Topology and robot motion planning

Mark Grant (University of Aberdeen)

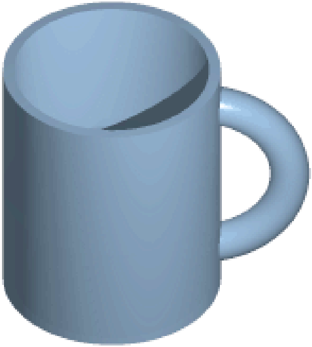
Maths Club talk
18th November 2015

Plan

1. What is Topology?
   - Topological spaces
   - Continuity
   - Algebraic Topology
2. Topology and Robotics
   - Configuration spaces
   - End-effector maps
   - Kinematics
   - The motion planning problem

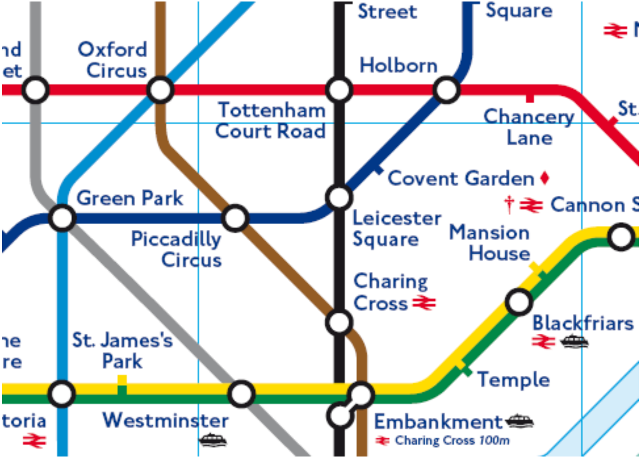
What is Topology?

It is the branch of mathematics concerned with properties of shapes which remain unchanged under continuous deformations.



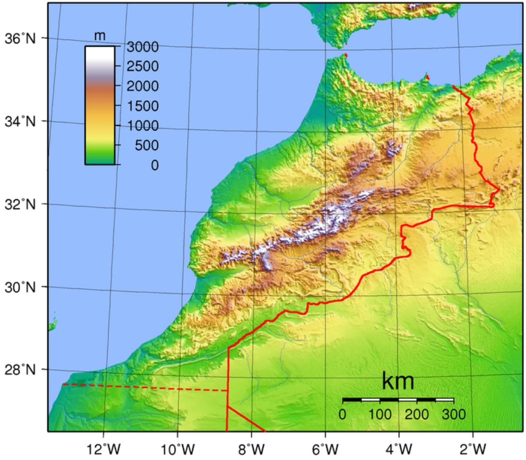
What is Topology?

Like Geometry, but exact distances and angles don't matter.



What is Topology?

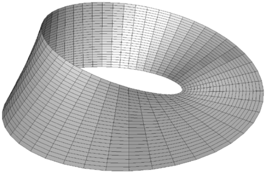
The name derives from Greek ($\tau\acute{o}\pi o\zeta$ means place and $\lambda\acute{o}\gamma o\zeta$ means study). Not to be confused with topography!



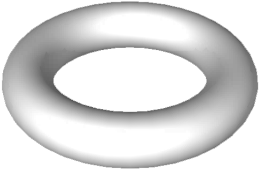
Topological spaces

Topological spaces arise naturally:

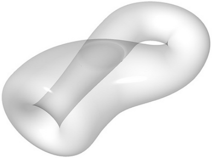
- as configuration spaces of mechanical or physical systems;
- as solution sets of differential equations;
- in other branches of mathematics (eg, Geometry, Analysis or Algebra).



Möbius strip



Torus



Klein bottle

Topological spaces

A topological space consists of:

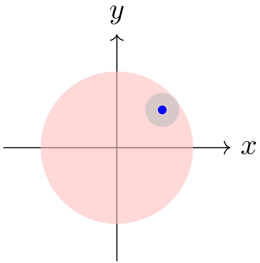
- a set X of points;
- a collection of subsets of X which are declared to be open. This collection must satisfy certain axioms (not given here).

The collection of open sets is called a topology on X. It gives a notion of "nearness" of points.

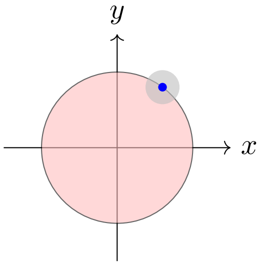
Topological spaces

Usually the topology comes from a metric—a measure of distance between points.

For example, the plane $\mathbb{R}^2$ has a topology coming from the usual Euclidean metric.



$\{(x, y) \mid x^2 + y^2 < 1\}$ is open.



$\{(x, y) \mid x^2 + y^2 \leq 1\}$ is not.

