Musings for the maths averse

Maybe how it came about

The subject of Mathematics finds it hard to throw off the image of being the least liked topic in the school curriculum. At best it's a bit like Marmite – some really like it, many would rather give it a wide berth. It's not entirely the fault of teachers or maths curriculum writers. They've been well aware of the problems and have tried a variety of approaches over the decades. Maths has many faces and I'm not just thinking about geometry, algebra, number theory and so on. If you mention the word mathematics, probably most people just think of arithmetic, which is certainly a very useful bit of maths and one of the many branches of the tree of mathematics that has applications.

Mathematics used to be taught as a useful subject, with lots of exercises that pleased future engineers, physicists and even would-be modern chemists. It's an 'ideal' examination subject, with many questions having a unique correct answer that will score 100% and mistakes in an argument that are not controversial can be counted and hence scored without much argument. Compare this to an English essay that has no correct answer and whose quality is quite subjective. So teaching the mathematics of practical problems was a pretty self-sustaining occupation, even though the problems often had to be simplified unrealistically to allow a pupil or student to come up with an answer. These simplifications tended to make the subject seem 'dry' for those who wanted mathematics for its applications – objects became points or balls, planes became perfectly smooth, air resistance was wished away and so on. The motivation may have been practical but many people didn't see past the simplifications to real problems. Indeed, the subject was often reduced to learning standard techniques, so it became a subject of pure learning and not problem solving at all. For example one learnt how to differentiate or integrate a wide range of different functions just because …, just because it could be done. Fine for those who liked this but no fun it you didn't.

Mathematicians also objected to maths being taught just as a practical tool. Mathematics, they said, was all about concepts, about structures held together by logic, about relationships between abstract ideas, about rigorous proofs. So lots of the then routine problem solving was dropped from maths syllabi so that pupils and students could appreciate more of what mathematics was 'really about'. This successfully distanced another group of people. The engineers and physicists howled that students couldn't 'do' the problems that their predecessors tackled. This isn't a new issue. Set a modern physical science student a Mathematical Tripos paper of the mid-19th century and I think they'd find it very tough. The modern students have other skills that their predecessors didn't have, but that's a digression.

Where's all this going? It's intended to be a pre-amble for saying that there are good reasons why many people have been put off maths and having been put off don't look in this direction again. If you're one of them then this piece is for you.

If I can reminisce a bit, I remember having a theoretical physics lecturer for whom technique was everything. He might open a lecture with something like "*Consider a potential V of such and such form. Now inserting this into Schrödinger's equation produces the following.. . We first make a transformation…*" and this went on for a few blackboardfuls of symbols until we got to "..*and now you see the solution is this*…". End of his piece. It was only then that some of us saw what the whole problem was about and why he wanted to start with that particular potential. I suspect some in the class remained totally mystified. The context wasn't explained, the implications of the result weren't drawn out. This was my unintentional lesson in science of how important it is for the relevance of the maths to be understood before any technical argument is started and how it's also necessary to draw out the conclusions implicit in what can be very powerful results.

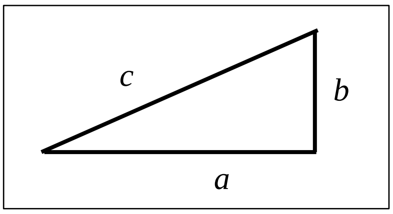
A different look at mathematics

I think one good way to look at mathematics is as a puzzle solving activity. Lots of non-mathematical people enjoy puzzles: crosswords, Sudoku, Countdown, Only Connect, 'brainteasers' and a wide range of games that involve finding and creating patterns like Chess, Draughts, Halma, Go, Bridge, Rummy or one of its predecessors Mahjong, … the list is very long. They all involve assessing what you've got and trying to create something from it that isn't given directly. Sometimes a random component is thrown in, often not. Think of mathematics as a puzzle playground.

