DESCRIPTION LOGICS:
REASONING SUPPORT FOR
THE SEMANTIC WEB

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Abstract

DL-based ontologies play a key role in the Semantic Web. They can be used to describe the intended meaning of Web resources and can exploit powerful Description Logic (DL) reasoning tools, so as to facilitate machine understandability of Web resources. Their wider acceptance, however, has been hindered by (i) the semantic incompatibilities between OWL, the W3C Semantic Web standard ontology language that is based on Description Logics, and RDF(S), the W3C proposed foundation for the Semantic Web, as well as by (ii) the lack of support for customised datatypes and datatype predicates in the OWL standard.

This thesis proposes a novel modification of RDF(S) as a firm semantic foundation for many of the latest DL-based Semantic Web ontology languages. It also proposes two decidable extensions of OWL that support customised datatypes and datatype predicates. Furthermore, it presents a DL reasoning framework to support a wide range of decidable DLs that provide customised datatypes and datatype predicates.

In our framework, datatype groups provide a general formalism to unify existing ontology-related datatype and predicate formalisms. Based on datatype groups, datatype expressions can be used to represent customised datatypes and datatype predicates. The framework provides decision procedures for a wide range of decidable DLs that support datatype expressions, including those that are closely related to OWL and its two novel extensions mentioned above. A remarkable feature of the proposed framework is its flexibility: the hybrid reasoner is highly extensible to support new datatypes and datatype predicates, new forms of datatype expressions and new decidable Description Logics.

This thesis constitutes an advance in the direction of Description Logic reasoning support for the Semantic Web. It clarifies the vision of the Semantic Web and particularly the key role that Description Logics play in the Semantic Web; it should, therefore, be of value to both communities. It is hoped that this work can provide a firm foundation for further investigations of DL reasoning support for the Semantic Web and, in general, ontology applications.
Declaration

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Acronyms

\( \mathcal{AL} \) = Attributive Language, a Description Logic that provides atomic concept, universal concept (\( \top \)), bottom concept (\( \bot \)), atomic negation, intersection (\( \sqcap \)), value restriction (\( \forall R.C \)) and limited exist restriction (\( \exists R.\top \))

\( \mathcal{ALC} \) = \( \mathcal{AL} \) extended with full concept negation (\( \neg \))

\( \mathcal{ALC}(\mathcal{D}) \) = \( \mathcal{ALC} \) extended with an admissible concrete domain (\( \mathcal{D} \))

\( \text{DAML+OIL} \) = the result of merging DAML-ONT and OIL

\( \text{DAML-ONT} \) = an early result of the DARPA Agent Markup Language (DAML) programme

\( \text{DL} \) = Description Logic

\( \text{HTML} \) = HyperText Markup Language

\( \mathcal{L}(\mathcal{G}) \) = a \( \mathcal{G} \)-combined DL, where \( \mathcal{L} \) is a \( \mathcal{G} \)-combinable DL, and \( \mathcal{G} \) is a conforming datatype group

\( \text{OIL} \) = Ontology Inference Layer

\( \text{OWL} \) = Web Ontology Language, a W3C recommendation

\( \text{OWL-E} \) = OWL extended with datatype Expressions

\( \text{OWL-Eu} \) = an unary restriction of OWL-E

\( \text{RDF} \) = Resource Description Framework

\( \text{RDFS} \) = RDF Schema

\( \text{RDFS(FA)} \) = RDFS with Fixed metamodeling Architecture
\[ S = \mathcal{ALC}_{R^+}, \text{i.e., } \mathcal{ALC} \text{ extended with transitive role axioms } (R^+) \]

\[ \mathcal{SH} = S \text{ extended with role hierarchy } (\mathcal{H}) \]

\[ \mathcal{SH}f = \mathcal{SH} \text{ extended with functional role axioms } (f) \]

\[ \mathcal{SH}I = \mathcal{SH} \text{ extended with inverse roles } (I) \]

\[ \mathcal{SH}IO = \mathcal{SH} \text{ extended with nominals } (O) \text{ and inverse roles } (I) \]

\[ \mathcal{SH}IQ = \mathcal{SH}Q \text{ extended with inverse roles } (I) \]

\[ \mathcal{SH}OQ = \mathcal{SH}Q \text{ extended with nominals } (O) \]

\[ \mathcal{SH}OQ(D) = \mathcal{SH}OQ \text{ extended with a conforming universal concrete domain } (D) \]

\[ \mathcal{SH}OIQ = \mathcal{SH}Q \text{ extended with nominals } (O) \text{ and inverse roles } (I) \]

\[ \mathcal{SH}OIQ(\mathcal{G}) = \mathcal{SH}OIQ \text{ extended with a conforming datatype group } (\mathcal{G}) \]

\[ \mathcal{SH}Q = \mathcal{SH} \text{ extended with qualified number restrictions } (Q) \]

\[ SI = S \text{ extended with inverse roles } (I) \]

\[ SW = \text{Semantic Web} \]

\[ XML = \text{Extensible Markup Language} \]

\[ W3C = \text{World Wide Web Consortium} \]
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To my parents
Chapter 1

Introduction

In *Realising the Full Potential of the Web* [15], Tim Berners-Lee identifies two major objectives that the Web should fulfil. The first goal is to enable people to work together by allowing them to share knowledge. The second goal is to incorporate tools that can help people analyse and manage the information they share in a meaningful way. This vision has become known as the *Semantic Web* (SW) [16].

The Web’s provision to allow people to write online content for other people is an appeal that has changed the computer world. This same feature that is responsible for fostering the first goal of the Semantic Web, however, hinders the second objective. Much of the content on the existing Web, the so-called *syntactic Web*, is human but not machine readable. Furthermore, there is great variance in the quality, timeliness and relevance [15] of Web resources (i.e., Web pages as well as a wide range of Web accessible data and services) that makes it difficult for programs to evaluate the worth of a resource.

The vision of the Semantic Web is to augment the syntactic Web so that resources are more easily interpreted by programs (or ‘intelligent agents’). The enhancements will be achieved through the *semantic markups* which are machine-understandable annotations associated with Web resources.

1.1 Heading for the Semantic Web

Encoding semantic markups will necessitate the Semantic Web adopting an annotation language. To this end, the W3C (World Wide Web Consortium) community has developed a recommendation called Resource Description Framework (RDF) [81]. The development of RDF is an attempt to support effective creation, exchange and use of
1.1. *HEADING FOR THE SEMANTIC WEB*

Example 1.1 Annotating Web Resources in RDF

As shown in Figure 1.1, we can associate RDF annotation\(^1\) to http://example.org/Ganesh.html and state that it is the homepage of the resource Ganesh, which is an elephant and eats grasses.

We invite the reader to note that the above RDF annotations are different from HTML \([125]\) mark-ups in that they describe the contents of Web resources, instead of the presentations of Web pages.

Annotating alone do not establish semantics of what is being marked-up. For example, the annotations presented in Figure 1.1 does not explain what elephants mean. In response to this need for more explicit meaning, ontologies \([46, 142]\) have been proposed to provide shared and precisely defined terms and constraints to describe the meaning of resources through annotations — such annotations are called *machine-understandable annotations*.

An ontology typically consists of a hierarchical description of important concepts in a domain, along with descriptions of the properties of each concept, and constraints about these concepts and properties. Here is an example of an ontology.

Example 1.2 An Elephant Ontology

An elephant ontology might contain concepts, such as animals, plants, elephants, adult elephants (elephants with their ages greater than 20) and herbivores (animals that eat only plants or parts of plants), as well as constraints that elephants are a kind of animal, and that adult elephants eat only plants. These constraints allow the concept

\(^1\)See Section 3.1.1 for precise definitions of RDF syntax.
‘adult elephants’ to be unambiguously interpreted, or understood, as a specialisation of the concept ‘herbivores’ by, e.g., an animal feeding agent.

Note that when we define complex concepts, such as ‘adult elephants’, we can refer to not only abstract entities, such as the concept ‘elephants’, but also concrete entities, such as ‘greater than 20’, which are called datatypes. Both concepts and datatypes are very useful in ontologies; therefore, the DL reasoning framework to be presented in this thesis support both of them.

Besides playing an important role in the Semantic Web, ontologies are popular through other applications, such as information retrieval and extraction [48, 13, 96, 145, 80, 137, 30], information integration [14, 97, 92], enterprise integration [47, 143], electronic commerce [83, 95, 84] and medical and bioinformatics terminology systems [127, 122, 43, 134, 136, 44, 133]. Standard SW ontology languages can benefit users of these fields by improving inter-operability of ontologies and by enabling them to exploit the reasoning services for these languages.

