How much direct sunlight falls on differently angled surfaces during one day?

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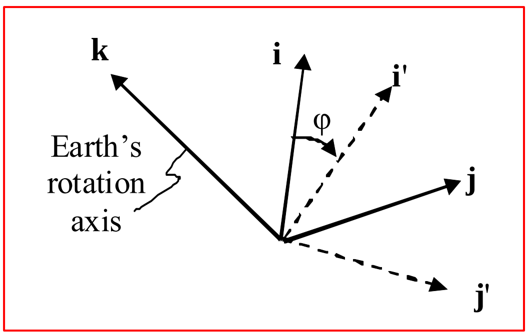
I've written this piece because I really wanted to know the answer. It's a piece for those who would like to see some numbers. To follow the workings you'll need some mathematical knowledge but the conclusions can be picked up by anyone who is prepared to take the working on trust.

The title question is an important one for siting passive solar energy collectors, either photovoltaics or heaters, that don't track the Sun's motion across the sky. Both these solar energy converters can produce some output from diffuse radiation, coming from the whole sky or reflected from the ground, but they produce most of their output from the sun shining directly onto them. They give maximum output when directly facing the sun and a reduced output when inclined. How much difference does inclination make? In one sense the answer is easy. All one needs to know is the angle between the perpendicular to the surface (known as *the normal*) and the direction of the sunlight. The reduction factor is the cosine of this angle. Problem solved. Finding the correct angle, though, isn't trivial.

Plenty of fine graphics programs will draw nice pictures of walls, roofs or complete houses and show the sun shining on them but they don't give you the numbers for the changing sunlight intercepted as the day progresses. Most domestic passive solar collectors will be fixed on walls or roofs whose orientations are themselves fixed. How much sunlight will be collected over a day? If there is the opportunity to change the orientation of a collector, what is the best orientation to make it? Over two months in the winter, for example, the sunlight in these parts (latitude 57º) is within 20º of the horizontal. Surely an almost vertical collector is best? We'll see.

Sunlight comes at different angles during the day because of the rotation of the Earth about its axis. The first problem we face is to calculate the effect of this. Mid-day will be taken as the time the Sun is due south (i.e. solar time is implied) and the approximation of 15º of Earth's rotation per hour is good enough to convert rotation angles to times on either side of mid-day. The rotation angle, called φ in the working below, is known as the *hour angle*.

Hand-waving arguments won't produce definite numbers. We need 3D geometry to get all the angles right and the discussion will use vectors to keep track of changing directions during the day. If you can follow the mathematics, do check it through for it's not copied from a book. What are you going to get for your pains? You'll see how the calculations can be made and there's nothing like seeing how to do something in detail to give understanding. Indeed, I got some myself while writing this. You'll see some northern hemisphere example results and be able to draw conclusions. You'll see how to make your own calculations for any different cases you would like to try, perhaps with figures more relevant to your own interests.



The axes

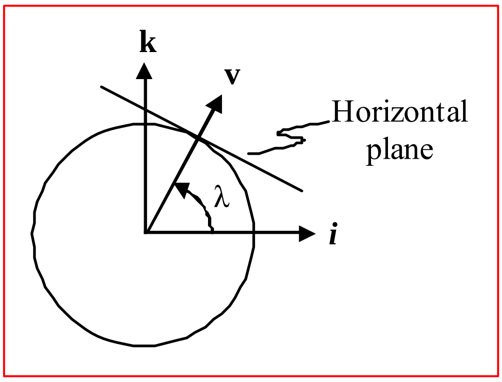
Choose 3D axes (**i**,**j**,**k**) so that **k** lies along the Earth's axis and at mid-day **i** is in the meridian plane. Looked at from afar (e.g. from a Sun-centred view) a rotation of the Earth about an angle φ (leaving **k** unchanged) will alter the direction of the **i,j** axes on the Earth to **i'**,**j'**. A fixed direction in space with components (x,y,z) relative to (**i**,**j**,**k**) will become ($x\cos\varphi + y\sin\varphi$, $-x\sin\varphi + y\cos\varphi$, z) relative to (**i'**,**j'**,**k'**). From a geocentric view, a direction that rotates with (**i'**,**j'**,**k'**) will retain its coordinates (x,y,z) with respect to (**i'**,**j'**,**k'**) but relative to (**i**,**j**,**k**) will now have coordinates ($x\cos\varphi - y\sin\varphi$, $x\sin\varphi + y\cos\varphi$, z).

We're going to consider (**i**,**j**,**k**) as fixed in space and (**i'**,**j'**,**k'**) as fixed on the Earth.

Direction of sunlight

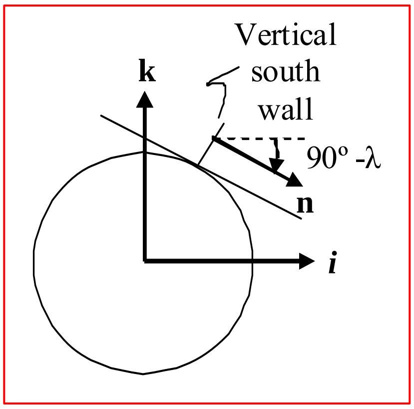
Notice that the **i,j** plane is the equatorial plane. Sunlight makes an angle to the equatorial plane called its declination (δ). The declination can be taken as constant over one day but varies through the year from about +23° at midsummer in the northern hemisphere to -23° at midwinter. [±23.45° if you need a more precise figure]. Hence the unit vector representing the direction of the sunlight is **S** = ($-\cos\delta$, 0, $-\sin\delta$) along axes (**i**,**j**,**k**).

