ABSTRACT

Duality theories are proposed as a basis for the construction of certain state spaces that appear in Decision Theory. This is considered in more detail in the particular case of Stone Duality. Topological cancellation axioms for preference are introduced. These are used to produce a form of representation for preference between particular acts over state spaces that are Stone spaces. The well-known connection between Stone spaces and logic allows them to serve as a site for the interpretation of a syntactic theory of acts, following ideas of Blume, Easley and Halpern.

1. INTRODUCTION

Certain types of decision studied in Decision Theory are concerned with choices between acts, where the outcome of each act is dependent on the state of the world [27, 4, 23]. Following Savage [27]: the world is the object about which the decision-maker (DM) is concerned, and a state is a description of the world, leaving no relevant aspect undescribed. The subjective uncertainty of a DM about the state can be incorporated by using a probability distribution on states.

An important question is how the DM — and any modeller of the DM — determines the state space. In most treatments it is simply given, subject to the constraint that states must describe all the relevant data.

A very different approach has recently been pioneered by Blume, Easley and Halpern (BEH) [6, 7]. This is a key starting point for the present work. In their work, the state space can be explicitly constructed from a propositional language of properties (tests, observations) of states. The construction is very familiar to logicians — the state space is the usual canonical model that underpins the completeness theorem for classical propositional logic with finitely many basic propositions. Thus states include all relevant information in the sense that they determine the truth or falsity of all basic propositions of the given language; moreover, they contain only such information. This means that the state space is given objectively, at least relative to the test language used by the DM.

The use of a logical language to describe states enables BEH to give a syntactic formulations of acts in formal languages that are, essentially, simple programming languages. In particular, there are conditional statements over a Boolean language of tests on the state. A very appealing aspect of this approach is that syntactic acts may then be interpreted on different spaces, perhaps considered by different DMs. An axiomatization of preference relations using cancellation axioms is introduced that gives existence theorems for utility representations over the constructed state spaces.

There is a highly-developed theory of dualities that construct spaces from algebraic structures [19]. In the setting
of Logic this is manifested as a close relationship between topological spaces and algebraic structures related to logical languages. Canonical models of various kinds are used to construct spaces from algebras. A particular instance of this captures the main state space construction used by BEH. Part of the strength of these duality theories comes from the fact that they constitute well-behaved relations between categories of structures [24].

The importance of duality in interpreting programming languages and relating them to logics was demonstrated by Abramsky [1], building on Scott’s semantics [29, 30] that interprets programs as certain continuous functions over a sophisticated notion of state. An accessible explanation of the connection between computations over test languages and duality is the subject of the book by Vickers [31].

It seems natural to ask if duality theories can be used as a mathematical basis over which logical and semantical theories of decision such as those of BEH can be developed. This paper presents the results of some initial investigations towards such a programme.

An immediate benefit of this approach is in being able to state in precise, categorical terms, the sense in which state space constructions are canonical.

Only finite spaces and finitely-generated test languages are treated by BEH. A substantial part of this paper is concerned with relaxing these finiteness constraints. This is a rather technical goal, but it facilitates the exploration of the use of duality and the representation of preference in the case where the state space carries a non-trivial topology. Indeed, as well as the above logical consideration, the semantics of acts as computations, and the infinite dimensional nature of acts both point towards a critical role for topology in the theory.

It is worth noting that a further related approach to state spaces is given by Fishburn [12] Chapter 12 and Section 13.4. There the set of acts is given, and the set of states is constructed as a certain dual space (a set of functions from acts — and appropriate additional data — to the real numbers). This also applies to acts over infinite sets of states.

Section 2 describes the acts of interest. Section 3 describes how dualities can be used to construct state spaces. Section 4 contains axioms for preference relations, including topological cancellation axioms. Section 5 contains several forms of state-dependent representation for preference and also considers notions of state-independent representation. Section 6 contains a discussion including further work. All non-trivial proofs are relegated to an appendix.

2. THE OBJECTS OF CHOICE

Most of the decision problems studied will be those in which the objects of choice are essentiallyAnscombe–Aumann acts [4, 23], also known as horse-race lotteries, and henceforward referred to as AA acts. Recall that these are acts which, in different states of the world, result in probability distributions over a fixed set of outcomes. Savage acts will also be discussed. These are acts which yield a single outcome, dependent upon the state.

In the present development, both the state and outcome spaces both will be equipped with the structure of topological spaces. There are, of course, cases of interest where the topology is discrete — standard developments of the theory of choice of AA acts often require no more than sets of states and outcomes. For any space X, let Δ(X) be the set of probability measures on X.
Many state and outcome spaces will be considered throughout this paper. However, at various points, attention will be focussed upon particular spaces. The convention will then be that the letter $S$ is reserved for the state space, that $s$ ranges over elements of $S$, that $O$ is reserved for the space of outcomes, and that $o$ ranges over elements of $O$.

