Cosmology 3 – General Relativity, geometry and the density of the Universe

On a big scale, on the scale of the Universe, gravity, space and time obey the rules of General Relativity, as far as we can tell. Einstein introduced the theory over the few years prior to 1916 as a description of the interaction of matter via gravity. His theory was based on the cosmological principle that basic statements about the laws of physics, including gravity, should be the same no matter where in the Universe you take as your reference point and how you are moving. This led Einstein to formulate a theory involving the 'curvature' of space-time ('curvature' in this context being a well-defined but relatively novel mathematical concept) and use the mathematical notation of 'covariance' and the then comparatively new mathematics of tensor calculus to describe his ideas. These were the appropriate mathematics to frame laws that are independent of changes of origin into frames of reference that might be accelerating. One result of the use of this appropriate but more advanced mathematics is that the detailed workings of his theory have never migrated to University courses that are at a lower level than Hons Maths or Physics courses. The details of how General Relativity works are therefore inaccessible to many people but, that said, it's not just a very few people who understand it. Many, many people have followed through the details and the results of General Relativity are widely known, much more widely than in Einstein's day. Moreover, Einstein himself showed that the theory predicted effects that Newton's theory of gravity didn't. It is therefore a theory that is testably different from Newtonian gravitation and, pretty remarkably, it has stood up to every test thrown at it. I'll come to some of these tests shortly.

Because the mathematics behind General Relativity is a bit esoteric, physics courses often leave the subject for advanced students only. This is a mistake. As the astrophysicist D. W. Sciama said in the preface to his monograph on 'The Physical Foundations of General Relativity' *General Relativity must not be regarded as a beautiful theory that is irrelevant to the needs of physics. On the contrary, it is needed to give full meaning to the most basic and elementary concepts on which physics is built*. He might have added to this that General Relativity is needed to understand the Universe at large, which is the main reason it comes into our course.

What is General Relativity all about? It is a theory where matter messes with space and time, to put it crudely. In our 'normal science' matter exists **in** space and time but neither time nor space are affected by matter. Objects move from one place to another against a background of a frame of reference that exists whether the objects are present or not. Objects move **through** a pre-existing space, and **in** a pre-existing time. The distance between two points, for example, does not depend on whether we place any matter at either of the points, or indeed, in between them. It seems almost absurd to suggest it might. What kind of place would that make the world? Einstein's view is not simply that matter messes with space and time but that matter **defines** the measurement of space and time: indeed, it defines the very existence of space-time. If there is no matter, there is no space-time.

If the above sounds a bit weird I should add that most basic physics courses don't spend any time discussing what 'space' and 'time' are. I haven't done this social experiment but I suspect if you polled people in Union Street and asked them what 'space' is they would perhaps say that space is 'nothing'. Well, space isn't nothing. Think of the inverse square law of gravitation. The force of attraction of one mass on another depends on the amount of space between them. It can't depend on the amount of 'nothing' between them. Odd when you think about it that way. Space can't be 'nothing'. Clearly space is something that needs to be understood. I'll borrow here a quotation of Charles Lamb used by Stephen Hawking.

Lamb wrote '*Nothing puzzles me like time and space. And yet nothing troubles me less than time and space, because I never think about them*'. If you want to understand the Universe, you need to think about time and space.

Einstein's definitive 1916 paper

If I had to make a judgement on what was the greatest intellectual achievement of the twentieth century, then I'd think hard about nominating Quantum Mechanics. Quantum mechanics has had a huge practical impact but people still haven't come up with the definitive interpretation of quantum mechanics. Instead, my accolade for the greatest intellectual achievement of the 20th century goes to General Relativity. If you have the capacity to learn something about it, and everyone in this class has, then you should do so.

Einstein didn't set out to change perceptions of the fabric of space and time just for the sake of doing so. The story had begun over 10 years prior to 1916. He had set up his theory of Special Relativity, published in 1905, to describe how you transform coordinates and times from one frame of reference to another frame of reference moving at constant speed with respect to you. It sounds a rather abstract and not particularly practical exercise to worry about. You know something of Special Relativity from the earlier part of this course. The theory turned out to have enormous implications, not the least of which was to show that the everyday way we do this (called the Galilean transformations) is just an approximation, and even Newton's law of motion in its widely quoted form of F = ma is likewise an approximation. That was in the past and Einstein turned his attention to the meaning of Newton's first law of motion and the special case of gravity. We'll get to some of the problems that worried Einstein shortly but it's worth saying first that the concepts of Special Relativity underlie those of General Relativity, particularly the concept of space-time that Special Relativity introduced.

General Relativity in a nutshell

Although I'll spend more than a lecture talking about General Relativity, I'm not going to derive anything from the basic equations. There's a good reason for this. It takes something of a lecture course on its own to describe the concepts that have been used to formulate these equations. However, I couldn't let you away without actually showing what the fundamental equations of General Relativity look like. They can be written in one line:

$$R_{\mu\nu} - (1/2)g_{\mu\nu}R - \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$$

The left-hand-side describes the attributes of space-time, particularly the curvature properties of space-time. I'll mention curvature later in this section but not in sufficient detail to describe precisely how the left-hand-side is defined. Λ is the cosmological constant that I'll also mention later. The right-hand-side consists of another constant, within the round brackets, which involves both the fundamental constant of gravity G and the speed of light c, and a quantity T called the stress-energy tensor.

Einstein's equations (the little Greek letters describe components that run over the 4 dimensions of space and time) are not at all like Newton's law of gravity. They essentially replace gravity by curvature components of space-time and state what it is that creates the curvature, namely the quantities on the right-hand-side that are described by the stress-energy tensor. If you think about it, you can't discus the motion of anything without having a space-time framework so it is general relativity that is behind the whole of physics, not just problems explicitly involving gravity. A key implication, which I'm not going to dwell on here, is that it is not just mass that is the source of gravity but all energy. That is why the thing called the 'stress-energy tensor' appears on the right hand side and not just an expression involving masses. It's not at all obvious that in 'ordinary circumstances' the whole thing reduces pretty well to Newton's law of gravity. It certainly looks plausible that there is a lot more going on than Newton described, and indeed there is.

The result of Einstein's work was this set of equations relating space-time to the matter within it. *They are the most valuable discoveries of my life* Einstein said to fellow theoretical physicist Arnold Sommerfeld in 1915, and most people agree. Einstein became more convinced that he had discovered, or perhaps you'd prefer to say 'invented', a fundamental truth about nature. He later commented *Nature rarely surrenders one of her magnificent secrets* and he felt privileged to have found such a secret in the equations of General Relativity.

The intention of this chapter is to try to explain what Einstein's theory is about. I hope you won't think at the end what Byron thought after reading Coleridge's 'popular' explanation of metaphysics *I wish he would explain his Explanation.* Let's look first at accelerations, for accelerations are a central issue in General Relativity.