Like any playground where there are objects to play with, in maths there are objects and they are frequently represented by letters. Some students get hung up here. They have no problem with statements like 17 + 8 = 25 but if I write $x + 8 = 25$ and ask 'what is x?' then I have a puzzle. One technique for trying to solve puzzles is just to guess an answer. How about '12'? Well 12 + 8 = 20 and that's too small; 15 + 8 = 23, still too small. Guessing is 'ground zero' in puzzle solving, the absolute rock bottom method if you haven't a clue how to do better. It can work and if you really need an answer then it's better than nothing (unless the price for coming up with the wrong answer is steep). With a modern spreadsheet you can sometimes rapidly make lots of guesses and home-in on a good enough answer. However, in the case above, no guessing is needed for the logical thing to do is to subtract 8 from both sides of the relationship $x + 8 = 25$ to give $x = 25 – 8 = 17$. The at first unknown number represented by the letter x is 17.

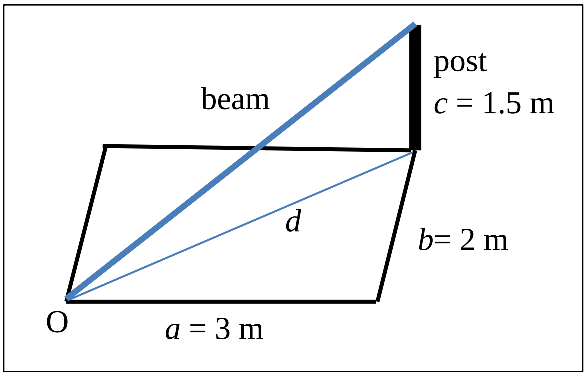
Maths and logic are closely connected. Indeed you could say that all of mathematics is logic but finding a solution to a puzzle may well require more than an application of logic. Personal curiosity is vital. You need to be emotionally involved at least to the extent of wanting to find the answer. You also need to have some hope that you will succeed and be prepared to put in some time and effort. If what to do isn't obvious, it may be a challenge that needs some perseverance. You need to have in the back of your mind the rules of logic as they apply to the unknown you want to find. If it's a number then adding or subtracting the same to both sides of a relationship (equation) leaves it unchanged; multiplying or dividing both sides by the same also leaves it unchanged; switching the order of addition or multiplication makes no difference (e.g. $xy = yx$), and so on. There's nothing new or strange there.

Maths isn't always about numbers. Geometry is about shapes, for example. Euclid's *Elements* in the traditional 13 books used to be standard fare in the school syllabus, or at least a fair chunk of the *Elements*. It's all about shapes and their relationships, and recipes. For example, how to draw a circle through three points not in a straight line. Quite useful on occasions. It included the famous '*Theorem of Pythagoras*' for right-angled triangles that says that the area created by a square whose length is the longest side of a triangle equals the sum of the corresponding square areas created on the other two sides.



In the 17th century Rene Descartes developed *Cartesian geometry* that converted geometrical relationships into equations. Referring to the sketch here, the *Theorem of Pythagoras* says $a^2 + b^2 = c^2$ where a and b are the lengths of the short sides and c is the length of the long side. You almost certainly knew this. It's very useful.

This piece isn't intended as a maths lesson but I'll give an example of a puzzle. Suppose I've laid out a rectangle 3 m × 2 m on a level floor. At one corner I've stood an upright post 1.5 m high and I need to put in place a beam from the top of this post to the opposite corner. How long a beam do I need? Beams are expensive so I can't just get one that's obviously too long and saw a piece off.



This is a geometric problem so I'll begin by drawing out a sketch of what I know. What exactly is that? I know some lengths and that various lines are at right angles to each other. Lengths and right-angles suggest I root out the *Theorem of Pythagoras* from the recesses of my brain. Can this be applied to the puzzle?
Looking at the attempt (above) to draw a three-dimensional plan, remember that the lengths a and b are sides of a right-angled triangle on the floor. The diagonal d on the floor from the corner O to the foot of the post is therefore given by $d^2 = a^2 + b^2$. Now it will hopefully click that the beam, the post and d also form a right angled triangle so $beam^2 = c^2 + d^2 = c^2 + a^2 + b^2$. Eureka, the puzzle is solved because a, b, and c are all given, hence $beam^2 = 1.5^2 + 3^2 + 2^2 = 15.25$ by calculator or mental arithmetic or spreadsheet if you prefer. Hence I need a beam of length $\sqrt{15.25} = 3.9$ m.

