Lusternik-Schnirlemann Category and Moment-Angle Complexes

Piotr Beben (with Jelena Grbić)

Aberdeen

May 2016

Polyhedral Products

K-simplicial complex on n vertices,
$(\underline{X}, \underline{A}) = ((X_1, A_1), \dots (X_n, A_n))$, $A_i \subseteq X_i$,

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} Y_1^\sigma \times \cdots \times Y_n^\sigma \quad \subseteq \quad X_1 \times \cdots \times X_n,$$

where

$$Y_i^\sigma = \begin{cases} X_i, & \text{if } i \in \sigma \\ A_i, & \text{if } i \notin \sigma, \end{cases}$$

example

- $K = \Delta^n \Rightarrow (\underline{X}, \underline{A})^K = X_1 \times \cdots \times X_n$.
- K *disjoint points* $\Rightarrow (\underline{X}, *)^K = X_1 \vee \cdots \vee X_n$.

Moment-angle complexes

Davis-Januszkiewicz spaces

$$\mathrm{DJ}(K) = (\mathbb{C}P^\infty, *)^K$$

Moment-angle complexes

$$\begin{aligned}\mathcal{Z}_K &= (D^2, \partial D^2)^K \\ &\simeq (\mathbb{C}^n, \mathbb{C}^n - \{0\})^K \\ &= \mathbb{C}^n - \bigcup_{\sigma \notin K} \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in \sigma\}.\end{aligned}$$

More generally, $\mathcal{Z}_K^\ell = (D^{\ell+1}, \partial D^{\ell+1})^K$ and $\mathbb{R}\mathcal{Z}_K = \mathcal{Z}_K^0 = (D^1, \partial D^1)^K$.

- The orbit space of a certain free torus action on $\mathcal{Z}_K$ for K boundary of dual simple polytope is a **quasi-toric manifold** $M = \mathcal{Z}_K / T^{f_0 - n}$.
- A **subspace arrangement** $\mathcal{A}$ is a finite set of affine subspaces of $\mathbb{R}^n$ (or $\mathbb{C}^n$), $V_\mathcal{A}$ is their union, and $M_\mathcal{A} = \mathbb{R}^n - V_\mathcal{A}$ (or $M_\mathcal{A} = \mathbb{C}^n - V_\mathcal{A}$) its complement.

  Thus $\mathcal{Z}_K$ is homotopy equivalent to the complement $M_\mathcal{A}$ of the arrangement $\bigcup_{\sigma \notin K} \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = 0 \text{ if } i \in \sigma\}$. Similarly for $\mathbb{R}\mathcal{Z}_K$.

Complexity Theory

k-equal Problem

Given a set of real numbers $\{x_1, \ldots, x_n\}$ and $k \geq 2$, how many comparisons $x_i - x_j \geq 0$ are needed (by the best algorithm in the worst case) to determine if any k of them are equal?

More Generally

Given an arrangement $\mathcal{A}$ and any x in $\mathbb{R}^n$, how many comparisons of the form $\ell(x) \geq 0$ (for each linear equation ℓ associated to each affine subspace in $\mathcal{A}$) are needed to determine if $x \in V_{\mathcal{A}}$?

The arrangement associated to the k-equal problem is the k-equal arrangement

$$\mathcal{A} = \{(x_1, \ldots, x_n) \mid x_{i_1} = \cdots = x_{i_k} \text{ for each } \{i_1, \ldots, i_k\} \subseteq [n]\}.$$

Theorem (Björner, Lovász, Yao,...)

The minimum number of comparisons $\ell(x) \geq 0$ are given asympototically in terms of the Betti numbers of $M_{\mathcal{A}}$ and the intersection lattice of $\mathcal{A}$.

Consequently, the minimum number of comparisons needed for the k-equal problem are asymptotically

$$\Theta(nlog\frac{2n}{k}).$$

Combinatorics

Let S be a **simplicial n-polytope** (a polytope whose codimension 1 faces are $(n-1)$-simplices, i.e. the dual of a simple polytope).

The f-**vector**

$$f(S) = (f_0, \ldots, f_{n-1})$$

where f_i is the number of i-dimensional faces of S.

Its h-**vector**

$$h(S) = (h_0, h_1, \ldots, h_n)$$

is given by

$$h_0 t_n + \cdots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}$$

The g-Theorem (Stanley, Birella, Lee)

$(f_0, \ldots, f_{n-1})$ is f-vector of some simplicial (or simple) polytope S iff h-vector $(h_0, \ldots, h_n)$ satisfies:

(1) $h_i = h_{n-i}$ *for* $i = 0, \ldots, n$ *(the* **Dehn-Sommerville relations***);*

(2) $1 = h_0 \leq h_1 \leq h_{\lfloor \frac{n}{2} \rfloor}$*;*

(3) $h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}$ *(the* i^{th} *pseudopower) for* $i = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. □

- Proof of the necessity part of g-theorem involves cohomology of quasitoric varieties.
- Betti numbers of quasitoric manifold

$$M = \mathcal{Z}_{\partial S}/T^{f_0 - n}$$

  satisfy $\beta^{2i}(M) = h_i(S)$, so Dehn-Sommerville relations follow by Poincaré Duality.
- g-Theorem is still a conjecture for general triangulations of spheres.
- But Dehn-Sommerville equations hold also for arbitrary triangulations K of closed compact manifolds, and this can be shown using the bigraded betti numbers of $\mathcal{Z}_K$.

Chemistry

A *fullerene* spherical molecules of carbon, each carbon atom belongs to three carbon rings, each carbon ring is either a pentagon or hexagon.

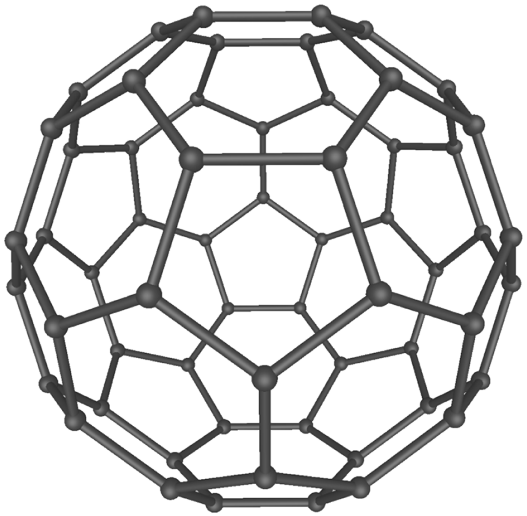


Figure: C60 fullerene

Mathematically, a fullerene is a simple 3-polytope all of whose 2-faces are pentagons and hexagons.