Accelerations

Acceleration occurs when speed changes. Acceleration is the rate of change of speed (we have used the symbol v for speed). Using a notation we introduced in the last section of the course, if a_c stands for acceleration (unfortunately cosmologists have already used a for the cosmic scale factor) then by definition $\boldsymbol{a_c} = \dot{\mathrm{v}} = \ddot{r}$, with a dot to represent the rate of change, as before. Accelerations occur in a vehicle when you put your foot on the accelerator or the brake. They occur when a lift starts or stops. They also occur when you let something fall. In this last case the cause is gravity and something about this puzzled Einstein.

Gravity causes bodies to accelerate. We're used to calling freely falling bodies 'weightless'. Astronauts that are freely falling around the Earth in the Space Station can be seen on TV to be floating as if they were in empty space. Astronaut training planes that plummet to the ground with acceleration g create 'weightless' conditions for a minute or so before they have to flatten out their flight-paths. Free-fall frames of reference are just the same as *inertial* frames, said Einstein. It doesn't matter what the local strength of gravity is. This idea is the basic physical principle that Einstein put at the foundation of General Relativity, now called **the strong principle of equivalence**.

Principle of Equivalence

The Principle of Equivalence states: *to an observer in free fall in a gravitational field the results of all local experiments are completely independent of the magnitude of the field.* I don't want to lecture on how General Relativity evolves from this statement but I just want to point out that General Relativity is based on a physical principle rather than any obscure mathematical postulates. The principle applies not just to mechanical experiments but to those involving light. The net result is that General Relativity is a theory that intimately links together the concepts of space and time, the gravitational effects of matter and, as we'll see, the propagation of light.

You'll find the principle of equivalence stated in all sorts of different ways. One way is that the effects observed locally in a gravitational field and an accelerated frame of reference are just the same and the two kinds of situations can't be distinguished, locally. (Almost repeating what I said above, this 'locally' business is a way of flagging that the effects of changes in gravitational field across any observations or experiments are not considered). Another is that the acceleration due to gravity is independent of the mass accelerating (and what the mass is made of).

Einstein's thought experiments – example 1

Einstein devised a number of 'thought experiments' carried out in lifts, or elevators as they are referred to in the US. The 'lift' is supposed to represent a small space where the observer can carry out experiments. Rockets, space-stations and the like weren't part of the currency of ideas in the early 20th century and so Einstein imagined a laboratory in a lift that could be accelerated or could drop freely. This was his local environment.

Our first imaginary experiment is to project a ball in a lift that is accelerating upwards. The ball arcs forward in a parabolic curve if air resistance is ignored, as can be shown from basic mechanics. Now imagine the projection of the ball taking place in a stationary lift sitting in a gravitational field, such as on or near the surface of the Earth. The result will be exactly the same. Einstein concluded that gravity and accelerations in space are just the same thing, and in fact can't be distinguished.

There is a well-known way of treating accelerating frames of reference in mechanics. One assigns a force ma to everything, a force directed in the opposite direction to the acceleration a. Such a force is called an *inertial force*. If you are in a lift accelerating upwards then you can apply Newton's law to objects in your lift so long as you add a downwards force of ma to everything. If your lift is accelerating upwards with acceleration g, then this force is just mg. What struck Einstein as fundamental that other people had overlooked was that the inertial force mg was a result coming from Newton's law of motion ("$F = ma$") whereas the gravitational force mg comes from the law of gravity. In short, what Einstein was saying is that gravitational mass and inertial mass are identical. They could have been quite different quantities but of course by using the same symbol there is an implication that they are the same. Einstein drew some deep conclusions from the fact that these separately defined masses were the same, as we'll see.

Various experiments have been done to try to detect the difference between these two kinds of masses, starting with the famous balance experiment of the Hungarian Loránd (Roland) Eötvös who began his experiments in 1890. You can look this up on the web. In Budapest there is a University named after Eötvös and precious few physicists have had that honour. The experiment has been repeated on several occasions and, like the Michelson Morley experiment, tries to measure a difference that relativity says will be zero. Experimentally, any difference in the two different kinds of mass is less than 1 part in a million million, which is zero in my money.

Space-time diagrams

Space-time diagrams are very helpful in both Special Relativity and General Relativity. In the simplest such diagrams, one spatial dimension is plotted along one axis, say the horizontal axis and time at right angles. In fact it is very common to plot not time t but ct on the time axes so that the line x = ct representing light starting out at the origin is a straight line at 45° to both axes.

The path of an object as it travels through different points in space and time is called the *world line* of the object. If something is travelling at constant speed then its world line will be a straight line. Objects that have no forces on them travel at constant speed, as Newton's first law says and hence their world lines are straight lines. In this sense straight-lines have a special significance and there is a special word to describe straight lines in any mathematical space. They are called *geodesics*. Light, as you know, travels in straight lines and hence follows a geodesic path.

If a body is in free fall we say it is accelerating towards the ground (or centre of the Earth if it is a satellite). The world lines of accelerating bodies will be curved because the body doesn't cover equal intervals of distance in equal times but changing distances in a given time interval. The second diagram on the slide shows a curved world line.

Gravity curves space-time - example 2

What on earth does curved space-time mean? If I drive a golf ball down the fairway, then I can see the ball curving through space as it flies on its way towards the green. You might therefore think that space-time on the fairway is surely hugely curved but in fact the curvature of space-time is measured by what happens to a light beam as seen in a free-fall frame, and not what happens to a golf-ball as seen in a frame fixed to the Earth. In fact the curvature of space-time near the surface of the Earth is very small. What is important is the new way of thinking about gravity that Einstein introduced.

Einstein's idea of curved space-time is introduced so that a body freely falling in the presence of gravity obeys the same law as a body when no gravity is present. He recognized that gravity causes accelerations but wanted to build the curvature of the world-line of a freely falling body into the underlying space-time geometry instead of into its path. Thus freely falling bodies follow geodesics in curved space-time. Remember that geodesics are the straight lines of space-time geometry. Einstein's viewpoint is that just as a body given an initial velocity in distant space will follow a straight world line, so the same body released near a mass like the Earth will also have a straight world-line because the gravitational effect of the Earth results in re-defining space-time so that the geodesic is still straight in the new space-time geometry. I should make it clear that this straight-line trajectory of a body is as seen in *a free-fall frame*. In a frame at rest relative to the Earth the body will of course be accelerating.

I'll put it another way, that may (or may not) help. It was Newton's first law that worried Einstein, the law that says a body will travel at uniform speed in a straight line in the absence of any forces on it. It sounds fine enough but you can never get away from gravity and hence you can never have no forces acting on a body. Ok, said Einstein, supposing we think of gravity as altering the way distance and time is measured in space-time, 'warping', if you like to put it that way, the underlying fabric of space-time such that in the absence of all non-gravitational forces a particle will follow a geodesic. Can this be done consistently? If so, we have got around the 'problem of gravity' and preserved Newton's first law, albeit in a modified way. Einstein was looking at gravity in a completely novel way. By describing the geometry of curved space-time with suitable curvature induced by the presence of masses (and other energy), you have effectively described the influence of gravity. I'm going to come back to curved space-time later in this section.