The advent of RDF Schema (RDFS) [22] represented an early attempt at a SW ontology language based on RDF. RDF and RDFS, or simply RDF(S), are intended to provide the foundation for the Semantic Web [94, Sec. 7.1]. As the constructors that RDFS provides for constructing ontologies are very primitive, more expressive SW ontology languages have subsequently been developed, such as OIL [57], DAML+OIL [68] and OWL [12], which are all based on Description Logics.

1.2 Description Logics and the Semantic Web

Description Logics (DLs) [7] are a family of class-based knowledge representation formalisms, equipped with well-defined model-theoretic semantics [6]. They were first developed to provide formal, declarative meaning to semantic networks [123] and frames [99], and to show how such structured representations can be equipped with efficient reasoning tools. The basic notions of Description Logics are classes, i.e., unary predicates that are interpreted as sets of objects, and properties, i.e., binary predicates that are interpreted as sets of pairs.

Description Logics are characterised by the constructors that they provide to build complex class and property descriptions from atomic ones. For example, ‘elephants

\footnote{Following the XML Schema type system, ‘greater than 20’ can be seen as a customised, or derived, datatype of the base datatype \textit{integer}; cf. Section 3.3.1.}
with their ages greater than 20’ can be described by the following DL class description:\(^3\)

$$\text{Elephant} \sqcap \exists \text{age.} > 20,$$

where Elephant is an atomic class, age is an atomic datatype property, \(> 20\) is a customised datatype (treated as a unary datatype predicate) and \(\sqcap, \exists\) are class constructors. As shown above, datatypes and predicates (such as \(=, >, +\)) defined over them can be used in the constructions of class descriptions. Unlike classes, datatypes and datatype predicates have obvious (fixed) extensions; e.g., the extension of \(> 20\) is all the integers that are greater than 20. Due to the differences between classes and datatypes, there are two kinds of properties: (i) object properties, which relate objects to objects, and (ii) datatype properties, which relate objects to data values, which are instances of datatypes.

Class and property descriptions can be used in axioms in DL knowledge bases. DL Axioms are statements that describe (i) relations between class (property) descriptions, (ii) characteristics of properties, such as asserting a property is transitive, or (iii) instance-of relations between (pairs of) individuals and classes (properties). We can use DL axioms to represent concepts and constraints in an ontology. For example, we can define the class AdultElephant with the following DL axiom

$$\text{AdultElephant} \equiv \text{Elephant} \sqcap \exists \text{age.} > 20;$$

we can represent the constraint ‘Elephant are a kind of Animal’:

$$\text{Elephant} \sqsubseteq \text{Animal};$$

we can also assert that the object Ganesh is an instance of the class description ‘Elephants who are older than 25 years old’:

$$\text{Ganesh} : (\text{Elephant} \sqcap \exists \text{age.} > 25).$$

In general, we can represent an ontology with (part of) a DL knowledge base.

Description Logics have distinguished logical properties. They emphasise on the decidability of key reasoning problems, such as class satisfiability and knowledge base satisfiability. They provide decidable reasoning services [39], such as tableaux algorithms (cf. Section 2.2), that deduce implicit knowledge from the explicitly represented

\(^3\)Precise definitions of syntax and semantics are presented in Chapter 2.
knowledge. For example, given the above axioms, a DL reasoner (with datatype support) should be able to infer that \texttt{Ganesh} is an \texttt{AdultElephant}. Highly optimised DL reasoners (such as FaCT [59], Racer [51] and DLP [115]) have showed that tableaux algorithms for expressive DLs lead to a good practical performance of the system even on (some) large knowledge bases.

High quality ontologies are pivotal for the Semantic Web. Their construction, integration, and evolution crucially depend on the availability of a well-defined semantics and powerful reasoning tools (cf. Section 3.1.3). Description Logics address both these ontology needs; therefore, they are ideal logical underpinnings for SW ontology languages (Baader et al. [9]). Unsurprisingly, the SW ontology languages OIL, DAML+OIL and OWL use DL-style model-theoretic semantics. This has been recognised as an essential feature in these languages, since it allows ontologies, and annotations using vocabulary and constraints defined by ontologies, to be shared and exchanged without disputes as to their precise meaning.

DLs and insights from DL research had a strong influence on the design of these Web ontology languages. The influence is not only on the formalisations of the semantics, but also on the choice of language constructors, and the integration of datatypes and data values. OIL, DAML+OIL and OWL thus can be viewed as expressive DLs with Web syntax.

Among these SW ontology languages, OWL is particularly important. OWL has been adopted as the standard (W3C recommendation) for expressing ontologies in the Semantic Web. There are three sub-languages of OWL: OWL Lite, OWL DL and OWL Full. In this thesis, when we mention ‘OWL’ we usually mean ‘OWL DL’ because OWL Lite is simply a sub-language of OWL DL, while OWL Full can be seen as an unsuccessful attempt at integrating RDF(S) and OWL DL (cf. Section 3.2.2).

1.3 The Two Issues

The wider acceptance of DL-based ontologies in the Semantic Web has been hindered, however, by the following issues.

**Issue 1: Semantic Mismatch Between RDF(S) and OWL**

Semantic Web ontology languages are supposed to be extensions and thus compatible with RDF(S) [16]. There are, however, at least three known problems in extending the
1.3. THE TWO ISSUES

RDF(S) semantics [53] with OWL constructors [116, 117, 74], i.e., the problems of ‘too few entailments’, ‘contradiction classes’ and ‘size of the universe’, all of which stem from some unusual characteristics of RDF(S) (cf. Section 3.2.2). In short, the intended foundation of the Semantic Web and DL-based SW ontology languages are not compatible with each other.

Here comes our worrying vision: the annotations of Web resources are represented in RDF(S), while the ontologies of various SW application are represented in DL-based Web ontology languages; between them is their semantic mismatch, which discourages existing RDF(S) users from adopting OWL, the newer standard [21].

**Issue 2: No Customised Datatypes and Datatype Predicates**

Although OWL is rather expressive, it has a very serious limitation; i.e., it does not support customised datatypes and datatype predicates. It has been pointed out that many potential users will not adopt OWL unless this limitation is overcome [126]. This is because it is often necessary to enable users to define their own datatypes and datatype predicates for their applications. In what follows, we give some examples to illustrate the usefulness of customised datatypes and datatype predicates in various SW and ontology applications.

**Example 1.3 Semantic Web Service: Matchmaking**

*Matchmaking* is a process that takes a service requirement and a group of service advertisements as input, and returns all the advertisements that may potentially satisfy the requirement. In a computer sales ontology, a service requirement may ask for a PC with memory size greater than 512Mb, unit price less than 700 pounds and delivery date earlier than 15/03/2004.

Here ‘greater than 512’, ‘less than 700’ and ‘earlier than 15/03/2004’ are customised datatypes of base datatypes integer, integer and date, respectively.

**Example 1.4 Ontology Merging: Unit Mapping**

Suppose we need to merge two map ontologies A and B: A describes distance in miles, while B describes distance in kilometres. When we merge A and B, we need to set up a bridging constraint to make sure that the data value of the distance in miles (in ontology A) should be equal to 1.6 times the data value of distance (between the

---

4Take the length units for example: there are so many of them (actually there are more than one hundred length units according to http://www.chemie.de/) that it is very likely that different ontologies might use different units.
same pairs of locations) in kilometres; otherwise, there will be data inconsistencies in
the merged ontology.

Here, ‘multiplication’ is a datatype predicate defined over the rationals, while ‘mul-
tiply by 1.6’ is a customised datatype predicate.

Example 1.5 Electronic Commerce: A ‘No Shipping Fee’ Rule

Electronic shops may need to classify items according to their sizes, and to reason
that an item for which the sum of height, length and width is no greater than 15cm
belongs to a class in their ontology, called ‘small-items’. Then they can have a rule
saying that for ‘small-items’ no shipping costs are charged. Accordingly, the billing
system will charge no shipping fees for all the instances of the ‘small-items’ class.

Here ‘greater than 15’ is a customised datatype, ‘sum’ is a datatype predicate, while
‘sum no greater than 15’ is a customised datatype predicate.

