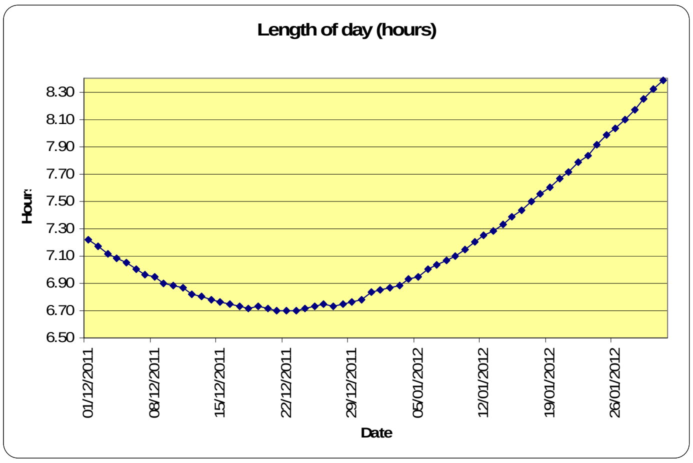
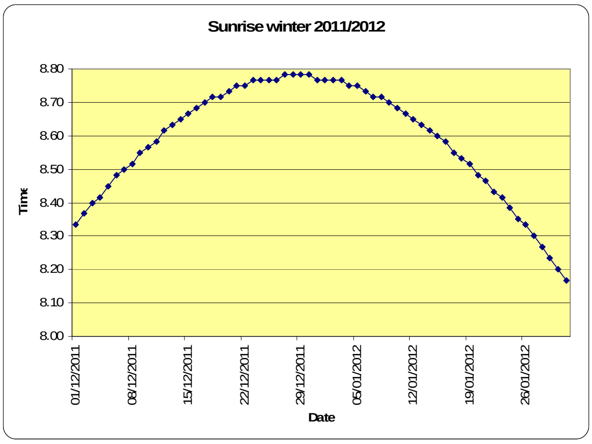
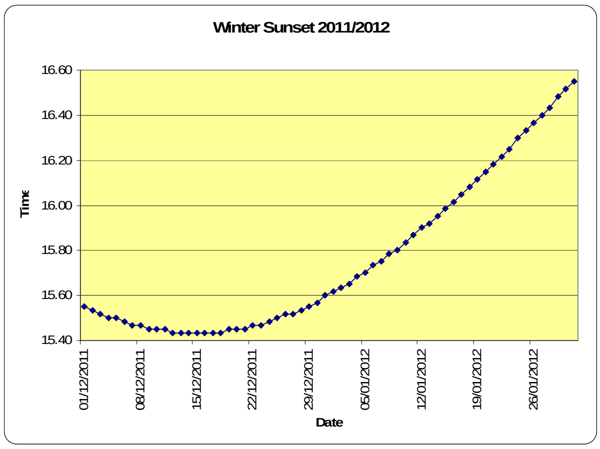
Midwinter days and other stories

John S. Reid

This opens with a curiosity about the hours of daylight. The subject seems pretty simple, the hours being mainly between sunrise and sunset. Of course there's twilight too, the amount of which depends on the angle the Sun makes with the horizon at dawn and dusk and the path of the Sun below the horizon. I'll look at that later. I guess what's prompted this piece was being asked by different people why the latest sunrise and earliest sunset don't coincide with the shortest day. In fact they're some way off. The first graph shows the length of the day at latitude 57°N 2°W (approximately Aberdeen) calculated over December 2011 and January 2012. The slight wiggles are just because the tabulated data plotted was given only to an accuracy of 1 minute. The shortest day of 6 hours 41 minutes is on December 22nd, as expected. Now look at sunrise and sunset over the same period. The latest sunrise is at 08.47on the 29th and 30th of December; the earliest sunset at 15.43 on 15th and 16th of December, a fortnight earlier than the latest sunrise. Put another way: the earliest onset in the afternoon of the dark winter nights occurs a week before the shortest day. After that the sun sets later. So what's happening? It is, of course, all to do with the orbit of the Earth and the tilt of its axis. These subjects are astronomical yet many astronomy texts don't talk about such a practical result of our astronomical knowledge. The reason for the odd behaviour is all to do with 'the equation of time'.