Belts

- p_k *is the number of* k*-gonal* 2*-faces of* 3*-polytope* P*.*
- *A* k-**belt** *of a* 3*-polytope* P *is a sequence of* 2*-faces* $(F_0, \ldots, F_{k-1})$ *s.t.* $F_{i-1} \cap F_{(i \mod k)}$ *is an edge, and all other intersections are* $\varnothing$*.*

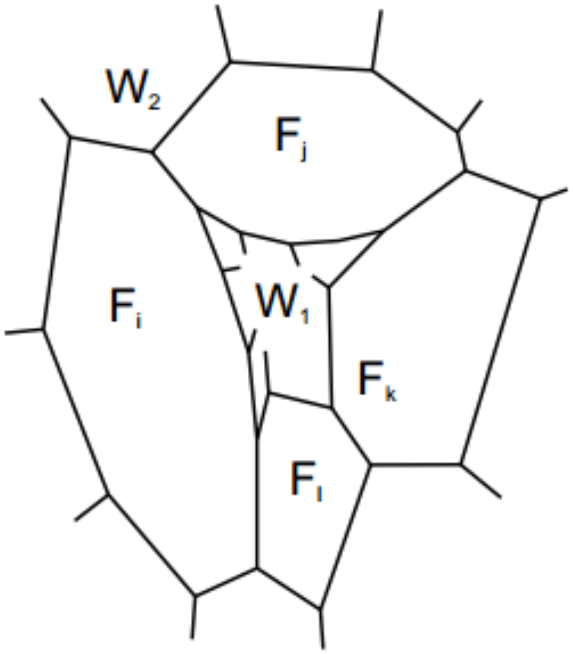


Figure: A 4-belt

Dual $K = \partial P^*$ of simple polytope P is a simplicial complex. $\exists$ bigrading on $H^*(\mathcal{Z}_K)$ defining:

$\beta^{i,j}(P)$ the **bigraded Betti number** of $H^*(\mathcal{Z}_{\partial P^*})$ (more on this later).

Buchstaber, Erokhovets

Theorem

For a fullerene P

- $\beta^{-1,6} = 0$ – *the number of* 3*-belts.*
- $\beta^{-2,8} = 0$ – *the number of* 4*-belts.*
- $\beta^{-3,10} = 12 + k$, $k \geqslant 0$ – *the number of* 5*-belts.*
  If $k > 0$, *then* $p_6 = 5k$.

Thus fullerenes have no 3*-belts and* 4*-belts.*

Theorem

For any fullerene

- $\beta^{-1,4} = \frac{(8+p_6)(9+p_6)}{2}$;
- $\beta^{-2,6} = \frac{(6+p_6)(8+p_6)(10+p_6)}{3}$;
- $\beta^{-3,8} = \frac{(4+p_6)(7+p_6)(9+p_6)(10+p_6)}{8}$.

For any simple 3-polytope, p_k for $k \neq 6$ satisfy

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k\geq 7}(k-6)p_k \text{ (Eberhard).}$$

Chemists are interested symmetries of molecules such as fullerenes P, e.g. $Aut(P)$.

Fan, Ma, Wang

If P and Q are fullerenes:

$$P \approx Q \Longleftrightarrow H^*(\mathcal{Z}_{\partial P^*}) \underset{rings}{\cong} H^*(\mathcal{Z}_{\partial Q^*})$$

Corollary

$$Aut(P) \approx Aut(H^*(\mathcal{Z}_{\partial P^*}))$$

(polytope automorphisms and ring automorphisms, respectively).

Question

- *To what extent does $H^*(\mathcal{Z}_G)$ determine a graph G? What other invariants, if any, are needed?*
- *How does commutative algebra in the form of $H^*(\mathcal{Z}_G)$ fit into algebraic graph theory?*

Commutative and Combinatorial Algebra

(Buchstaber, Panov)

For $\mathbf{k}$ *a field or* $\mathbb{Z}$*:*

- $H^*(\mathrm{DJ}(K);\mathbf{k}) \cong \mathbf{k}[K] = \frac{\mathbf{k}[v_1,\ldots,v_n]}{I_K}$ (**Stanley-Reisner Ring**),

  where ideal

  $$I_K = \langle \text{square-free monomials } v_{i_1}\cdots v_{i_k} \text{ s.t. } \{i_1,\ldots,i_k\} \notin K \rangle,$$

- $H^*(\Omega\mathrm{DJ}(K);\mathbf{k}) \cong Tor_{\mathbf{k}[K]}(\mathbf{k},\mathbf{k})$ (*Homology ring of* $\mathbf{k}[K]$),
- $H^*(\mathcal{Z}_K;\mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_n]}(\mathbf{k}[K],\mathbf{k})$.

- (Buchstaber, Baskakov, Franz, Panov, Hochster):

$$H^*(\mathcal{Z}_K;\mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_n]}(\mathbf{k}[K],\mathbf{k}) \cong \bigoplus_{I\subseteq[n]} \tilde{H}^{*-|I|-1}(|K_I|;\mathbf{k}).$$

cup product multiplication induced by canonical inclusions for disjoint I and J:

$$\iota_{I,J}\colon |K_{I\cup J}| \longrightarrow |K_I * K_J| \cong |K_I| * |K_J| \simeq \Sigma|K_I| \wedge |K_J|.$$

Bigraded Betti number:

$$\beta^{i,2j}(\mathcal{Z}_K) = \sum_{|I|=j} rank(\tilde{H}^{i+j-1}(|K_I|)).$$

The Golod Property

Golod: algebraic definition

A ring $R = k[v_1, \ldots, v_n]/\mathcal{I}$, $\mathcal{I}$ a homogenous ideal, is **Golod** over field $\mathbf{k}$ if $\mathrm{Tor}^+_{\mathbf{k}[v_1,\ldots,v_n]}(R, \mathbf{k})$ has trivial multiplication and all Massey products vanish.

$$\Downarrow$$

Golod: combinatorial definition

A simplicial complex K is **Golod** over field $\mathbf{k}$ if Stanley-Reisner ring $\mathbf{k}[K]$ is Golod.

$$\Downarrow$$

Golod: topological definition

A simplicial complex K is **Golod** over ring F if all cup products (of positive degree elements) and Massey products vanish in $H^+(\mathcal{Z}_K; F) \cong \mathrm{Tor}^+_{F[v_1,\ldots,v_n]}(F[K], F)$.

