A brief look at 4 dimensions

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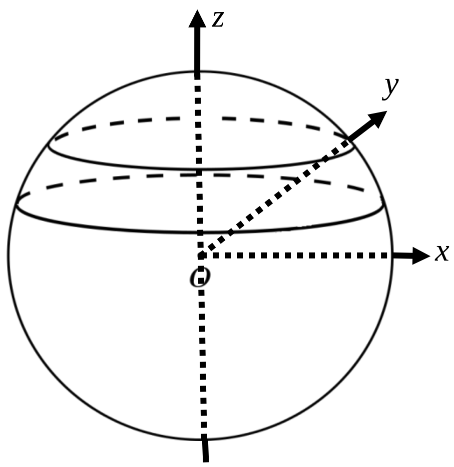
I was going to entitle this 'Thinking about 4 dimensions' or even 'Thinking in 4 dimensions' but that sounded a little pretentious, at least implying more insight than I really have. It's true that every student of physics and mathematics is introduced to multi-dimensional spaces quite early on, at least in university courses. 'Vector spaces' can be many dimensional, each dimension having a unit vector and vectors having a magnitude in each dimension. The usual rules that apply in 2 and 3 dimensions may be extended with extra symbols to more 'dimensions'. The square of a distance, for example, can be written as $w^2 + x^2 + y^2 + z^2$ as easily in 4 dimensions as $x^2 + y^2$ in 2 dimensions. What's the problem? As a manipulation of symbols, there seems to be no problem here. However, extra dimensions introduce extra possibilities.

First, imagine going from 2 to 3 dimensions. Let's start with a sphere in three dimensions, centred on the origin [$x^2 + y^2 + z^2 = r^2$, if you would like symbols, where r is the radius]. A 2 D slice by the plane at $z = 0$ is simply a circle of radius r. What does the sphere look like for 2 D beings in the xy plane transported up the z axis? It looks like a circle of decreasing radius $(r^2 - z^2)^{½}$ until it reduces to a point at $z = r$. Likewise if the xy plane with the 2 D beings is transported down the z axis, below the origin.



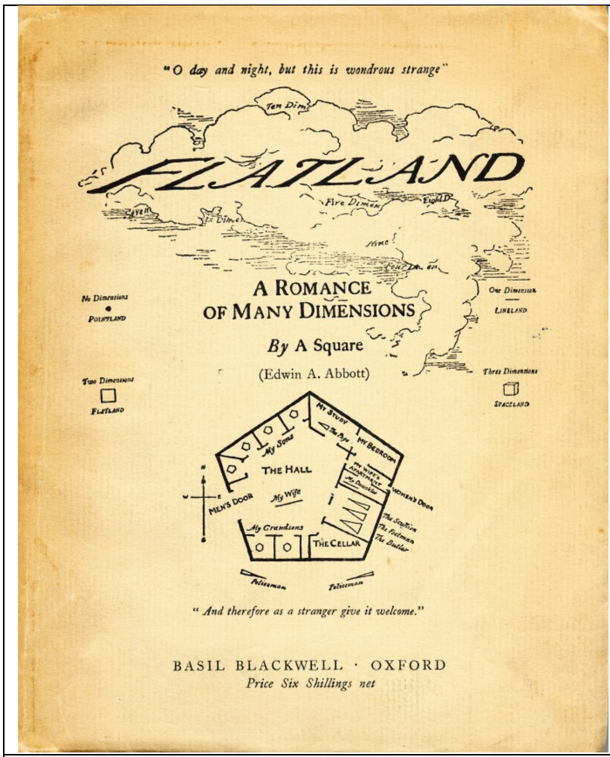
Now consider a hypersphere at the origin in 4 D space, where the extra dimension is labelled w. [$w^2 + x^2 + y^2 + z^2 = r^2$ in our symbols]. In 3 D space corresponding to $w = 0$, it looks like and ordinary sphere of radius r. If we could move in the 4th dimension up the w axis, the sphere would be seen to shrink to radius $(x^2 + y^2 + z^2)^{½} = (r^2 - w^2)^{½}$ until it seems reduced to a point. Now, each three-dimensional 'slice' is a 3 D figure, a sphere in this case.