Continuity

In order to compare topological spaces, we study the continuous functions between them.

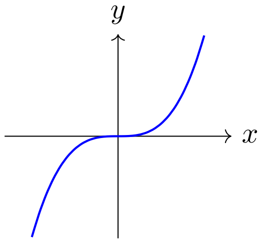
A function $f : X \to Y$ is a rule which assigns to each point x of X a unique point $f(x)$ of Y.

Informally, such a function f is continuous if it "sends nearby points in X to nearby points in Y".

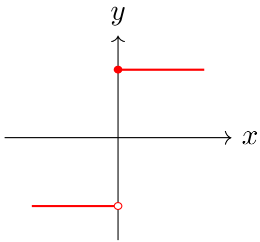
Formally, we ask that the pre-image under f of every open set in Y is open in X.

Continuity

For example, consider the real numbers $\mathbb{R}$ with their usual topology.



The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is continuous. We can sketch its graph without lifting our pen from the paper.



The function $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

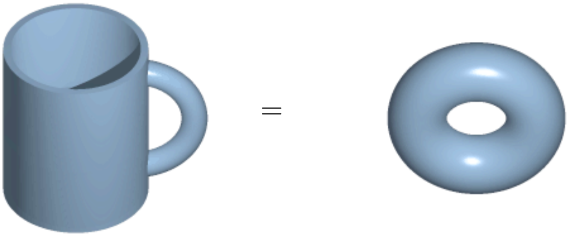
is not continuous. There are nearby points which get sent far apart.

Continuity

Topological spaces X and Y are homeomorphic if there are continuous functions $f : X \to Y$ and $g : Y \to X$ such that

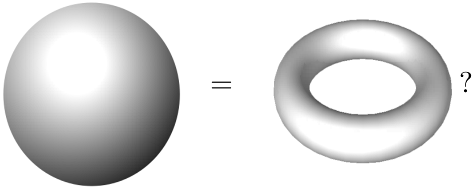
$$f\big(g(y)\big) = y \quad \text{for all } y \text{ in } Y \qquad \text{and} \qquad g\big(f(x)\big) = x \quad \text{for all } x \text{ in } X.$$

Homeomorphic spaces are considered "the same" in topology.



Algebraic Topology

It can be very difficult to decide whether two topological spaces are homeomorphic.

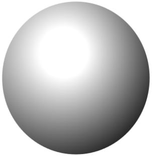


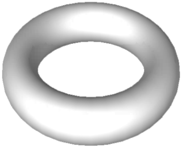
In algebraic topology we assign algebraic quantities to topological spaces, in such a way that homeomorphic spaces get assigned the same quantities.

These topological invariants can sometimes be used to prove that spaces are not homeomorphic.

Algebraic Topology

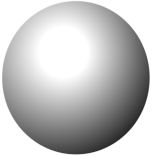
One such topological invariant is the fundamental group, which assigns a group $\pi_1(X)$ to each space X. It is defined using loops in X (continuous functions from the circle to X).

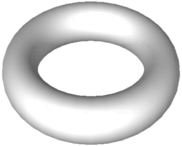
$\pi_1\Bigg($  $\Bigg) = 0.$

$\pi_1\Bigg($  $\Bigg) = \mathbb{Z} \times \mathbb{Z}.$

Algebraic Topology

Another such is the second homotopy group, which is a commutative group $\pi_2(X)$. It is defined using continuous functions from spheres to X.

$\pi_2 \Bigg($  $\Bigg) = \mathbb{Z}.$

$\pi_2 \Bigg($  $\Bigg) = 0.$

Algebraic Topology

An important feature of these constructions is their functoriality.

For example, a continuous function $f : X \to Y$ induces a homomorphism of groups $f_* : \pi_2(X) \to \pi_2(Y)$.

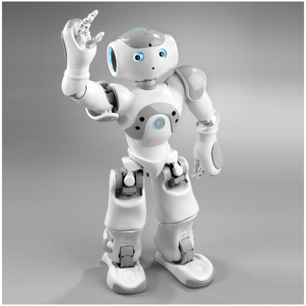
If $f : X \to Y$ and $g : Y \to Z$ are continuous, their composition $g \circ f : X \to Z$ induces the composition $g_* \circ f_* : \pi_2(X) \to \pi_2(Z)$.

The identity function $\mathrm{Id}_X : X \to X$ induces the identity homomorphism $\mathrm{Id}_{\pi_2(X)} : \pi_2(X) \to \pi_2(X)$.

Robotics

The Oxford English Dictionary defines a robot as

a machine capable of carrying out a complex series of actions automatically, especially one programmable by a computer.