The usefulness of customised datatypes and datatype predicates in SW and ontol-
ogy applications means that it is necessary to provide DL reasoning support for some
extension of OWL that supports them.

To sum up, the SW standards RDF(S) and OWL are results of restricted techniques
available when they were designed as well as political compromises within the corre-
spanding W3C working groups. We will provide more details about the two issues in
Chapter 3.

1.4 Objectives

The aim of the thesis is, therefore, to provide further techniques to solve the two is-
ssues presented in the last section. This aim is further developed into the following
objectives:

1. To propose a novel modification of RDF(S) as a firm semantic foundation for the
   latest DL-based SW ontology languages.

2. To propose some extensions of OWL, so as to support customised datatypes and
datatype predicates.

3. To provide a DL reasoning framework, which (i) supports customised datatypes
and datatype predicates, (ii) integrates a family of decidable (including very expressive) DLs with customised datatypes and datatype predicates, and (iii) provides decision procedures and flexible reasoning services for some members of this family that are closely related to OWL and the proposed extensions.

To fulfill these objectives, this thesis makes the following main contributions:

- the design of RDFS(FA),\textsuperscript{5} a sub-language of RDFS with DL-style model-theoretic semantics, which provides a firm foundation for using DL reasoning in the Semantic Web and thus solidifies RDF(S)’s proposed role as the foundation of the Semantic Web (Chapter 4);

- the datatype group approach, which specifies a formalism to unify datatypes and datatype predicates, and to provide a wide range of decidable Description Logics (including very expressive ones) integrated with customised datatypes and datatype predicates (Chapter 5);

- the design of OWL-E, a decidable extension of OWL that provides customised datatypes and datatype predicates based on datatype groups, and its unary restriction OWL-Eu, which is a much smaller extension of OWL (Chapter 6);

- the design of practical tableaux algorithms for a wide range of DLs that are combined with arbitrary conforming datatype groups, including those of a family of DLs that are closely related to OWL, DAML+OIL, OWL-Eu and OWL-E (Chapter 6);

- a flexible framework architecture to support decidable Description Logic reasoning services for Description Logics integrated with datatype groups. (Chapter 7).

We invite the reader to note that although the above contributions are made in the context of the Semantic Web, they can be applied to general ontology applications.

By fulfilling these objectives, this thesis shows that Description Logics can provide clear semantics, decision procedures and flexible reasoning services for SW ontology languages, including those that provide customised datatypes and datatype predicates.

\textsuperscript{5}RDFS with Fixed metamodeling Architecture.
1.5 Reader’s Guide

The remainder of the thesis is organised as follows.

Chapter 2 Introduces the basic DL formalisms, reasoning services, reasoning techniques and, most importantly, the existing approaches to integrating Description Logics with datatype predicates.

Chapter 3 Explains the importance of DL-based ontologies in the Semantic Web, presents RDF(S), OWL, and their datatype formalisms, and clarifies the two issues that the rest of the thesis is going to tackle.

Chapter 4 Presents RDFS(FA) as a strong connection between OWL and RDF(S), defining its DL style model theoretic-semantics and RDF style axioms, explaining how it solves the problem of layering OWL on top of RDF(S), and therefore establishing a firm foundation for using DL reasoning in the Semantic Web and clarifying the vision of the Semantic Web.

Chapter 5 Proposes a general formalism to unify existing ontology-related data-type and predicate formalisms and investigates, in the unified formalism, a family of decidable DLs that provide customised datatypes and datatype predicates.

Chapter 6 Proposes decidable extensions of OWL, i.e., OWL-E and OWL-Eu, that support customised datatypes and datatype predicates, and designs practical decision procedures for a family of Description Logics that are closely related to OWL and the proposed extensions.

Chapter 7 Introduces a flexible architecture for the framework, which is based on an extension of the current general DL interface DIG/1.1, for providing DL inferencing services for OWL and OWL-E.

Chapter 8 Describes a prototype implementation of the tableaux algorithm for the $SHIQ(G)$ DL based on the FaCT system, along with two simple type checkers.

Chapter 9 Reviews the work presented and the extent to which the stated objectives have been met. The significance of the major results is summarised, and perspectives for further research are sketched.

Some results in this thesis have previously been published: some aspects of RDFS(FA) in Chapter 4 appeared in [108, 110, 113, 112]; an early version of the datatype group
approach presented in Chapter 5 has been published in [114]; a brief introduction of OWL-E presented in Chapter 6 appeared in [107]; finally, the $SHOQ(D_n)$ DL, which is closely related to the family of DLs presented in Chapter 6, and its reasoning support for the Web ontology languages were published in [106, 111, 109].
Chapter 2

Description Logics

Chapter Aims

- To introduce the basic formalisms of Description Logics (DLs).
- To introduce basic DL reasoning techniques.
- To describe the existing approaches to integrating description languages with datatype predicates and to discuss their limitations.

Chapter Plan

2.1 Foundations (26)
2.2 Reasoning Algorithms (37)
2.3 Description Logics and Datatype Predicates (39)

2.1 Foundations

Description Logics (DLs) [7] are a family of logic-based knowledge representation formalisms designed to represent and reason about the knowledge of an application domain in a structured and well-understood way. They are based on a common family of languages, called description languages, which provide a set of constructors to build concept (class) and role (property) descriptions. Such descriptions can be used in axioms and assertions of DL knowledge bases and can be reasoned about w.r.t. DL knowledge bases by DL systems.
2.1. FOUNDATIONS

2.1.1 Description Languages

The foundations of description languages are concept and role descriptions (or concepts and roles for short). Intuitively, a concept represents a class of objects sharing some common characteristics, while a role represents a binary relationship between objects, or between objects and data values. We do not cover Description Logics integrated with datatype predicates in this section but will introduce them in Section 2.3.

A description language consists of an alphabet of distinct concept names (C), role names (R) and individual (object) names (I); together with a set of constructors to construct concept and role descriptions. For a description language L, we call \text{Cdsc}(L) and \text{Rdsc}(L) the set of concepts and roles of L, respectively.

Description Logics have a model theoretic semantics, which is defined in terms of interpretations. An interpretation (written as I) consists of a domain (written as \Delta^I) and an interpretation function (written as \cdot^I), where the domain is a nonempty set of objects and the interpretation function maps each individual name a \in I to an element a^I \in \Delta^I, each concept name CN \in C to a subset CN^I \subseteq \Delta^I, and each role name RN \in R to a binary relation RN^I \subseteq \Delta^I \times \Delta^I. The interpretation function can be extended to give semantics to concept and role descriptions (see, e.g., Table 2.1). In the rest of the thesis, we will adopt the above notations (i.e., font styles) of individual names (a), class names (CN) and role names (RN).

Description languages are distinguished by the constructors they provide; each language is named according to a convention introduced in [131], where the language \text{AL} (acronym for ‘attributive language’) was introduced as a minimal language that is of practical interest. The other languages of this family are extensions of \text{AL}. Each additional constructor is associated with a special capital letter (e.g., C for general negation and U for disjunction, see Table 2.1).

In modern DLs, the language \text{S} is often used as a minimal language, which has previously been called \text{ALC}_R^+ according to the above convention; we call this language \text{S} because it relates to the propositional (multi) modal logic \text{S4} [129], and because we want to avoid names becoming too cumbersome when adding letters to represent additional features [67]. Well known \text{S}-family languages include \text{SI} [64], \text{SH}, \text{SHI} [63], \text{SHf} [58, 138], \text{SHIQ} [76, 67] and \text{SHOQ}(\text{D}) [75].

Now we show how we build \text{S}-roles and \text{S}-concepts from role names and concept names. \text{S}-roles are simply role names; i.e., \text{S} provides no constructor to build \text{S}-roles. On the other hand, according to Definition 2.1, \text{S} provides quite a few constructors to

\footnote{\text{SHf}, \text{SHIQ} and \text{SHOQ}(\text{D}) are pronounced as ‘chef’, ‘shick’ and ‘shock D’, respectively.}
build $S$-concepts, viz. concept descriptions of $S$. This is common in many DLs, and that is why description languages are sometimes also called concept languages.

