Sectional category weight and topological complexity

(joint with Michael Farber)

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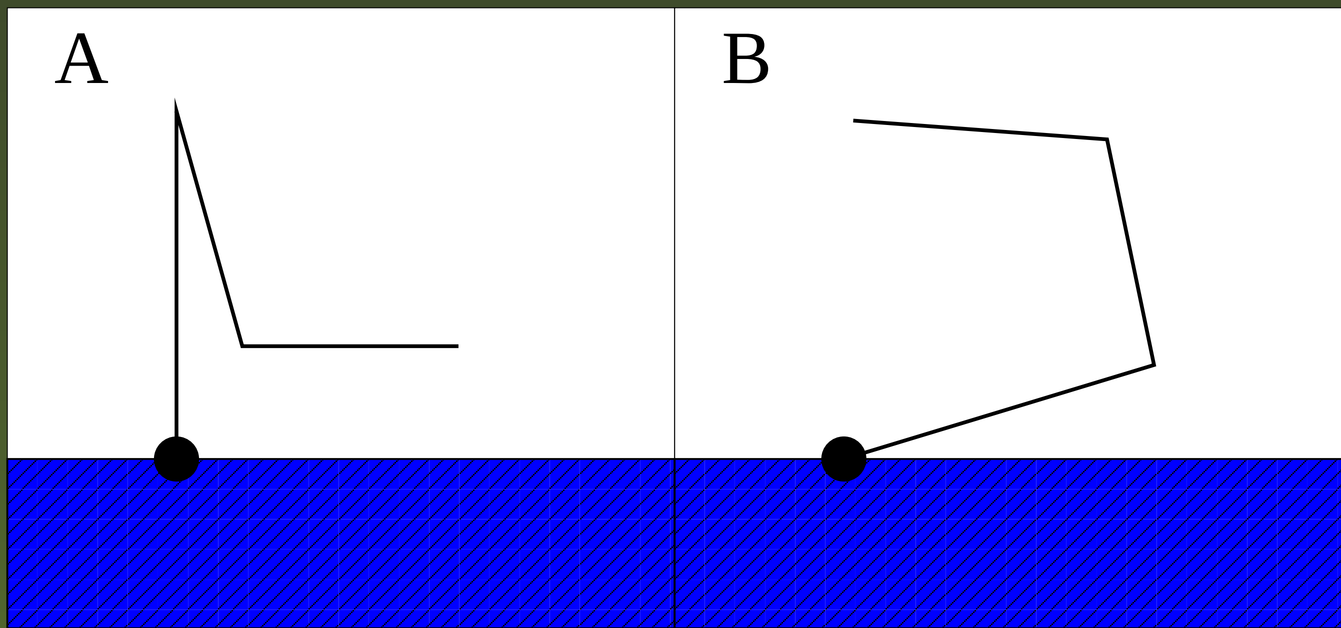

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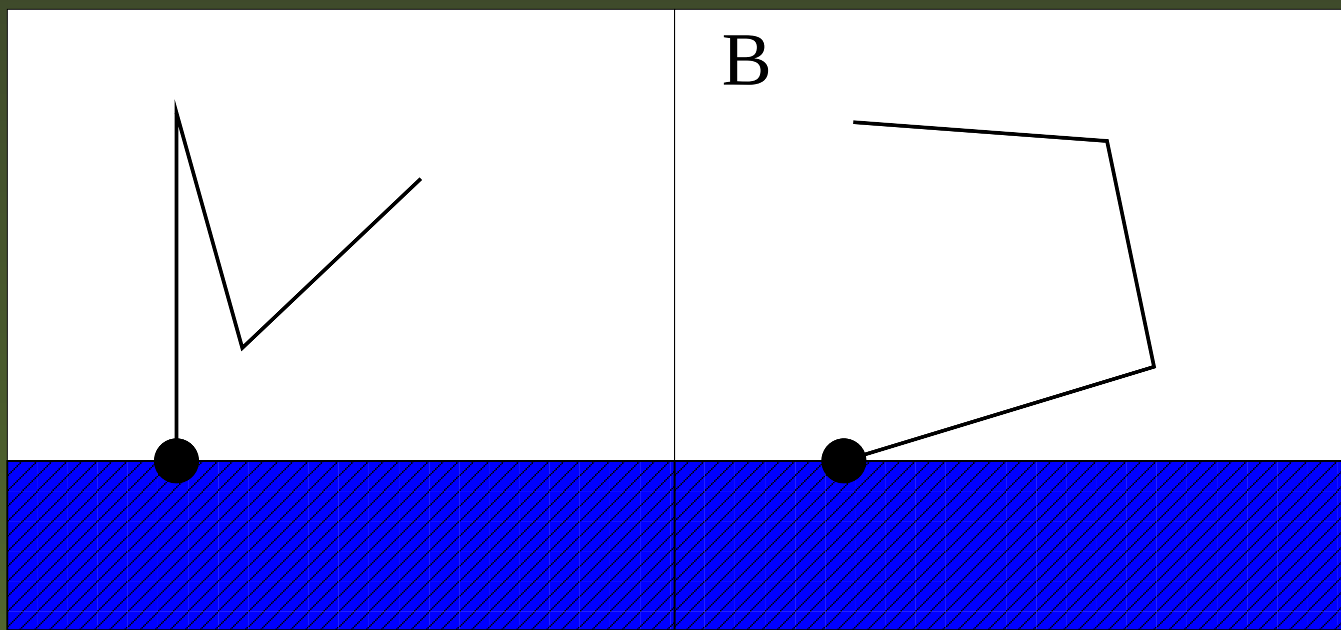