The answer being a number, you can check that it's reasonable. You'd be amazed at the number of ludicrous answers handed in in homework problems. The underlying reason is that the writer hasn't engaged with the problem and doesn't really care what answer has come out of their working. Engagement is essential. Another thing you can do with an equation is check that it's reasonable in special cases where the answer is pretty obvious. For example if any one of *a, b, or c* is zero then the '*beam*' length is clearly correct and if any two are zero it's also trivially right. Yet another check to make is that the formula gives the right kind of quantity. In this case I wanted a length and the formula is indeed a length. If it were an area or a speed or something else it would be wrong without any more investigating.

This puzzle question shows one of the great strengths of maths – the solution is generic and doesn't just apply to my specific case. Because it's in terms of letters that represent any number then it can be applied to any similar case. A second strength of mathematics is that a wide range of problems are equivalent to this one. What the answer shows is that the length of the diagonal of a rectangular box whose sides are of sizes a, b and c is given by the square root of $(a^2 + b^2 + c^2)$. Any puzzle that can be reduced to the same idea is solved.

The result above is not a formula that needs to be remembered, for once you have the idea of how to get the answer it can be found again reasonably quickly. Of course my answer came from the *Theorem of Pythagoras* and strictly speaking you don't need to remember that either, for it can be proved in several ways. However, most of the ways are harder than remembering this very simple result and since it's so useful most of us do remember it. This highlights another problem with traditional maths education: there seemed to be lots to remember, especially if you found it hard to see the links between separate items. Life is now different, particularly away from exams. Relationships can be looked up quickly on the web; mathematical calculations can be made using spreadsheets without the need to find an explicit formula for the answer; mathematics can even be derived using software like *Mathematica*, *Maple*, *Matlab* and indeed several dozen other computational algebra programs. Changed times indeed. Get hold of basic mathematical ideas for solving puzzles and you'll be surprised how far you can go.

Coming back to my theme, mathematics is puzzle solving – interesting, useful, often fun, something humans do better than any other animal. Puzzle solving can be quite intuitive but in the maths applied to science the puzzle usually has to be solved with some explicit intellectual effort. It's been said that there are four phases in trying to solve a puzzle. First, **understand it** – the context and what is given. Think about these before rushing ahead. This is where you start. Secondly, **look for connections** between the data given and any connections that might exist between the data and the answer. Thirdly, **devise a plan** and carry it out using the knowledge available or any that can be found. The strength of mathematics is that each step in a solution is provable. Don't introduce statements that can't be proved. Fourthly, **review the alleged solution** and draw out its consequences.

Of course it's not that easy. If there were a recipe for solving all problems we could just program computers to do the work and sit back and read off the answers. There are other guidelines. Can you solve a simplified version of the puzzle? Can you solve a puzzle that seems to be related? Do you know the solution of an analogous problem? Can you change the given problem to another problem that might be soluble? Do I need more data than is given? Can I work back from an answer to see the connection to the data? I can't make this an exhaustive list of guidelines for it would become a chore to read. My take home message is that maths is about puzzle solving and almost everyone likes puzzle solving. If you've been put off the subject in the past, then if you have an enquiring mind (and who doesn't underneath it all) forget about any previous difficulties you may have had and try again to acquire some expertise. You will solve puzzles you never thought you could and become knowledgeable about connections in science that you couldn't see before.

Yes, that's true and leads me on to a more serious cadenza. Physics has penetrated almost every science these days and in physics mathematics is more than a game. It illuminates almost every branch of the subject. Physics with hand-waving and no maths is a bit like air-guitar playing. It can look good but ultimately a very important part of the experience is missing. There is no sound. *JSR*