Let $\mathbb{R}_+$ be the non-negative real numbers, and let $\nu_O$ be a Borel measure on $O$. A \textit{semantic AA act} over $S$, $O$ and $\nu_O$ is a continuous map $a : S \times O \rightarrow \mathbb{R}_+$ such that $\hat{a}_s \in \Delta(O)$ for all $s \in S$, where

$$\hat{a}_s(F) = \int_F a_s \, d\nu_O$$

for all measurable $F$, and where $a_s : O \rightarrow \mathbb{R}_+ ; o \mapsto a(s,o)$.

A similar convention is used elsewhere for fixing an argument. In the finite, discrete setting, $\nu_O$ is usually assumed to uniformly take the value 1. The above definition restricts attention to acts that at each state can be represented as a density for the given $\nu_O$. More general acts require a more sophisticated approach to representation than the one outlined below.

Often, only some subset, $\mathbb{A}_A$, of the above set of maps will be of interest. In this case, the term ‘semantic AA act’ is reserved for elements of this subset.

A central feature of the BEH approach is the ability to treat acts as syntactic objects. For example, the \textit{syntactic AA acts}, $\alpha$, are generated by a grammar

$$\alpha ::= \alpha^0 \mid \text{if } b \text{ then } \alpha \text{ else } \alpha \mid w_1\alpha + w_2\alpha ,$$

where $\alpha^0$ ranges over some atomic, syntactic acts, $b$ ranges over a syntactic category of Boolean tests on $S$ and $w_1, w_2$ are computable, non-negative reals with $w_1 + w_2 = 1$. The conditional statements enable one to build decision trees.

The weighted sums allow for random decision nodes in those trees. There is also a relation here with the \textit{options} employed by Ramsey [25]. The \textit{syntactic Savage acts} are formed by the same grammar, but with the weighted sums removed and with only deterministic atomic acts $\alpha^0$.

The particular syntax is not important for this paper — all that matters here is the denotation. An interpretation for a syntactic Savage act $\alpha$ on a given state space $S$ and outcome space $O$ is function $[\alpha] : S \rightarrow O$. The interpretation of an AA act on $S$ and $O$ should be as a map that defines a probability measure on $O$ for every $s \in S$. Instead the more restrictive requirement is made that each syntactic AA acts must be interpreted as semantic AA acts in $\mathbb{A}_A$ for a given $\nu_O$. Thus each act is interpreted by a particular continuous density $[\alpha] : S \times O \rightarrow \mathbb{R}_+$. Note that this means that the set $\mathbb{A}_A$ must contain the closure of the set of interpretations of basic acts under the usual semantics of conditionals, and additionally under the evident semantics of probabilistic mixtures in the case of AA acts. The set $\mathbb{A}_A$ is not itself required to be so closed.

Syntactic acts are intended to be programs, and so to be computable — in particular, over the Boolean tests. Bearing in mind the close relationship between computability and continuity [29], some form of continuity restriction on semantic acts seems reasonable.

### 3. Dualities and Acts

Many logics have a sound interpretation on certain topological spaces. Logics are also closely related to algebraic structures: for example, classical propositional logics give rise to Boolean algebras via the Lindenbaum construction.

A \textit{Stone space} is a topological space that is compact, Hausdorff and zero-dimensional (has a basis of clopen sets). The
logical significance of Stone spaces come from the fact that they provide a representation of Boolean algebras. Indeed, the category of Boolean algebras and homomorphisms is dual (categorically) to the category of Stone spaces and continuous maps [19]. This encapsulates the soundness and completeness of Stone space models of classical propositional logic via the standard canonical model construction.

BEH use a classical propositional language $\mathcal{L}$ of Boolean tests, generated from a finite set $\mathcal{L}_0$ of basic propositions, to generate the finite state space. States are taken to be atoms of the logical language. An atom $s$ can be presented simply as a subset of $\mathcal{L}_0$, indicating which basic propositions hold in that state. Let $At(B)$ be the set of atoms of $B$. Assuming that $\mathcal{L}$ is subject to the axioms of classical propositional logic, a state then also corresponds to a morphism of Boolean algebras $s : B \to \{\top, \bot\} = 2$, where $B$ is the Boolean algebra corresponding to $\mathcal{L}$. At an algebraic level this is part of the finite case of Stone representation [9].

Morphisms (structure-preserving maps) appear often not to be considered to be of much interest in Decision Theory — although they are sometimes present implicitly. However, if one takes the view that acts can have many different interpretations, then morphisms should not be ignored. They describe translations between syntactic theories, and also important relationships between interpretations.