**Definition 2.1. ($S$-concepts)** Let $CN \in C$ be an atomic concept name, $R \in \text{Rdsc}(S)$ an $S$-role and $C, D \in \text{Cdsc}(S)$ $S$-concepts. Valid $S$-concepts are defined by the abstract syntax:

$$C ::= \top | \bot | \text{CN} | \neg C | C \sqcap D | C \sqcup D | \exists R.C | \forall R.C$$

The semantics of $S$-concepts is given in Table 2.1

<table>
<thead>
<tr>
<th>Constructor</th>
<th>Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>$\top$</td>
<td>$\Delta^I$</td>
</tr>
<tr>
<td>bottom</td>
<td>$\bot$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>concept name</td>
<td>$\text{CN}$</td>
<td>$\Delta^I \subseteq \Delta^I$</td>
</tr>
<tr>
<td>general negation</td>
<td>$\neg C$</td>
<td>$\Delta^I \setminus C^I$</td>
</tr>
<tr>
<td>conjunction</td>
<td>$C \sqcap D$</td>
<td>$C^I \cap D^I$</td>
</tr>
<tr>
<td>disjunction</td>
<td>$C \sqcup D$</td>
<td>$C^I \cup D^I$</td>
</tr>
<tr>
<td>exists restriction</td>
<td>$\exists R.C$</td>
<td>${ x \in \Delta^I \mid \exists y. (x, y) \in R^I \land y \in C^I }$</td>
</tr>
<tr>
<td>value restriction</td>
<td>$\forall R.C$</td>
<td>${ x \in \Delta^I \mid \forall y. (x, y) \in R^I \rightarrow y \in C^I }$</td>
</tr>
</tbody>
</table>

Table 2.1: Semantics of $S$-concepts

**Example 2.1 Building an $S$-Concept** Let $\text{Plant} \in C$ be a concept name, $\text{eat}, \text{partOf} \in \text{Rdsc}(S)$ $S$-roles,

$$\forall \text{eat}. (\text{Plant} \sqcup \exists \text{partOf}. \text{Plant})$$

is an $S$-concept, following the abstract syntax in Definition 2.1:

1. $\text{Plant}$ is an $S$-concept;
2. $\exists \text{partOf}. \text{Plant}$ is an $S$-concept;
3. hence $\text{Plant} \sqcup \exists \text{partOf}. \text{Plant}$ is an $S$-concept;
4. finally, $\forall \text{eat}. (\text{Plant} \sqcup \exists \text{partOf}. \text{Plant})$ is an $S$-concept.

Table 2.1 shows how the interpretation function is extended to give semantics to $S$-concepts. An interpretation $\mathcal{I}$ is said to be a model of a concept $C$, or $\mathcal{I}$ models $C$, if the interpretation of $C$ in $\mathcal{I}$ (viz. $C^I$) is not empty.
Example 2.2 Interpretation of an $S$-Concept $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I})$ is a model of $\forall\text{eat}.(\text{Plant} \sqcup \exists\text{partOf. Plant})$ where
\[
\Delta^\mathcal{I} = \{\text{Ganesh, Bokhara, Balavan, grass1, stone1}\}
\]
and the interpretation function $\mathcal{I}$ is defined by:
\[
\begin{align*}
\text{Plant}^\mathcal{I} &= \{\text{grass1}\} \\
\text{eat}^\mathcal{I} &= \{(\text{Ganesh, grass1}), (\text{Bokhara, stone1})\} \\
\text{partOf}^\mathcal{I} &= \emptyset.
\end{align*}
\]
According to Table 2.1, we have
\[
\begin{align*}
(\exists\text{partOf. Plant})^\mathcal{I} &= \emptyset \\
(\text{Plant} \sqcup \exists\text{partOf. Plant})^\mathcal{I} &= \{\text{grass1}\} \\
(\forall\text{eat}.(\text{Plant} \sqcup \exists\text{partOf. Plant}))^\mathcal{I} &= \{\text{Ganesh, Balavan, grass1, stone1}\}
\end{align*}
\]
Note that Balavan, grass1, stone1 do not relate to anything via the role eat, according to the semantics of the value restriction (see Table 2.1), all of them are instances of the concept $\forall\text{eat}.(\text{Plant} \sqcup \exists\text{partOf. Plant})$.

In addition to the constructors presented in Table 2.1, $S$ provides transitive role axioms, which will be introduced in the next section.

2.1.2 Knowledge Base

Typically, a DL knowledge base is composed of two distinct parts: the intensional knowledge (TBox and RBox), i.e., general knowledge about the problem domain, and extensional knowledge (ABox), i.e., knowledge about a specific situation.

TBox

Intuitively, a T erm inological Box is a set of statements about how concepts are related to each other. For example,
\[
\exists\text{hasStudent. PhDStudent} \sqsubseteq \text{AcademicStaff} \sqcap \text{FullTimeStaff},
\]
says only full time AcademicStaff can have PhDStudents.

Formally, a $TBox$ is defined as follows.

**Definition 2.2.** (TBox) Let $\mathcal{L}$ be a Description Logic, $C, D \in \text{Cdesc}(\mathcal{L})$ $\mathcal{L}$-concepts, a TBox $\mathcal{T}$ is a finite, possibly empty, set of statements of the form $C \sqsubseteq D$, called concept
2.1. FOUNDATIONS

Inclusions. \( C \equiv D \), called a concept equivalence, is an abbreviation for \( C \sqsubseteq D \) and \( D \sqsubseteq C \). Statements in \( T \) are called terminological axioms.

An interpretation \( \mathcal{I} \) satisfies a concept inclusion \( C \sqsubseteq D \), or \( \mathcal{I} \) models \( C \sqsubseteq D \) (written as \( \mathcal{I} \models C \sqsubseteq D \)), if \( C^\mathcal{I} \subseteq D^\mathcal{I} \), and it satisfies a concept equivalence \( C \equiv D \) (written as \( \mathcal{I} \models C \equiv D \)), if \( C^\mathcal{I} = D^\mathcal{I} \). An interpretation \( \mathcal{I} \) satisfies a TBox \( T \) (written as \( \mathcal{I} \models T \)), iff it satisfies all the terminological axioms in \( T \).

Reasoning in the presence of a TBox is much harder than that without a TBox \[103\], especially when terminological cycles are allowed. A terminological cycle in a TBox is a recursive concept inclusion, e.g., \( \text{Person} \sqsubseteq \forall \text{hadParent} \cdot \text{Person} \), or one or more mutually recursive concept inclusions, e.g.,

\[
\{ \text{Person} \sqsubseteq \exists \text{hasParent} \cdot \text{Mother}, \text{Mother} \sqsubseteq \exists \text{hasChild} \cdot \text{Person} \}.
\]

Note that, however, for concept names \( A, B \in C \), \( \{ A \sqsubseteq B, B \sqsubseteq A \} \) is not a terminological cycle, since it is equivalent to a concept equivalence \( A \equiv B \).

It can be shown that a knowledge base without such cycles and containing only unique introductions, i.e., terminological axioms with only concept names appearing on the left hand side and, for each concept name \( \text{CN} \), there is at most one axiom in \( T \) of which \( \text{CN} \) appears on the left side, can be transformed into an equivalent one with an empty terminology by, roughly speaking, substituting all the introduced concept names in concept descriptions with their unique equivalence introductions; this transformation is called unfolding \[138\].

Although terminological cycles in a TBox present computational, algorithmic and even semantic problems \[101\], experiences with terminological knowledge representation systems in applications show that terminological cycles are used regularly \[79\].

It has already been shown that we can handle general terminologies, or TBoxes (possibly) with terminological cycles, theoretically and practically. For DLs with the disjunction (\( \mathcal{U} \)) and general negation (\( \mathcal{C} \)) constructors, general terminologies can be transformed into (possibly cyclical) terminologies with only introductions \[24\], which can be further transposed into meta constraints \[58\] in tableaux algorithms (cf. Section 2.2). If such DLs also provide, e.g., transitive closure, or both transitive role

\[2\]Following \[101\], the semantics we have studied so far is called descriptive semantics. Even though it can produce counter-intuitive results when the terminology contains cycles \[101, 58\], while the so-called fixpoint semantics will not \[101, 34\], the descriptive semantics is adopted in this thesis because of its wide acceptance as the most appropriate one \[24, 39\], and because existence of fixpoint cannot be guaranteed for the expressive languages studied here.
axioms and role inclusion axioms (see Definition 2.3), general terminologies can be ‘internalised’ [65] (see Theorem 2.8), so that reasoning w.r.t. the TBox can be reduced to pure concept reasoning. The extensive use of the disjunctive constructor in the above processes inevitably leads to intractability, which can be overcome, to some extent, by the optimisation technique called absorption that reduces the number of concept inclusion axioms by absorbing them into inclusion definitions whenever possible [58]. As a result of the above theoretical and empirical investigations, all modern DL systems provide unrestricted concept inclusion axioms.