Long-standing problem:

Compute the Poincaré series

$$P(R) = \sum b_i t^i$$

of a commutative ring R, where

$$b_i = \mathrm{Tor}_R^i(\mathbf{k}, \mathbf{k}).$$

Conjecture (Kaplansky, Serre)

$R = k[v_1, \ldots, v_n]/\mathcal{I}$ *($\mathcal{I}$ a homogenous ideal)* $\Longrightarrow$ *then $P(R)$ is a rational function.*

Theorem (Golod)

$R = k[v_1, \ldots, v_n]/\mathcal{I}$ *is Golod* $\Longrightarrow$ *$P(R)$ is a rational function.*

Theorem (Grbić, Theriault)

$\mathbf{k}[K]$ *is Golod (i.e. K is Golod)* $\Longrightarrow$ $P(\mathbf{k}[K]) = \frac{t(1-t)^n}{t - P(H^*(\mathcal{Z}_K))}$.

Problem:

Characterize Golod rings R. In particular, characterize Golod complexes K.

Homotopy Theory

***Theorem** (Grbić, Theriault, Panov, Wu, Berglund, Jöllenbeck)*
If K is a flag complex, then the following are equivalent:

(1) K *is Golod;*

(2) prod. vanishes in $H^+(\mathcal{Z}_K) \cong \mathrm{Tor}^+_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z}[K],\mathbb{Z})$*;*

(3) 1*-skeleton of* K *is a chordal graph;*

(4) $\mathcal{Z}_K$ *is a* co*-*H*-space (wedge of spheres).* □

$(4) \Rightarrow (1)$, since in general:

$$X \text{ a } co-H-space \Longrightarrow \text{cup prod. and Massey prod. vanish in } H^+(X)$$

co-H-spaces are precisely LS-category 0 and 1 spaces...

Lusternik-Schnirelmann Category

The **LS-category** $cat(X)$ is smallest number m of open sets $U_1, \ldots, U_{m+1}$ that cover X and are contractible in X.

The **cup length** $cup(X)$ of X is largest number m s.t. there is a nonzero cup product $u_1 \cdots u_m$ for some $u_i \in \tilde{H}^*(X)$.

- $cat(X) \leq m \Longrightarrow cup(X) \leq m$;
- $cat(X) \leq 1 \Longrightarrow$ Massey prod. vanish in $H^+(X)$

Theorem (Rudyak,...)

If $cat(X) \leq m$ *then:*

(i) $cup(X) \leq m$;

(ii) *Massey products* $\langle v_1, \ldots, v_k \rangle \in H^+(X)$ *vanish whenever for some odd* i, *even* j, *and* $m_i + m_j > m$ *we have:*

$$v_i = a_1 \cdots a_{m_i} \text{ and } v_j = b_1 \cdots b_{m_j} \text{ for some } a_s, b_t \in H^+(X).$$

□

Definition

Say K *is* m-**Golod** *if* (i) *and* (ii) *above hold for* $X = \mathcal{Z}_K$.

Thus:

- $cat(X) \leq m \Longrightarrow K$ is m-Golod.

Problem 1

*Characterize those K for which K is m-**Golod**.*

and:

Problem 2

Characterize those K for which $cat(\mathcal{Z}_K) = m$.

- As we saw, both these problems are solved when $m = 1$ and K is a flag complex.
- Problem 2 is solved for $(X, *)^K$ for certain *nice* X (Felix, Tanre).

LS-category ≤ 1 (co-H-spaces) and 1-Golod K

- Since $cat(X) \leq 1 \Longrightarrow K$ is Golod, knowing homotopy types of $\mathcal{Z}_K$ tells us when K is Golod.
- This has had some success, e.g. shifted complexes and chordal flag complexes.
- In the other direction, since $H^*(\mathcal{Z}_K)$ is simplest when K is Golod, homotopy type of $\mathcal{Z}_K$ should be simplest here as well.

Homotopy type $\mathcal{Z}_K$ known for the following Golod K (increasing generality, due to Iriye, Kishimoto, Grbić, Theriault, Panov, Wu,...):

- K is n disjoint vertices;
- K is *shifted*;
- K is chordal flag complex or graph;
- K is a $\frac{n}{2}$-neighbourly or a 1-neighbourly 2-dim surface;
- K is Alexander dual of *shellable* or *sequential Cohen-Macaulay*;
- K *extractible* (most general);

In all cases they are co-H-spaces, often a wedge of spheres.

Conjecture A

K is Golod $\Longleftrightarrow$ $\mathcal{Z}_K$ is a co-H-space ($cat(\mathcal{Z}_K) \leq 1$).

(equivalenty) Conjecture A

Massey products vanish and $\iota_{I,J} : |K_{I\cup J}| \longrightarrow |K_I * K_J|$ trivial on cohomology for all $I \cap J = \varnothing$ $\Longleftrightarrow$ $\mathcal{Z}_K$ is a co-H-space.

Theorem (Iriye, Kishimoto)

$\mathcal{Z}_K$ is a co-H-space if and only if $\mathcal{Z}_K \simeq \bigvee_{I\subseteq[n]} \Sigma^{|I|+1}|K_I|$. $\square$

This is a desuspension of a general splitting of $\Sigma\mathcal{Z}_K$ due to Bahri, Bendersky, Cohen, Gitler (the BBCG splitting).

Conjecture is true for flag complexes (Grbić, Panov, Theriault, Wu), and rationally (Berglund)

Conjecture A seems to be false (Iriye, Yano).

Question A

To what extent is it true?

Question A - Main Idea

Ganea

A space Y is a co-H-space ($cat(Y) \leq 1$) $\Longleftrightarrow$ the evaluation map $\Sigma\Omega Y \xrightarrow{ev} Y$ has a right homotopy inverse.

- To see if $\mathcal{Z}_K$ is a co-H-space, we construct what looks like a configuration space model

$$\gamma \colon \mathcal{C}(\mathbb{R}\mathcal{Z}_K) \longrightarrow \Omega\mathcal{Z}_K.$$

and use $\Sigma\gamma$ to help us find right homotopy inverses for $\Sigma\Omega\mathcal{Z}_K \xrightarrow{ev} \mathcal{Z}_K$.