As an illustration of the significance of morphisms, consider Figure 1. This figure captures in precise terms the canonical nature of the construction of the interpretation of syntactic Savage acts, in the case of a total preference $\geq$ in Theorem 4 of [7] (under the assumption of classical axioms). The triangle on the left lives in the category of finite Boolean algebras. The rectangle lives in the category of finite sets. The set $O$ is a given set of outcomes that includes, via some given $\omega$, a set $A_0$ (giving an interpretation of basic acts). Each morphism $[\alpha]_{\text{canonical}}$ is the obvious interpretation for acts formed from conditionals [7]. The functor $\mathcal{P}$ is the contravariant power-set. The map $\eta$ gives the representation of $B$ as a finite power-set; it also gives the canonical interpretation of the test language on the set of atoms. The set $S$ is any other finite set. The morphism $\beta = [\cdot]_S$ gives any interpretation of the test logic in $S$, and $[\cdot]_{S,O}$ is any interpretation of syntactic acts on $S$ and $O$ (with conditionals interpreted in the usual way). With this data there is a unique function $\beta'$ as shown such that the triangle commutes. Moreover, for every $\alpha$, the rectangle commutes, provided this is true for all basic $\alpha = \alpha^0 \in A_0$.

In the more general setting of Stone duality $B$ is not required to be finite. The set $At(B)$ is then replaced by a Stone space $\text{spec}(B)$ of points, which generalize the logical atoms. The points are Boolean morphisms $s : B \to 2$, and the topology generated by opens $O_b = \{ s \in \text{spec}(B) \mid s(b) = \top \}$, where $b \in B$. The contravariant power set functor $\mathcal{P}$ is replaced by the functor $B$ that maps a space to its Boolean algebra of clopen sets and a continuous map to its inverse image. The details of these constructions [19] are not required here.

In later parts of this paper, it will also be assumed that $O$ is compact Hausdorff, just as it would be if it was either finite or a Stone space. If one argues that the language used
to describe the state space should precisely match the construction of that state space, then perhaps the same should be true of the outcome space. Certain pathological examples appear to rely upon this not being the case [23] pp.35–36.

Duality allows one to consider algebraic versions of semantic Savage acts (as maps of Boolean algebras), and give an algebraic interpretation to syntactic acts.

4. AXIOMS FOR PREFERENCE

A standard part of decision theory is the consideration of axioms for preference, often with a view to the production of utility representations. This will now be done for semantic AA acts in the set $AA$.

The traditional approach to axiomatization for AA acts is to work with a given (but often abstract) $S$ and $O$, a given set $AA$ of semantic AA acts and to set out axioms for the preference relation $\succeq \subseteq AA \times AA$. Of course, this is really understood to apply to any $S$, $O$ and $AA$ that are suitable for interpretation of the axioms. For BEH, this, together with the fact that the sets $At(B)$ are canonical for the test language $B$ and that interpretations are constrained to follow the syntax of syntactic AA Acts, allows the axioms to be re-stated in a different way: it is done simply for the space $At(B)$.

The approach taken below will be the traditional one. However, it should be understood that this is essentially equivalent (at least mathematically) to stating the axioms for the points of the Stone space of the Boolean algebra of tests because of the canonicity of the construction. One could re-state axioms using the points of $spec(B)$. The axioms have been stated in an elementary (rather than topological) way to make this clear.

The axioms below will explicitly use the functional form of (semantic) acts: the notation $a_1 = a_2$ for a pair of acts $a_1, a_2$ signifies their equality as functions. If $n \geq 0$ is an integer, $a_0, \ldots, a_n$ are AA acts and $\beta_0, \ldots, \beta_n \in \mathbb{R}$, then $\sum_{i=0}^{n} \beta_i a_i$ is the (pointwise) linear combination of functions.

BEH introduce and study a number of axioms for decision. Several of these are closely related to cancellation axioms [28, 22]. For comparison with what is to follow, one of these axioms is now re-stated:

**Axiom 1. Extended Mixture Cancellation (EMC)**

Let $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$ be two sequences of AA acts, with $a_0 = \ldots = a_m$ and $b_0 = \ldots = b_m$. If there is an integer $n > 0$ and sequence of acts $a_{m+1}, \ldots, a_{m+n}$ such that $a_{m+i} \succeq b_{m+i}$ for all $1 \leq i \leq n$, and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$, then $b_0 \succeq a_0$.

In this paper, slightly stronger forms of cancellation axiom are employed, in which the premise holds only up to a particular topology.