Angle of the Sun to the horizontal



We must obviously consider the sunlight only when it is above the horizon and it is another 3D problem to determine the angle of the sunlight to the local horizontal. The horizontal plane turns with the Earth. At mid-day the geometry is as on the figure to the right, where λ is the latitude of our location. Vector **v** is the unit normal to the horizontal plane (i.e. the local vertical) and **v** is given by ($\cos\lambda$, 0, $\sin\lambda$) at mid-day. At mid-day, the angle the sunlight makes to the local vertical is λ-δ and hence *the angle the sunlight makes to the horizontal* is 90° - (λ - δ). For latitude 57°, for example, this works out at (90 – 57 + 23) = 56° at midsummer.

The local vertical rotates with the Earth and hence after rotating through angle φ the vertical direction **v** will have coordinates ($\cos\lambda\cos\varphi$, $\cos\lambda\sin\varphi$, $\sin\lambda$) in the frame (**i**,**j**,**k**). The angle between the local vertical **v** and the sunlight direction **S** is given by $\cos^{-1}(\mathbf{v}.\mathbf{S})$. Hence the angle sunlight makes to the horizontal is 90° - $\cos^{-1}(\mathbf{v}.-\mathbf{S})$ = $\sin^{-1}(\cos\lambda\cos\varphi\cos\delta + \sin\lambda\sin\delta)$. When this angle reduces to zero, the Sun sets. When the angle is zero, its sine is zero and hence sunset occurs when $\sin\lambda\sin\delta = -\cos\lambda\cos\varphi\cos\delta$, i.e. the hour angle of sunset is given by $\cos\varphi = -\tan\lambda\tan\delta$. This allows a useful check that we are on the right lines because sunset times can be checked for any declination. They won't correspond exactly to reality because we have ignored the effect of refraction of sunlight by the Earth's atmosphere, which should be taken account in proper sunset tables. However, the effect is a few minutes, at most. We've also ignored the effect of landscape on sunset times.



The figures given by the formula above tell us between what rotation angles φ we have to sum up the solar energy falling on a given surface during the day, i.e. the angles for which the Sun is above the horizon.

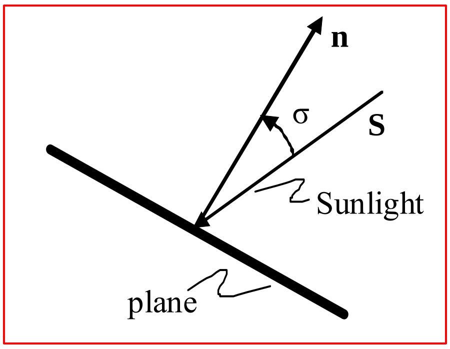
The angle of a surface to sunlight

Suppose unit area has a unit normal in the direction **n** relative to (**i'**,**j'**,**k'**), i.e. its direction components ($n_{i'}$, $n_{j'}$, $n_{k'}$) are ($\sin\theta\cos\psi$, $\sin\theta\sin\psi$, $\cos\theta$). i.e. in spherical coordinates centred on **k**' its coordinates are (1, θ, ψ). In less mathematical jargon, θ represents the angle the perpendicular to the surface makes with the Earth's rotation axis; ψ the twist of this perpendicular about the rotation axis from a due north-south orientation. E.g. a south facing vertical wall or window will have its normal in the direction relative to (**i**',**j**',**k**') of (sinλ, 0, -cosλ); a flat roof (cosλ, 0, sinλ); an east facing vertical wall (0, sinλ, -cosλ); a south facing roof angled up at 30° to the horizontal (cos(λ-30°), 0, sin(λ-30°)), etc.

The surface normal will rotate with the Earth. After rotating through angle φ from the mid-day position, the earlier working tells us that the **n** will have components ($n_{i'}$cosφ - $n_{j'}$sinφ, $n_{i'}$sinφ + $n_{j'}$cosφ, $n_{k'}$) relative to (**i**,**j**,**k**).

Reduction of sunlight over a surface

If the direction -**S** makes an angle σ with the normal to a surface, then the amount of sunlight falling on unit area is reduced by cosσ. Now cosσ is just the scalar product **n**.(-**S)**, since both are unit vectors. i.e. the solar reduction factor is given by (cosδ×($n_{i'}$cosφ - $n_{j'}$sinφ) + 0 + sinδ×$n_{k'}$). This is the key expression to evaluate for a surface in any orientation given by **n,** for all times that the Sun is above the horizon, as determined from the working above.



For a surface normal with general (**i**',**j**',**k**') components (sinθcosψ, sinθsinψ, cosθ) it is easier to work from the geocentric view where the surface normal stays fixed and the Sun's direction rotates through angle –φ as the Earth rotates through φ. The sunlight direction **S** becomes (-cosδcosφ, cosδsinφ, -sinδ) relative to (**i**',**j**',**k**') and the reduction factor **n**.(-**S**) = sinθcosψcosδcosφ –sinθsinψcosδsinφ + cosθsinδ.

Example

A south facing window (in the northern hemisphere) will receive sunlight reduced from a surface perpendicular to the Sun's rays by (cosδsinλcosφ - sinδcosλ). [A south facing window is one of the most common architectural features in buildings in the northern hemisphere. Indeed, if you become misoriented in a European town then looking to see where the majority of houses are facing will soon locate due south]. At mid-day φ =0, giving the reduction as (cosδsinλ - sinδcosλ) = sin(λ - δ), as expected. When φ =90°, the reduction is – sinδcosλ. For positive declinations, this formula results in a negative value for all locations except the Pole. The implication is that the Sun is shining on the back of the window, if that were possible. 'Common sense' might suggest that the Sun shines on the front of a south facing window between the hours of 6 am and 6 pm (solar time) but this is only the case when the Sun's declination is zero. The mathematics calls it correctly, for the south facing window on the globe is generally tilted with respect to the polar rotation axis. For example, near mid-summer, the Sun is due West at our latitude just before 5 pm, solar time. When the Sun has a positive declination, the sunlight goes behind a south facing window before 6 pm and before sunset; when the Sun is on the equator (δ = 0) the sunlight is at right angles to the window at the 6 pm sunset and for negative declinations (between the autumn and spring equinoxes) the Sun sets while still shining on the front of a south window.