**RBox**

Intuitively, an R(ole)Box is a set of statements about the characteristics of roles. An RBox can include statements asserting that a role is functional (e.g., since one can have at most one father, we can hence say the role hasFather is functional) or transitive (e.g., the role partOf is transitive), or is in inclusions relationships (e.g., isComponent ⊑ partOf). An RBox is formally defined as follows.

**Definition 2.3. (RBox)** Let \( \mathcal{L} \) be a Description Logic, \( RN, SN \in R \) role names, \( R_1, R_2 \in \text{Rdsc}(\mathcal{L}) \) \( \mathcal{L} \)-roles, an RBox \( \mathcal{R} \) is a finite, possibly empty, set of statements of the form:

- \( \text{Func}(RN) \) or \( RN \in F \), where \( F \subseteq R \) is a set of functional roles, or
- \( \text{Trans}(SN) \) or \( SN \in R_+ \), where \( R_+ \subseteq R \) is a set of transitive roles, or
- \( R_1 \sqsubseteq R_2 \), called role inclusions; \( R_1 \equiv R_2 \), called role equivalences, are an abbreviation for \( R_1 \sqsubseteq R_2 \) and \( R_2 \sqsubseteq R_1 \).

Statements in \( \mathcal{R} \) are called role axioms. The kinds of role axioms that can appear in \( \mathcal{R} \) depend on the expressiveness of \( \mathcal{L} \).

An interpretation \( \mathcal{I} \) satisfies a functional role axiom \( RN \in F \) (written as \( \mathcal{I} \models RN \in F \)), if, for all \( x \in \Delta^\mathcal{I} \), \#\{\( y \in \Delta^\mathcal{I} \mid \langle x, y \rangle \in RN^\mathcal{I} \} \leq 1 \) (\# denotes cardinality). An interpretation \( \mathcal{I} \) satisfies a transitive role axiom \( SN \in R_+ \) (written as \( \mathcal{I} \models SN \in R_+ \)), if, for all \( x, y, z \in \Delta^\mathcal{I} \), \{\( \langle x, y \rangle, \langle y, z \rangle \}\} \subseteq SN^\mathcal{I} \rightarrow \langle x, z \rangle \in SN^\mathcal{I} \).

An interpretation \( \mathcal{I} \) satisfies a role inclusion \( R_1 \sqsubseteq R_2 \) (written as \( \mathcal{I} \models R_1 \sqsubseteq R_2 \)), if \( R_1^\mathcal{I} \subseteq R_2^\mathcal{I} \), and it satisfies a role equivalence \( R_1 \equiv R_2 \) (written as \( \mathcal{I} \models R_1 \equiv R_2 \)), if \( R_1^\mathcal{I} = R_2^\mathcal{I} \). An interpretation \( \mathcal{I} \) satisfies an RBox \( \mathcal{R} \) (written as \( \mathcal{I} \models \mathcal{R} \)), iff it satisfies all the role axioms in \( \mathcal{R} \). \( \Box \)
2.1. FOUNDATIONS

Many DLs, e.g., $\mathcal{ALC}$, do not provide any role axioms at all. The RBox is, therefore, usually disregarded as a part of a DL knowledge base.\footnote{Sometimes role axioms are regarded as parts of the TBox, which is not quite proper, since when we internalise a TBox, we do not internalise any role axioms.} For $\mathcal{S}$-family DLs, however, the RBox is a very important component in a DL knowledge base, since $\mathcal{S}$ itself provides transitive role axioms.

Like constructors, each kind of role axiom is associated with a special letter, i.e., $f$ for functional role axioms, $\mathcal{R}+$ for transitive role axioms and $\mathcal{H}$ for role inclusion axioms. We can extend the $\mathcal{S}$ DL, which is also called $\mathcal{ALC}_{R+}$, with role inclusion axioms to have the $\mathcal{SH}$ DL, which can be further extended to $\mathcal{SHf}$ by also providing functional role axioms.

Interestingly, there are some restrictions and interactions among the three kinds of role axioms. Most importantly, the set of functional roles ($\mathcal{F}$) and the set of transitive roles ($\mathcal{R}_+$) should be disjoint. The reason for this restriction is that it is still an open problem whether DLs with transitive functional roles are decidable or not.\footnote{Allowing transitive roles in number restriction leads to undecidability [66], and a functional role $\mathcal{RN} \in \mathcal{F}$ can be encoded by a concept inclusion using a number restriction, i.e., $\top \sqsubseteq 1\mathcal{RN} \cdot \top$.} With the presence of role inclusions, e.g., for $\mathcal{RN}, \mathcal{SN} \in \mathcal{R}$, if $\mathcal{RN} \sqsubseteq \mathcal{SN}$ and $\mathcal{SN}$ is functional, then obviously $\mathcal{RN}$ must be functional as well; $\mathcal{RN}$ can not, therefore, be a transitive role.

ABox

The third component of a DL knowledge base is the A(assertional)Box, or world description, where one describes a specific state of affairs, w.r.t. some individuals, of an application domain in terms of concepts and roles. An ABox is formally defined as follows.

**Definition 2.4. (ABox)** Let $\mathcal{L}$ be a Description Logic, $a, b \in \mathcal{I}$ individual names, $C \in \text{Cdsc}(\mathcal{L})$ an $\mathcal{L}$-concept and $R \in \text{Rdsc}(\mathcal{L})$ an $\mathcal{L}$-role. An ABox $\mathcal{A}$ is a finite, possibly empty, set of statements of the form $a : C$, called concept assertions, or $\langle a, b \rangle : R$, called role assertions. Statements in $\mathcal{A}$ are called assertions (or individual axioms).

An interpretation $\mathcal{I}$ satisfies a concept assertion $a : C$ (written as $\mathcal{I} \models a : C$) if $a^\mathcal{I} \in C^\mathcal{I}$, and it satisfies a role assertion $\langle a, b \rangle : R$ (written as $\mathcal{I} \models \langle a, b \rangle : R$) if $\langle a^\mathcal{I}, b^\mathcal{I} \rangle \in R^\mathcal{I}$. An interpretation $\mathcal{I}$ satisfies an ABox $\mathcal{A}$ (written as $\mathcal{I} \models \mathcal{A}$), iff it satisfies all the assertions in $\mathcal{A}$.
Figure 2.1: A world description (ABox)

Figure 2.1 shows an example of an ABox. The interpretation $I$ given in Example 2.2 on page 29 is a model of the above ABox.

There are two common assumptions about ABoxes. The first one is the so-called unique name assumption (UNA), i.e., if $a, b \in I$ are distinct individual names, then $a^I \neq b^I$. Without the UNA, we may need an explicit assertion to assure that $a, b$ are different individuals. Note that we do not need a new form of assertion though, as we can use, e.g., $a : (=1 indentity), b : (=2 indentity)$, where $indentity$ is a new role name that is not used in the knowledge base, and $(=n indentity)$ is a concept, each instance of which relates to exactly $n$ different individuals via the role $indentity$. Even though it is not necessary, a special form of assertion may, however, be useful and convenient.\footnote{For example, the OWL language to be introduced in Chapter 3 provides such a form of assertion.}

The second assumption is called the open world assumption, i.e., one cannot assume that the knowledge in the knowledge base is complete. This is inherent in the fact that an ABox (or a knowledge base in general) may have many models, only some aspects of which are constrained by the assertions. For example, the role assertion $\langle Ian, Jeff \rangle : hasPhDStudent$ expresses that Ian has a PhD student Jeff in all models; in some of these models, Jeff is the only one PhD student of Ian, while in others, Ian may have some other PhD students.

Interestingly, individual names can be used not only in an ABox, but also in the description language, where they are used to form concepts called nominals, i.e., concepts interpreted as sets consisting of exactly one individual. Nominal is also used to refer to the singleton set constructor (indicated by $O$ in the name of a description language), written as $\{a\}$, where $a \in I$ is an individual name. As one would expect, such a nominal is interpreted as $\{a\}^I = \{a^I\}$.

Knowledge Base

Having discussed all its three components, let us now formally define a DL knowledge base as follows.
2.1. FOUNDATIONS

Definition 2.5. (Knowledge Base) A knowledge base $\Sigma$ is a triple $\langle T, R, A \rangle$, where $T$ is a TBox, $R$ is an RBox, and $A$ is an ABox.