**Axiom 2. Extended Topological Mix. Canc. (ETMC)**

Let $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$ be non-empty sequences of acts, with $a_0 = \ldots = a_m$ and $b_0 = \ldots = b_m$. If, for all $\epsilon > 0$ there exists an integer $n > 0$ and sequences of acts $a_{m+1}, \ldots, a_{m+n}$ and $b_{m+1}, \ldots, b_{m+n}$ such that $a_{m+i} \succeq b_{m+i}$ for $1 \leq i \leq n$, and $|\sum_{i=0}^{m+n} a_i(s, o) - b_i(s, o)| < \epsilon$ for all $s \in S$ and $o \in O$, then $b_0 \succeq a_0$.

The integer $n$ and the acts $a_{m+i}$, $b_{m+i}$ are not required to be uniform as $\epsilon$ varies. A variant is useful:


Let $a_0$ and $b_0$ be acts. If, for all $\epsilon > 0$ there exists an
integer \( n > 0 \), sequences of acts \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) and rationals \( k_1, \ldots, k_n \geq 0 \) and \( k_i > 0 \) such that: \( a_i \geq b_i \) for \( 1 \leq i \leq n \), and \( |\sum_{i=0}^{n} k_i(a_i(s,o) - b_i(s,o))| < \epsilon \) for all \( s \in S \) and \( o \in O \), and \( \sum_{i=0}^{n} k_i = 1 \), then \( b_0 \geq a_0 \).

A well-known axiom states that the graph of the preference relation should be closed. This is often known as \textit{continuity}.

\textbf{Axiom 4. Continuity (CONT.)}

Let \( a \) and \( b \) be acts. Suppose that there are sequences \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \), such that for all \( \epsilon > 0 \) there are some \( m, m' \) such that \( |a(s,o) - a_n(s,o)| < \epsilon \) for all \( n > m \), and \( |b(s,o) - b_n(s,o)| < \epsilon \) for all \( n > m' \). Suppose that \( a_n \geq b_n \) for all \( n \). Suppose further that, for all \( \epsilon > 0 \), there exists \( k \) such that for all \( n \geq k \) and \( s, o \), \( |a_n(s,o) - b_n(s,o)| < \epsilon \). Then \( a \succeq b \).

The following proposition collects together some relations between these axioms:

\textbf{Proposition 1.}

1. \textit{ETMC implies EMC and RCTMC}
2. \textit{EMC and RCTMC both imply reflexivity and transitivity of \( \succeq \)}
3. \textit{ETMC implies CONT}
4. \textit{RCTMC implies ETMC}.

If it had been required that the set \( \mathbb{A} \) was mixture-closed, then axioms leading to the applicability of the Herstein-Milnor Theorem \([12, 16, 23]\) could have been used below.

A preorder \( \succeq \) will be said to be \textit{total} if \( a \succeq b \) or \( b \succeq a \) holds for all acts \( a \) and \( b \), and \textit{entire} if \( a \succeq b \) for all \( a \) and \( b \).

\section{Representations}

The initial approach to the representation of acts that follows is a geometrical one \([5, 7, 15]\).

Semantic acts over \( S \times O \) (or rather their densities) are certain elements of the vector space \( \mathbb{R}^{S \times O} \) of functions from \( S \times O \) to the real numbers, with the usual pointwise operations. Throughout this section, suppose that both \( S \) and \( O \) are both compact Hausdorff spaces. The product \( S \times O \) is then also compact Hausdorff. This will simplify the analysis considerably.

Let \( \mathcal{C}(S \times O) \) be the set of all continuous functions in \( \mathbb{R}^{S \times O} \). Let \( \mathcal{C}_b(S \times O) \) be the set of such functions with both upper and lower bounds. Let \( \mathcal{C}(X) \) be the set of real-valued continuous functions with compact support on \( X \). The usual sup-norm, \( ||-||_{\infty} \), makes each of \( \mathcal{C}_b(S \times O) \) and \( \mathcal{C}(X) \) into a \textit{locally convex topological vector space} (TVS). Since \( S \times O \) is compact, \( \mathcal{C}_b(S \times O) = \mathcal{C}(S \times O) = \mathcal{C}(S \times O) \). The axioms of the previous section can be stated more concisely using the sup-norm, and their topological content made more transparent.

Let

\[ D = \{ a - b \mid a, b \in \mathbb{A} \} \subseteq \mathcal{C}(S \times O) \]

and, for any preference relation \( \succeq \), define

\[ D_{\succeq}^+ = \{ a - b \in D \mid a \succeq b \}. \]

A \textit{cone} is a subset of a vector space \( X \) that is closed under all non-negative linear combinations. Let \( W \subseteq X \). The \textit{cone} of \( W \), written \( \text{cone}(W) \), is the set of all non-negative linear combinations of elements of \( W \); this is the smallest cone containing \( W \). A \textit{closed cone} in a TVS is a cone which is
also closed in the topology. The closed cone of \( W \), written \( \text{cone}(W) \), is the topological closure of cone\((W)\), and is the smallest closed cone containing \( W \).

