The fundamental group of random simplicial complexes

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joint work with Michael Farber

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- *Topological phase transitions*. Random simplicial complexes exhibit intriguing topological phase transitions. The topology of a random complex changes abruptly once the probabilistic parameters cross certain critical levels.

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A property $\mathcal{P}$ is said to be satisfied *asymptotically almost surely* (a.a.s.) if

$$\mathbb{P}(G \in \mathcal{P}) \xrightarrow{n\to\infty} 1,$$

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Let $\omega(n) \to \infty$ *as* $n \to \infty$. *Then*

- *If* $p = \frac{\log(n) - \omega(n)}{n}$ *then* $G \in G(n,p)$ *is disconnected a.a.s.*
- *If* $p = \frac{\log(n) + \omega(n)}{n}$ *then* $G \in G(n,p)$ *is connected a.a.s.*

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A phase transition happens at $p \sim \frac{\log(n)}{n}$; this is a threshold for the connectivity of Erdős-Rényi random graphs.

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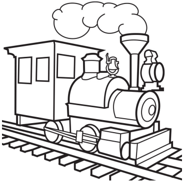

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Several topological and geometric properties have been established for the random spaces listed above.

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Clique (and independence) complexes are common objects in topological combinatorics.

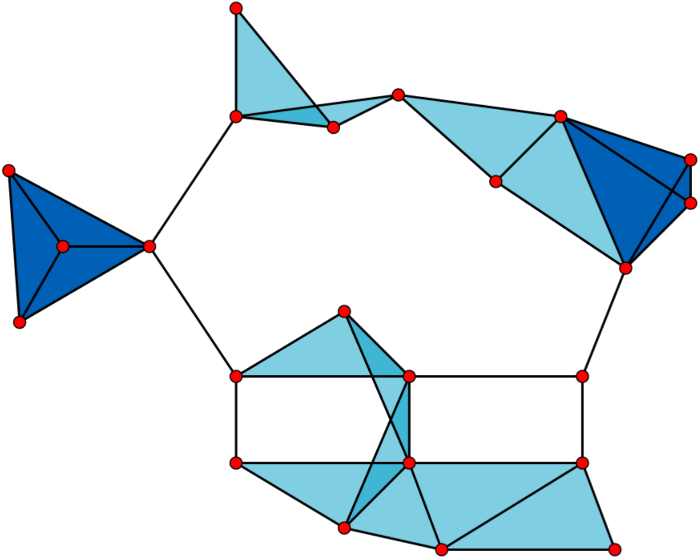

Figure: A clique complex.

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Very sparse LM complexes are close to being a wedge of $(k-1)$-dimensional spheres and very dense LM complexes are close to being a wedge of k-spheres.

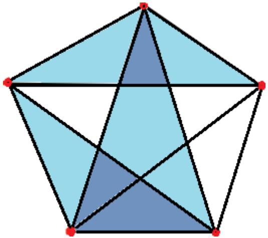

Figure: Linial-Meshulam 2-complex with $n = 5$.

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- Both models are controlled by a *single* probability parameter p.

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Formally we have a sequence of probability spaces Ω_n^r of subcomplexes of Δ_n with probability law

$$\mathbb{P}_r(X) = \prod_{i=1}^{r} p_i^{f_i(X)} \prod_{i=1}^{r} (1-p_i)^{e_i(X)}, \quad X \in \Omega_n^r$$

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Clearly $e_i(X)$ is equal to the number of boundaries of i-simplices minus $f_i(X)$. In particular two complexes with the same face numbers do not have necessarily the same probability of being generated.

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Theorem (C., Farber)

Let S be a finite simplicial complex and $Y \in \Omega_n^r$ a random simplicial complex.