Well, you may say, I'll take your word that this produces the same result as the usual view that gravity provides a force on the masses and that this force attracts them to the centre of the Earth. Surely Einstein's approach is a bizarre way of stating what's happening. You may not want to believe it but Einstein's way of looking at gravity is actually a better way. It produces results that are in fact **not** quite the same as the usual view and, when properly formulated, the laws satisfy the cosmological principle, whereas Newton's law of gravity in fact doesn't. The whole theory is completely consistent with Special Relativity. In one very obvious way, Newton's law of gravity has gravity propagating instantaneously, which is quite inconsistent with Special Relativity. In General Relativity gravity propagates at the speed of light. Another casualty of General Relativity is the strict application of the inverse square law of gravity. This only comes out of Einstein's viewpoint as a good approximation in 'usual circumstances', when gravity is weak. [I define when gravity is weak in a supplementary note to this chapter called '*On gravity*']. Nature, in fine detail, is more subtle.

The difference is seen in two ways. First, planetary orbits in General Relativity don't strictly repeat, as an ellipse does, but turn around in space, or precess to use the proper word. Precession arises because of the time it takes gravity to propagate from the central star (the Sun in the solar system) to the planet. For example, this is about 8 minutes as far as the Earth is concerned. The gravitational force acting on the Earth isn't directed to exactly where the Sun is relative to the Earth at any instant but is in the direction to where it was 8 minutes ago. Secondly, orbits are accompanied by a loss of energy due to the radiation of gravitational waves. This loss is usually very small but in some circumstances is substantial. I'll say more about this, too, in due course.

There was a young man who observed
I confess I am somewhat unnerved
I had never before
Seen the truth of the lore
That, where matter is, space must be curved

(Anon)

The rubber sheet model

I hesitate to show this illustration but pictures like this well-known one try to show how distances between points change in the region of a mass. The illustration shows how curvature looks when seen from outside. In General Relativity, curvature is a property of space-time and is related to how unit distance changes in different regions. You don't have to get 'outside' space-time to measure curvature. In reality, large distortions like the one idealised here would only become visible very close to highly concentrated masses. I'll talk about black-holes soon.

What the picture does illustrate is that if you imagine a ball rolling around the depression in the sheet, it can orbit quite happily not because of any gravity at the centre of the depression but simply because of the curvature of the sheet. Curved space-time does the job of gravity. That is the basic message of General Relativity.

Gravity affects light: example 3

Let me take a third example. Imagine yourself in a lift that is rapidly accelerating. In reality that would be a pretty dire situation so it's fortunately only a 'thought experiment' and not one to try at home. Now arrange that a light beam is shone in through a hole in the side of the lift. Because the lift is accelerating, the path of the light beam, if you could follow it showing in some haze, would appear bent. By the principle of equivalence, namely that gravitational fields are equivalent to accelerated frames of reference, the gravitational field must do the same thing. In other words, the light beam behaves as if it were a projectile with some mass following a curved path in space. Einstein showed that that mass m was the same as the mass you need to associate with light in the theory of Special Relativity, namely for each photon E/c^2, where E is the energy of a photon of light. So out of General Relativity comes very naturally the concept that gravity acts on light, too, and can bend light paths. With some hand-waiving, Newtonian physics can be made to predict a similar effect but the 'prediction', such as it is, gives a different value from that of Einstein.

Gravity red-shifts light

A related thought experiment compares the light emitted from a source on the surface of a mass M to the light emitted from a source that is accelerating by an amount g, the strength of the gravitational field at the surface of the mass M. Again, the light photons behave as if they have a mass E/c^2 (= hf/c^2, h is Planck's constant and f the light frequency) and lose energy as they 'escape' from the gravitational field of M. The result is a gravitational red-shift predicted by General Relativity. Applied to light from a massive star, this red-shift is over and above any Hubble red-shift. This red-shift increases the more massive the star is and the smaller is its radius. If the mass to radius ratio is big enough, the object becomes a black-hole with infinite red-shift and no light can escape from it. Thinking through the matter even further, you can see that light coming towards our Earth-based telescopes will be blue-shifted a tiny amount as it accelerates down the gravitational field we're sitting in on Earth. General Relativity would put it that the wavelength of the light changes because the very space around the Earth changes into a curved space, appropriately distorted by the mass of the Earth. Can you see the similarity of language here between this way of saying things and the cosmological red-shift that was due to a change in the space through which the light propagated? In the cosmological case, the change in space is due to the expansion of the Universe. Back to our gravitational red-shift, whatever the language used, the ratio of M/r for a star is greater, usually much greater, than the ratio of M/r for the Earth so in fact the stellar red-shift will win over the blue-shift of the Earth.

In these lectures I have usually quoted key relationships in the story, without proof. Gravitational red-shift is not exclusively predicted by the difficult equations of General Relativity. You can see how it might come about from physics you already know and the answer we are about to get is just the answer produced by General Relativity.

Einstein associated a mass m with a photon through the relationship involving his own $E = mc^2$ and Planck's expression for the energy of a photon in terms of its frequency f.

$$mc^2 = hf = hc/\lambda$$

$$\text{therefore } m = \frac{h}{c\lambda} .$$

The idea of the red-shift is that if a photon escapes from the surface of a star or planet of radius r and mass M, the photon has to travel against gravity and hence loses an amount of energy equal to the gravitational potential energy it gains. This potential energy is just $\mathrm{GM}m/r$. Hence the photon's change of energy $\Delta\mathrm{E} = \mathrm{GM}m/r$. Remembering that a photon energy E is given by hc/λ, the change in wavelength can now be worked out:

$$\Delta\left(\frac{hc}{\lambda}\right) = \frac{\mathrm{GM}}{r}.\frac{h}{c\lambda}$$

If the wavelength change $\Delta\lambda$ is small, then this becomes

$$hc\frac{\Delta\lambda}{\lambda^2} = \frac{\mathrm{GM}h}{c}.\frac{1}{r}.\frac{1}{\lambda}$$

$$i.e.\ \frac{\Delta\lambda}{\lambda} = \frac{\mathrm{GM}}{rc^2}\ .$$

The final line gives the gravitational wavelength red-shift ($\Delta\lambda/\lambda$) in terms of the physical constants G and c^2, and the mass and radius of the star or planet from which the photon comes. The sign of the wavelength change, which I've not kept track of above, is such that the energy of the photon is less when it has escaped from the gravitational pull and hence its wavelength is larger. The left-hand side above is just z in the notation of section 2 of these notes.

If you pause and look back at the argument above you'll see that there is a strong similarity between the gravitational red-shift and the loss of kinetic energy of a projectile thrown upwards (against gravity). For the projectile this energy loss appears as a loss of speed. For a photon, the quantum of light energy, the speed is necessarily fixed as c, as you know from Special Relativity, and hence the loss of energy appears as a loss of the only property that determines the photon's energy, namely its frequency f ($E = hf$). A lower frequency means a longer wavelength, hence the red-shift.