Labelled Configuration Spaces

Let M be any path connected space, $N \subseteq M$ a subspace, and Y a basepointed space with basepoint $*$.

$$SP((M/N) \wedge Y) \cong \coprod_{i=0}^{k} M^{\times i} \times Y^{\times i} / \sim$$

where the equivalence relation $\sim$ is given by

- $(z_1, \ldots, z_i; y_1, \ldots, y_i) \sim (z_{\sigma(1)}, \ldots, z_{\sigma(i)}; y_{\sigma(1)}, \ldots, y_{\sigma(i)})$ for permutations $\sigma \in \Sigma_i$;
- $(z_1, \ldots, z_i; y_1, \ldots, y_i) \sim (z_1, \ldots, z_{i-1}; y_1, \ldots, y_{i-1})$ for $y_i = *$ or $z_i \in N$.

The pairs (z_j, y_j) are called *particles*, the y_j's their *labels*.

Classical Labelled Configuration Space

$C(M, N; Y) \subseteq SP((M/N) \wedge Y)$ subspace of all configurations $(z_1, \ldots, z_i; y_1, \ldots, y_i)$ such that $z_1 \neq \cdots \neq z_i$.

A Classical Result

Segal, Bödigheimer, McDuff,...

M smooth parallelizable ℓ-manifold, there exists a (weak) homotopy equivalence

$$\gamma\colon C(M,\varnothing;Y) \longrightarrow map(Z, Z - M; \Sigma^{\ell} Y).$$

where $Z = M \cup (\partial M \times [0,1))$.

- For $X_1 \times \cdots \times X_n$:

$$\Omega \prod_i \Sigma X_i \cong \Omega\Sigma X_1 \times \cdots \times \Omega\Sigma X_n \simeq C(\mathbb{R}, \varnothing; X_1) \times \cdots \times C(\mathbb{R}, \varnothing; X_n).$$

- For $X_1 \vee \cdots \vee X_n$:

$$\Omega \bigvee_i \Sigma X_i \cong \Omega\Sigma(X_1 \vee \cdots \vee X_n) \simeq C(\mathbb{R}, \varnothing; X_1 \vee \cdots \vee X_n).$$

Polyhedral products tend to sit between these two outliers.

Suppose $W \subseteq X_1 \times \cdots \times X_n$.

Represented sets

- A multiset of points $\mathcal{S} = \{x_1, \ldots, x_s\} \subset \coprod_i X_i$ is **represented** by a point in W if there exists a point in $p \in W$ such that some subset of coordinates of p is equal to $\mathcal{S}$.

Let $V = X_1 \vee \cdots \vee X_n$.

$\mathcal{C}(M, N; W)$ is the subspace of $SP((M/N) \wedge V)$ of configurations

$$y = (z_1, \ldots, z_k; x_1, \ldots, x_k) \in SP((M/N) \wedge V)$$

such that

- y non-degenerate, $i_1 \neq \cdots \neq i_s$, and $z_{i_1} = \cdots = z_{i_s}$, $\Longrightarrow \{x_{i_1}, \ldots, x_{i_s}\}$ is represented by some point in W.

Lemma
$\mathcal{C}(D^\ell, \partial D^\ell; \mathbb{R}\mathcal{Z}_K)$ *deformation retracts onto* $\mathcal{Z}_K^\ell$.

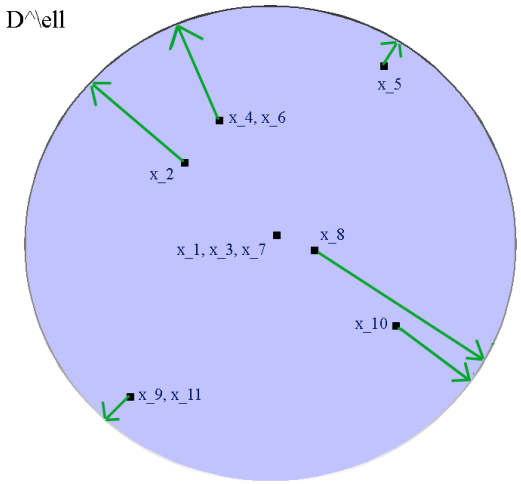
(1) Inclusion $\mathcal{Z}_K^\ell \xrightarrow{\iota} \mathcal{C}(D^\ell, \partial D^\ell; \mathbb{R}\mathcal{Z}_K)$

$$((t_1, x_1), \ldots, (t_n, x_n)) \mapsto (t_1, \ldots t_n; x_1, \ldots x_n)$$

homeomorphic onto its image;

(2) Two step radial expansion $R_t(x) = tx$:

(1)



(2)

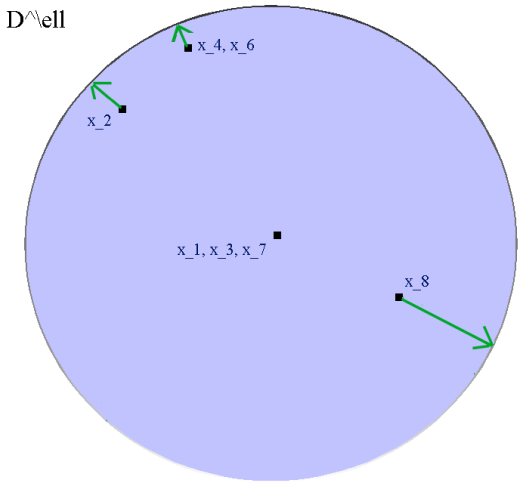


Figure: (1) Radially expand particles towards ∂D^ℓ to annihilation until there are no more than n particles left, each with a label in a distinct coordinate of $(D^1)^{\times n}$, at the same time attract labels near the basepoint towards it; (2) continue radially expanding until the remaining particles are a point in $\mathcal{Z}_K^\ell \subseteq \mathcal{C}(D^\ell, \partial D^\ell; \mathbb{R}\mathcal{Z}_K)$, meanwhile coordinate-wise attract the label set towards $\mathbb{R}\mathcal{Z}_K$.