An interpretation $I$ satisfies a knowledge base $\Sigma$, written as $I \models \Sigma$, iff it satisfies $T$, $R$ and $A$; $\Sigma$ is satisfiable (unsatisfiable), written as $\Sigma \not\models \bot$ ($\Sigma \models \bot$), iff there exists (does not exist) such an interpretation $I$ that satisfies $\Sigma$.

Given a terminological axiom, a role axiom, or an assertion $\varphi$, $\Sigma$ entails $\varphi$, written as $\Sigma \models \varphi$, iff for all models $I$ of $\Sigma$ we have $I \models \varphi$. A knowledge base $\Sigma$ entails a knowledge base $\Sigma'$, written as $\Sigma \models \Sigma'$, iff for all models $I$ of $\Sigma$ we have $I \models \Sigma'$. Two knowledge bases $\Sigma$ and $\Sigma'$ are equivalent, written as $\Sigma \equiv \Sigma'$, iff $\Sigma \models \Sigma'$ and $\Sigma' \models \Sigma$.

Knowledge base entailment and equivalence are very powerful tools: the former one will be used to describe DL reasoning services in the next section, and the latter one shows how to modify a knowledge base without affecting its logical meaning.

2.1.3 Reasoning Services

A DL system not only stores axioms and assertions, but also offers services that reason about them. Typically, reasoning with a DL knowledge base is a process of discovering implicit knowledge entailed by the knowledge base. Reasoning services can be roughly categorised as basic services, which involve the checking of the truth value for a statement, and complex services, which are built upon basic ones. Let $\mathcal{L}$ be a Description Logic, $\Sigma$ a knowledge base, $C, D \in \text{Cdesc} (\mathcal{L})$ $\mathcal{L}$-concepts and $a \in I$ an individual name, principle basic reasoning services include:

Knowledge Base Satisfiability is the problem of checking if $\Sigma \not\models \bot$ holds, i.e., whether there exists a model $I$ of $\Sigma$.

Concept Satisfiability is the problem of checking if $\Sigma \not\models C \equiv \bot$ holds, i.e., whether there exists a model $I$ of $\Sigma$ in which $C^I \neq \emptyset$.

Subsumption is the problem of verifying if $\Sigma \models C \subseteq D$ holds, i.e., whether in every model $I$ of $\Sigma$ we have $C^I \subseteq D^I$.

Instance Checking is the problem of verifying if $\Sigma \models a : C$ holds, i.e., whether in every model $I$ of $\Sigma$ we have $a^I \in C^I$. 
The above principle basic reasoning services are not independent of each other. If \( \mathcal{L} \) is closed under negation, i.e., the complement of any \( \mathcal{L} \)-concept is also an \( \mathcal{L} \)-concept, then all the basic reasoning services are reducible to knowledge base satisfiability [128].

In the thesis, we mainly consider concept satisfiability and subsumption services. Given that the DLs to be investigated are closed under negation, these two services are reducible to each other (\( C \) is subsumed by \( D \), if and only if \( C \cap \neg D \) is unsatisfiable; \( C \) is unsatisfiable, if and only if \( C \) is subsumed by \( \bot \)), and the undecidability of these services would imply that none of the above basic reasoning services are computable.

These two reasoning services can be transformed into reasoning with an empty TBox and ABox [2], which is usually accomplished in two steps:

1. to eliminate the ABox;
2. to eliminate the TBox.

The first step is based on an important characteristic of DLs, i.e., TBox reasoning is usually not influenced by ABox reasoning. More precisely, we have the following theorem (with a slightly different presentation).

**Theorem 2.6.** (Nebel [102], Schaerf [128]) If \( \mathcal{L} \) is a DL that does not provide the nominal constructor, and \( \Sigma = \langle T, R, A \rangle \) is a satisfiable knowledge base, then for every pair of \( \mathcal{L} \)-concepts \( C, D \in \text{Cds}(\mathcal{L}) \), we have

\[
\langle T, R, A \rangle \models C \sqsubseteq D \iff \langle T, R, \{\} \rangle \models C \sqsubseteq D.
\]

For DLs providing the nominal constructor, the above theorem does not apply. We can, however, encode the assertions in the ABox into concept inclusions in the TBox [128, 139].

**Theorem 2.7.** (Tobies [139, Lemma 5.3]) If \( \mathcal{L} \) is a DL that provides the nominal constructor, knowledge base satisfiability can be polynomially reduced to satisfiability of TBoxes and RBoxes.

For the second step, as mentioned in Section 2.1.2, we can either unfold or internalise a TBox. The following theorem addresses the internalisation of a TBox.

---

6 This is easy to understand, if one recalls that, for set \( A, B \), we have \( A \subseteq B \) if and only if \( A \setminus B = \emptyset \).

7 Nominal is the singleton set constructor; cf. page 33.
Theorem 2.8. (Horrocks et al. [65]) Let $\mathcal{L}$ be a DL providing general negation, transitive role and role inclusion axioms, $T$ a TBox and $C, D \in \text{Cdsc}(\mathcal{L}) \mathcal{L}$-concepts, $C_T := \prod_{C_i \sqcup D_i \in T} \neg C_i \sqcup D_i$ and $U$ be a transitive role with $R \subseteq U$ for each role $R$ that occurs in $T$, $C$, or $D$.

$C$ is satisfiable w.r.t. $T$ iff $C \sqcap C_T \sqcap \forall U. C_T$ is satisfiable. $D$ subsumes $C$ w.r.t. $T$ (written as $C \sqsubseteq_T D$) iff $C \sqcap \neg D \sqcap C_T \sqcap \forall U. C_T$ is unsatisfiable.

The most common complex services include classification and retrieval. Classification is a problem of putting a new concept in the proper place in a taxonomic hierarchy of concept names; this can be done by subsumption checking between each named concept in the hierarchy and the new concept. The location of the new concept, let us call it $C$, in the hierarchy will be between the most specific named concepts that subsume $C$ and the most general named concepts that $C$ subsumes. Retrieval (or query answering) is a problem of determining the set of individuals that instantiate a given concept; this can be done by instance checking between each named individual and the given concept.

After introducing some basic and complex reasoning services, we now briefly describe the complexity of reasoning services, which has been one of the main issues in DL research. Initially, the emphasis was on the reasoning services of tractable DLs, with an upper bound of polynomial complexity [19]. Unfortunately, only very primitive DLs are tractable in this sense, e.g., the satisfiability of $\mathcal{ALC}$-concepts w.r.t. general TBox is already EXPTIME-complete (Schild [129, 130], Lutz [88]). Interestingly, although the theoretical complexity results are discouraging, empirical analysis have shown that worse-case intractability rarely occur in practice [103, 54, 135, 61]; even some simple optimisation techniques could lead to significant improvement in the empirical performance of a DL system (Baader et al. [4]). More recently the Fact (Horrocks [59]), RACER (Haarslev and Möller [51]), and DLP (Patel-Schneider [115]) systems have demonstrated that, even with expressive DLs, highly optimised implementations can provide acceptable performance in realistic applications. In other words, thoughtful optimisation techniques (Horrocks [58], Horrocks and Patel-Schneider [62], Horrocks and Sattler [76], Horrocks [61]) have moved the boundaries of ‘tractability’ to somewhere very close to EXPTIME-hard, or worse (Donini [38]).
2.2 Reasoning Algorithms

In this section, we will briefly explain how to provide DL reasoning services for concept satisfiability and subsumption problems.

There are (at least) two options here: to reuse existing first order logic algorithms, or to design new DL algorithms. For the first option, since most DLs\(^8\) are within the two variable fragment of first-order predicate logic, we can reduce the two DL problems to known inference problems of first-order logics, e.g., \(L^2\) and \(C^2\), to provide DL reasoning services. The complexity of decision procedures obtained this way, however, is usually higher than necessary (Baader and Nutt [7]). This approach, therefore, is often used for obtaining upper bound complexity results of the two DL reasoning problems, rather than providing reasoning services in DL systems.\(^9\)

For the second option, in the early days people used so-called structural subsumption algorithms, i.e., algorithms that compare the syntactic structure of concepts, to solve the concept subsumption problem of some rather primitive DLs. These algorithms, however, are not complete for DLs with (full) negation and disjunction, e.g., the \(S\)-family DLs. For such languages, the so-called **tableaux algorithms** (first by Schmidt-Schauß and Smolka [131]) have turned out to be very useful to solve the concept satisfiability and subsumption problems [131, 56, 55, 3, 52].