I have programmed in Excel the expressions above for the angle of the Sun to the horizontal and the angle the sunlight makes to a south facing wall for a range of declinations from +23° to -23° and for our location at latitude 57°. The first three columns of the table here show the range of solar declinations used and when they occur during the year, within a day or so. Each declination occurs twice, once before the summer solstice and once afterwards.

Declination	Day/month	Day/month	kW h (no absorption)	kW h d^{-1} m^{-2}
23º	10/06	02/07	3.61	2.91
20º	20/05	23/07	3.99	3.14
15º	30/04	11/08	4.64	3.45
10º	15/04	27/08	5.26	3.65
5º	02/04	09/09	5.88	3.68
0º	20/03	22/09	6.46	3.51
-5º	07/03	05/10	6.91	3.13
-10º	23/02	19/10	7.08	2.54
-15º	09/02	03/11	6.93	1.80
-20º	21/01	21/11	6.41	0.97
-23º	01/01	11/12	5.95	0.51

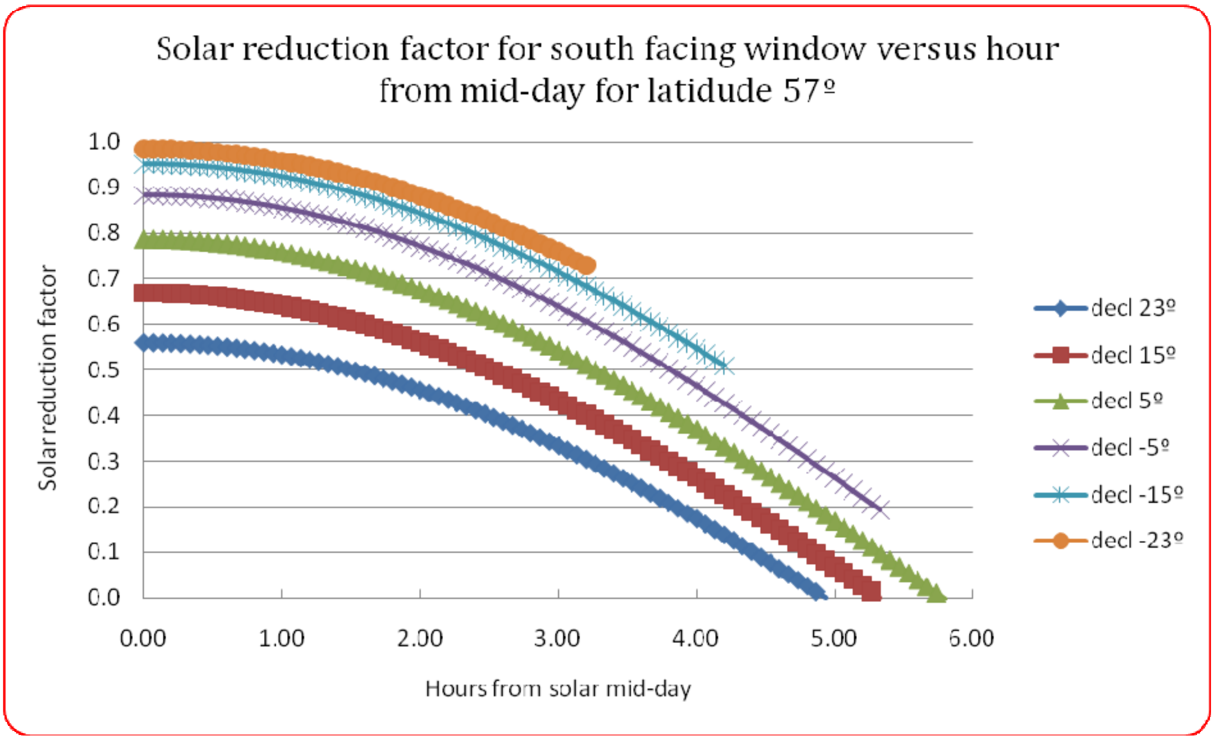
Consider an area of south facing window that would intercept a power of 1 KW when perpendicular to the Sun's rays. The 4th column of the table shows how many KW h that area will intercept over a full day's sunlight. The two factors at work are the length of time the Sun shines on the window and the changing angle of the Sun. That area will receive more energy in winter than in summer, basically because it is better oriented towards the Sun. If this result seems very odd, it's because no account has been taken so far of atmospheric absorption. The winter sunlight from a particular direction is weaker than the summer sunlight, for the altitude of the Sun is less in winter and the sunlight must pass through more absorbing atmosphere. It therefore takes a bigger area in winter to intercept 1 kW of sunlight. If we are to find the relative amounts of sunlight that fall on a solar energy collector at different times of the year, then this atmospheric absorption must also be included. We'll come to this shortly.

While we're still looking at the table, jumping ahead of the absorption discussion, the final column shows the calculated energy in kW h per square metre per day when the allowance is made for path length absorption that is discussed later. The column now compares like with like, i.e. 1 m^2 at different times of the year. We can now draw the non-obvious conclusion that the direct solar energy received on a south facing window is not highest at mid-summer, because the Sun shines on it for a shorter time than in the surrounding months and the inclination factor is less favourable. This will become clearer from the next graph. As expected, the energy falls away significantly between mid-October and mid-February but otherwise holds up well for the rest of the year.