Scanning Map

Theorem

M smooth parallelizable ℓ-manifold, exists a natural scanning map

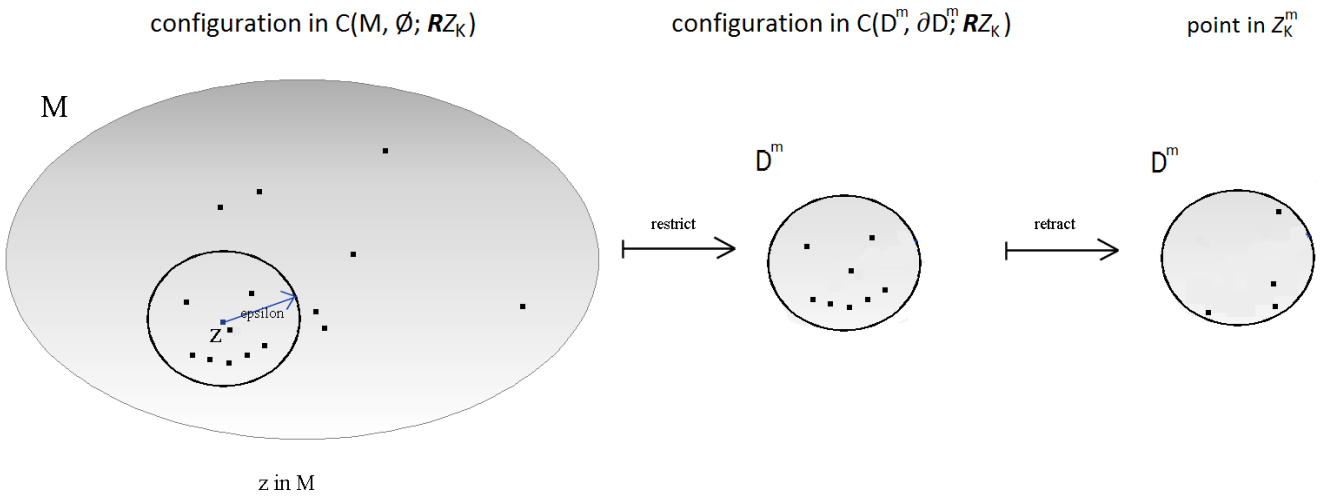
$$\gamma\colon \mathcal{C}(M,\varnothing;\mathbb{R}\mathcal{Z}_K) \longrightarrow map(Z, Z-M;\mathcal{Z}_K^{\ell}).$$

where $Z = M \cup (\partial M \times [0,1))$.

Proof.

$$\gamma\colon \mathcal{C}(M,\varnothing;\mathbb{R}\mathcal{Z}_K) \longrightarrow map(Z, Z-M\mathcal{C}(D^{\ell},\partial D^{\ell};\mathbb{R}\mathcal{Z}_K)) \longrightarrow map(Z, Z-M;\mathcal{Z}_K^{\ell})$$

When $\partial M = \varnothing$ $(Z = M)$:



In particular (taking $M = D^1$), we have a map

$$\gamma\colon \mathcal{C}(\mathbb{R}, \varnothing; \mathbb{R}\mathcal{Z}_K) \xrightarrow{\simeq} \mathcal{C}(D^1, \varnothing; \mathbb{R}\mathcal{Z}_K) \longrightarrow \Omega\mathcal{Z}_K.$$

Splittings

Let

$$\mathcal{C}(\mathbb{R}\mathcal{Z}_K) = \mathcal{C}(\mathbb{R}, \varnothing; \mathbb{R}\mathcal{Z}_K).$$

$$D_k(\mathbb{R}\mathcal{Z}_K) = \frac{\mathcal{C}_i(\mathbb{R}\mathcal{Z}_K)}{\mathcal{C}_{i-1}(\mathbb{R}\mathcal{Z}_K)}$$

Theorem B

$$\Sigma\mathcal{C}(\mathbb{R}\mathcal{Z}_K) \simeq \bigvee_{i\geq 1} \Sigma\mathcal{D}_i(\mathbb{R}\mathcal{Z}_K).$$

Moreover,

$$\mathcal{D}_k(\mathbb{R}\mathcal{Z}_K) \cong \bigvee_{(k,n)\text{–partitions } \mathcal{S}} \mathcal{D}_{\mathcal{S}}(\mathbb{R}\mathcal{Z}_K)$$

where $\mathcal{S} = (s_1, \ldots, s_n)$ non-negative integers s.t. $s_1 + \cdots + s_n = k$, and $\mathcal{D}_{\mathcal{S}}(\mathbb{R}\mathcal{Z}_K) \subseteq \mathcal{D}_k(\mathbb{R}\mathcal{Z}_K)$ subspace where s_i particles have labels in a ith-coordinate.

Let $\mathcal{A} = (1, \ldots, 1) \in \mathbb{N}^n$.

Proposition

(i) *homotopy commutative diagram of (homotopy) cofibration sequences*

$$\begin{tikzcd}
\mathbb{C}(\hat{\imath}) \arrow[r, "\partial"] \arrow[d, "\simeq"] & \Sigma\mathcal{D}_{\mathcal{A}}(\mathbb{R}\widetilde{\mathcal{Z}}_K) \arrow[r, "\Sigma\hat{\imath}", "include"'] \arrow[d] & \Sigma\mathcal{D}_{\mathcal{A}}(\mathbb{R}\mathcal{Z}_K) \arrow[r, "\hat{q}"] \arrow[d] & \Sigma\mathbb{C}(\hat{\imath}) \arrow[d, "\simeq"] \\
\Sigma^n|K| \arrow[r] & \widetilde{\mathcal{Z}}_K \arrow[r, "include"] & \mathcal{Z}_K \arrow[r, "q"] & \widehat{\mathcal{Z}}_K \cong \Sigma^{n+1}|K|,
\end{tikzcd}$$

(where $\Sigma^n|K| \simeq \mathbb{C}(\hat{\imath})$ similar to 2-step radial expansion argument, this time particles vanishing on tip of mapping cone).

(ii) *$\mathcal{Z}_K$ is a co-H-space when quotient map $\mathcal{Z}_{K_I} \xrightarrow{q} \widehat{\mathcal{Z}}_{K_I}$ has a right homotopy inverse for each $I \subseteq [n]$.*

Quotiented config. spaces $\mathcal{D}_{\mathcal{A}}(\mathbb{R}\mathcal{Z}_K)$ have nice combinatorial description (in analogy to quotiented moment-angle complexes $\widehat{\mathcal{Z}}_K = \mathcal{Z}_K/\widetilde{\mathcal{Z}}_K \cong \Sigma^{n+1}|K|$. having an obvious nice combinatorial description).

Homotopy Golod Condition

(1) *Homotopy version of trivial cup products:*

Require the inclusions

$$\Sigma\iota_{I,J} : \Sigma|K_{I\cup J}| \longrightarrow \Sigma|K_I * K_J|$$

to be nullhomotopic, for each $I \cap J = \emptyset$.