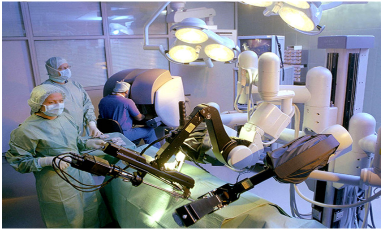


Often humanoid in appearance, robots have captured the imagination of the public.

Applications of Robotics

Industry

Biology and Chemistry

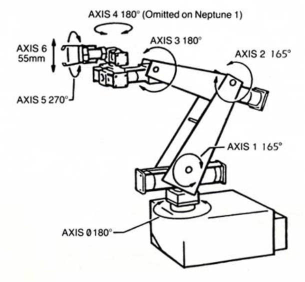


Medicine

Domestic

Configuration spaces

A typical robot mechanism consists of several rigid links, connected by moveable joints.



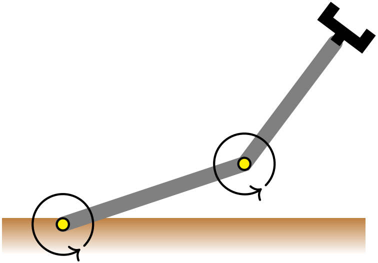
To specify a configuration of the robot, we must specify the positions of all of the moveable joints.

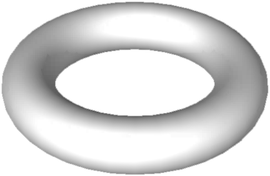
Configuration spaces

The configuration space of the robot is a topological space $\mathcal{C}$ whose points parameterize the possible configurations.

Below is a planar robot arm with two revolute joints.

Its configuration space is a torus.

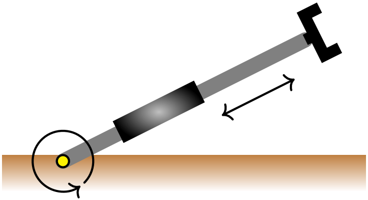


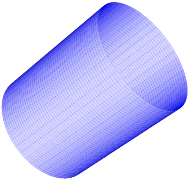


Configuration spaces

Below is a planar robot arm with one revolute and one prismatic joint.

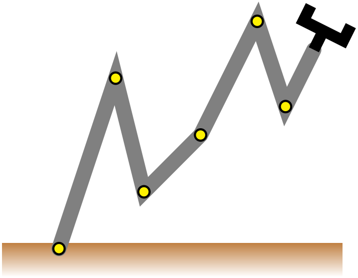
Its configuration space is a cylinder.





Configuration spaces

The dimension of the configuration space $\mathcal{C}$ will coincide with the number of degrees of freedom of the robot mechanism.



Note that this number may be arbitrarily large, even for planar mechanisms.

The theory of robot motion planning therefore requires an understanding of high-dimensional spaces. These are objects which geometers and topologists have been studying for centuries!

End-effector maps

One end of the robot arm typically has an end-effector, such as a hand for manipulating objects.

The possible positions of the end-effector are also parameterized by a topological space, the work space $\mathcal{W}$ of the robot.

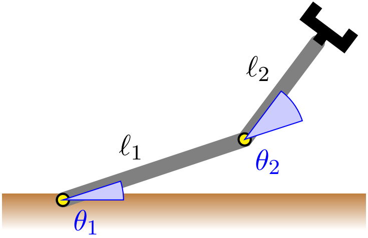
The position of the end-effector depends on the joint configurations in a continuous way. We therefore get a continuous function

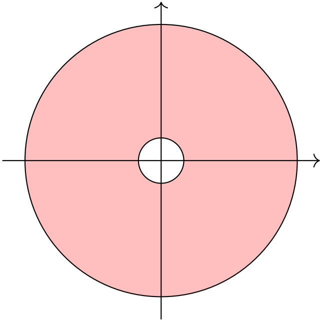
$$F : \mathcal{C} \to \mathcal{W},$$

called the end-effector map.

End-effector maps

For the planar arm with two revolute joints, the workspace is an annulus.



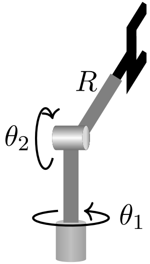


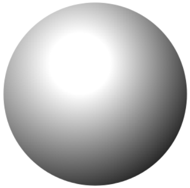
The end-effector map is given by

$$F(\theta_1, \theta_2) = \big(\ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2),\, \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2)\big).$$

End-effector maps

The universal joint or Cardan joint has two revolute joints with perpendicular axes. The workspace is a sphere.





The end-effector map is given by

$$F(\theta_1, \theta_2) = \big(R\cos\theta_1\cos\theta_2,\ R\sin\theta_1\cos\theta_2,\ R\sin\theta_2\big).$$

Kinematics

The Forward Kinematics problem is to calculate the position of the end-effector in terms of the joint configurations. This means to give a formula for the end-effector map

$$F : \mathcal{C} \to \mathcal{W}.$$

This is usually not too difficult, as we have seen.