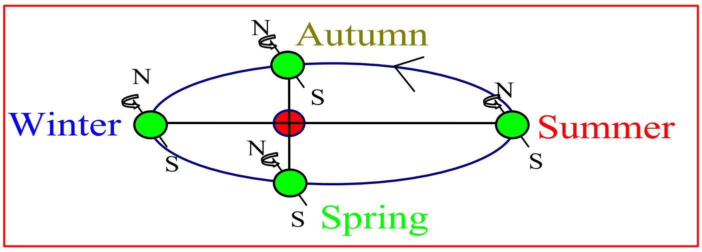
The Equation of Time

The 'equation of time' is a formula, obviously, but one worked out in the 1600s so the mathematics isn't advanced. To find out what's going on we don't even need to see the equation but we do need the concepts behind it. The equation of time is about the difference between time as shown on mechanical or electronic clocks and 'solar time'. Solar time is that shown on a well positioned sundial. The two are not the same, as we'll see, and it is the difference between them that's responsible for the curious timing of sunrises and sunsets. The difference between them is what is called 'the equation of time'. Mechanical or electronic clocks are set to show 'mean time', meaning they go at a constant rate throughout the year and that rate is adjusted so that a mean-time year and a solar year are the same. Once one starts looking at fractions of a second there are real issues about how time is defined but they don't influence this discussion.

Midday solar time is when the Sun is due south. This simple idea is behind the next few pages of discussion. The next midday is the next time the Sun is due south. A calibrated sundial shows solar hours as 1/24th of the solar day, at least when the Sun is shining on it. Sunrise and sunset are equally far on either side of solar midday in terms of solar hours. To say the same thing in another way, on the shortest day the sunrise is at its latest solar time and the sunset at its earliest. Everything is as you would expect. The shortest day with us just now is the 22nd December and hence this is the day with the latest solar time sunrise and earliest solar time sunset.

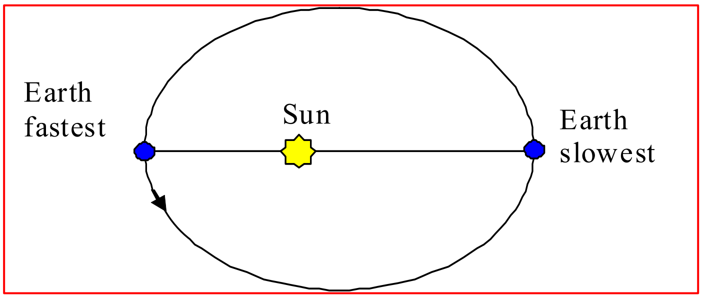
Wait a minute. Why is this the day of shortest sunlight? I've just plucked the result out of the air. Recall that the full length of a solar day is determined by two things. First the time it takes the Earth to rotate once on its axis as measured against the background stars. This is 23 hours 56 minutes 4.091 seconds of clock time. The second factor is the extra time needed for the Sun to be aligned due south again after one rotation, a time that is determined by how far the Earth has moved around its orbit during the rotation. This defines the full solar day. The daylight time is the time between sunrise and sunset, these times being defined when the Sun's upper limb appears on the ideal horizon, the ideal horizon being determined by modelling the Earth as a sphere. This isn't such a bad definition as it seems on paper. It's true that in hilly regions there are plenty of houses that never get the midwinter sun shining on them because it is blocked by hills to the south but this doesn't mean there is no daylight. Their daylight hours are more or less the same as for people living on the plains since the sunlight fills the sky and is scattered in all directions by the atmosphere.

The shortest daylight hours in the northern hemisphere occur when the Earth's axis, directed from south to north, makes the largest angle with a line from the centre of the Sun to the Earth. First remember that the direction of the Earth's axis is near enough fixed in space as the Earth orbits the Sun. Let's take its angle of tilt to the Earth's orbit as 23.5° in round numbers. That angle may be fixed but the angle between the direction from the Sun to the Earth and the Earth's South to North axis varies throughout the year. At midsummer it's 66.5°, at the equinoxes it's 90° and at midwinter it's 113.5°. I've drawn the diagram with mid-winter taking place when the Earth is near perihelion (closest to the Sun), which is almost the case just now, at least within a fortnight or so. Perihelion occurs early on 5th January in 2012.



Now to get to the nub of the matter. Why does clock time differ from solar time? There are 3 reasons.

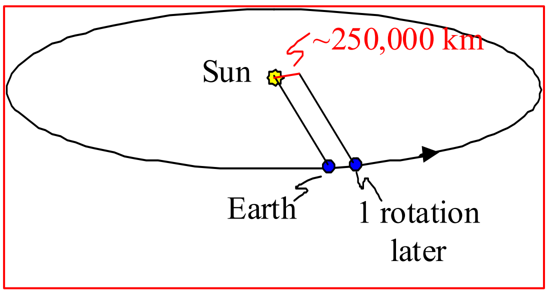
The simplest is the longitude effect. The Earth is divided into internationally agreed time zones more or less 15° of longitude wide, though with some local wiggles in places. Within each time zone the clocks are set the same. It takes an hour for the Sun to travel across 15° of longitude and hence at the western side of a time zone solar midday will occur an hour later than at the eastern edge, yet clock time will be the same for both places. The first difference between solar time and clock time is therefore the longitude effect. Aberdeen is 2° west of the Greenwich meridian, which is the defining longitude of our clock time zone. It takes 8 minutes for the Sun to move 2° of longitude and hence local solar time will differ from clock time because of this effect by 8 minutes. This will shift solar sunrise and sunset times by 8 minutes everyday of the year. They would be symmetrical not about clock midday but about 12.08 clock time if there were no other disturbing factors. This is a constant shift throughout the year and isn't usually considered part of the 'equation of time'. There are, however, other disturbing factors.