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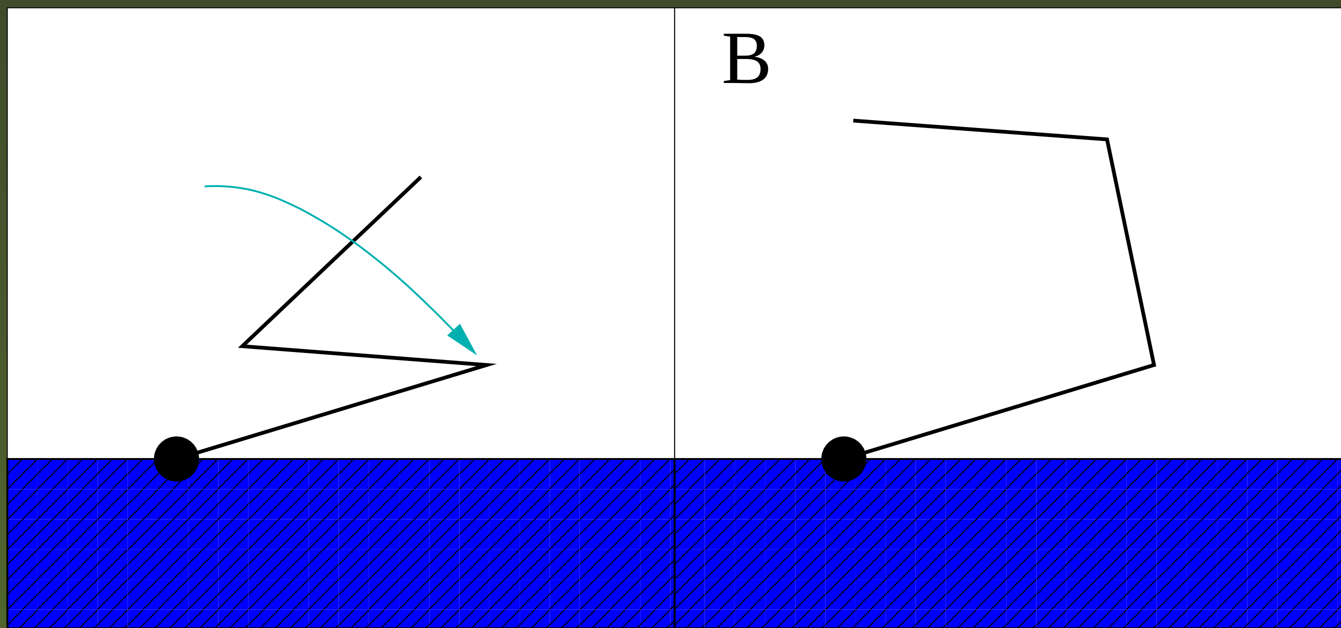