- *Suppose that*

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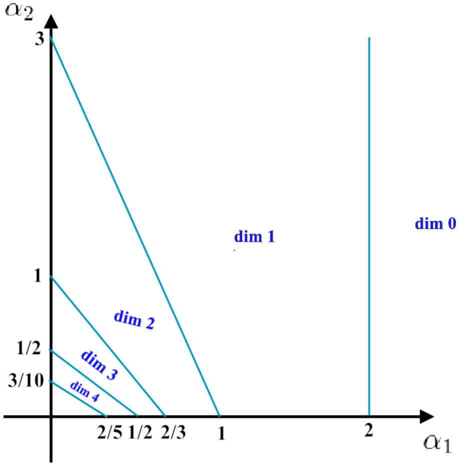


Figure: Dimension for $p_1 = n^{-\alpha_1}$, $p_2 = n^{-\alpha_2}$ and $p_i = 1$ otherwise

Consider the linear functions

$$\psi_k(\alpha) = \sum_{i=1}^{r} \binom{k}{i} \alpha_i,$$

where $\alpha_i = -\log p_i / \log n$. For simplicity assume that α_i are constant. It is easy to see that

$$\psi_1(\alpha) \le \psi_2(\alpha) \le \ldots \le \psi_r(\alpha)$$

and thus we can define the domains

$$\mathfrak{D}_k = \{\alpha \in \mathbb{R}_+^r : \psi_k(\alpha) < 1 < \psi_{k+1}(\alpha)\}$$

Theorem (C., Farber)

Suppose $p_i = n^{-\alpha_i}$ *and* $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathfrak{D}_k$. *Then*

1. *For a random complex* $Y \in \Omega_n^r$ *with Betti numbers* β_i *we have*

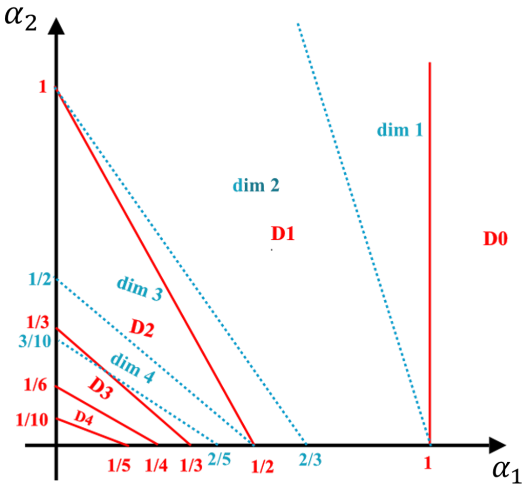
$$\mathbb{E}(\beta_k) \gtrsim n^{c_\alpha} \mathbb{E}(\beta_j), \quad j \neq k$$

   where c_α *is a positive constant depending on the limit of* α.

2. *The dominant Betti number satisfies*

$$\beta_k \sim f_k \sim \frac{n^{k+1-\sum_{i=0}^{k} \psi_i(\alpha)}}{(k+1)!}.$$

Dominant homological dimension



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Conjecture (C., Farber)

Conditioned to having dominant dimension $k \geq 3$ a multi-parameter random complex is a.a.s. homotopy equivalent to a wedge of spheres of possibly several dimensions $\geq k$.

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- *If $d > 1/2$, then a.a.s. G is e or $\mathbb{Z}_2$.*

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By taking the fundamental group $\pi_1(Y)$ of the random simplicial complex model Y we obtain a random group model with some similar features to Gromov's density model. However there are some key differences as well; eg torsion only appears in Gromov's model when the group is finite, contrary to the $\pi_1(Y)$ model.

As is the case of random Linial-Meshulam complexes and of clique complexes, multi-parameter random complexes have hyperbolic fundamental groups:

Theorem (C., Farber)

Let $Y \in \Omega_n^r$ be a multi-parameter random complex with $p_i = n^{-\alpha_i}$.

- *if $\alpha_0, \alpha_1, \alpha_2$ satisfy*

$$\alpha_0 + 3\alpha_1 + 2\alpha_2 < 1$$

  then Y has trivial fundamental group a.a.s.

- *conversely if $\alpha_0, \alpha_1, \alpha_2$ satisfy*

$$\alpha_0 + 3\alpha_1 + 2\alpha_2 > 1$$

  and $\alpha_1 < 1$ then $\pi_1(Y)$ is Gromov-hyberbolic and nontrivial a.a.s.

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- *let $m \geq 3$ be a fixed prime and $\alpha_0 + 3\alpha_1 + 2\alpha_2 > 1$ then $\pi_1(Y)$ has no m-torsion.*

Thank you for your attention!