The second effect is the variation of the speed of the Earth in its orbit, as described by Kepler's 2nd law. At perihelion the Earth is travelling at its fastest and its orbital arc is most strongly curved. The net result is that it takes more time for the Sun to come round due South again after the Earth has rotated once than it does at other times of the year. This lengthens the solar day a little. The effect is less than 10 seconds for a single day but it builds up near midwinter day after day, so that the mechanical clock keeps running fast relative to the solar day. The time of an event (say the next day's sunrise) on a mechanical clock is therefore later than on the solar clock. As midwinter approaches the solar time of sunrise hardly changes from one day to the next but mechanical clocks, which are running faster than solar time, show the event getting later and later. This continues after midwinter and the sunrise times continue to get later and later on the mechanical clock even though in solar time they are slightly earlier. Because of this effect the latest sunrise times are after midwinter.

If you think about it, the earliest sunset times must occur before midwinter. Once the daily change in the clock exceeds the daily decrease in solar sunset times, on the clock the sunsets will start to get later. For instance, if solar sunset is earlier by 3 seconds tomorrow but mechanical clocks are running 7 seconds fast per day relative to solar time, then sunset will appear on the clock 4 seconds later tomorrow. This happens before midwinter. The difference in rates between the clock and the sundial because of the orbital speed effect explain why the sunrise and sunsets change in the way they do, but the numbers don't come out correctly. There is a third effect going on and this is determined by the tilt of the Earth's axis.

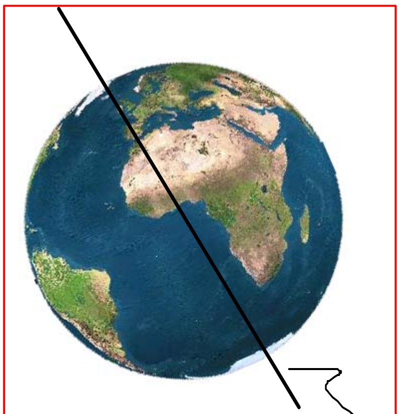
The axial tilt effect isn't nearly as obvious and you have to admire our ancestors who worked it out centuries ago. In fact it makes an even bigger difference than the orbital speed effect. Around midwinter the two effects work together to increase sunrise and sunset timing effects, giving rise to the fortnight's difference mentioned earlier.



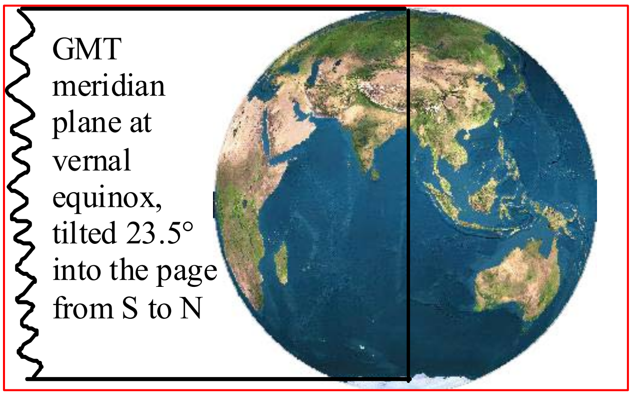
To see what's going on let's go back to the basic definition of the solar day. For just now we can ignore the added complication of the Earth's elliptical orbit and imagine the Earth going around the Sun uniformly in a circular orbit with the Sun at the centre. During the time the Earth rotates once on its axis it has moved a certain distance along its orbit. Quite a long way, in fact; about 250,000 km or 200 Earth diameters. It takes the Earth just over 7 minutes to move a distance along its orbit equal to its diameter so it is fairly zipping around the Sun. A point on the Earth's surface where the Sun is overhead at noon has a line from the centre of the Earth through the point going directly to the centre of the Sun. Exactly one rotation later that line will be pointing to a location in space some 250,000 km to the right of the Sun. The time it takes for the Earth to rotate further so that due south is in line with the centre of the Sun again determines the length of the solar day.

If the Earth rotated on an axis that was perpendicular to its orbit the time needed to make up the solar day would be just the 3 minutes 56 seconds the Earth's rotation is short of 24 hours. It would be the same all year round. Of course in this case the Sun would be permanently above the equator and there would be no seasons.