Now look in more detail at the effect of the changing angle of the Sun to the window over the year, as shown in the figure below (next page).

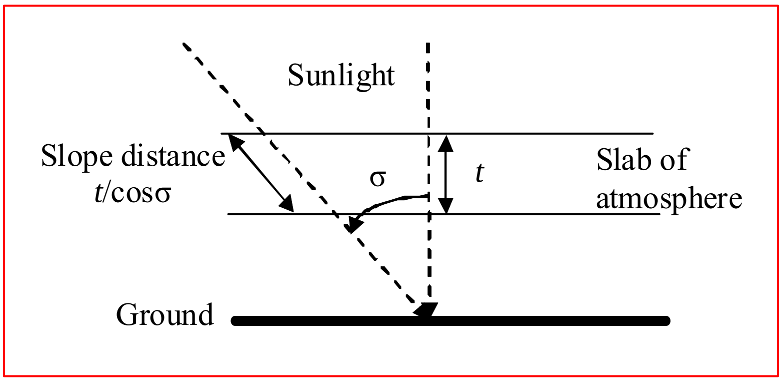
The 'reduction factor' in the graph is the loss of power caused by the tilt of the surface to the Sun's rays. The x-axis of the graph runs from mid-day to 6 O'clock (solar time). The curves are plotted for the range of solar declinations. They are symmetrical before and after mid-day. Some curves stop before the end because sunset occurs before 6 O'clock (or sunrise after 6 O'clock). In this example the lower curves with the greatest reduction factor are in summer.

The room I am in at the moment shows the effectiveness of a south facing vertical surface in intercepting sunlight in winter. I'm writing this in mid-February in the early afternoon. Outside, the air temperature is -2º C, the sun has been sparkling from crisp snow on the ground all day and we have had no heating on in the house since early morning. All the south facing rooms are a comfortable 22º C. Our north facing rooms, noticeably colder.



Absorption of the atmosphere

We won't get a 'ball-park' figure for the sunlight available to solar energy converters without taking some account of atmospheric absorption. It's obvious that the air itself scatters a significant amount of sunlight all over the sky, making the sky bright all round. In addition, dust both within clouds and outside clouds absorbs sunlight. So do some constituents of the atmosphere such as carbon dioxide and water vapour. To make matters worse from a calculational viewpoint, the absorption depends on wavelength within the solar spectrum and total effect depends on the height of ground level above sea level. Energy absorbed by atmospheric constituents is of course re-radiated, otherwise the absorbers would get ever and ever hotter, so the ground receives not only direct sunlight but re-radiated energy. To add yet another complication to our context, photovoltaic cells respond only to light whereas solar heaters also respond to the infra-red component in sunlight, which contains just as much energy as the light. In short, the situation is a mess of variables.



What all circumstances have in common is that the amount of absorption depends on the path length through the atmosphere, so the amount of direct solar radiation reaching the ground decreases as the altitude of the Sun decreases. Let's look at this effect and take some round figures for the sum total of all the other influences put together. We know that the atmosphere changes quite rapidly with height but horizontally it is much more uniform. The sketches earlier in this piece show the Earth as a little round ball but in fact it's such a big ball and the atmosphere is so thin in comparison with the Earth's radius that so long as the sunlight is making an angle to the vertical of less than about 75º, the influence of the curvature of the Earth on the amount of atmosphere the sunlight passes through can be ignored. That is a common assumption. The sketch above shows that in this approximation, where sunlight coming vertically down would go through a slab of atmosphere of thickness *t*, then sunlight inclined at an angle σ to the vertical must travel a thickness *t*/cosσ. In effect the atmosphere is increased in thickness for inclined rays by a factor of 1/cosσ. At the angle of 75º mentioned above, this factor is almost 4. σ is called the *zenith angle*. Clearly σ = 90º - (angle of the sunlight to the horizontal).

Now, about 80% of the atmosphere and pretty well all the weather is contained within the troposphere, whose height extends roughly 10 km from the ground. The horizontal distance away from you that sunlight enters the troposphere when it is inclined at 75º to the vertical is 10tan75º = 37 km. It's clear that on this kind of scale the curvature of the Earth (diameter 12,750 km) is a small influence even for angles as steep as 75º. For angles of 85º, the slope distance is 11.5 times the vertical distance on this model and the horizontal distance away of light entering the troposphere is about 115 km. At greater angles one certainly needs to take into account both curvature of the Earth and the curvature of light through the atmosphere caused by refraction. However, it's not worth the trouble because we all know that within 5º of the horizon the haze has reduced the power of the Sun to very little.

The reduction of direct sunlight can be taken as exponential with the path length through the atmosphere. i.e. the ratio of intensity on the ground to intensity above the atmosphere varies as $e^{-\mu T}$ where T is the path-length of the light through the atmosphere and the μ a constant that measures the absorption. Articles on solar radiation often introduce the atmospheric *clearness index*, denoted K, that measures this reduction. From what's been said, K depends on the atmospheric content (cloud, dust, etc.) and also on the time of day so average K values are often quoted, which are not what we want. To be definite, I will take the transmission of sunlight coming perpendicularly through the atmosphere as being 0.7 (i.e.70% is transmitted). For a sloping path the transmission will be $0.7^{1/\cos\sigma}$. For example, if the incident sunlight is 1300 W m^{-2} then sunlight reaching the ground at our latitude at mid-day in mid-summer on a bright summer's day when the zenith angle is 34º is $1300\times0.7^{1/\cos 34^\circ}$ = 845 W and the amount recorded on a horizontal surface will be 845×cos34º = 700 W. This is consistent with values recorded by our local solarimeter. On this model, the power of solar radiation (often called the *insolation*) when the incident angle is 75º to the vertical has fallen to 330 W. When this is spread onto a horizontal surface it gives only 85 W m^{-2}. Whatever the incident angle, interruption by thick cloud cuts off all direct sunlight, fading out shadows. In Aberdeen, the average amount of cloud in the daily sky is about 70%.