A subset \( W \) of a TVS \( X \) is a (closed, upper) half-space if there is a continuous linear functional \( F : X \rightarrow \mathbb{R} \) and a \( v \in \mathbb{R} \) such that \( W = \{ x \in X | F(x) \geq v \} \). A half-space is a homogeneous when \( v = 0 \).

Let

\[
\mathbb{H}_\subseteq = \mathbb{H}(\text{cone}(D_\subseteq^+))
\]

be the set of closed homogeneous half-spaces that contain \( \text{cone}(D_\subseteq^+) \), and let \( \mathcal{O}_\mathbb{H}_\subseteq \) be the set of all operators that give rise to these half-spaces. Note that \( \text{cone}(D_\subseteq^+) = C(S \times O) \) iff \( \mathbb{H}_\subseteq = \emptyset \).

The topological content of the axiom RCTMC gives a modification of the conical representation of preference of BEH [7], following the general technique of Aumann [5].

**Theorem 1.** A relation \( \succeq \) on a set \( \mathbb{A} \) of semantic AA-acts satisfies RCTMC if and only if it has a conical representation:

\[
D_\subseteq^+ = D \cap \text{cone}(D_\subseteq^+).
\]

If \( \succeq \) is entire, then \( \text{cone}(D_\subseteq^+) = C(S \times O) \). If \( \succeq \) is total but not entire, then \( \text{cone}(D_\subseteq^+) \) is a closed half-space.

The Hahn–Banach Theorem implies that every closed, convex cone is the intersection of the (homogeneous) closed half-spaces that contain it [14]. Hence,

\[
\text{cone}(D_\subseteq^+) = \bigcap \mathbb{H}_\subseteq .
\]

This allows the conical representation to be turned into a linear functional representation.

**Corollary 1.** A preference relation relation \( \succeq \) on a set \( \mathbb{A} \) of semantic AA acts that satisfies RCTMC is represented by the set \( \mathcal{O}_\mathbb{H}_\subseteq \) of all continuous linear operators that define homogeneous closed half-spaces that contain \( \text{cone}(D_\subseteq^+) \). For all acts \( a \) and \( b \), this implies that

\[
a \succeq b \quad \text{if and only if} \quad \forall T \in \mathcal{O}_\mathbb{H}_\subseteq. Ta \geq Tb .
\]

If \( \succeq \) is entire, then \( \mathcal{O}_\mathbb{H}_\subseteq = \emptyset \). If \( \succeq \) is total, then \( \succeq \) can be represented by a single linear operator.

Any set \( \mathcal{O} \) of continuous linear functionals from \( C(S \times O) \) into \( \mathbb{R} \) defines a preference relation \( \succeq \) that satisfies RCTMC, upon putting \( \mathcal{O} \) for \( \mathcal{O}_\mathbb{H}_\subseteq \) above.

Note that the set of operators suggested in Corollary 1 is the maximal set.

From this point on, the obvious converses to the existences of representations will not be stated.

A (real-valued, signed) Radon measure on a locally compact space \( X \) is a continuous linear functional from \( C_c(X) \) to \( \mathbb{R} \) [8, 10]. The functionals of Corollary 1 are all Radon measures. The fact that the acts \( a \) considered are all positive and compactly supported, means that the application of Radon measures in the previous corollary are instances of the integral with respect to the same measures [8, 10].

**Corollary 2.** A preference relation relation \( \succeq \) on a set \( \mathbb{A} \) of semantic AA acts that satisfies RCTMC is represented by the family \( \mathcal{O}_\mathbb{H}_\subseteq \) of Radon measures. For all acts \( a \) and \( b \), this implies that \( a \succeq b \) if and only if \( \int_{S \times O} adT \geq \int_{S \times O} bdT \)
for all $T \in \mathcal{O}$. 

A Riesz Representation Theorem allows each Radon measure $T$ on the compact space $S \times O$ to be represented by a signed Borel measure $\mu^T$ on $S \times O$.

**Corollary 3.** A preference relation relation $\succeq$ on a set $\mathcal{A}$ of semantic AA acts that satisfies RCTMC is represented by the family $M_\preceq = (\mu^T \mid T \in \mathcal{O})$ of signed Borel measures on $S \times O$. For all acts $a$ and $b$, this implies that:

$$a \succeq b \iff \int_{S \times O} a \, d\mu^T \geq \int_{S \times O} b \, d\mu^T \text{ for all } T \in \mathcal{O}.$$

A limitation of the above approach is that the measure $\nu_{\mathcal{O}}$ makes no contribution to the representation. If there was a preferred measure $\nu_S$ on the state space $S$ that made each action $a$ $p$-integrable with respect to $\nu_S \times \nu_{\mathcal{O}}$ and $1 < p < \infty$ and $p/(p-1) \in \mathbb{Z}_+$, then a standard theorem (see [11] IV.8 Theorem 1) would allow further expansion of the integral terms in this result. For example, the left-hand integral could be written in the form $\int_S \int_O u(s,o)(s,o) \, d\nu_{\mathcal{O}} \, ds$ for some $p/(p-1)$-integrable $u$. However, the usual point of AA acts is to derive the probability on $S$ along with the utility.