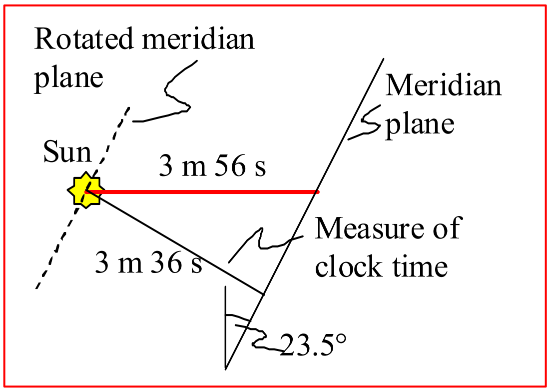
In reality the Earth's axis is tilted and the axis tilt effect arises because the amount of rotation needed depends on how this tilt is oriented in space relative to the Sun. The tilt makes it better to talk of a 'meridian plane' rather than a line pointing at the Sun. The meridian plane is the plane containing the Earth's axis and the place you are looking at. Solar noon is when the meridian plane passes through the Sun.



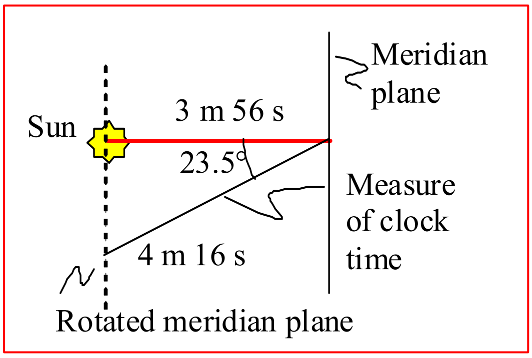
GMT meridian plane seen edge on from the Sun at vernal equinox



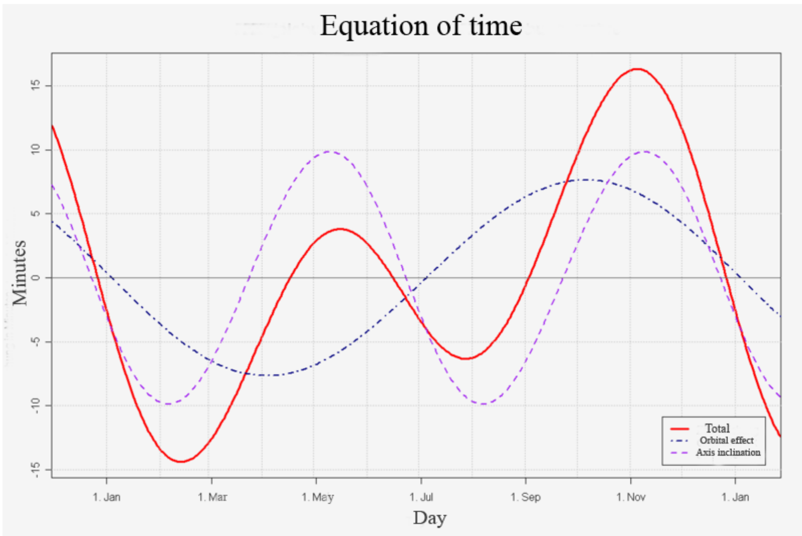
At the equinoxes the Sun is above the equator but the Earth's axis and hence the meridian plane is inclined, as in the illustration. To restore midday after a complete rotation of the Earth, the meridian plane must rotate until it again intersects the Sun. The time it takes to do this determines the length of the solar day. The rotation results in the meridian plane at the Sun moving as shown in the adjacent diagram. This take *less* time that it would do if the axis were perpendicular, because a different point on the plane intersects the Sun's position in only 3 min 36 secs, as shown. What this means in reality is that the Sun is at a slightly different height in the sky at midday the following day, which is correct at the equinoxes. It also means that the solar day is shorter than the clock day (it is 23 h 56 m 4 s + 3 m 36 s = 23 h 59 m 40 s) and hence solar clocks will appear to run fast because of this effect.



The opposite happens at the solstices when the Earth's axis is leaning directly away from the Sun at midwinter or directly towards the Sun at midsummer. Let's look at northern midwinter. The meridian plane is perpendicular to the Earth's orbit but the rotation axis is tilting down at 23.5°. As the Earth rotates, every point on the meridian plane moves down in a direction perpendicular to the Earth's axis. This downward motion is a measure of how far the Earth has rotated and hence of clock time. The adjacent diagram shows that it takes about 4 min 16 secs after a complete rotation to reach midday again. The solar day is now longer by about 20 seconds.



The two contributions to the equation of time are therefore the orbital effect and the Earth's axis tilt effect. The orbital effect gives rise to a contribution that varies sinusoidally throughout the year. The fastest change is at perihelion and aphelion, implying that the zeros of the corresponding sine wave contribution are determined by these points (see illustration below). The amplitude of the cumulative difference between solar time and clock time is a bit less than 8 minutes. Over the centuries the perihelion is advancing through the calendar, though it wobbles by a day or two every year due to such effects as leap years and the fact that a calendar year isn't quite the length of time the Earth orbits the Sun. These effects alter the phase of the sine wave. The amplitude alters over millennia as the eccentricity of the Earth's orbit alters. In spite of this, for practical purposes the equation of time doesn't change much over anyone's lifetime.