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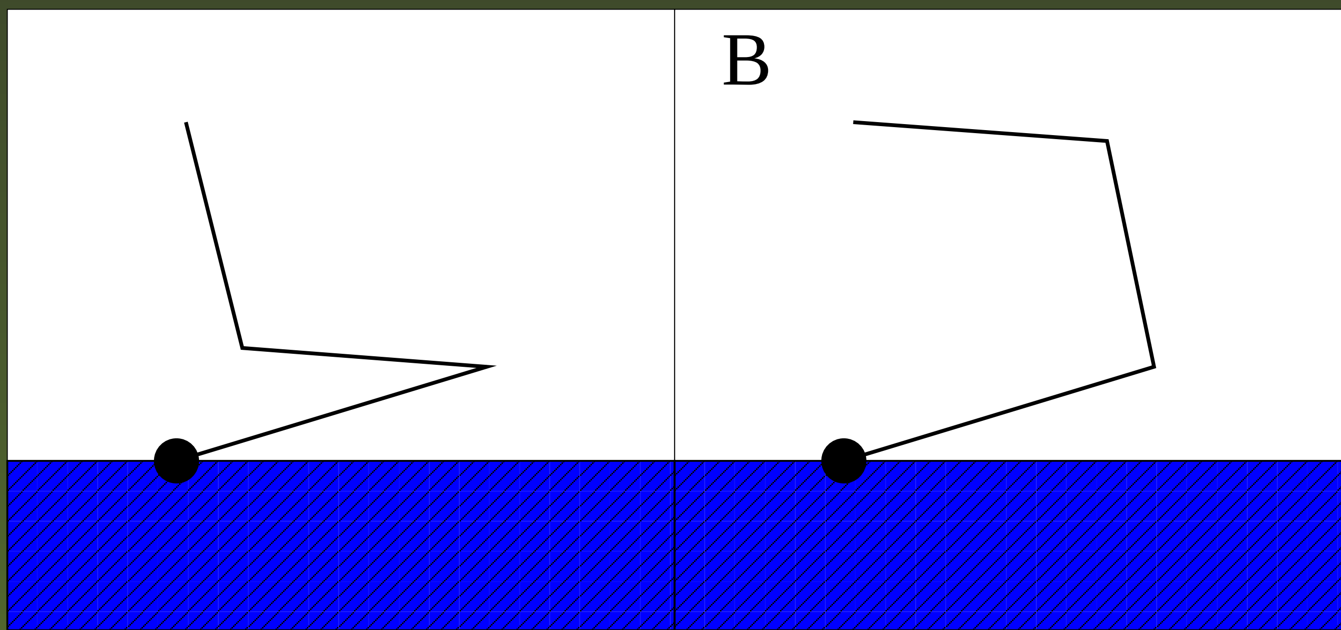

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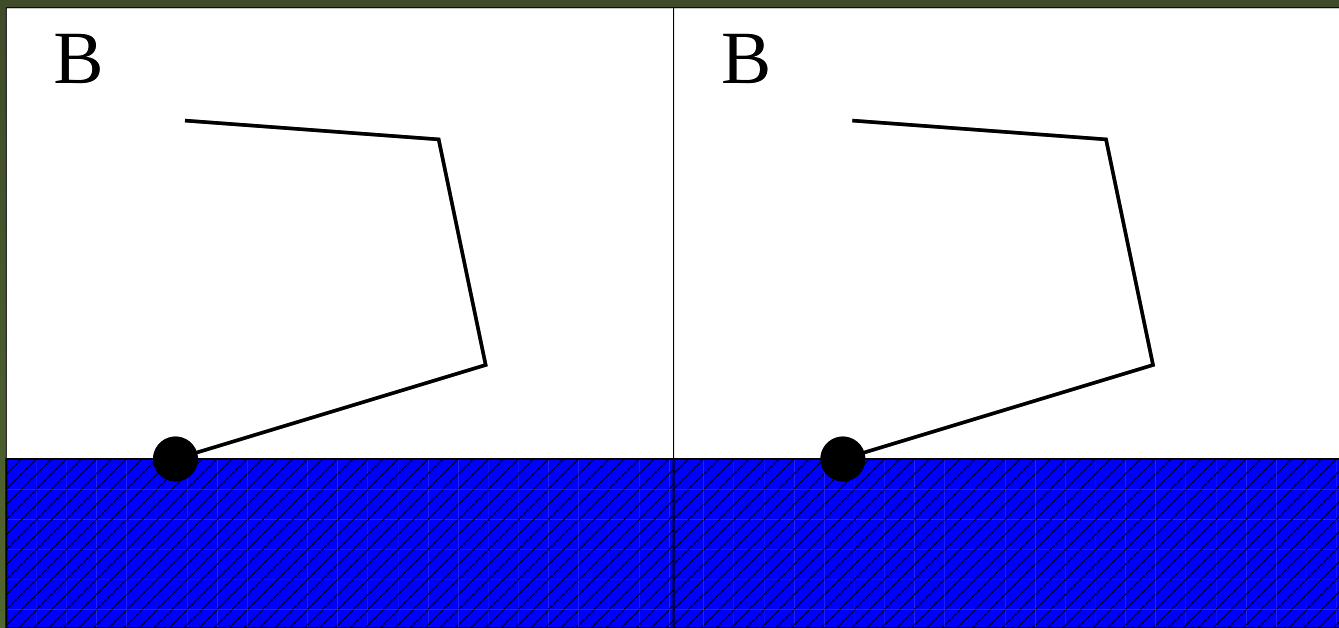