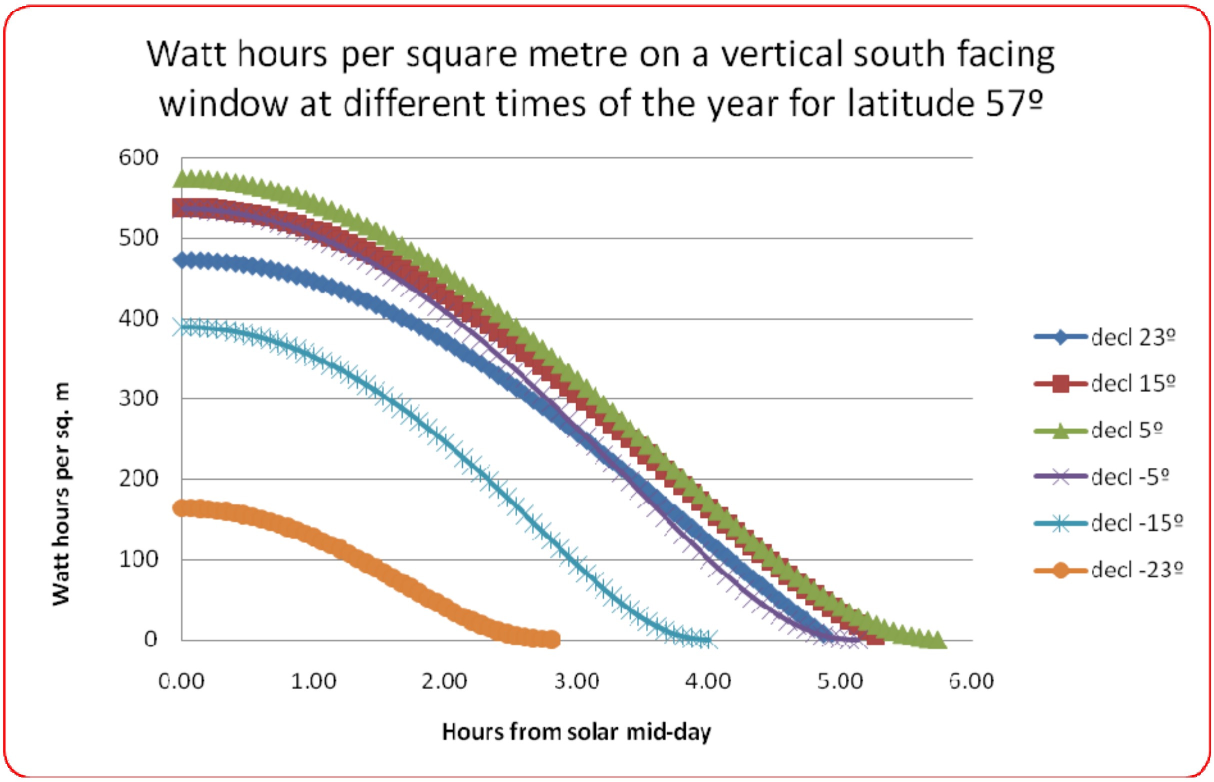
Including the effect of atmospheric absorption

Looking back to one of the early sections you will see that we have already obtained the formula for the zenith angle: $\cos\sigma = (\cos\lambda\cos\varphi\cos\delta + \sin\lambda\sin\delta)$. The previous paragraph shows how to estimate how much direct sunlight reaches ground level (on a clear day) and the earlier working shows how to calculate the fraction of this that reaches any chosen inclined surface. All that is needed to estimate the solar power available is in place.

More results

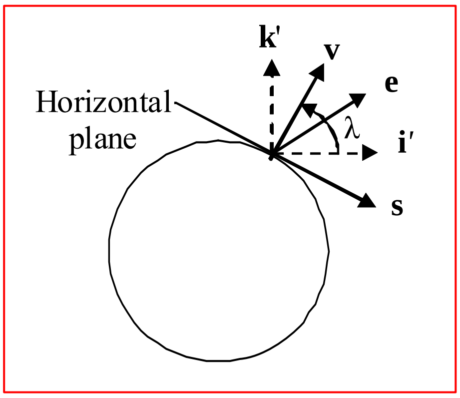
The previous graph showed the insolation on a south facing vertical window that would be received in the absence of the atmosphere during a sunny day at different times of the year for our latitude of 57°. Winter did better than summer inspite of the shorter days because the window was better oriented with respect to the sunlight direction. The next graph now takes into account (with all the provisos mentioned above) the stronger summer sunlight and the weaker winter sunlight. The most favourable time of year is still not mid-summer but early April or early September, when the Sun is still just over 5° north of the equator. The longer exposure and more favourable angle more than make up for the slightly weaker Sun. This isn't a result one could have guessed. It needed the actual numbers to make it apparent.

The figures for the sum total energy that could be received during a day have been given in the earlier table and discussed then.



Angles relative to horizontal and vertical axes

The axes system used above gives the simplest expressions but finding the correct angles to use for a roof or wall whose position you know relative to the surroundings needs an appreciation of where the Earth's rotation axis lies relative to where you are. The previous results can be converted to a new axes system (**v**,**s**,**e**) defined with **v** the local vertical, **s** the horizontal pointing due south and **e** is the horizontal pointing due east.