The axis inclination effect varies sinusoidally with a period of 6 months. It also isn't constant over the centuries but again the changes are slow as the tilt of the Earth's axis wobbles in size and direction. The graph alongside shows the current version of the two components of the equation of time and the total effect (solid red line). It is an edited version of that appearing in Wikipedia. What is plotted here is how much solar time is in advance of clock time. In some books the reverse is plotted, namely how much must be added to sundial time to give clock time.

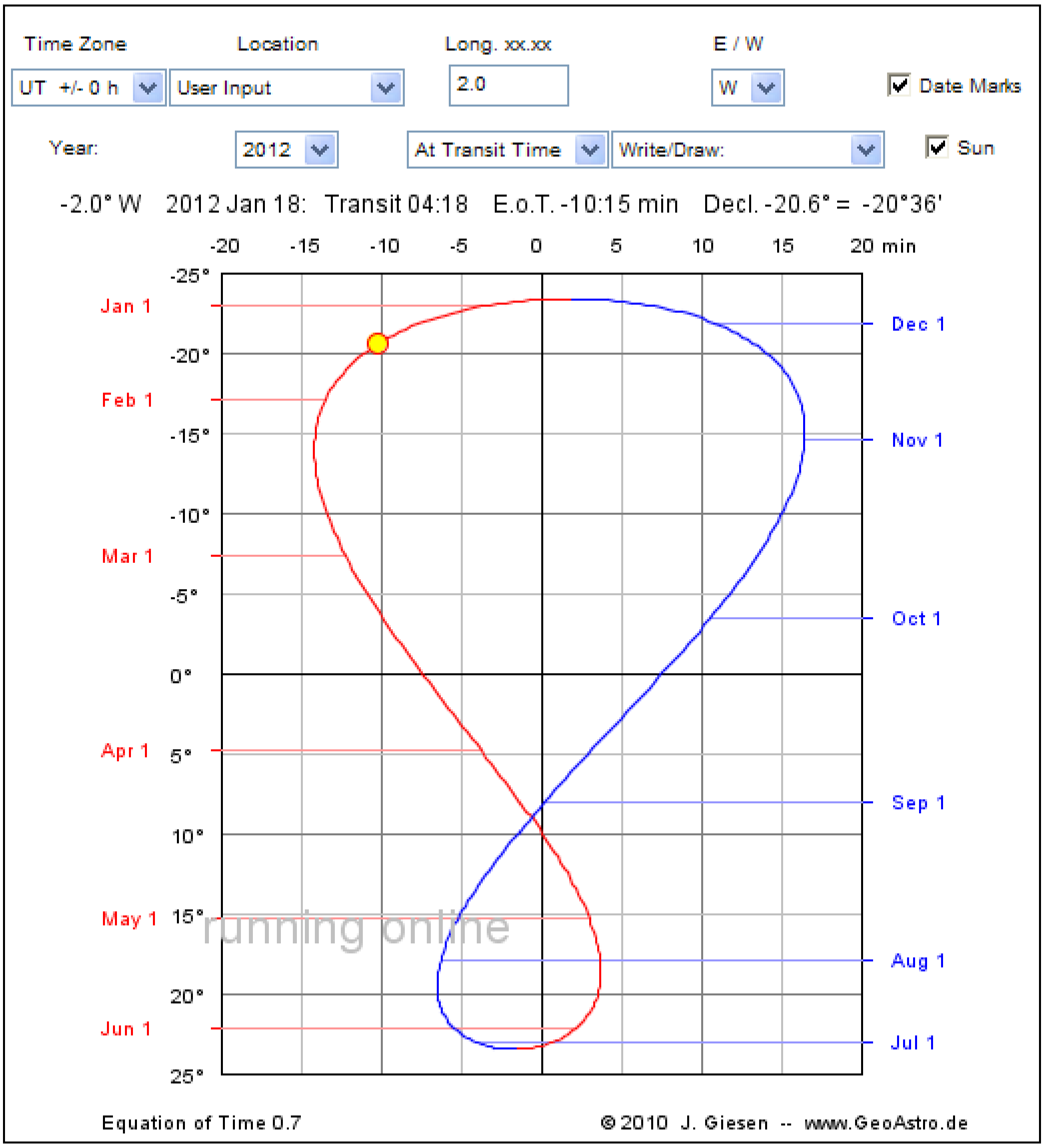
Perhaps I should add that if you find an astronomy book that explains the equation of time it will almost certainly do it differently. The textbook will talk of the mean Sun moving uniformly along the equator while the actual Sun moves round the ecliptic. This is all very well but the explanation is one step removed from the definition of a solar day and the story I have given above goes back to the fundamental definition of a solar day.