(2) *Coherence of these nullhomotopies:*

consider that there is a commutative diagram

$$\begin{tikzcd}
 & |K_{I\cup J_1} * K_{J_2}| \arrow[dr, "(\iota_{I,J_1})*\iota_{J_2}"] & \\
|K_{I\cup J_1\cup J_2}| \arrow[ur, "\iota_{I\cup J_1,J_2}"] \arrow[dr, "\iota_{I,J_1\cup J_2}"'] & & |K_I * K_{J_1} * K_{J_2}| \\
 & |K_I * K_{J_1\cup J_2}|. \arrow[ur, "\iota_I*(\iota_{J_1,J_2})"'] &
\end{tikzcd}$$

Then we require the composites

$$Cone(\Sigma|K_{I\cup J_1\cup J_2}|) \xrightarrow{\hat{\iota}_{I\cup J_1,J_2}} \Sigma|K_{I\cup J_1} * K_{J_2}| \longrightarrow \Sigma|K_I * K_{J_1} * K_{J_2}|$$

$$Cone(\Sigma|K_{I\cup J_1\cup J_2}|) \xrightarrow{\hat{\iota}_{I,J_1\cup J_2}} \Sigma|K_I * K_{J_1\cup J_2}| \longrightarrow \Sigma|K_I * K_{J_1} * K_{J_2}|$$

to be homotopic to to each other via a homotopy that is fixed on the base $\Sigma|K_{I\cup J_1\cup J_2}|$ of the cone $Cone(\Sigma|K_{I\cup J_1\cup J_2}|)$, where $\hat{\iota}_{I\cup J_1,J_2}$ and $\hat{\iota}_{I,J_1\cup J_2}$ are the extensions given by the nullhomotopies of $\Sigma\iota_{I\cup J_1,J_2}$ and $\Sigma\iota_{I,J_1\cup J_2}$. One then continues this process for longer joins.

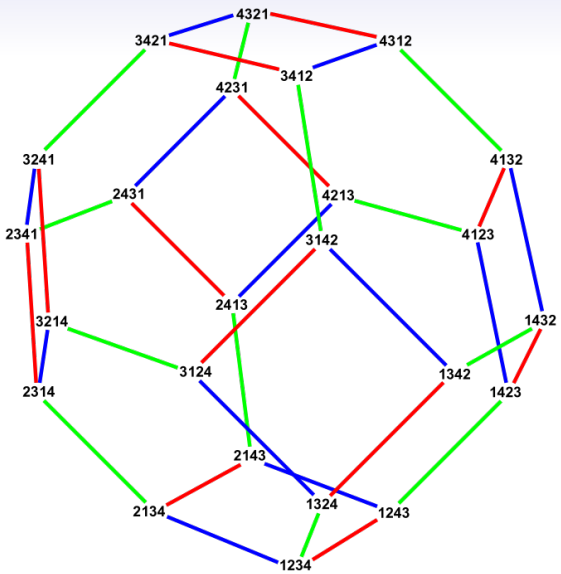


Figure: Skeleton of an order 4 permutohedron

Take the dual of the order n permutohedron: the delta set

$$\mathcal{K}_n = \{\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}\}$$

with $(m-2)$-faces the set

$$\mathcal{F}_{m-2} = \{\text{all length } m \text{ ordered partitions } (I_1, \ldots, I_m) \text{ of } [n]\}$$

and face maps $d_i : \mathcal{F}_{m-2} \longrightarrow \mathcal{F}_{m-3}$ given by

$$d_i((I_1, \ldots, I_m)) = (I_1, \ldots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \ldots, I_m).$$

$\mathcal{K}_n$ is a simplicial complex of dimension $n-2$, $|\mathcal{K}_n| \cong S^{n-2}$.

A simplicial complex K on vertex set $[n]$ is **homotopy Golod** if K is a single vertex, or (recursively) $K\backslash\{i\}$ is homotopy Golod for each $i \in [n]$ and there is a map

$$\Psi_K : |\mathcal{K}_n| \times Cone(\Sigma|K|) \longrightarrow \Sigma|\Delta^{n-1}|$$

such that for any $\gamma \in |\mathcal{K}_n|$:

(1) $\Psi_K(\gamma, *)$ is the basepoint $*_{-1} \in \Sigma|\Delta^{n-1}|$

(2) the restriction

$$(\Psi_K)_{|\{\gamma\}\times(\{0\}\times\Sigma|K|)} : \Sigma|K| \longrightarrow \Sigma|\Delta^{n-1}|$$

is the suspended inclusion;

(3) if $\gamma \in |\mathcal{S}|$ for some $\mathcal{S} = (I_1, \ldots I_m) \in \mathcal{F}_{m-2}$, then

$$\Psi_K(\{\gamma\} \times Cone(\Sigma|K|)) \quad \subseteq \quad \Sigma|K_{I_1} * \cdots * K_{I_m}| \subseteq \Sigma|\Delta^{n-1}|.$$

Theorem D

K is homotopy Golod $\Longrightarrow$ $\mathcal{Z}_K$ is a co-H-space (i.e. $cat(\mathcal{Z}_K) \leq 1$).

Extractible complexes are homotopy Golod:

K is **extractible** if $K\backslash\{i\}$ is a simplex for some i, or (recursively) each $K\backslash\{i\}$ is extractible and

$$\bigvee_i \Sigma|K\backslash\{i\}| \xrightarrow{include} \Sigma|K|$$

has a right homotopy inverse. This is one of the most general classes of Golod complexes for which $\mathcal{Z}_K$ is a co-H-space.

$\lfloor \frac{n}{2} \rfloor$-neighbourly complexes are homotopy Golod:

K is $\lfloor \frac{n}{2} \rfloor$-**neighbourly** if every subset of at most $\lfloor \frac{n}{2} \rfloor$ vertices is a face of K, where n is the number of vertices of K.

$\lfloor \frac{n}{2} \rfloor$-neighbourly but not extractible:

Let

$$K = (\partial\Delta^2 * \partial\Delta^1) \cup_{\partial\Delta^1} \Delta^1.$$

K has 5 vertices, and every pair of vertices is connected by an edge, so K is highly connected, therefore homotopy Golod. But $H_2(|K|) \cong \mathbb{Z}$, while $H_2(|K\backslash\{i\}|) = 0$ for each $i \in [5]$, so K cannot be extractible.

K is **homotopy Golod (without coherence)** if each suspended inclusion

$$\Sigma^{|I\cup J|}\iota_{I,J}\colon \Sigma^{|I\cup J|}|K_{I\cup J}| \longrightarrow \Sigma^{|I\cup J|}|K_I * K_J|$$

is nullhomotopic for each $I \cap J = \varnothing$.

Theorem E

$\mathcal{Z}_K$ is a co-H-space $\Longrightarrow$ K is homotopy Golod without coherence.