Pound & Rebka experiment

A convincing experimental demonstration of the gravitational red-shift was made by Pound and Rebka at Harvard in 1960. They used the newly discovered Mössbauer effect (for which Rudolf Mössbauer won the Nobel Prize for Physics in 1961) that certain radioactive elements emit and absorb gamma rays at extremely precise energies. Placing a detector at the top of a 22 m high tower at Harvard, they found that it would not absorb these precisely defined gamma rays as well as it would when placed next to the source kept at the bottom of the tower. The reason was that the very small red-shift of the gamma rays between the foot of the tower and the top changed their wavelength enough to spoil the absorption. To measure the effect they moved the source slowly upward at a speed where the Doppler shift should just compensate for the gravitational red-shift, according to Einstein's equation. This effectively measured the gravitational red-shift and Pound & Rebka found it in agreement with Einstein's prediction to within the accuracy of about 1%. In the picture, Prof. Robert V. Pound is on the right and Glen A. Rebka on the left, with the source.

It's worth looking quickly at some of the numbers involved. The period, and hence the frequency, of a pendulum clock is controlled by the gravitational constant g. If you take a pendulum clock upstairs, it will run a little slower because the strength of gravity is less upstairs than downstairs - not by a lot but quite measurably. First year Physics lets you work out exactly how much. The fractional change in frequency of the clock ($\Delta f/f$) is given by $-(\Delta r/r)$, where Δr is the height the clock is raised and r is the radius of the Earth. If the clock is raised up 22 m, as in the Pound & Rebka experiment, then the change in frequency of the clock is about 3.4 parts per million, as much as a couple of seconds a week. If instead of a pendulum clock you have an atomic clock whose second is determined by electromagnetic oscillations, then there is also a change in frequency with height, determined by the General Relativity result given above. This works out as $(\Delta f/f) = -(\mathrm{GM}/c^2r)\times(\Delta r/r) \approx 8\times10^{-10}$, around a four thousandth of the pendulum clock change. Nevertheless, to the accuracy of today's standard clocks it's an effect that must be taken into account. I'll add a modern take on this. The next generation of precision clocks that will come to replace the Cs atomic clock in providing the definition of a second are 'optical clocks'. They will be accurate to a few parts in 10^{-18}. It is already possible to see that two of these clocks in the same laboratory run at different rates if one is only a metre higher than the other. In future the definition of a second will need to include the mention of gravity in some way or another. In short, gravity affects the measurement of time. It's not a new concept and we now know through General Relativity that it's a concept that's here to stay.

Black-holes

Black-holes were a predicted outcome of General Relativity, though people pointed out afterwards that a similar idea was suggested by Newtonian gravity. (The term 'black-hole' was coined as late as 1967 by the impressive theoretical physicist John A Wheeler (1911 – 2008) who was one of the leading lights in developing General Relativity in the mid 20th century). The black-hole idea itself came via Karl Schwarzschild (1873 – 1916), a brilliant astronomer and mathematician who in 1916 produced the first solution to Einstein's field equations of general relativity, showing how much space is curved around a point mass. To solve the equations under any conditions is impressive but to do so in comparatively short order while you are in the German army in Russia was even more so. The unfortunate Schwarzschild contracted a fatal disease in Russia and was sent home, but died in mid 1916. He had pointed out that there was a radius around a point mass, now called the Schwarzschild radius, from within which light cannot escape. The Schwarzschild radius (r in the next expression) can be calculated from the Newtonian argument of setting the escape velocity equal to the speed of light. i.e.

$$\mathrm{c} = (2\mathrm{GM/r})^{1/2}, \text{ giving } r = 2\mathrm{GM/c}^2\,. \qquad (3.1)$$

In so much as Schwarzschild had time to think about the implications of his solution, he believed that his radius was just a mathematical abstraction and not a physically reality. There is an important difference between the Newtonian and Schwarzschild results. With the Newtonian idea, a body within the radius r can get further away than r but will necessarily fall back again, since it cannot have the escape velocity. The Schwarzschild (General Relativity) result is that a body within the Schwarzschild radius can never escape beyond the radius r.

Astronomy in the second half of the twentieth century has confirmed for us that black-holes are an astronomical reality, though clearly you have to infer their existence out there rather than see them directly. The average density of matter over the Schwarzschild radius decreases with increasing mass involved. You can see this from simple physics:

$$\text{density} = \text{mass/volume} = M \Big/ \left[\frac{4}{3}\pi r^3\right] = c^6 \Big/ \left(\frac{32\pi G^3 M^2}{3}\right) \qquad (3.2)$$

Observational evidence suggests that there is a black-hole at the centre of many galaxies, including our own, of mass at least a million times the mass of our Sun. Say the mass is 10^6 $M_\odot = 2\times10^{36}$ kg. Taking $G = 6.67\times10^{-11}$ m^3 kg^{-1} s^{-2} and $c = 3\times10^8$ m s^{-1}, gives the density as 1.8×10^7 kg m^{-3}, less than the density of a neutron star. The Schwarzschild radius for such a mass is 2.96×10^9 m, or about 4.75 times the radius of our Sun. Black-holes are formed in astronomy by the self gravitational collapse of mass. That collapse doesn't stop when the mass is within the Schwarzschild radius but continues towards a singularity. A black-hole with 'only' the mass of the Earth has a Schwarzschild radius of about 9 mm. Such black-holes are unlikely to exist, for the following reason.

If you think about the nature of atoms, the astonishing vision you should have is that we are composed almost entirely of space, not matter. Everything around us is. If you could remove the space within the actual matter that we see, the entire contents of the Earth, its oceans and continents, the rocks, mantle and core beneath our feet, would be compressed to the density of nuclear matter, protons and neutrons. This is about 10^{18} kg m^{-3}. The total mass of the Earth is 6×10^{24} kg and hence if you imagine all the space removed from all the matter of the Earth then that matter at nuclear density would occupy $6\times10^{24}/10^{18} = 6\times10^6$ m^3. That's a sphere of radius 113 m. The self-gravity of such a sphere is not nearly enough to compress it further so no force could reduce a mass of this size to the Schwarzschild radius of 9 mm, a reduction in volume by a factor of 2 million. So you have deduced that such objects aren't likely to exist in the Universe. That's a relief. Section 6 of this course will show that stellar mass black-holes and larger can exist, and indeed do exist. They are many km across.

Before moving on from black-holes, let me dispel one popular misconception. Black-holes don't suck matter into them like a vacuum cleaner, unless that matter is extremely close. If there were a black-hole at the centre of the solar system of mass equal to that of the Sun we would continue to orbit exactly as we do now. We just wouldn't get any heat and light. Orbiting matter that isn't by chance going more or less directly towards a source of gravity keeps on orbiting. For it to get closer, it must lose energy either by radiation or by collision. The Earth isn't doing either, which is why we don't spiral into the Sun.