Just about everything we do in society is controlled by clock time these days but solar time is not an invention of astronomers, it is part of everyday experience. Anyone who has solar panels on their roof will know that the electricity generated is determined by solar time, not clock time. In February even at Greenwich, with clocks set to GMT, halfway between sunrise and sunset is nearer 12.30 than 12 O'clock. If you're having a winter holiday in Tenerife at the time, longitude 16.5° W but still using GMT (or UT as it's now called), then halfway between sunrise and sunset, i.e. solar midday, will be about 13.30. This is great on holiday. You can get up late and still experience most of the day. It is clock time that is the astronomers' invention. The link between the two times is the *equation of time*.

The analemma

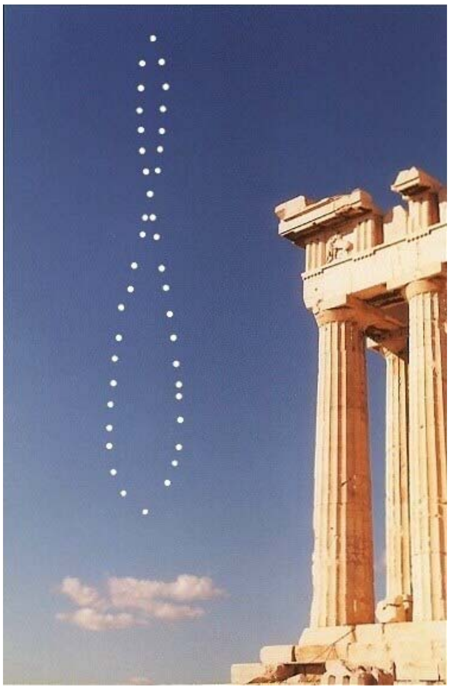
Anyone who reads a bit about sundials will soon come across the *analemma*, an apparently curious figure showing the position of the shadow of a small disk at midday throughout the year. It looks like a slightly distorted figure of 8. Without putting in some effort, it's not at first obvious what this is all about. Equipped with an understanding of the equation of time, the analemma is now obvious.

If I presented the equation of time as an actual equation giving time difference for a given day in the year then anyone with Excel skills could calculate the analemma. The figure below



shows the analemma as calculated with the java applet at http://www.jgiesen.de/deceot/. What it corresponds to is the position of the Sun in the sky throughout the year at midday, clock time. The position at midday solar time traces out a vertical straight line in the sky about 47° long, the Sun obviously being highest at midsummer and lowest at midwinter. As we now know, relative to clock time the Sun arrives due south sometimes before midday and sometimes afterwards. The analemma here plots the declination of the Sun vertically; along the horizontal is plotted the number of minutes fast or slow the Sun is at midday. To the right of the zero line the Sun is fast on the clock by the number of minutes shown. The horizontal coordinate is just the equation of time. In the picture above the red line represents the first half of the year, the blue line the second half. The yellow dot the day I made the plot (18th Jan 2012). Because the Sun really does behave like this, the change in position of the shadow of a small disk at 12 O'clock will show the analemma over a year.

A similar figure of 8 pattern is obtained if you choose any other clock time. On the web there are several photographs of the Sun in a cloudless sky made at a fixed time, not necessarily midday. It would be hard to do this in Aberdeen, which has cloud cover around 70% of the time. The accompanying illustration is from the site of Anthony Ayiomamatis at http://www.perseus.gr/Astro-Solar-Analemma-102816.htm which is well worth a visit for he has a set of analemma taken at 2 hour intervals over a year, the result of an impressive project. The figure is upside down compared with the applet calculation, which has January at the top whereas of course January is at the bottom in the northern hemisphere sky. This is because the applet is aimed at sundial enthusiasts. The higher the Sun, the nearer its shadow is to the base of the gnomon.



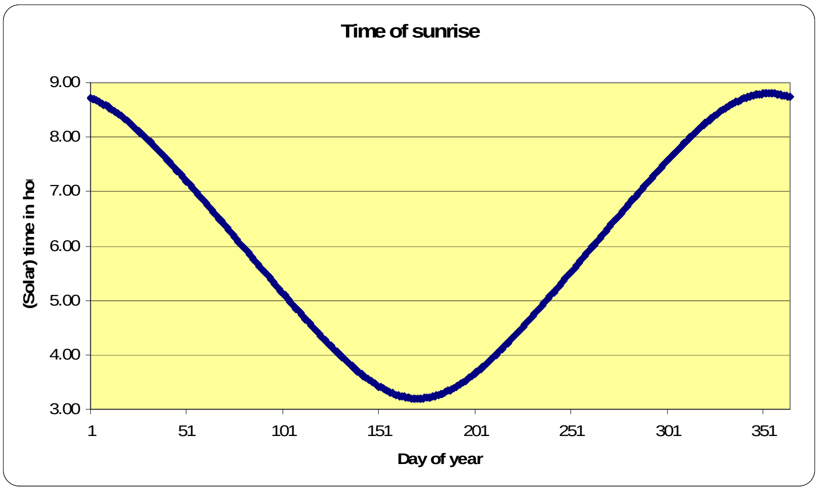
From an observation of the analemma over a year one could work out the equation of time and see that it decomposed into two contributions. From this one could deduce details of the Earth's orbit around the Sun. The analemma on other planets in the solar system will be different and I'll leave it for interested readers to investigate these!