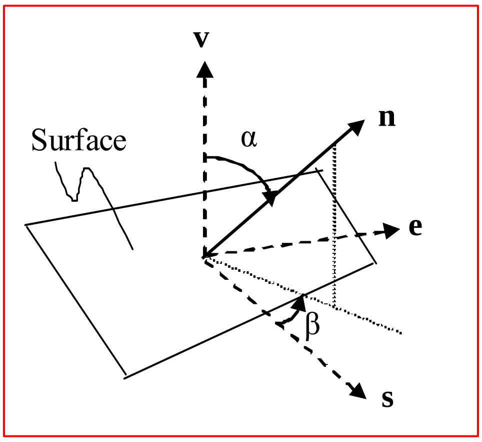


The relationship between the new axes (**v**,**s**,**e**) the old axes (**i'**,**j'**,**k'**) is:

$\mathbf{v} = \cos\lambda\, \mathbf{i}' + \sin\lambda\, \mathbf{k}'$; $\mathbf{s} = \sin\lambda\, \mathbf{i}' - \cos\lambda\, \mathbf{k}'$; $\mathbf{e} = \mathbf{j}'$

Sunlight relative to (**v**,**s**,**e**) *axes*
Measured in the (**v**,**s**,**e**) system the sunlight direction at mid-day has coordinates **S** = (-cos(λ-δ), -sin(λ-δ), 0) ≡ (-sinλsinδ – cosλcosδ, cosλsinδ – sinλcosδ, 0) and after the Earth has turned through angle φ the (**v**,**s**,**e**) coordinates of the sunlight direction are (-cosλcosδcosφ – sinλsinδ, –sinλcosδcosφ + cosλsinδ, cosδsinφ). The direction the Sun is seen from the Earth is the reverse direction, **-S**.

Surface normal relative to (**v**,**s**,**e**) *axes*
A horizontal surface has normal direction (1, 0, 0) and a south facing wall has normal direction (0, 1, 0). Let α represent the angle a surface normal is tilted from the vertical. Let β represent the angle the surface is twisted out of the north-south plane. The direction of a general surface normal therefore has components (cosα, sinαcosβ, sinαsinβ). For example a roof oriented SE and inclined forwards at 30º to the horizontal has components in the (**v**,**s**,**e**) system of (cos30º, sin30ºcos45º, sin30ºsin45º).



Sunlight reduction with (**v**,**s**,**e**) *surface coordinates*
As before, the sunlight reduction factor is **n**.(-**S**) which can be evaluated from the components of **n** and **S** given in the previous two sections. The expression becomes:
cosα(cosλcosδcosφ + sinλsinδ) +sinαcosβ(sinλcosδcosφ - cosλsinδ) - sinαsinβcosδsinφ.

To re-cap, the above expression allows one to work out the reduction of full sunlight that falls on each square metre of surface compared that with sunlight that falls on a square metre oriented perpendicular to the sunlight. α and β define the orientation of the surface relative to local axes that are vertical, horizontal south and horizontal east, as in the previous section; λ is the latitude of where you are; δ is the declination of the Sun and φ is the rotation angle of the Earth from its solar mid-day position.

Yet more results
How much better is it to place your solar collectors on a tilted roof? When the Sun is on the celestial equator, at the equinoxes, and is therefore half-way in its north-south travel across the sky, then a roof whose normal points directly at the Sun must be angled up by the latitude λ to the horizontal, by 57º in our case. That's a very steep roof. Roofs on the houses around where I live have concrete tiles and a pitch of 30º. A green energy web site I have just visited claimed that the optimum pitch for a south facing roof in the UK on which to secure a photovoltaic collector is 30º. Is their figure reasonable? We need to use the earlier formulae to obtain some numbers for pitched roofs. I've modified the spreadsheet I used to produce the early graphs and table entries for a south facing surface to include a variable pitch of roof, α, that I can select. The reduction factor simplifies to cos(λ – α)cosδcosφ + sin(λ – α)sinδ. When the pitch is equal to the latitude, the reduction factor becomes simply cosδcosφ.

Usually a graph can show trends better than a table but in this case the difference in the numbers is quite small and the table shows these differences more clearly. *The values in the nearby table are the total incident energy in kWh day^{-1}m^{-2} that the Sun has the declination on the left, calculated for different roof pitch angles α for a south facing roof at latitude 57º.* A pitch of 0º is a flat roof. This will see the Sun at all times it is above the horizon (ignoring obstructions) and as expected does well in summer but very poorly in winter. It is for this reason that snow melts much more quickly on a south facing inclined roof than on the flat. A pitch of 90º is a vertical wall, like the gable end of a house. It gets more or less the same total over the year as a flat roof but does better in winter. The figures for the vertical wall are of course just the same as those in the earlier example of the vertical south facing window. Summed over the year, there is very little to chose between pitches of 30º, 45º and 57º (our latitude) but the 57º pitch does noticeably better in winter than the 30º pitch and it is the winter that one needs solar energy the most.

declinationº	pitch 0º	pitch 30º	pitch 45º	pitch 57º	pitch 90º
23	6.13	6.58	6.17	5.57	2.91
20	5.58	6.26	6.03	5.54	3.14
15	4.7	5.76	5.71	5.41	3.45
10	3.84	5.14	5.29	5.14	3.65
5	3.03	4.47	4.75	4.74	3.68
0	2.28	3.73	4.09	4.18	3.51
-5	1.6	2.95	3.34	3.49	3.13
-10	1.01	2.15	2.51	2.68	2.54
-15	0.54	1.36	1.65	1.8	1.8
-20	0.2	0.66	0.83	0.92	0.97
-23	0.08	0.33	0.42	0.47	0.51
total	28.99	39.39	40.79	39.94	29.29

Conclusion

This whole piece has been an example of mathematical modelling in action. What's the point? Can't we just measure the Sun's radiation where we want to know what it is? Most solar radiation measurements have historically been done with horizontal sensors that don't distinguish direct sunlight from diffuse energy. In fact there is no large historical databank of direct solar energy measurements. Most places in the world have lots of cloud, the world average being around 50%. Hence to get direct sunlight measurements for every day of a year would take decades. Indeed, the average number of clear sky days here over a year is less than one a month.