Matter only disappears into a black-hole if it approaches close to the Schwarzschild radius. Objects orbiting within twice the Schwarzschild radius no longer have recognisably elliptical .orbits but have spiral orbits, ending within the Schwarzschild radius. They will lose significant energy emitting gravitational waves. Any charged matter will also loose energy by radiation. Indeed, the conversion of mass into energy close to a black bole is one of the most efficient ways of turning matter into energy, much exceeding the efficiency that the Sun does this. Any particle coming in from afar and passing within 1.5 times the Schwarzschild radius will be captured. The black-hole will get a bit more massive and the Schwarzschild radius will expand a bit.

For a stellar-mass black-hole, say $5M_\odot$, the Schwarzschild radius is about 15 km and the tidal forces close to the Schwarzschild radius will tear any objects apart. For a galactic centre sized black-hole, the tidal forces near the Schwarzschild radius aren't big and you could fly your spacecraft intact into the black-hole. Of course you could never tell anyone outside what happened after that and once within the Schwarzschild radius, theory predicts that it will not be long before you're ripped apart too.

There was a young fellow named Cole
Who ventured too near a black-hole
His dv by dt
Was quite wondrous to see
But now all that's left is his soul

(A. P. French)

What might happen inside a black-hole? That is another story. Most black-holes will have two other properties beside mass, namely charge and spin. For a black-hole that has either of these properties then matter doesn't continue to compress indefinitely within the hole but there is an inner horizon. There's more to a black-hole than you might think.

Another 'more than you might think' feature of black-holes is that they will evaporate, in principle, as discovered by Stephen Hawking. They turn into electromagnetic radiation, a process called *Hawking radiation*. The lifetime of a black-hole is proportional to its mass M cubed (M^3). A 1 kg black-hole (of Schwarzschild radius 1.5×10^{-27} m) evaporates in about 8×10^{-17} seconds, from which you can work out that a stellar mass black-hole will last much longer than the Universe is old but a mini black-hole of mass say 1.0×10^{-20} kg can never in principle be seen. Generating black-holes in particle accelerators like the LHC is a non-starter.

Before getting back to cosmology proper, I'll mention a few more experimental tests of General Relativity. Experiment is the 'supreme court of appeal', as Eddington said.

Starlight is bent by the Sun

The bending of light by a star was predicted by Einstein in his 1916 paper, and had been discussed in earlier papers. This paper was published in German by a German-born scientist, in a German periodical during the First World War. When you've been fighting a country for two years in a singularly bloody and dirty conflict trying to belittle the status of your enemy there is good reason for ignoring everything they have to say. However, in Cambridge Arthur Eddington (1882 – 1944) was Professor of Astronomy, Director of the Cambridge Observatory and a singularly bright spark, known for his mathematical brilliance and physical insight. He was also a pacifist and hence somewhat immune to wartime propaganda and the painting of stereotypes. Earlier papers by Einstein had alerted him to the thrust of General Relativity. He saw how to mount an experimental check of the predicted bending of starlight. He would exploit the forthcoming total solar eclipse of May 1919. By an amazing co-incidence, during this eclipse the Sun would be in front of the particularly rich star cluster of The Hyades, about the best place in the sky it could be for this test of General Relativity.

With the Moon temporarily screening out the disc of the Sun, the eclipse provided an opportunity to see a star whose light came to us along a path that went close by the Sun. If the starlight grazing the Sun was bent, then the position of the star should be altered relative to stars more distant from the Sun. The angle through which the starlight is bent is just given by $\frac{4GM_{Sun}}{c^2R}$, where R is the distance of closest approach of the light to the centre of the Sun.

It's not a difficult expression. If there were no bending, then the stars should be in the same relative positions as if the Sun weren't there. Stars both near and further from the Sun could be recorded at the same time on a photographic plate. Actually, the experiment wasn't quite that simple, for a hand-waiving argument using classical physics and assigning a mass to photons predicted a bending half that given by General Relativity. [The General Relativity result included both the effect of gravitational time dilation **and** the effect of the curvature of space-time near the Sun]. The actual experiment was therefore to try to distinguish the larger General Relativistic prediction from the non-relativistic prediction of half the amount.

Eddington organised two expeditions under the auspices of the Royal Society of London and the Royal Astronomical Society, one to Brazil and one that he went with to the island of Principe off West Africa. As everyone knows who has tried to see an eclipse of the Sun, any attempt is at the mercy of the weather and hence mounting two expeditions was a prudent strategy. The deflection of the starlight looked for was very small, about 1.75" arc close to the Sun. In Principe the weather almost clouded out the event but Eddington managed to get one successful photographic plate. In Brazil, the observations were better. The average of the two results confirmed the predictions of General Relativity within reasonable limits and certainly excluded the pseudo-Newtonian result. The successful outcome of this experiment was highly influential in raising Einstein to celebrity status. *Lights all askew in the Heavens. Einstein theory triumphs* as The New York Times trumpeted. *Revolution in Science. New Theory of the Universe. Newtonian ideas overthrown* was how the London Times put it. The results also put Eddington into the public eye as an expositor of General Relativity.

'*One thing is certain and the rest debate*
Light rays, when near the Sun, do not go straight'

as Eddington wrote on one occasion.

[I'll splice two anecdotes into the notes here that I won't cover in the lecture. The first comes from the reminiscences of the Nobel prize winning astrophysicist Subramanyan Chandrasekar who as a young man was among a small after-dinner group in the Senior Common Room of Trinity College, Cambridge, in 1933, a group that included Rutherford and Eddington. The question of why Einstein was so famous came up. Rutherford turned to Eddington and said '*You are responsible for Einstein's fame. The war had just ended…. The people felt that all their values and all their ideals had lost their bearings. Now, suddenly, they learnt that an astronomical prediction by a German scientist had been confirmed by expeditions to Brazil and West Africa and, indeed, prepared for already during the war, by British Astronomers. Astronomy had always appealed to public imagination; and an astronomical discovery, transcending worldly strife, struck a responsive chord……*". Rutherford's assessment is the more believable in the light of a similar response to NASA's landing a man on the Moon in 1969. People were fed up with the conflict ridden 1960s and saw the moon landing as an event that united people in spite of conflict around the world that seemed to be always in the news.

Eddington was quite happy to play the rôle of 'brilliant academic'. In November 1919 the Royal Society of London held a meeting at which the Astronomer Royal reported the results of the eclipse expeditions. As the meeting was dispersing, Ludwig Silberstein came up to Eddington and said '*Professor Eddington, you must be one of the three persons in the world who understand General Relativity*'. As Eddington demurred at this statement, Silberstein responded '*Don't be modest Eddington*'. Eddington replied '*On the contrary, I am trying to* *think who the third person is!*' Silberstein himself wrote one of the first books on General Relativity in 1922. Of course there were more than three people but it is true that Einstein's papers were not available in Britain or America during the First World War and for a good while afterwards. Eddington had received his copies personally from the Dutch astronomer de Sitter.]