I have a photograph of the analemmic sundial taken at the door of the Physics Dept. of Utrecht University that I will look for. Meanwhile if you look on the web you will see a common variant which is a dial where the person standing at the centre is the gnomon (the shadow casting object). Think of your head as the disk casting the shadow. Because of the analemma, if the shadow of your head is to fall at the correct place, you will have to move your feet to a position dictated by the analemma. The analemma is therefore translated from the shadow to the base of the gnomon. It was a triumph of mediaeval thought to work out how to make a sundial that showed the correct time on a single scale without having to move the gnomon during the year. The secret is to incline the gnomon parallel to the Earth's axis.

Sunrise (*and sunset*)

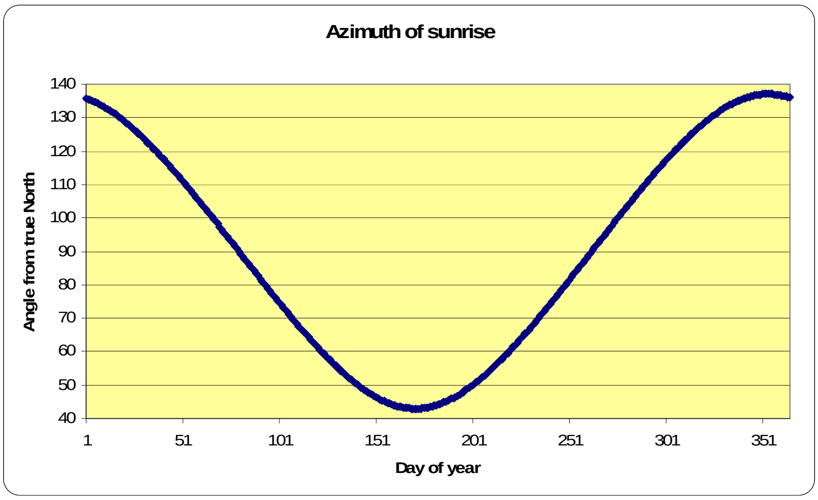
Having begun by considering events at sunrise and sunset, it's worth saying a bit more about where on the horizon sunrise and sunset occur throughout the year, and when. This leads to the topic of twilight. The bearing from true north (the *azimuth*) of sunrise or sunset is given from spherical geometry as $\cos(Az) = \sin(decl)/\cos(\lambda)$, where Az is the azimuth of the sunrise, $decl$ is the declination of the Sun and λ the latitude of the place. The declination of the Sun varies sinusoidally over the year from a zero at the vernal equinox, so putting this into Excel with a latitude of 57° the result can be plotted. It would be better to show the result on a circular chart but my Excel doesn't offer this choice so I've plotted it as a simple graph. Remembering that

45° corresponds to NE and 135° to SE you can see that from the winter solstice (near the right-hand end of the graph) the sunrise is close to SE and at midsummer the sunrise is close to NE. Correspondingly, sunset varies between SW and NW.



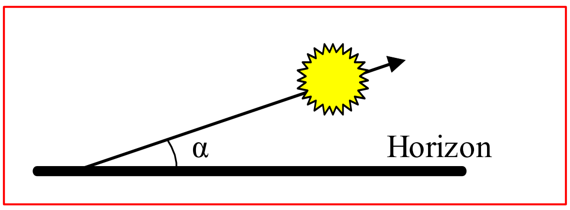
I have to confess that the simplified formula just calculated is for the centre of the Sun on the horizon whereas sunrise is normally defined as the upper limb of the Sun just reaching the horizon and hence the centre below the horizon. The plotted results are less than half a degree out because of this. Another small difference is caused by atmospheric refraction but the accurate answer is still within the thickness of the line of points on the plot.