Tableaux algorithms test the satisfiability of a concept \(D\) by trying to construct a model (witness) for \(D\). A model of \(D\) is usually represented by a **completion tree** \(T\): nodes in \(T\) represent individuals in the model; each node \(x\) is labeled with \(L(x)\), a set of sub-concepts\(^{10}\) of \(D\); each edge \((x, y)\) is labeled with \(L((x, y))\), a set of role names in \(D\). A tableaux algorithm starts from the root node \(x_0\) labeled with \(L(x_0) = \{D\}\). \(T\) is then expanded by repeatedly applying the completion rules (see Figure 2.2) that either extend the node labels or add new leaf nodes; such completion rules correspond to the logical constructors as well as role axioms provided by a particular description language.

The algorithm terminates either when \(T\) is **complete** (no further completion rules can be applied or when an obvious contradiction, or **clash** (see Figure 2.3), has been revealed). Non-determinism is dealt with by searching different possible expansions: the concept is unsatisfiable if every expansion leads to a contradiction and is satisfiable.

---

\(^8\)With a few exceptions like DLs introduced by Calvanese et al. [25], Lutz [86].

\(^9\)It is worth pointing out that Hustadt et al. [77] have presented an Exptime resolution-based decision procedure to decide the \(SHIQ^-\)knowledge base satisfiability problem.

\(^{10}\)A is a sub-concept of a concept description \(B\), if \(A\) appears in \(B\); e.g., \(C, D\) and \(\exists R.D\) are all sub-concepts of \(C \sqcap \exists R.D\).
2.2. REASONING ALGORITHMS

\(\square\)-rule: if \(1. C_1 \sqcap C_2 \in \mathcal{L}(x), \) and
\(2. \{C_1, C_2\} \not\subseteq \mathcal{L}(x), \)
then \(\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{C_1, C_2\}\)

\(\square\)-rule: if \(1. C_1 \sqcup C_2 \in \mathcal{L}(x), \) and
\(2. \{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset, \)
then \(\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{C\} \) for some \(C \in \{C_1, C_2\}\)

\(\exists\)-rule: if \(1. \exists R.C \in \mathcal{L}(x), \) and
\(2. \) there is no \(y\) s.t. \(R \in \mathcal{L}(\langle x, y \rangle)\) and \(C \in \mathcal{L}(y)\)
then create a new node \(y\) with \(\mathcal{L}(\langle x, y \rangle) = \{R\}\) and \(\mathcal{L}(y) = \{C\}\)

\(\forall\)-rule: if \(1. \forall R.C \in \mathcal{L}(x), \) and
\(2. \) there is some \(y\) s.t. \(R \in \mathcal{L}(\langle x, y \rangle)\) and \(C \notin \mathcal{L}(y), \)
then \(\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{C\}\)

Figure 2.2: The completion rules for \(\mathcal{ALC}\)

A node \(x\) of a completion tree \(T\) contains a clash if (at least) one of the following conditions holds:
(1) \(\bot \in \mathcal{L}(x)\); (2) for some \(A \in C, \{A, \neg A\} \subseteq \mathcal{L}(x)\).

Figure 2.3: The clash conditions of \(\mathcal{ALC}\)

if any possible expansion leads to the discovery of a complete clash-free tree (Horrocks [58]).

For expressive DLs, e.g., the \(S\)-family DLs, a technique called blocking [24] is used to ensure termination of the tableaux algorithms. To explain blocking, we shall introduce the terms of predecessor and ancestor: the predecessor of a node \(y\) is the node \(x\) if \(\langle x, y \rangle\) is an edge in \(T\); ancestor is the transitive closure of predecessor. A node \(y\) is said to be blocked if there is some ancestor node \(x\) s.t. \(\mathcal{L}(y) \subseteq \mathcal{L}(x)\); in this case, \(x\) is called the blocking node. The use of blocking (in the case of \(S\)) is that once a node is blocked, the completion rules no longer apply on it. Figure 2.4 shows the completion rules for \(S\), i.e., \(\mathcal{ALC}_{R^+}\), which also provides transitive role axioms and thus requires blocking. Note that different DLs might use different kinds of blocking techniques.

In Chapter 6, we will describes the tableaux algorithms for a family of new Description Logics that are designed to provide reasoning services for OWL, DAML+OIL and OWL-E, which is a decidable extension of OWL that provides customised datatypes and predicates. In these tableaux algorithms, models can not only be represented by trees, but also forests; the blocking techniques used in them are a bit more complex.
than the one we introduced in this section.

### 2.3 Description Logics and Datatype Predicates

A major limitation of many DLs, such as the $S$ DL, is that datatypes (such as integers and strings), data values and datatype predicates cannot be adequately represented. Without them, we have to use concepts to represent datatype constraints. For example, if we want to describe Elephants younger than 10 years old, we have to use concepts, such as YoungerThan10:

$$\text{Elephant} \sqcap \text{YoungerThan10}.$$ 

The approach, however, is not satisfactory at all, since it does not really describe what we meant. We can make the following assertion for an instance Ganesh of the above concept:

$$\text{Ganesh} : \text{23YearsOld},$$

and it does not imply any contradiction.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>□-rule</td>
<td>if $1.C_1 \sqcap C_2 \in \mathcal{L}(x)$ and $x$ is not blocked, and $2.{C_1, C_2} \not\in \mathcal{L}(x)$, then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup {C_1, C_2}$</td>
</tr>
<tr>
<td>⊓-rule</td>
<td>if $1.C_1 \sqcap C_2 \in \mathcal{L}(x)$ and $x$ is not blocked, and $2.{C_1, C_2} \cap \mathcal{L}(x) = \emptyset$, then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup {C}$ for some $C \in {C_1, C_2}$</td>
</tr>
<tr>
<td>⊔-rule</td>
<td>if $1.\exists R.C \in \mathcal{L}(x)$ and $x$ is not blocked, and $2.$ there is no $y$ s.t. $R \in \mathcal{L}(\langle x, y \rangle)$ and $C \in \mathcal{L}(y)$, then create a new node $y$ with $\mathcal{L}(\langle x, y \rangle) = {R}$ and $\mathcal{L}(y) = {C}$</td>
</tr>
<tr>
<td>∀-rule</td>
<td>if $1.\forall R.C \in \mathcal{L}(x)$ and $x$ is not blocked, and $2.$ there is some $y$ s.t. $R \in \mathcal{L}(\langle x, y \rangle)$ and $C \not\in \mathcal{L}(y)$, then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup {C}$</td>
</tr>
<tr>
<td>∀+-rule</td>
<td>if $1.\forall R.C \in \mathcal{L}(x)$ and $x$ is not blocked, and $2.R \in R_+$, and $3.$ there is some $y$ s.t. $R \in \mathcal{L}(\langle x, y \rangle)$ and $\forall R.C \not\in \mathcal{L}(y)$, then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup {\forall R.C}$</td>
</tr>
</tbody>
</table>

Figure 2.4: The completion rules for $S$
2.3. DESCRIPTION LOGICS AND DATATYPE PREDICATES

2.3.1 The Concrete Domain Approach

It was Baader and Hanschke [8] who first presented a rigorous treatment of datatype predicates within so called ‘concrete domains’.

Intuitively, a concrete domain provides a domain and a set of \( n \)-ary predicates (such as ‘\(<\)’) with obvious (fixed) extensions over the domain. A concrete domain is formally defined as followed:

**Definition 2.9. (Concrete Domain) (Lutz [88])** A concrete domain \( \mathcal{D} \) consists of a pair \((\Delta_D, \Phi_D)\), where \( \Delta_D \) is the domain of \( \mathcal{D} \) and \( \Phi_D \) is a set of predicate names. Each predicate name \( p \) is associated with an arity \( n \), and an \( n \)-ary predicate \( p^D \subseteq \Delta^n_D \). Let \( V \) be a set of variables. A predicate conjunction of the form

\[
    c = \bigwedge_{j=1}^{k} p_j(v_{1}^{(j)}, \ldots, v_{n_j}^{(j)}),
\]

where \( p_j \) is an \( n_j \)-ary predicate and the \( v_{i}^{(j)} \) are variables from \( V \), is called satisfiable iff there exists a function \( \delta \) mapping the variables in \( c \) to data values in \( \Delta_D \) s.t. \( (\delta(v_{1}^{(j)}), \ldots, \delta(v_{n_j}^{(j)}) \in p_j^D \) for \( 1 \leq j \leq k \). Such a function \( \delta \) is called a solution for \( c \).