The calculated numbers are intended to give an idea of what the direct sunlight would be in the absence of cloud, fog, haze and other natural factors that interfere with sunlight. There are very many papers in the scientific literature that take this process further. For example the modelling can be extended to include diffuse energy from the rest of the sky and energy reflected from the ground. Papers routinely take account of the changing incident solar energy resulting from the elliptical shape of the Earth's orbit with the Sun off-centre. This is easily done, but it only makes a difference of a few percent. Mathematical modelling puts understanding to the test. If the understanding is good then the numbers will be realistic.

I believe that the numbers here are good enough to draw reasonable conclusions. One of these is that the annual direct solar energy received by a fixed surface is not particularly sensitive to tilt angle over the range 30º to 60º for our latitude on a south facing surface but that a steeper angle is better in winter, though it sacrifices some power in summer. Another conclusion is that the tools are all here for you to calculate any special case you wish. All you need is an Excel spreadsheet, or any equivalent tool.

Home solar electricity example

This is a question I'd like to know the answer to. Our house could readily support 10 m^2 of solar electricity generating panels on an almost south facing roof inclined at 30° to the horizontal. Estimate the value of electricity I can expect to generate annually.

1) *The solar radiation*. The adjacent table amalgamates information already calculated, the third last column being the number of KW h d^{-1} for 1 m^2 from a cloudless sky for the given solar declination, the second last column an estimate of the number of days when the Sun has about the declination shown and the final column the total solar energy in kilowatt hours to fall on the roof. Notice how the kilowatt hours fall substantially when the Sun has southerly declinations (below 0), i.e. between the autumn and spring equinoxes. These 6 months receive only 20% of the annual energy. The annual total from the calculation is close to 1300 kW h per square m, or 13000 kW h for a 10 m^2 panel.

Declination	Day/month	Day/month	pitch 30°	~No. of days	Total KW h
23°	10/06	02/07	6.58	49	322
20°	20/05	23/07	6.26	37	232
15°	30/04	11/08	5.76	33	190
10°	15/04	27/08	5.14	25.5	131
5°	02/04	09/09	4.47	25.5	114
0°	20/03	22/09	3.73	25	93
-5°	07/03	05/10	2.95	25.5	75
-10°	23/02	19/10	2.15	25.5	55
-15°	09/02	03/11	1.36	33	45
-20°	21/01	21/11	0.66	37	24
-23°	01/01	11/12	0.33	49	16

2) *The reduction factors*. There are two main reduction factors: clouds and inefficiency of solar cells. I've given a figure for our local cloud cover as being 70%. It almost certainly varies throughout the year but let's suppose that clouds reduce the availability to 30% of the total. That reduces the annual amount to 3900 kW h, although photovoltaic panels will generate some electricity in cloudy conditions. Solar cells are not that efficient in generating electricity, for technical reasons. The very best might approach 50% under ideal conditions but 15% is a better realistic estimate. That implies 585 kW h of electricity will come from my 10 m^2 of panels. There are other inefficiencies too: dust and dirt will settle on the panels and birds will soil them; some of the solar cells will pack up over time; there will be some losses in the wiring before the electricity reaches the house, the meter and the distribution board. All in all, the modelling suggests that I'll not get much more than 500 kW h annually from my 10 m^2 of panels.

3) *The income*. The government used to offer about 40 p per kW h fed into the grid but the figure is being reduced to 21p on 1st April 2012 (this is no April fool). Taking this as the return on energy generated whether it is used in the house or fed-in then the value of 500 kW h is £105. The current electricity price as I write is about 12 p per kW h, with no capital expense required by the customer.

4) *The conclusion*. Sunlight may be free but the current price for installing 10 m^2 of solar electrical generating panels is around £10,000. At current costs and current payback rates, the figures above suggest that I will only get an annual return of about 1% on the capital cost. I really need to find someone with a meter to give me actual annual figures to compare with the

above. I suspect that the great returns implied in solar panel ads are, first, for locations further south and secondly prefaced by 'up to……', meaning on a brilliant clear day. Meanwhile, there are several conclusions based on the above, apart from "I'm out", as the Dragons say, for the time being. Moreover, writing off the capital cost of £10,000 over the 25 year life of the installation implies that even on a linear basis one is loosing £400 per year in asset depreciation. It's hard to see how the current 'business model' makes sense for the customer. Yes, there will be a better return for those located nearer the equator than I am but solar panels are too expensive as judged against today's energy costs. One can scarcely watch a news program these days without hearing about the high cost of energy but in fact 12 p per kW h is very cheap. I know it was only 1 p per kW h a few decades ago but electricity generated by renewable means is going to be quite a bit more expensive than coal/oil generated electricity. Society will either have to pay more or use less and since fossil fuels will not be available in a few decades time in the quantity they now are for heating or transport, then society as a whole is going to use more electricity in future. I guess we'd better get used to it paying more for it.

Appendix – More Examples

These examples show how the expressions and numbers in the previous sections can be used to provide information in different circumstances. The following calculations have all the previous reasonable approximations implicit, e.g. cloudless days and the same solar radiation incident on the outer atmosphere of the Earth for all seasons. The results in the table at the end are for our latitude of 57° but the given formulae can be applied for any latitude.

Tracking panel

A solar panel is mounted on a tracking device so that it is always pointing directly towards the Sun. Compare the energy it receives during the day with the fixed panels discussed earlier.