In the context of syntactic AA acts generated over an arbitrary classical propositional logic of Boolean tests, the state space $S$ generated will be a Stone space and so compact Hausdorff. Further assume that $O$ is also compact Hausdorff. Thus, if the RCTMC axiom is then stated just for the elements of $S$ and $O$, and for the semantic acts $\mathcal{A}$ occurring as interpretations of syntactic acts, then all of the representation results of this section apply.

It is common to ask for a state-independent representation for AA acts. In a standard treatment such as Kreps’ book[23], and where all spaces are finite, one has, after imposing further axioms, a representation of the form: $a \succeq b$ iff

$$\sum_{s \in S} \mu(s) \left( \sum_{o \in O} u(o)a(s,o) \right) \geq \sum_{s \in S} \mu(s) \left( \sum_{o \in O} u(o)b(s,o) \right)$$

(1)

for all AA acts $a, b : S \rightarrow \Delta(O)$, where $u : O \rightarrow \mathbb{R}$, $\mu \in \Delta(S)$ and $\succeq$ is total.

In the present treatment, following an idea of BEH, the state and outcome spaces $S, O$ present in the state-dependent representation (Theorem 1 and its corollaries) have taken to have been constructed by dualities. If only some state-independent utility representation is required, then one can use modified state spaces $S'$ and $O'$ for the state-independent representation.

Suppose that there is a representation as in Theorem 3. Define a new outcome space $Q = S \times O$. For all $s, s' \in S$ and $o \in O$, and every $a \in \mathcal{A}$, define $\pi(s',(s,o)) = a(s,o)$. It is then trivial that any non-empty set of probability measures on $S$ gives a state-independent representation as follows:

**Theorem 2.** If $\succeq$ on $\mathcal{A}$ is not entire and satisfies the RCTMC axiom, then there is a state-independent representation of $\succeq$. Let $Q = S \times O$, let $M_\succeq = (\mu^T \mid T \in \mathcal{O})$ be as in Corollary 3 and let $\emptyset \not\subseteq N \subseteq \Delta(S)$. Then for all $a, b \in \mathcal{A}$: $a \succeq b$ iff

$$\int_S \left( \int_Q \pi_s(q) \, d\mu(q) \right) \, d\rho(s) \leq \int_S \left( \int_Q \pi_s(q) \, d\mu(q) \right) \, d\rho(s)$$

for all $\mu \in M_\succeq$ and $\rho \in N$.

The above shifting of goal posts and makes the above result rather unsatisfactory, and leads to the independence the inner integral from the variable $s \in S$. BEH have an
alternative approach is used is which the outcome space $O$ is untouched, but the state space is modified.

A measure theoretic state-dependent representation of acts closely related to that of Corollary 3 is derived by Hill [18] in the case of Savage acts from more traditional axioms and using positive measures. The transformation of this representation into a state-independent representation over the same spaces has also been studied by Hill [17].

6. DISCUSSION

It should be clear that this paper has barely made a dent in the programme mentioned in the introduction.

The limitations in the representation theorems appear to stem from the decision to restrict to acts given by continuous densities $a : S \times O \rightarrow \mathbb{R}_+$. This needs to be generalised to a representation that makes use of acts as maps $S \rightarrow \Delta(O)$.

The representation of Savage acts over Stone spaces has not yet been considered, but the BEH theory of Bayesian updating (information about the state) can be understood as defining assignments between fibres of a presheaf built over state spaces.

The use of Stone spaces is merely convenient in its close correspondence to a well-known logic, and in the analytic simplicity of working on compact spaces. However, it is not really justifiable if one is serious about interpreting acts as programs. It would be more natural to work over Scott domains [2, 29, 30]. This would require significant changes to the language of tests [1].

Richer decision theories often require spaces that are not compact and so not Stone spaces, for example $\mathbb{R}^n$. Furthermore, it is a little odd to use Stone duality to construct spaces before applying analysis to functionals on spaces of continuous functions over that space. It may be that the use of real-valued Gelfand, or related dualities, are more appropriate for the construction of such spaces. Logics with multiplicative connectives (such as Linear Logic) can provide languages of tests in this kind of setting [3].