Kinematics

The Inverse Kinematics problem is to find all configurations of the joints which achieve a particular position of the end-effector.

This means to find all solutions c in $\mathcal{C}$ of the equation

$$F(c) = w_0,$$

for a given w_0 in $\mathcal{W}$. This is usually much harder.

Kinematics

An inverse kinematic map is a function

$$I : \mathcal{W} \to \mathcal{C}$$

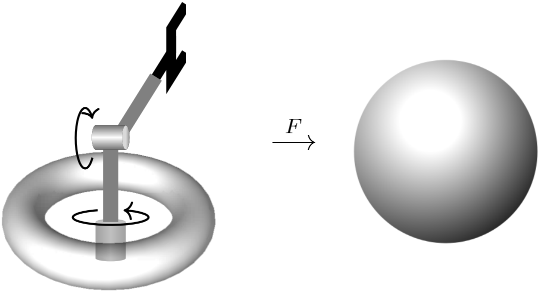
satisfying $F\big(I(w)\big) = w$ for all w in $\mathcal{W}$.

Such a function gives a particular solution $I(w)$ to the inverse kinematics problem for all w in the workspace.

Sometimes we can use algebraic topology to show that a continuous inverse kinematic map cannot exist.

Kinematics

For example, recall that for the universal joint, $\mathcal{C}$ is a torus and $\mathcal{W}$ is a sphere.



Kinematics

Suppose there was a continuous inverse kinematic map $I : \mathcal{W} \to \mathcal{C}$. Then $F \circ I = \mathrm{Id}_{\mathcal{W}}$, and so on second homotopy groups we have that the composition

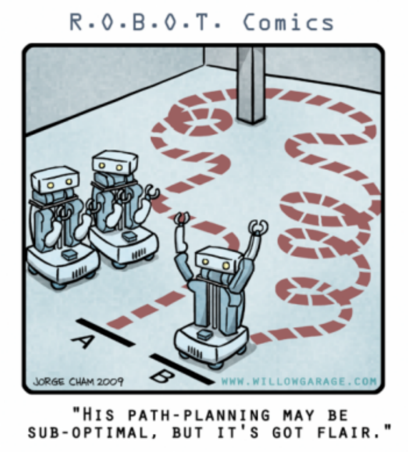
$$\pi_2(\mathcal{W}) \xrightarrow{I_*} \pi_2(\mathcal{C}) \xrightarrow{F_*} \pi_2(\mathcal{W})$$

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

is the identity homomorphism. This is a contradiction.

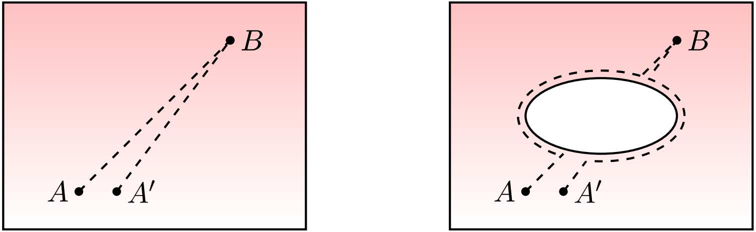
The motion planning problem

Find an algorithm which, given configurations A and B of the system, outputs a motion from A to B.



The motion planning problem

The topology of the configuration space $\mathcal{C}$ plays an important role. It dictates whether motion planning algorithms exist which are continuous in the input configurations.



Premise

It is desirable to find motion planning algorithms with fewest domains of continuity, since these will be optimally stable.

The motion planning problem

Definition (Michael Farber)

The topological complexity $\mathsf{TC}(\mathcal{C})$ of the configuration space $\mathcal{C}$ is the minimum number of domains needed to cover $\mathcal{C} \times \mathcal{C}$, on each of which there is a continuous motion planning algorithm.

The number $\mathsf{TC}(\mathcal{C})$ is an interesting topological invariant, whose computation has practical relevance in engineering problems.

The motion planning problem

More recently, Petar Pavešić has defined the complexity $\text{cx}(F)$ of the end-effector map $F : \mathcal{C} \to \mathcal{W}$, using ideas of Alexander Dranishnikov.

This invariant promises to be even more relevant to Robotics (and even more challenging to compute)!

Thank you for your attention!

- https://colorcolourcouleur.wordpress.com/2011/06/03/the-london-underground-map/
- https://www.youtube.com/watch?v=2STTNYNF4lk
- http://business-reporter.co.uk/2013/03/nick-clegg-praises-manufacturing-industry-supply-chains/
- http://www2.lbl.gov/Publications/Currents/Archive/May-05-2000.html
- http://www.wired.com/2009/09/surgical-robots/2/