BG))$, so it is not possible to admit both checkmarks in the first and second columns. Similarly, if $\mathcal{F}_*$ admits the operations determined by U_I and C_I then $\mathcal{F}_*(I)$ is finite-dimensional. For Godin $\mathcal{F}_*(I) = H_*(M)$. But for us $\mathcal{F}_*(I) = H_*(BG)$, and so we can never hope to admit both U_I and C_I. So at least in this sense, our new theorem admits the best possible class of surfaces.

What is the use of these string topology HCFTs? In principle this is clear: they give us information about the diffeomorphism groups $\mathrm{Diff}(\Sigma)$ and about the manifold M or the classifying space BG. But in practice it is not so clear. For it is hard to pin down *any* classes in $H_*(B\mathrm{Diff}(\Sigma))$ outside the stable range, and within the stable range general results of Tamanoi state that the resulting operations vanish. (At least if M or G has positive dimension.) I take this as an indication that it is sensible to seek *structural* results in string topology before launching into serious computations. The next part of the talk will explain such a development: we will extend the field theory to one in which the geometry of surfaces is replaced by something homotopy-theoretic.

Part II. The aim in this second part is to present a homotopy-theoretic variant of HCFTs, and show that string topology of classifying spaces is in fact one of these new field-theories.

An *h-graph* is a space with the homotopy type of a finite graph. An *h-graph cobordism* $S\colon X \nrightarrow Y$ is a zig-zag

$$X \xrightarrow{\ i\ } S \xleftarrow{\ j\ } Y$$

in which X, Y and S are h-graphs, i and j are closed cofibrations, i is *positive* in the sense that it is surjective on path components, and j is an *h-embedding*, meaning that up to homotopy S can be obtained from Y by adding finitely many points and arcs. The *homotopy automorphism monoid of* S is the topological monoid $\mathrm{hAut}(S) = \{f\colon S \xrightarrow{\simeq} S \mid f|X \sqcup Y = \mathrm{Id}\}$. H-graphs, h-graph cobordisms, and their homotopy automorphisms are our homotopy-theoretical replacement for 1-manifolds, cobordisms, and their diffeomorphism groups.

A *homological h-graph field theory of degree d*, or HHGFT for short, is an assignment that sends each h-graph X to a graded vector space $\mathcal{F}_*(X)$, and that sends each h-graph cobordism $S\colon X \nrightarrow Y$ to a linear map

$$\mathcal{F}_*(S)\colon H_*(B\mathrm{hAut}(S)) \otimes \mathcal{F}_*(X) \longrightarrow \mathcal{F}_{*+d\chi(S,X)}(Y),$$

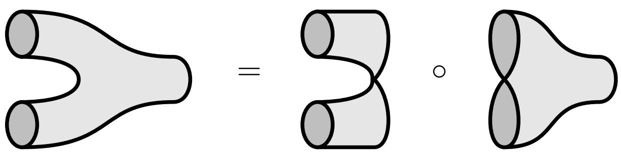
satisfying disjoint union and gluing conditions.

An HHGFT naturally restricts to an HCFT. For an open-closed cobordism Σ from X to Y, for which each component of Σ meets X, determines an h-graph cobordism $\Sigma\colon X \nrightarrow Y$ and a homomorphism $\mathrm{Diff}(\Sigma) \to \mathrm{hAut}(\Sigma)$.

H-Lahtinen '12. *Let G be a compact Lie group. There is a HHGFT whose value on an h-graph X is $H_*(BG^X)$. (Here BG^X denotes the space of all maps from X to BG.) It extends the HCFT of the previous theorem.*

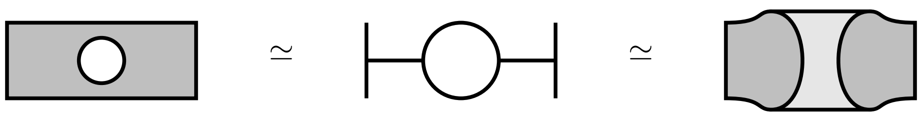
There are many consequences for an HCFT when it can be extended to an HHGFT in this way. Here I will list just a few of them, as an advertisement for the utility and novelty of h-graph methods.

First, we can obtain new factorisations of existing cobordisms, like so:



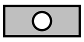
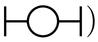
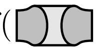
This decomposition is a reflection of the familiar definition of the Chas-Sullivan product: one goes from general pairs of loops to composable pairs of loops, and then one composes the loops.

Second, one obtains new equivalences between existing surfaces. Indeed, the two open-closed cobordisms below are not diffeomorphic, but they are both homotopy-equivalent to the TIE-fighter h-graph cobordism in the middle.



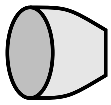
By a property of our h-graph field theories that I have not mentioned, it follows that the two open-closed cobordisms define the the same operations in the HCFT.

Third, and far more importantly, by passing from diffeomorphisms to homotopy automorphisms we obtain many more automorphisms:

Diff() $\simeq *$ hAut() $\simeq \mathbb{Z} \rtimes \mathbb{Z}/2$ Diff() $\simeq \mathbb{Z}$

In particular, after taking the homology classifying spaces we obtain many new homology classes, and so many new operations on $H_*(BG^I)$.

Fourth and finally, there are brand-new cobordisms, even between 1-manifolds:



Here the ordinary disc serves as an h-graph cobordism from S^1 to I.

Some of these new features express facts about string topology that we may already have known. The advantage is that we have expressed these facts *in terms of* the HHGFT structure. In other words h-graphs capture essential features of string topology, and they do so in a systematic way.

Part III. Let me end with some speculation.

First, a conjecture: *Godin's string topology of manifolds HCFT extends to an HHGFT.*

Second, Tamanoi has shown that there are often large families of elements of $H_*(B\mathrm{Diff}(\Sigma))$ that vanish identically in string topology. Such results are proved by combining functoriality of HCFTs with the degree-shifts present. Since HHGFTs give us significantly more functoriality than HCFTs, it is natural to ask: *Are there new Tamanoi-style vanishing results?*

Third, a famous theorem of Costello allows us to construct an HCFT from the data of a Calabi-Yau A_∞-algebra A. The value on S^1 is the Hochschild homology $HH_*(A)$. Wahl-Westerland have placed this result in a general framework applicable to any A_∞-algebra 'with structure'. *Can we obtain HHGFTs from A_∞ algebras with appropriate structure?*

Finally, Poirier-Rounds have shown how to define 'compactified' chain-level operations on $C_*(LM)$, using 'string diagrams'. These will hopefully recover and extend Godin's string topology operations. On this other hand this talk has explained how to extend the Chataur-Menichi operations in a quite different direction. *Can the notions of string diagram and h-graph be combined? Is this combined structure present in string topology of manifolds or of classifying spaces?*