The points made by BEH of the value of constructivity in decision theory are persuasive. However, one could take this to mean that not only should the spaces, utilities and probabilities be guaranteed to exist by some abstract construction, but that this should be done within some acceptable constructive mathematics. Many constructions of (infinite) spaces from algebras via duality (for example Stone duality) are not constructive. Gilboa and Schmeidler [13] and Karni [21] have produced theories of acts in which the use of state spaces can be avoided. It would be interesting to know whether the methods of Pointless Topology [20] could be used to produce an intermediate position with acts, the axiomatization of preference and the representation of preference living purely on the algebraic side — that is, directly on tests on states and outcomes.

7. REFERENCES


APPENDIX

Proof. (Proof of Proposition 1)

1. Every pair of sequences $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$ that satisfy the premise of the EMC property, also satisfies the premise of ETMC. Similarly if a pair $a_0$ and $b_0$ satisfy the premise of RCTMC, then they constitute sequences (of length 1) that satisfy the premise of ETMC.

2. EMC and RCTMC both imply the Cancellation axiom (Definition 5) of [7]. Proposition 1 of [7], and this implies reflexivity and transitivity.

3. Suppose that there are converging sequences $a_1, a_2, \ldots \rightarrow a$ and $b_1, b_2, \ldots \rightarrow b$ (in the sup-norm), and that $a_i \geq b_i$ for all $i \geq 1$. For any $\epsilon > 0$ there are some $m$, $n$ such that $\|a - a_m\|_{\infty} < \epsilon/2$ and $\|b - b_n\|_{\infty} < \epsilon/2$. Let $a'_0 = b$, $a'_1 = a_m$, $b'_0 = a$ and $b'_1 = b_n$. Then

$$\|(a'_0 + a'_1) - (b'_0 + b'_1)\|_{\infty} = \|(a'_0 - b'_0) + (a'_1 - b'_1)\|_{\infty} \leq \|b - b_n\|_{\infty} + \|a_m - a\|_{\infty} \leq \epsilon/2 + \epsilon/2 = \epsilon.$$ 

Applying the ETMC axiom gives $b'_0 \geq a'_0$. That is, $a \geq b$.

4. This is dealt with separately in Corollary 4 below.

□

Proof. (Proof of Theorem 1)

Only if.

Suppose that $\geq$ satisfies RCTMC. It is immediate that $D^+_{\geq} \subseteq D \cap \text{cone}(D^+_{\geq})$. Let $d = a - b \in D \cap \text{cone}(D^+_{\geq})$. For all $\epsilon > 0$ there is a $d' \in \text{cone}(D^+_{\geq})$ with $\|d - d'\|_{\infty} < \epsilon$. So $d' = \sum_{i=1}^{n} k_i (b_i - a_i)$ for some integer $n$, real coefficients $k_1, \ldots, k_n \geq 0$, and acts $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$, with $k_i \geq 0$ and $b_i \geq a_i$ for $1 \leq i \leq n$. Now if $\|d - \sum_{i=1}^{n} k_i (b_i - a_i)\|_{\infty} < \epsilon$ for some real coefficients $k_i \geq 0$, then one can, without loss of generality, replace these by rational coefficients (henceforth those which the $k_i$ denote) and retain the same properties. Define $r = 1 + \sum_{i=1}^{n} k_i$ and $k'_i = k_i / r$ for $1 \leq i \leq n$ and $k'_0 = 1/r$ and $a_0 = a$ and $b_0 = b$. Applying the RCTMC property (with these $a_i$, $b_i$, $k'_i$) gives that $a_0 \geq b_0$, so that $a \geq b$, and $d \in D^+_{\geq}$. Hence $D \cap \text{cone}(D^+_{\geq}) \subseteq D^+_{\geq}$.

□

Proof. (Proof of Lemma 1)

Consider any semantic AA acts $a_0$ and $b_0$. So $a_0 - b_0 \in C(S \times O)$. Then for all $\epsilon > 0$, there are integer $n$, reals $k_1, \ldots, k_n \geq 0$ and sequences of acts $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$,
such that $||(a_0 - b_0) + \sum k_i (a_i - b_i)||_\infty < \epsilon$ and $a_i \succeq b_i$ for all $1 \leq i \leq n$. Replacing the $k_i$ with rationals and re-scaling to make the sum convex allows us to apply RCTMC and conclude that $b_0 \succeq a_0$. □

Lemma 2. Suppose that $\succeq$ satisfies RCTMC, but is not entire. Then $\succeq$ is total if and only if $\overline{\text{cone}}(D^+\succeq)$ is a half-space.

Proof. (Proof of Lemma 2)

If $\overline{\text{cone}}(D^+\succeq)$ is a half-space, then for all acts $a$ and $b$, either $a - b$ or $b - a \in D^+\succeq = D \cap \overline{\text{cone}}(D^+\succeq)$. Hence $\succeq$ is total.