In this case there is no spreading factor, only the atmospheric absorption factor $0.7^{1/\cos\sigma}$, with $\cos\sigma = (\cos\lambda\cos\varphi\cos\delta + \sin\lambda\sin\delta)$.

The results table later compares the daily power received in KW h day^{-1} m^{-2} with that given earlier for a fixed south facing vertical panel (or window).

In brief, tracking doesn't bring big rewards in winter, since the low altitude Sun (here) is never all that far from shining straight onto a fixed south-facing vertical panel but between the equinoxes in summer the gain in using tracking is considerable, rising from 50% to 330% at mid-summer.

Vertical, turning panel

To track the Sun precisely requires motion adjustment in two dimensions. A simpler version would be to have a vertical panel that turns about a vertical axis so it faces the Sun as best as a vertical panel can do. i.e. the normal to the panel is kept in the plane containing the vertical and the Sun. Put another way, the compass bearing that the panel normal points to is the same as the compass bearing of the Sun.

In this circumstance, the spreading factor is determined only by the inclination of the Sun to the vertical (namely σ in our discussion) and is given by $\sin\sigma$, since the normal to a vertical panel is horizontal. The results are given in the table at the end.

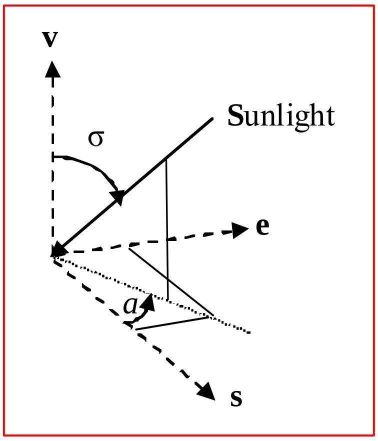
It is only in the 2 months on either side of mid-summer that there is a significant loss of power collected over full tracking. Hence this much simpler tracking would in practice be pretty effective.

Vertical, spinning panel

My personal problem at the moment is that we are planning to fit a solar panel on our boat to keep the batteries charged when we are absent. One possibility is to fit the panel vertically onto the side of the vessel, where it is out of the way. The boat swings on its moorings due to wind, waves and tide and could be pointing in any direction, with the panel sometimes directly facing the Sun but usually facing in another direction. How efficient will the panel be, compared with the various options already calculated? The answer isn't on the internet or in textbooks, but one of the pay-backs of working out what is going on is to be able to tackle one's own problems.

If the panel happens to face the Sun, the power is that received by the turning panel in the previous example. The panel will receive a lower amount of sunlight if it is turned away by an angle up to 90°, and for greater angles it will generate no power at all since the Sun will fall on its back. We need to find the average power over the 360° of possible orientations. For 180°, the power is zero. The direction of Sun and panel (normal) are best treated in the **v**,**s**,**e** coordinate system. The angles β that the panel will generate power will be 90° on either side of the Sun's azimuth, the direction of the Sun measured in the horizontal **s**,**e** plane.

The direction of sunlight, **S**, in the **v**,**s**,**e** system has already been found, namely (-cosλcosδcosφ – sinλsinδ, –sinλcosδcosφ + cosλsinδ, cosδsinφ). The direction that the Sun is seen in from Earth is **–S**. The tangent of the azimuth *a* is given by the ratio of the east and south components of **-S**. i.e. tan*a* = -cosδsinφ/(sinλcosδcosφ – cosλsinδ).



The vertical panel has its normal **n** in the **s**,**e** plane, hence the orientation angle α = 90°, making cosα = 0 and sinα = 1. We must find the average value of the spread function of the sunlight for angles $\beta_{min} < \beta < \beta_{max}$, i.e. *a*-90° < β <*a*+90°.

Now for the slightly tricky bit. As far as β is concerned, the spread function F = Pcosβ +Qsinβ, where P and Q are determined by the other factors involved, not β, as in the main discussion. Looking back, P = (sinλcosδcosφ - cosλsinδ) and Q = - cosδsinφ. Hence for each value of the time φ, the average value of the spread function F over the 180° (equivalent to π radians) where the Sun hits the face of panel is $\bar{F} = \frac{1}{\pi}\left[P\left(\sin\beta_{max} - \sin\beta_{min}\right) - Q\left(\cos\beta_{max} - \cos\beta_{min}\right)\right]$. Since the other values of the panel orientation β give no signal at all (the Sun falls on the back of the panel) then the spread function for the panel averaged over 360° is half the above value. It looks a bit messy but Excel will cope.

See the following table for the results. In brief, the vertical panel on the side of the boat is not a particularly good idea and it is better, at least from March through to end September, to place the panel lying horizontal, as shown by the last column.

Results

Sun's declination	When	*Tracking* kW h d^-1 m^-2	*Turning* kW h d^-1 m^-2	*Fixed* kW h d^-1 m^-2	*Spinning* kW h d^-1 m^-2	*Horizontal* kW h d^-1 m^-2
23	end June	9.68	7.15	2.91	2.28	6.13
20	mid May; mid July	9.09	6.9	3.14	2.2	5.58
15	end April; early August	8.12	6.45	3.45	2.05	4.7
10	mid April; mid August	7.16	5.93	3.65	1.89	3.84
5	early April; early Sept.	6.19	5.33	3.68	1.7	3.03
0	mid March; mid Sept.	5.19	4.62	3.51	1.47	2.28
-5	early March; end Sept.	4.16	3.82	3.13	1.22	1.6
-10	end Feb; mid Oct.	3.11	2.93	2.54	0.93	1.01
-15	early/mid Feb; end Oct.	2.04	1.97	1.8	0.63	0.54
-20	mid Jan.; mid Nov.	1.04	1.02	0.97	0.33	0.2
-23	early Dec; early Jan.	0.53	0.53	0.51	0.17	0.08

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