K is k-**neighbourly** if every k vertices in K span a $(k-1)$-simplex in K.

Corollary

If K is $\frac{n}{3}$-neighbourly then:

$$\mathcal{Z}_K \text{ is a co-H-space} \iff \text{each } \Sigma^{|I\cup J|}|K_{I\cup J}| \longrightarrow \Sigma^{|I\cup J|}|K_I * K_J| \text{ nullhomotopic.}$$

$\square$

LS-Category > 1

Question

What about $cat(\mathcal{Z}_K) > 1$.

Flag complexes

Problem

Repeat the classification of $cat(\mathcal{Z}_K) = 1$ (equiv. 1-Golod) flag complexes K for $cat(\mathcal{Z}_K) = m > 1$ and m-Golod K.

- The $m = 1$ case involved knowledge about topology of $\mathcal{Z}_L$ for 1-dimensional cycles L.
- Thus we need to know more about $\mathcal{Z}_L$ for higher dimensional cycles.
- Start with the simplest cycles: triangulated spheres K (still difficult). Here $\mathcal{Z}_K$ is a manifold.

Spheres

The above can be used to show:

Theorem

- *If K on vertex set $[n] = \{1, \ldots, n\}$ is any triangulated d-sphere for $d = 0, 1, 2$,*
- *or (under some conditions) K is obtained from these spheres via join, connected sum, and vertex double operations,*

then the following are equivalent.

(1) *K is m-Golod over $\mathbb{Z}$;*

(2) $Nill(\mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z}[K], \mathbb{Z})) \leq m + 1$ *(equivalently $cup(\mathcal{Z}_K) \leq m$);*

(3) *for any filtration of full subcomplexes*

$$\partial\Delta^{d+2-\ell} = K_{I_\ell} \subsetneq K_{I_{\ell-1}} \subsetneq \cdots \subsetneq K_{I_1} = K$$

such that $|K_{I_i}| \cong S^{d+1-i}$, we have $\ell \leq m$;

(4) $cat(\mathcal{Z}_K) \leq m$.

Moreover, $1 \leq m \leq d + 1$; that is, K satisfies any of the above for some m which cannot be greater than $d + 1$. □

Question: General sphere triangulations? Triangulations of manifolds?

Gluings

- **Geometric category:** $gcat(X) \leq k$ if open cover $X = \bigcup_{i=0}^{k} U_i$ s.t. U_i (self) contractible.
- **Strong category:** $\mathcal{C}at(X) = \min\{gcat(Y) \mid Y \simeq X\}$.
- $\mathcal{C}at(X) - 1 \leq cat(X) \leq \mathcal{C}at(X) \leq gcat(X)$.

Proposition

If C is a (possibly empty) full subcomplex common to $K_1, \ldots, K_m$,

$$cat(\mathcal{Z}_{K_1 \cup_C \cdots \cup_C K_m}) \leq \max\{1, cat(\mathcal{Z}_{K_1}), \ldots, cat(\mathcal{Z}_{K_m})\} + \mathcal{C}at(\mathcal{Z}_C).$$

Connected sums

Let $\dim L_i = d$, σ_i is a d-face common to L_i and L_{i+1}, and $\sigma_i \cap \sigma_j = \varnothing$ when $i \neq j$.

Proposition

Let

$$K = L_1 \#_{\sigma_1} L_2 \#_{\sigma_2} \cdots \#_{\sigma_{k-1}} L_k$$

Then

$$cat(\mathcal{Z}_K) \leq \max\{1, cat(\mathcal{Z}_{L_1^{(d-1)}}), \ldots, cat(\mathcal{Z}_{L_k^{(d-1)}})\} + 1.$$

Dimension and Coord. Suspension

Proposition

- $cat((\Sigma X, \Sigma A)^K) \leq cat((X, A)^K)$.
- $\Rightarrow$ for $\ell \geq 1$, $cat(\mathcal{Z}_K^\ell) \leq cat(\mathcal{Z}_K) \leq cat(\mathbb{R}\mathcal{Z}_K)$.
- $cat(\mathcal{Z}_K) \leq \dim K + 1$.

example

If G is a graph,

$$cat(\mathcal{Z}_G) \leq 2.$$

Thus, G is (at most) 2-Golod, and all Massey products of decomposible elements in $\mathrm{Tor}^+_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z}[G], \mathbb{Z}))$ vanish.

Many interesting spaces found amongst (or derived from) moment-angle complexes

$$\mathcal{Z}_K = (D^2, \partial D^2)^K \simeq (ES^1, S^1)^K$$

and other polyhedral products of the form $(BG, *)^K$ and $(EG, G)^K$:

- Intersections of quadrics, complements of arrangements.
- K a sphere $\Longrightarrow \mathcal{Z}_K$ a smooth closed manifold.
- K a triangulated manifold $\Longrightarrow \mathcal{Z}_K - *$ open manifold ($\mathcal{Z}_K$ has 1 singularity).
- $(B\mathbb{Z}, *)^K$ and $(E\mathbb{Z}, \mathbb{Z})^K$ are classifying spaces for the **right-angled Artin group** $A(K)$ and its commutator subgroup.
- $(B\mathbb{Z}_2, *)^K$ and $(E\mathbb{Z}_2, \mathbb{Z}_2)^K$ are classifying spaces for the **right-angled Coxeter group** $C(K)$ and its commutator subgroup.
- Every complex cobordism class has as a representative a quasitoric manifold $M = \mathcal{Z}_{\partial P^*}/T^{n-\dim P}$ (Panov).

So extending above bounds might answer certain questions about these spaces, e.g.

- group cohomology of commutators subgroups of Right-angled Artin/Coxeter groups;
- counting minimal number of critical points of a Morse function, since

$$1 + cat(M) \leq crit(M) = \min\{\# \text{ critical points } f \mid f\colon M \longrightarrow \mathbb{R} \text{ smooth }\}.$$

for smooth manifolds M.

Question

Given m-Golod K, when is $cat(\mathcal{Z}_K) = m$? Is $cat(\mathcal{Z}_K)$ even bounded? If not, how fast does it grow asymptotically w.r.t. vertex set $[n]$?

- If such a bound exists, we get a generalization of the basic and well-known fact:

$$\text{the identity map } |K| \xrightarrow{\mathbb{1}} |K| \text{ induces trivial map on cohomology}$$
$$\iff \Sigma|K| \xrightarrow{\Sigma\mathbb{1}} \Sigma|K| \text{ is nullhomotopic.}$$

(e.g. for $m = 1$, the inclusions $|K_{I\cup J}| \xrightarrow{\iota_{I,J}} |K_I * K_J|$ in place of $|K| \xrightarrow{\mathbb{1}} |K|$, together with vanishing Massey products, since $cat(\mathcal{Z}_K) = 1$ implies they are nullhomotopic).