In the 1990s a very successful satellite was launched called the **Hipparcos mission**. Our first-year course mentioned that it was designed to measure the distance of several hundred thousand stars by measuring their parallax to 0.001″ arc. Notice from the formula given two paragraphs back for starlight bending that the bending effect decreases only as $1/R$, R being the distance the starlight passes from the Sun. If the effect is 1.75″ arc close to the Sun's surface then if we want measurements of stellar positions accurate to 0.001″ arc then you can see that the bending of light from stars that are well away from the Sun has to be taken into account too. The success of the Hipparcos mission for many stars depended on taking account of this General Relativistic light-bending effect.

Gravitational lensing

Gravitational lensing experiments have confirmed the effect more accurately. Indeed the bending of starlight by the gravity of other stars located almost directly between us and the source of light is an effect that is now in the forefront of astronomy. If distant starlight can be bent, then gravity has the effect of providing a refractive index gradient around stars. Such a refractive index can provide imaging, just as in optics. The effect is known as gravitational lensing. I'll say a bit more about this when I'm talking about the search for missing mass in the universe.

The effect is quite widely seen. Multiple images of very distant galaxies situated behind nearer clusters of galaxies are seen on image plates. One result is that we see galaxies we wouldn't otherwise have done because a bigger range of light reaches our telescopes than would do so in the absence of lensing. It's a bit like the increase in brightness of an image that you get by using a lens in a camera instead of a having a pin-hole. However, an intermediate galactic cluster is irregular in detail so the lensed images show a variety of irregular effects.

So called gravitational microlensing is already used and being widely developed as a tool to detect whether distant stars have planets circling them. In this kind of lensing, a lensing star lies exactly on the line of sight between a remote star and us. As the lensing star passes in front of the remote star, taking a time somewhere between days and months, the light from the remote star is seen to brighten in a characteristic way at all wavelengths as the lens effect kicks in. If the lensing star has a planet, then this planet alters the characteristics of the light received by us from the remote star in a distinctive way that allows the presence of the planet to be inferred. In short, the influence of gravity on light is now considered a well-known effect. The technique is being developed on an international scale to look for extra-solar planets in projects such as OGLE (the optical gravitational lensing experiment) and the Microlensing Planet Search project (MPS). See the web.

Precession of Mercury's orbit

I'll be briefer about two other tests. Einstein also predicted that the orbit of the planet Mercury would twist around in space, *precess* is the technical word I used already, in the gravitational field of the Sun at a rate not predicted by Newtonian gravitation. The orbit is observed from the Earth to precesses quite fast, about 5600" arc per century. Most of this can be explained by the shift in our stellar co-ordinate system (the precession of the equinoxes) and the influence of the other planets on Mercury. Einstein predicted an additional contribution over and above the other known contributions. The additional rate is not very much (43" arc per century) but such an additional precession had already been seen and had not been satisfactorily explained. Further experiments have completely confirmed the General Relativity value, against different values offered by other theories attempting to account for it. There is now no unexplained discrepancy in the precession rate of Mercury (or any of the other planets).

Gravity probe B

The Gravity Probe B is an experiment with a satellite launched in 2004 after a 40 year development. You should know of its existence because the Physics Department here under the steering of Prof Mike Player has had an input into this experiment. The probe had an extremely technically challenging experiment on board that measured the gyroscopic precession in the gravitational field about 700 km above the Earth's surface, which is slightly different from the field at the surface of the Earth. The experiment aims to test not one but two predictions of General Relativity that have not been tested before. One involves the very slight effect on space-time produced by the rotation of the Earth. This effect is known as 'frame dragging'. I'll leave those interested to read more about this on the Gravity Probe B web-page except to say that one result of this impressive experiment was the announcement that Einstein's curvature of space effect (the geodetic effect) had been verified to better than 1%. In 2009, the microscopically small 'frame dragging' effect had been isolated in the data and verified to within 14% standard error. The final results can be found on the Gravity Probe B website.

The stronger the gravity, the slower the clock

There are indeed other reasons to believe the predictions of General Relativity. The gravitational red-shift is an example of another effect of General Relativity, namely that clocks run slow in a gravitational field. The stronger the gravity, the slower the clock. You can look on the gravitational red-shift as this effect. The 'clock' that marks the frequency of emission of light runs slow if the emission takes place in a strong gravitational field. The effect is very small on the surface of the Earth. It is much bigger on a white dwarf star and has been verified to better than 1%.

One consequence of the gravitational effect on clocks is the need to correct the atomic clock time of the GPS satellites by the amount predicted by General Relativity because they are orbiting in a slightly different gravitational field from us on the ground. Without this correction, the positions given by the GPS would be systematically wrong. With the corrections, they are not.

Another very convincing test turns out to result from the observation of certain pulsars in binary systems. Pulsars are rapidly rotating neutron stars, usually spinning on their axis faster than once per second with a regularity that matches the regularity of our best atomic clocks. If a pulsar is part of a close binary system, the rate of change of the regular pulses emitted can be measured. These changes are determined by a range of orbital effects, including the General Relativity orbital precession resulting from the influence of the very strong gravitational field of a companion star. Observing one such pulsar binary system and related ones over many years not only provided experimental evidence for the effect of gravity on the rate at which a clock runs in a strong gravitational field but gave direct evidence of the rate of loss of energy of this binary system through radiation of gravitational waves, exactly as predicted by General Relativity. For this work Russell Hulse and Joseph Taylor won the 1993 Nobel Prize in Physics.

Even stronger evidence has been uncovered since 2003 by the discovery at the Parkes radio-astronomy facility in Australia of the double pulsar system PSRJ0737-3039A & B. In this system, two pulsars orbit each other every 2.4 hours with a separation of less than the radius of the Sun. We see their orbits almost edge-on and receive beams from both pulsars. This ‘find’ is very lucky and the observed orbital parameters of the pulsars has provided an even stronger test of General Relativity than Hulse and Taylor’s binary pair, a test General Relativity has passed without difficulty.

I hope the above sections have given you a flavour of General Relativity and left you with the impression that it is not simply a conjecture but a powerful technique whose predictions are strongly backed by that *ultimate Court of Appeal*, to use again Eddington’s phrase mentioned in an earlier lecture, namely experiment. I want to go on to talk about the curvature of space.

One anecdote about Einstein relates that when a doctoral student Ilse Rosenthal-Schneider asked him in 1919 how he would have felt if Eddington hadn’t confirmed the prediction of General Relativity, he is said to have replied “*I would have felt sorry for the good Lord. The theory is correct anyway*.” It was probably said in humour but coming from anyone else would have been interpreted as evidence of a swollen head. The fact is that almost a century later and with more than 7 billion people now in the world, no-one has yet proved Einstein wrong.

Curved space-time?

The curvature of space-time is at the heart of General Relativity. You’ll see the words flash past you in popular articles and the mere possibility that the geometry of our Universe is different from the Euclidean geometry that had been the staple diet of mathematical teaching for 2000 years has fascinated people.

First, a brief word on what ‘curvature of the universe’ is **not** about. It’s not about the size of the universe. Balls have a radius of curvature, the space around us is spherical and hence you might think we’re talking about the size of the universe. We’re not, and it’s not that simple.