Only if. This follows directly the second half of the proof of Lemma 2 of [7], but using the closed cone $\overline{\text{cone}}(D\succeq)$, and noting that the standard Archimedean property holds by Proposition 1 above combined with Proposition 7 of [7]. □

The proof of Corollary 1 is an immediate consequence of Theorem 1 and the fact that half-spaces correspond to linear operators. A unique operator suffices in the case where the preference order it total but not entire.

Proof. (Proof of Corollary 1)

Suppose that RCTMC holds. First of all $C(S \times O)$ is a locally convex TVS, so the representation of convex sets by half-spaces of Theorem 1 applies. Then

$$a \succeq b \iff a - b \in D^+\succeq = D \cap \overline{\text{cone}}(D^+\succeq)$$

$$\iff a - b \in \overline{\text{cone}}(D^+\succeq)$$

$$\iff a - b \in H \text{ for all } H \in \mathbb{H}_\succeq$$

$$\iff T(a - b) \geq 0 \text{ for all } T \in \mathcal{O}\,\mathbb{H}_\succeq$$

$$\iff Ta \geq Tb \text{ for all } T \in \mathcal{O}\,\mathbb{H}_\succeq$$

for any acts $a$ and $b$.

For the converse, suppose that the set $\mathcal{O}$ of continuous linear operators is given. Let $\mathbb{H}$ be the set of all homogeneous closed half spaces of $C(S \times O)$ corresponding to $\mathcal{O}$. Define $\Gamma = \bigcap \mathbb{H}$. Note that $a \succeq b$ if and only if $a - b \in \Gamma$, for all $a$, $b$.

Furthermore, $D^+\succeq = D \cap \Gamma = D \cap \overline{\text{cone}}(D^+\succeq)$. So RCTMC holds by Theorem 1. □

The proof of Corollary 3 is a just a direct application of a standard Riesz Representation Theorem for continuous linear operators.

Proof. (Proof of Corollary 3)

A Riesz Representation Theorem applies to this situation and allows every positive Radon measure on $S \times O$ to be represented by a regular Borel measure on $S \times O$ [26]. Let $\text{Bor}(S \times O)$ be the Borel $\sigma$-algebra over the topology of $S \times O$ induced by the sup-norm, and $\mathbb{R}^\infty = \mathbb{R} \cup \{-\infty, +\infty\}$ be the usual extended reals. Therefore every signed Radon measure $T$ is represented by a pair $\mu^+ T, \mu^- T : \text{Bor}(S \times O) \to \mathbb{R}^\infty$ of Borel measures. That is, $T(a) = \int_{S \times O} \text{ad} \mu^+ T - \int_{S \times O} \text{ad} \mu^- T$ for every semantic AA act $a$. Since $S \times O$ is compact, it follows that $\mu^T = \mu^+ T - \mu^- T$ is a single signed Borel measure that represents $T$ (see also [11] Section IV.6, Theorem 3). This gives immediately the representation of $\succeq$ using pairs of Borel regular measures, or using single signed Borel measures on $S \times O$. □
The next result is a Corollary to Theorem 1, and is the content of the fourth point of Proposition 1.

**Corollary 4.** RCTMC implies ETMC.

**Proof.** (Proof of Corollary 4)

Suppose RCTMC, and the premise of ETMC. Let \( a_0, \ldots, a_m \) and \( b_0, \ldots, b_m \) be non-empty sequences of acts, with \( a_0 = \ldots = a_m \) and \( b_0 = \ldots = b_m \). If, for all \( \epsilon > 0 \) there exists an integer \( n > 0 \) and sequences of acts \( a_{m+1}, \ldots, a_{m+n} \) and \( b_{m+1}, \ldots, b_{m+n} \) such that \( a_{m+i} \succeq b_{m+i} \) for \( 1 \leq i \leq n \), and \( |\sum_{i=0}^{m+n} a_i(s,o) - b_i(s,o)| < \epsilon \) for all \( s \in S \) and \( o \in O \). So \( |\sum_{i=0}^{m+n} (b_i - a_i) - \sum_{i=1}^{n} (a_{m+i} - b_{m+i})|_\infty < \epsilon \). Now \( \sum_{i=1}^{n} (a_{m+i} - b_{m+i}) \in \text{cone}(D^+_\subseteq) \). So \( (\sum_{i=0}^{m} (b_i - a_i))/(m + 1) = b_0 - a_0 \in \text{cone}(D^+_\subseteq) \). By Theorem 1, the conical representation gives \( b_0 - a_0 \in D^+_\subseteq \). That is \( b_0 \succeq a_0 \). \( \square \)