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- Find algorithm which has as input $A, B \in X$ (initial and final states) and as output a path $\gamma\colon I \to X$ with $\gamma(0) = A$ and $\gamma(1) = B$
- Corresponds to finding a section $s\colon X \times X \to X^I$ of the path fibration

$$\pi\colon X^I \to X \times X, \quad \pi(\gamma) = (\gamma(0), \gamma(1))$$

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- **Definition** The *Topological Complexity* of X, $\mathbf{TC}(X)$ is the min. k s.t. $X \times X = U_1 \cup \cdots \cup U_k$

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- **Theorem** (Farber) $X \simeq Y \Rightarrow \mathbf{TC}(X) = \mathbf{TC}(Y)$

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- Connection with immersion problem:
  Thm (Farber, Tabachnikov, Yuzvinsky)

$$\mathbf{TC}(\mathbb{R}P^n) = \begin{cases} I_n & n = 1, 3, 7, \\ I_n + 1 & \text{else} \end{cases}$$

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- Interesting in its own right, for example
  Problem (Farber) For an abstract group G compute $\mathbf{TC}(G) = \mathbf{TC}(K(G,1))$

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Other examples include $\mathrm{cat}(X) = \text{genus}(p\colon PX \to X)$ where p is Serre path fibration, and work of S Smale and V Vassiliev on complexity of algorithms for solving polynomial equations

Lower bounds for TC

- Assume $H^*(X \times X) \cong H^*(X) \otimes H^*(X)$ as algebras, where product on the right is

$$(\alpha \otimes \beta)(\gamma \otimes \delta) = (-1)^{|\beta||\gamma|} \alpha\gamma \otimes \beta\delta$$

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- More generally $\mathrm{genus}(p) > \text{cup-length}(\ker p^*)$ for any fibration p (Schwarz)

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- This can be generalised to $\mathrm{genus}(p\colon\ E \to B)$ and applied to $\mathbf{TC} = \mathrm{genus}(\pi)$

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- **Proposition** $\mathrm{wgt}_p(u) \geq 1$ if and only if $p^*(u) = 0$

Applications to TC

- In particular the $\mathbf{TC}$-*weight* of $u \in H^*(X \times X)$ is $\mathrm{wgt}_\pi(u)$, where $\pi\colon\ X^I \to X \times X$ is the path fibration. If $u_1 \cdots u_k \neq 0 \in H^*(X \times X)$ then $\mathbf{TC}(X) > \sum \mathrm{wgt}_\pi(u_i)$

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- Hence the previous lower bound can be sharpened by finding indecomposables u with $\mathrm{wgt}_\pi(u) \geq 2$

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- e.g. $e(\mathrm{Sq}^i) = i$, $e(\mathrm{P}^i) = 2i$ and $e(\beta) = 1$
- **Theorem** (Farber, G) Suppose $e(\theta) = n$ and $u \in H^n(X;R)$. Then the element

$$\overline{\theta(u)} = 1 \times \theta(u) - \theta(u) \times 1 \in H^{n+i}(X \times X;S)$$

has $\mathrm{wgt}_\pi(\overline{\theta(u)}) \geq 2$

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- **Lemma** Let $f = (\varphi, \psi)\colon\ Y \to X \times X$ be a map where φ, ψ denote the projections of f onto the factors of $X \times X$. Then $\mathrm{genus}(f^\star \pi) \leq 2$ if and only if $Y = A \cup B$, where A and B are open in Y and $\varphi|_A \simeq \psi|_A$, $\varphi|_B \simeq \psi|_B$

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- Hence $\mathbf{TC}(L_p^n) > 1 + 2(2n) = 4n + 1$ provided $p \nmid \binom{2n}{n}$. In fact $\mathbf{TC}(L_p^n) = 4n + 2$ in such cases

Further work

- **Theorem** (G) Let $p\colon E \to B$ be a fibration, and suppose the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined and non-zero. Then
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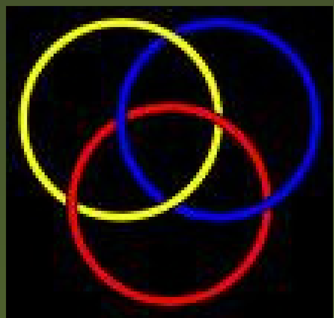

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- When $X = S^3 - B$, complement of Borromean rings, this gives $\mathbf{TC}(X) > 3$ while cup-length$(I) = 2$
- **Conjecture** If $u \in H^n(X)$ has $\mathrm{cwgt}(u) \geq 2$ then $\bar{u} \in H^n(X \times X)$ has $\mathrm{wgt}_\pi(\bar{u}) \geq 2$, for n in a range depending on $\mathrm{conn}(X)$
(true if X is simply-connected)

Thanks for listening!