Rational Homotopy

Theorem (Avramov, Berglund)

Localized at the rationals $\mathbb{Q}$, the following are equivalent:

(1) K *is Golod over* $\mathbb{Q}$;

(2) $H^+(\mathcal{Z}_K;\mathbb{Q}) \cong \mathrm{Tor}^+_{\mathbb{Q}[v_1,\ldots,v_n]}(\mathbb{Q}[K],\mathbb{Q})$ *trivial mult./vanishing Massey prod.;*

(3) $H^+(\mathcal{Z}_K;\mathbb{Q})$ *has trivial mult. and* $(\Lambda(u_1,\ldots,u_n)\otimes\mathbb{Q}[K],d)$ *is formal;*

(4) $\mathcal{Z}_K \simeq$ *a wedge of spheres;*

(5) $cat(\mathcal{Z}_K) = 1$.

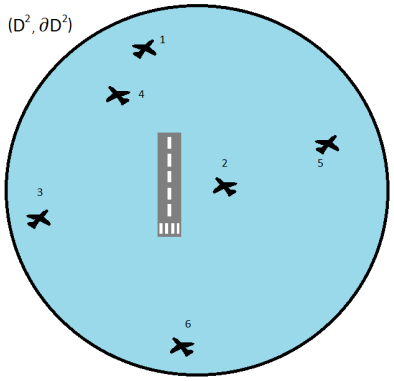
Question

How does this generalize to $cat(\mathcal{Z}_K) > 1$? In particular, $(3) \Leftrightarrow (2)$? Note: there is a generalization of formality due to Stasheff and Halperin.

Related Invariants

Problem: Compute $TC(\mathcal{Z}_K)$, sectional category, and related invariants. What are the implications of these in commutative algebra?

- $TC((S^n, *)^K)$ is known (González, Gutiérrez, Yuzvinsky).
- $cat(X) \leq TC(X) \leq 2cat(X) - 1$.
- $\mathcal{Z}_K$ can be thought of as configuration space of n points in $(D^2, \partial D^2)^K$, where tuples of points are allowed in $interior(D^2)$ depending on K.



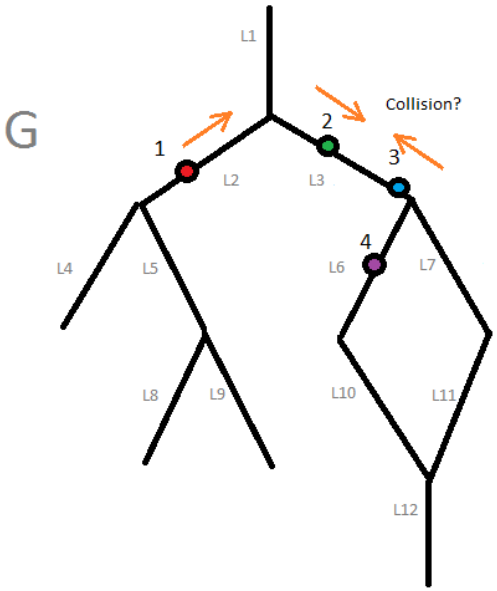
  Thus $\mathcal{Z}_K$ (or $\mathcal{Z}_K \cap Config(D^2, n)$) are simple models for motion planning for air traffic control, or robotic n-arm motion with restraints.
- An n-arm 2-leg **arachnoidal linkage** in $\mathbb{R}^d$ is precisely $\mathcal{Z}_K^d = (D^d, \partial D^d)^K$ for K the n-gon. A similar result for certain planar m-arm linkages. (Kamiyama, Tsukuda)

Connections to Motion Planning on Graphs

Take a graph G with m edges, a collection of simplicial complexes

$$\mathcal{L} = \{L_1, \ldots, L_m\}$$

each on n vertices. We have a configuration space $C_n(G; \mathcal{L})$:



modelling factory-floor motion planning: n robots, certain tuples of robots (of right or shape size) can bypass one another in i^{th} corridor, depending on L_i.

- When each L_i is n disjoint points: there are no collisions, and (usually) $C_n(G;\mathcal{L}) = K(\pi_1, 1)$ (Abrams, Ghrist).
- When $m = 1$, $C_n(G; L_1)$ is special case ($K = G$) of certain diagonal arrangements $\mathbf{C}_K(\mathbb{R}; I)$, appearing in summands of a stable splitting

$$\Sigma^\infty \Omega(\Sigma X, *) \simeq \Sigma^\infty \bigvee_{I \in \mathbb{N}^n} \mathbf{C}_K(\mathbb{R}; I)_+ \wedge X^I$$

  and in the homology of $\Omega(\mathbb{C}P^\infty, *)^K$ (Dobrinskaya).
- When $m = 1$, $I = (1, \ldots, 1) \in \mathbb{N}^n$, we have Euler characteristic

$$\chi(\mathbf{C}_K(\mathbb{C}P^{\ell-1}; I)) \quad = \quad \ell^{th} \text{ chromatic number of } G$$

  where K is the flag complex of complement graph $\bar{G}$ of G (Eastwood, Huggett).

Anders Björner - a Number Theoretic Complex:

Let $P(k)$ be the set of prime factors of integer k. Simplicial complex:

$$K(m) = \{P(k) \mid k \text{ is square-free and } k \leq m\}.$$

Theorem

Euler characteristic $\chi(K(m))$ gives:

- *Prime Number Theorem true $\iff |\chi(K(m))| \leq \epsilon m$ all $\epsilon > 0$, large m.*
- *Riemann hypothesis true $\iff |\chi(K(m))| \leq m^{\frac{1}{2}+\epsilon}$ all $\epsilon > 0$, large m.*

- Thus, one should study Betti numbers of $\beta_i(K(m))$.
- Björner estimated $\beta_i(K(m))$ by showing $K(m)$ is a shifted complex, and using properties of shifted complexes.

Problem

What can stronger topological/combinatorial invariants such as $H^*(\mathcal{Z}_{K(m)})$ tell us about number theory? Do they raise the right questions?

- Since shifted complexes are Golod, cup/Massey prod. vanish in $H^+(\mathcal{Z}_{K(m)})$, and $|K_{I\cup J}| \longrightarrow |K_I * K_J|$ trivial on $H^*()$. What does this tell us?

Thank you!