The word ‘curvature’ is used in this context in a technical, mathematical sense. Curvature of space-time is a concept intimately connected with how you measure the interval between two points in space-time. If space-time is **flat**, it has no curvature and the interval $\mathrm{d}S$ between closely spaced points is given by $\mathrm{d}S^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$. This is just the formula you may meet in Special Relativity, applied to 3 spatial dimensions x, y and z and to time. In reality, it is little more than the theorem of Pythagoras dressed up in fine clothes. If space-time is curved, then the interval formula is not as simple. You shouldn’t be surprised that cosmologists look for the most general formula that will describe a homogeneous, isotropic space. I’ll not write it down but if you want to look it up the buzz-word is the *Robertson-Walker metric*. Sometimes Friedman’s name gets in there too.

Built into General Relativity is the possibility that space-time is curved and this is the bottom line of why the mathematics of General Relativity is more complicated than the mathematics of Special Relativity. I'm reminded of the leap that Kepler made in planetary orbit descriptions when he introduced ellipses in place of circles. The mathematics of ellipses is significantly harder than that of circles. For example, finding the length of a planetary orbit, the circumference of an ellipse, is not at all straightforward. However, Kepler had introduced the right concept for the job and likewise we believe that Einstein has done so with the concept of the curved space-time of General Relativity.

Curved space-time comes into the story in two ways. First, Einstein used the concept in a moderately local context to remove the need for the traditional 'force of gravity'. Masses (and energy), he said, curved space-time locally and the resulting trajectories of bodies differed from what they would be in the absence of the masses. Once you have calculated the trajectories in curved space-time you have deduced fully the effects of the masses. The result does not always come out to be the same as Newton's law of gravity predicts and Einstein's result seems so far to be verified by experiment. However, cosmologists use the word 'curvature' in the context of the universe as a whole. The curvature of the Universe is a concept that applies everywhere, at all times. In mathematical terms it boils down to "how do we calculate the general interval between two points in space-time?". Just as Kepler could say some very useful things about planetary elliptical orbits without knowing all the maths of ellipses, so we can say something useful about curved spaces without the full maths of General Relativity.

One final point to make here is that when mathematicians talk about curved spaces they don't distinguish between the concepts of space and time. A 'space' to the mathematician is simply a structure in which there are co-ordinates. They don't care if the coordinates in reality refer to physical space, time or indeed anything else – very confusing really for ordinary people who use the word 'space' to mean where objects are located and move around.

The geometry of the Universe

The appreciation of what curved spaces were all about was pretty new in Einstein's day. Relevant concepts had not long before been developed by European 19th century mathematicians like Gauss, Riemann, Christoffel and Ricci. What is the difference between a 2-dimensional coordinate system covering a plane and one covering a sphere? The plane is infinite and flat. The sphere is finite and curved. The infinite aspect of one case and the finiteness in another is a matter of topology and is not the issue at present. Our issue is with curvature. Curvature is something an arc of a circle has that a straight line doesn't have. Curvature is a concept that's used for surfaces, volumes and objects with more dimensions than a line. Mathematicians usually measure the curvature of a surface by the parameter $k = 1/R^2$, where R is a radius that just fits the surface in a chosen direction. Hence a plane has curvature zero ($k = 0$) since it is equivalent to a sphere of infinite radius. This makes sense.

What you also know is that a piece of flat graph paper can be nicely marked out in a regular grid of small equi-spaced steps of dx and dy at right angles. If you've ever tried it you'll know that there is no way you can wrap this graph paper neatly over a sphere. The sphere has curvature that the plane hasn't got. A regular system of what pass for straight lines on a sphere isn't going to cover the sphere with little boxes of identical size. Hence the curvature of the surface of a sphere requires that in this case a more complicated formula is needed to measure the interval between two nearby points than simply $dx^2 + dy^2$. This is why the interval formula is intimately connected with curvature. This mathematical view of curvature is not exactly obvious from common sense. For example, you can wrap your graph paper around a cylinder and hence as far as the mathematics is concerned, an open cylinder has no curvature and is 'flat'. Of course even mathematicians know that a can of beans isn't flat in the ordinary sense. The can isn't an open cylinder but this just highlights that the word 'curvature' in General Relativity has a carefully defined, technical meaning.

Flatlanders who lived in a 2 dimensional world on the surface of a sphere could tell whether their world had the curvature of a plane or a sphere. They would do this by applying tests to the geometry of the figures they drew.

Everyone in the room will know that if you draw a triangle on a flat sheet then the sum of the angles within the triangle is 180º. You may not know that if you draw a triangle on a sphere, then the sum of the angles exceeds 180º. It's easy to see that this is true in a special case. Draw a large triangle on the Earth starting at the pole and going round to the equator down lines of longitude 90º apart. The final side is along the equator. The angles in this triangle are clearly all 90º and hence the sum of the three angles is 270º, not 180º. A smaller triangle showing a smaller sum for the interior angles that is still greater than 180º is shown on the slide.

Curved spaces have another feature that the circumference of a circle drawn on them is less than the circumference of a circle with the same radius drawn on a flat plane. The circumference of a circle of radius r is just $2\pi r$ if the circle is drawn on a plane. If the circle is drawn on a sphere of radius R, then the circumference is $2\pi R\sin(r/R)$. I shan't prove this. If you were a flatlander and wanted to find the curvature of the space you were in, then if you drew a circle that was a very small fraction of the curvature R and measured it then you'd just get the answer $2\pi R\sin(r/R) = 2\pi r$, since $\sin(r/R) = r/R$ for small r. Hence you couldn't tell. However, if you drew a big circle then you'd find the circumference was not $2\pi r$ and you could deduce the curvature of your space. If you were a flatlander who really wanted to convince your fellow flatlanders that your space is curved without messing about with small differences between flat space and curved space, then you would draw a circle whose radius was the distance between pole and equator of your space. Measured in your space, its radius is $\pi R/2$. The circumference of your circle is $2\pi R$ and hence its circumference measured in your world is just 4 times its radius. QED, your space is curved.

Notice that to us looking at the sphere and plane from a higher dimension, it's obvious what the difference is between the two surfaces but the mathematical tests for curvature can be done from within the space. I hope you can see where this argument is going. Even the flatlanders who live in a 2D world have a means of telling what kind of space they live in, whether it is flat or what's called a 2-sphere. We live in 4D space-time. We can't see what it looks like from a higher dimension. So do we live in flat space-time, or a space-time that's curved like a higher dimensional sphere or in the opposite way to a sphere, namely one with negative curvature? A 2-surface with negative curvature looks hyperbolic when viewed from the third dimension. See the diagram.