'Time' is often described as another 'dimension'. Indeed, 'relativity' uses 'space-time' as a unifying concept. Well, the space-time of relativity is more complicated than simply adding time as another dimension but let's ignore this issue at the moment. With classical time, if the w introduced above is seen as 'time' then the idea does seem to make sense. A slice of the 4 dimensions at a given time is 3 dimensional. A spherical balloon deflating might be described roughly as a hypersphere, though it doesn't reduce to a point, and before the deflation it was more of a hypercylinder in that its 3 D shape was constant with time.

I suspect four dimensional space-time was the idea behind the 'worm-hole aliens' that featured in the series Deep Space Nine, or as the locals called them 'the prophets'. These beings existed in space-time and so could experience what anything looked like over a span of time, analogous to the way we can experience extended 3 D objects. Just as we can change the distribution of pieces on a chessboard by lifting them up into a third dimension and moving them, these aliens could move objects through time if they chose. Seeing the future rather takes the fun and the drive out of life, so I'm not sure what motivated them! It was fiction.

The idea of time being a dimension does introduce the concept that all 'dimensions' may not refer to the same kind of quantity. Another example is 'phase space' introduced by physicists as containing a momentum coordinate associated with each spatial coordinate of every particle. For example, a single particle moving in a plane defined by coordinates q_1 and q_2 with momenta p_1 and p_2 would be said to have motion described in 4-dimensional phase space whose axes are p_1, p_2, q_1 and q_2. In this circumstance, one has to be very careful about defining even simple concepts like 'distance'. In fact in the space-time of relativity, the concept of 'distance' is replaced by that of the 'metric'. I've discussed this in another piece on [Space and Space-time](). This is a digression here so we'll leave metrics.

What would it be like to experience 4 spatial dimensions? I'm going to draw on a famous book written by "A. Square" (aka Edwin A. Abbott) in 1884 called '*Flatland: a romance of many dimensions*'. The first part of the book is largely a description of the imaginary world of beings who live entirely in two dimensions. It is as much a social satire of late Victorian attitudes, clothed in fiction, than any novel statement of the mathematics of two dimensions. Later, though, the story looks at how a 2-dimensional being is challenged to see in three dimensions. Its ideas can be translated into how a 3-dimensional being might see a 4-dimensional world.



My facsimile copy of *Flatland*, printed 60 years after the original

To a flatlander, a cube is a hypersquare – a square moved parallel to itself in a 3rd but unseen dimension. To us, a cube moved parallel to itself in an unexperienced orthogonal direction is a hypercube. The square has 4 points (2^2), 4 linear sides (2×2) and 1 area. The cube has 8 points (2^3), 12 linear edges ($3*2^2$), 6 areal sides (3×2^1) and 1 volume (2^0); our 4 D hypercube (known as a tesseract) has, by analogy, 16 points (2^4) and 32 linear edges ($4*2^3$), 24 areal sides ($6*2^2$), 8 volume 'faces' (4×2^1) and 1 hypervolume (the 4-space enclosed by all the faces). Hmm, it's not easy to see what's going on. A hypersphere cut by a plane (kw + lx + my +nz = d in 4 D) will always be a sphere but a cube cut by a plane can be a square, a rhombus or even a triangle, depending on the orientation of the plane. So a hypercube cut by a plane will create variously shaped solids, the most regular of which is a cube. A drawing is a projection in two dimensions. It's difficult enough to visualise a cube in 3 D from a drawing of one. It's not really surprising that we can't visualise a tesseract from a projective drawing, for the drawing is 2 dimensions lower than the object. How could you visualise a cube from its projection into 1 dimension, even if the projection was animated by the rotation of the cube?

The most significant feature that Abbott realised is that enclosures don't exit if you have access to a higher dimension. In the story, flatlanders live in houses some of which are like the one in the centre of the cover, shown above. To see the contents of the house, the flatlander has to go into the house; to see inside a cupboard, the flatlander has to open the cupboard door. A 3 D being can see all from above. The concept of inside/outside is dimensionally restricted. Of course we can't see inside lichen growing on the surface of a rock because the lichen isn't truly 2-dimensional. It has a skin in the third dimension. If it were truly 2-dimensional (and had enough substance to absorb light) then we would be able to see its insides.