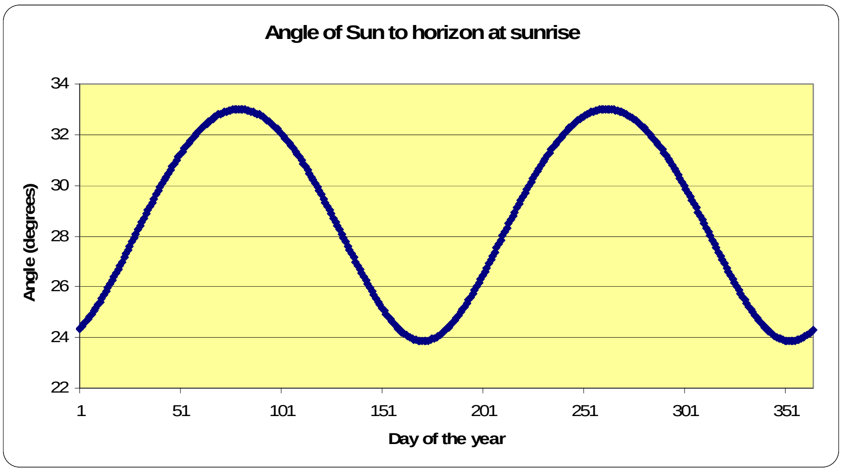
The time of sunrise (at least the solar time) is given by the hour angle of the Sun and an expression for this at sunrise was derived in a companion piece of mine connected with solar panels entitled 'How much sunshine?'. The hour angle (HA) is given by cos(HA) = -tan(λ) tan(*decl*), again for the centre of the Sun. The result shows a similar kind of variation throughout the year, sunrise changing from just after 3 am at the summer solstice until about 8.45 near midwinter. Because of the hour of 'summertime', the summer sunrise is an hour later by the clocks, and of course there is the equation of time to consider when looking at a clock and the longitude effect.



Twilight

The boundaries of *civil twilight* are when the Sun is 6° below the horizon. This will extend before sunrise or after sunset by a length of time that depends on the angle the Sun makes with the horizon when rising and setting. This angle, call it α, can be calculated from the simple formula cos(α) = sin(*λ*)/cos(*decl*). The Sun rises most quickly at the equinoxes, when the angle is (90-*λ*) or 33° at Aberdeen, and is at its shallowest at midsummer and midwinter, when the angle at Aberdeen is just under 24°. The change is not big and probably not noticeable for the casual watcher of sunrises.





At midnight the Sun is an angle below the northern horizon of (90° – λ – *decl*). At midsummer this is just 9.5° at Aberdeen. The so called *nautical twilight* is the time when the horizon at sea ceases to be visible, generally put at when the Sun is 12° below the horizon. At midsummer nautical twilight officially lasts throughout the night at Aberdeen, though thick stratus cloud may dim out the horizon on some days. *Astronomical twilight* extends to when the Sun is 18° below the horizon, which it needs to be to get a dark sky against which one can see the fainter stars. As we know, none but the brightest stars are visible near midsummer at Aberdeen and the local astronomical society does little stargazing from home for a few months over the summer. At midnight (90° – λ – *decl*) = 18° when *decl* = 15°. With greater declinations, astronomical twilight extends through the night at Aberdeen, which it does at our latitude for 50 days on either side of midsummer. Of course the Moon is also a spoiler for the dark skies that astronomers need.

Twilight, then, depends on the angle the Sun goes down or rises up and its path below the horizon. If we could see through the Earth, we'd see the Sun daily going round in a circle with angular radius (90° - 23.5°) = 66.5° due to the Earth's rotation. Of course we can't see through the Earth so only part of this circle is visible, the part that results in daytime. In addition, because of the orbit of the Earth around the Sun, the tilt of the axis of rotation changes direction over a year. This results in the daily circle changing its inclination over the year and hence the part we can see getting larger in the half year leading to midsummer and smaller afterwards. The combination of the two motions, rotational and orbital, means that the Sun spirals up the sky for the first half of the year and spirals down for the second half. In about 180 days (half the year) the Sun moves up or down by 47°, giving an average daily motion of 0.26°, or half the Sun's apparent diameter. The daily change up or down the sky is greatest at the equinoxes, when it is about 0.36° per day (about three-quarters of the Sun's apparent diameter per day), but zero at the solstices when the Sun changes direction. This daily change in height of the Sun mightn't concern us in everyday life but it is crucial for any navigator taking the altitude of the Sun at noon to find their latitude. If you don't know which day it is you'll typically get your north-south position wrong by 15 nautical miles per day out you are, more near the equinoxes. 15 miles is a lot to be wrong at sea if you're anywhere near a coast; 15 miles is a lot to be wrong in many places on land, even in a desert.

What started as a simple account of a winter curiosity about sunrise and sunset has morphed into a broader look at the Sun in the sky, a topic relevant to everyday experience, to the growing day of plants, to solar power, to sundials and even to navigation. Much of this you won't find in modern astronomy texts but the undeniable fact is that the Sun is by far the most important star to us on Earth and knowing where it is in the sky and when it shines its light upon us is pretty fundamental astronomy. It's well worth the effort thinking about that.

JSR