I should add for the sake of honest description that the curvature involved in Einstein's equations that I showed near the beginning of this section is not measured by determining the circumference of circles or the angles of a triangle but is measured by a procedure due to the mathematician Riemann. It involves imagining a vector taken around a small closed loop in a curved space so that it remains parallel to itself and determining that after this procedure it will be pointing in a slightly different direction. Such a direct measure of curvature is what the orbiting gyroscopes in the Gravity B experiment did. This isn't the place to elaborate on the details but I just wanted to reinforce my earlier hint that even a century after Einstein's work it is still true that anyone wanting to 'do' serious General Relativity needs to equip themselves with some serious mathematics.

Our Universe

General Relativity says that matter curves space-time. That's what the equation near the beginning of this chapter is all about. The Universe certainly contains matter and hence space-time is curved on the largest scale. However, it's possible for the spatial dimensions on their own to have positive curvature, no curvature (described as "flat") or negative curvature. The $64,000 question, or perhaps nowadays it is the 15th question for £1,000,000, is "what is the intrinsic curvature of the space of our Universe as a whole?" As it turns out, this has a direct relevance to the ultimate fate of the Universe because both are dictated by the density of matter in the Universe. If there is a high density of matter in the Universe then that will provide enough gravitational force to bring the expansion of the Universe to a halt and to reverse it. In this scenario, our descendants long into the future, if we have any, will see galaxies coming towards them in what will end in a big crunch. This state of affairs is described as a **closed Universe**. Such a Universe has positive curvature. If the density of matter in the Universe is very low, then the expansion of the Universe will never be halted. Such a situation is described as an **open universe**. Such a Universe has negative curvature. In a **flat universe**, which has zero curvature, then the expansion of the Universe will come to a halt only after an infinite time. That means it will get slower and slower but in any finite time never quite stop.

Evolution of the cosmic scale factor

This slide summarises my previous comments on how the future of the universe can depend on its geometry.

The flat universe scenario is a very special case. For this to happen, the density of the Universe must be finely balanced at a critical value, denoted by cosmologists as ρ_c.

Critical density

General Relativity tells us that the **critical density** ρ_c (Greek rho, subscript c) is given by:

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{3.1}$$

If you put in the current value of H, and the value for G, then ρ_c works out at about 10^{-26} kg m^{-3}, a very small number in terms of day-to-day densities. Just how small is it? The mass of a proton is 1.7×10^{-27} kg and hence this corresponds to about 5 hydrogen atoms per m^3. That's all you need, spread over the Universe, to provide in total enough gravitational attraction to stop the Universe from expanding in the very, very long term. Compare 5 atoms per m^3 with the number of atoms in the air around us per m^3, which is about 3×10^{25}. If you find it hard to think in terms of atoms, then think about the volume of space occupied by the entire Earth, Scotland to New Zealand and everything in between. Now take a single grain of salt and imagine its constituents somehow spread throughout this entire volume, with absolutely nothing else present. Now you have picture of the density of matter represented by

ρ_c. Compared with anything we experience in the solar system, it is utterly, incredibly miniscule.

Cosmologists define the density parameter Ω as the ratio of the average density of the Universe $\rho(t)$ to the critical density, i.e.

$$\Omega = \frac{\rho(t)}{\rho_c} . \tag{3.2}$$

The most accurate answer we have to the question of the geometry of the Universe is that Ω_0, the value of Ω now, is close to 1 and hence **the Universe is flat**. In other words, the geometry of space on the largest of scales is just Euclidean geometry, the same geometry as one uses on a flat piece of paper. That is very convenient! This value is predicted by the Big Bang theory of the origin of the Universe along with the concept of **inflation**. Experimentally, the flatness of the Universe is deduced from a study of the fluctuations in the microwave background that permeates the Universe. The curvature of space-time is therefore all in the time 'dimension'.

There is one unexpected twist in the story above. In all the scenarios, the expansion of the Universe should be slowing. However, the recent evidence from more than one source points to the expansion of the Universe accelerating. Nothing in the Big Bang theory predicted this. General Relativity can handle this situation by the addition of an extra term that Einstein didn't have in his earlier attempts. This extra term is scaled by a constant called the **cosmological constant**. Usually the cosmological constant is given the symbol Λ (Greek 'lambda', in upper case) and I showed this term in the equation quoted near the beginning of this section. Einstein introduced Λ into General Relativity for a different reason and then regretted it, calling its introduction *the biggest mistake I ever made*. He didn't know about the accelerating expansion of the Universe. Now we recognise that the inclusion of Λ allows for an accelerating expansion of the Universe. The modern interpretation is to associate the acceleration of the expansion with a concept called **dark energy**. I'll postpone talking about that until later.

In summary, the density of the Universe controls its spatial geometry and is measured by the cosmological parameter Ω_0. The best measurements give $\Omega_0 = 1$ within a few percent uncertainty, showing that the geometry of the Universe is flat.

Parameters that specify the Universe

What properties has the Universe got as a whole that we can measure? The last chapter introduced the Hubble expansion constant H_0. This chapter has introduced the density parameter Ω_0. A third parameter is the acceleration parameter q that measures the acceleration of the cosmic scale factor or, what is effectively the same thing, the rate of change of the Hubble constant. Cosmologists define a dimensionless quantity q_0, the value of q at the present time, so that the change in the scale factor with time in the near future is written:

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{q_0}{2} H_0^2 (t - t_0)^2 . \tag{3.3}$$

The minus sign is put there in anticipation that the rate of expansion of the universe will be slowing because of the gravitational attraction of all the mass in the universe. This makes q_0 directly related to the second derivative of *a* by:

$$q_0 = -\frac{a(t_0) \times \ddot{a}(t_0)}{\dot{a}^2(t_0)} \quad . \tag{3.4}$$

You might have spotted that if we know what the mass density is in the universe then we should be able to predict q_0 from the known law of gravity. This is true and it gives a value of q_0 that is rather simple, namely:

$$q_0 = \frac{\Omega_0}{2} \quad . \tag{3.5}$$

However, if we're not quite sure of what's in the universe then you can think of q_0 as another measurable parameter. Observations are telling us that Ω_0 is close to 1 and hence q_0 should be ½. However, recent measurements over the last two decades0 or so have hit the headlines because q_0 looks as if it is negative. This means that the observed expansion of the Universe is speeding up, not slowing down as all the models suggest. Truly, there is something else going on that I have only hinted at. I'll come back to the topic in section 5 of this course.

Notice the point that the *q* parameter is about the changing expansion of the Universe, how its size changes with time. Hence it is about curvature that there is along the time axis and it does not alter the conclusion that the spatial dimensions of the Universe are flat.

In summary, $H_{0,}$ Ω_0 and q_0 are simple numbers that characterise the universe as a whole. The universe has other characteristics too and we are beginning to see that cosmology is looking like a science rather than just a philosophy, with characteristics that can be defined, measured and checked against the predictions of competing theories.

The next section will discuss the Big Bang theory of the origin and evolution of our universe.

Some in the class may appreciate the final verse of 'The Einstein and the Eddington':

But thank you very, very much,
For troubling to explain;
I hope you will forgive my tears,
My head begins to pain;
I feel the symptoms coming on
Of softening of the brain.

JSR