Suppose there was a fourth spatial dimension, *w*. If our three dimensional sphere at the origin [$x^2 + y^2 + z^2 = r^2$] was in fact filled with a hidden cache of chocolate, then even though its surface was opaque to us in 3-dimensions, light could travel from both inside and outside up the fourth dimension to the eye of a 4 D being. The chocolate would not be hidden. Such a being looking at one of us, say, would see both our outside shape and our innards. If the being was in fact only 3-dimensional, say living in *w*, *x*, *y* space, then we would see only a 2-dimensional cross-section of it.

Like us lifting chessmen into a third dimension and placing them back on the board, if a fourth dimension exists then objects moved into the fourth dimension and out again will disappear and re-appear somewhere else, without apparent cause. Mercifully, this doesn't seem to happen in everyday life (excepting perhaps the mysterious disappearance of socks in the washing machine) or even in laboratory conditions. On a very small scale, could quantum tunnelling be explicable in terms of a higher dimension, since particles appear on the other side of a potential barrier that is too high for them to cross? Actually the effect is well calculated with our 3 D quantum mechanics. The existence of another dimension will need a lot more evidence than a single phenomenon before it can be accepted. In Abbott's story, flatlanders were so sure that ideas of 3 D Spaceland were corrupting that by law no-one could speak of them, on pain of imprisonment in an asylum or death. One flatlander having been shown the apparent marvels of 3 D Spaceland asked to be taken to 4 dimensional space. His guide grew angry and said that 4 D space was mere Thoughtland and couldn't exist.

All this is fantasy, surely. We know the universe is only 3-dimensional, 4 D if we include space-time. Well, do we? It's true we haven't seen any evidence of 4 spatial dimensions. The universe is big, very big, and if something is a long way away then it has to be enormous in some way before we can see it. Maybe 4 D phenomena are not present near the Earth. Would we even recognise 4 spatial dimensions if they were on our doorstep? The flatlanders had no concept of up and down so couldn't conceive a third dimension. We can write symbols for four dimensions but we can't really visualise a fourth spatial dimension. In the science fiction trilogy 'The 3 body problem' by Cixin Liu, travellers encounter a bubble of 4-

dimensional space far from Earth. Is that possible? Looking for an analogy, could a three-dimensional bubble intersect the flatlanders' world? Well, yes it could. We haven't directly experienced the existence of more than three dimensions but multi-dimensional reality has certainly entered theoretical concepts aimed at describing the universe at large.

Perhaps the first serious introduction was a century ago by Kaluza, joined a few years later Klein. Kaluza-Klein space has 5 dimensions (including time), one of which, according to Klein, is curled up. Kaluza-Klein space was an attempt to unify General Relativity and Maxwell's electrodynamics, with the addition of quantum effects. It was a bold attempt at unification, now seen as one of the historical antecedents of string theory, which is the modern attempt to describe the structure of the universe. Included in its aims is unifying quantum phenomena and gravity.

I'm straying now from the 4 dimensions of my title. The strings of string theory are 11-dimensional. They can be closed (like a loop) or open and have a finite but very small length. It seems at first like a theory with more basic room to explain the complexity of the physical world than fundamental particles that are mere points. The strings can vibrate in various spaces, giving rise to the observed properties of what we see as fundamental 'particles'. At least that's what the theory aims to describe but to have any link to reality 11 dimensions are needed. That is what the mathematics implies. Physics has learnt to be very respectful of the implications of mathematics. It was Dirac's mathematics that suggested the possible existence of antimatter. It is the mathematics of quantum mechanics that works, not the philosophy. Since the universe doesn't appear to have 11 dimensions then it's postulated that most of the dimensions are compacted. Compaction can be done in a stupendous number of ways described by mathematicians as Calabi-Yau manifolds. Choosing different ways of compaction generates different fundamental particles and their properties. I don't know how it's done either but my point is that more than 3 spatial dimensions are an important concept in modern physics. The dimensions may be 'compacted' to generate the particles that we see in the 4-dimensional space time that is our total experience but maybe there are regions of the universe where things are different and perhaps, in the famous words, 'we ain't seen nothing yet'.

JSR