A concrete domain \( \mathcal{D} \) is called admissible iff

1. \( \Phi_D \) contains a name \( \top_D \) for \( \Delta_D \) and if \( p \in \Phi_D \), then \( \overline{p} \in \Phi_D \), and

2. the satisfiability problem for finite conjunctions of predicates is decidable.

By \( \overline{p} \), we denote the name for the negation of the predicate \( p \); i.e., if the arity of \( p \) is \( n \), then \( \overline{p}^D = \Delta^n_D \setminus p^D \).

To illustrate concrete domains, we present some numerical concrete domains as follows. As for non-numerical concrete domains, readers are referred to, e.g., the temporal and spacial concrete domains discussed in Lutz [88, 89].

**Example 2.3 The Concrete Domain** \( \mathcal{N} = (\Delta_N, \Phi_N) \), where \( \Delta_N \) is the set of non-negative integers and \( \Phi_N := \{ \geq, \geq_n \} \).\(^{11}\) \( \mathcal{N} \) is not admissible since \( \Phi_N \) does not satisfy condition 1 of admissibility in Definition 2.9; in order to make it admissible, we would have to extend \( \Phi_N \) to \( \{ \top_N, \bot_N \} \cup \{ <, \geq, <_{n}, \geq_n \} \). \( \mathcal{N} \) is admissible as there exists a polynomial time algorithm to decide the satisfiability problem of predicate conjunctions over \( \mathcal{N} \) (Lutz [88, Sec. 2.4.1]).

\(^{11}\)Note that \( \geq \) is a binary predicate and \( \geq_n \) is a unary predicate.
It can be argued that the concrete domain approach is not user-friendly in that it disallows negated predicates used in predicate conjunctions (2.1). For example, it disallows users to use ‘not less than’ (\(<\)) and forces them to use ‘greater than or equal to’ (\(\geq\)).

Example 2.4 The Concrete Domain

\[ Q = (\Delta_Q, \Phi_Q), \]
\[ \text{where } \Delta_Q \text{ is the set of rational numbers and } \Phi_Q \text{ is } \{\top_Q, \bot_Q\} \cup \{<, \geq, =, \neq, \leq, \succ, \preceq, \exists, \forall, \text{int}, \text{int}^{-1}\}. \]

Note that + is a ternary predicate
\[ +^Q = \{ (t_1, t_2, t_3) \in \mathbb{Q}^3 \mid t_1 + t_2 = t_3 \}, \]
int is a unary predicate
\[ \text{int}^Q = \mathbb{Z} \text{ (where } \mathbb{Z} \text{ denotes the integers) } \]
and \(\exists, \forall, \text{int}^{-1}\) are the negations of + and int, respectively.

It has been proved that \(Q\) is admissible (Lutz [90, p.29-30]). If we extend \(Q\), however, with the unary ternary predicate * where
\[ *^Q = \{ (t_1, t_2, t_3) \in \mathbb{Q}^3 \mid t_1 \cdot t_2 = t_3 \}, \]
and its negation \(\exists, Q + '•'\) becomes undecidable because it can easily be proved by using a reduction of Hilbert’s 10-th problem (Lutz [89, Sec. 4]).

Again, it can be argued that the concrete domain approach is not user-friendly in that it does not support predicates with unfixed arities, such as the predicate
\[ +'^Q = \{ (t_1, \ldots, t_n) \in \mathbb{Q}^n \mid t_1 + \ldots + t_{n-1} = t_n \}, \]
which can easily represent the constraint \(x_1 + x_2 + x_3 = x_4\) as \(+'(x_1, x_2, x_3, x_4)\), while for the + predicate there is no way to represent the above constraint without the help of concept languages.

In the concrete domain approach, there is a very useful result about the combination of admissible concrete domains. That is, if \(D_1\) and \(D_2\) are admissible concrete domains with \(\Delta_{D_1}\) being disjoint with \(\Delta_{D_2}\), then a new concrete domain \(D_1 \oplus D_2\) is also admissible [8]. This indicates that it will not be a restriction that we integrate only one concrete domain into a description language in the next section.

The \(ALC(D)\) DL

Baader and Hanschke [8] showed that we can integrate an arbitrary admissible concrete domain \(D\) into the ALC DL and give the \(ALC(D)\) DL, with the abstract domain (\(\Delta^I\)) being disjoint with the domain of \(D\) (\(\Delta_D\)). The combination of a concrete domain into
DLs is achieved by adding datatype-related constructors and features, or functional roles, in the description languages, and allowing the use of predicates defined in the concrete domain with these constructors and features. For example, Elephants younger than 10 years old can be described using the concept

\[ \text{Elephant} \sqcap \exists \text{age. } < 10, \]

where \text{age} is a feature, \(< 10\) is a unary integer predicate, and \(\exists \text{age. } < 10\) is an application of the datatype-related constructor called a predicate exists restriction.

The \(\mathcal{ALC}(\mathcal{D})\)-roles and feature chains are formally defined as follows.

**Definition 2.10.** (\(\mathcal{ALC}(\mathcal{D})\)-roles) Let \(RN \in \mathcal{R}\) be a role name, \(f_1, \ldots, f_n \in \mathcal{F}\) functional role names, \(\mathcal{D}\) a concrete domain and \(R \in \mathcal{R}_{\text{dsc}}(\mathcal{ALC}(\mathcal{D}))\) an \(\mathcal{ALC}(\mathcal{D})\)-role, valid \(\mathcal{ALC}(\mathcal{D})\)-roles are defined by the abstract syntax:

\[ R ::= RN \mid f_1 \circ \cdots \circ f_n, \]

where \(R = f_1 \circ \cdots \circ f_n\) is called a chain of functional roles, or a feature chain. The interpretation function is extended to give semantics to feature chains as follows:

\[ (f_1 \circ \cdots \circ f_n)^I = \{(a_0, a_n) \in \Delta^I \times (\Delta^I \cup \Delta_D) \mid \exists a_1, \ldots, a_{n-1} \in \Delta^I. \]

\[ f_1^I(a_0) = a_1 \land \cdots \land f_n^I(a_{n-1}) = a_n \}. \]

\[ \diamond \]

Obviously, the above definition indicates that \(\mathcal{ALC}(\mathcal{D})\) provides functional role axioms. Note that in \(\mathcal{ALC}(\mathcal{D})\), a feature chain connects an individual to either an individual or a data value.\(^{12}\)

Now we formally define the \(\mathcal{ALC}(\mathcal{D})\)-concepts as follows.

**Definition 2.11.** (\(\mathcal{ALC}(\mathcal{D})\)-concepts) Let \(CN \in \mathcal{C}\) be an atomic concept name, \(R \in \mathcal{R}_{\text{dsc}}(\mathcal{ALC}(\mathcal{D}))\) a role name, \(u_1, \ldots, u_n \in \mathcal{R}_{\text{dsc}}(\mathcal{ALC}(\mathcal{D}))\) \(\mathcal{ALC}(\mathcal{D})\) feature chains, \(C, D \in \mathcal{C}_{\text{dsc}}(\mathcal{ALC}(\mathcal{D}))\) \(\mathcal{ALC}(\mathcal{D})\)-concepts, \(\mathcal{D}\) an admissible concrete domain and \(p \in \Phi_D\) a \(n\)-ary predicate name, valid \(\mathcal{ALC}(\mathcal{D})\)-concepts are defined by the abstract syntax:

\[ C ::= \top \mid \bot \mid CN \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists R.C \mid \forall R.C \mid \exists u_1, \ldots, u_n, p, \]

\[ ^{12}\text{Lutz [88] gives a slightly different definition on } \mathcal{ALC}(\mathcal{D}) \text{ feature chains, where feature chains are disjointly divided into abstract feature chains and concrete feature chains, which allows a clearer algorithmic treatment and clearer proofs.} \]
where $\exists u_1, \ldots, u_n.p$ is called *predicate exists restriction*.

The interpretation function can be extended to give semantics to predicate exists restriction as follows.

$$(\exists u_1, \ldots, u_n.p)^I = \{ x \in \Delta^I \mid \exists t_1, \ldots, t_n \in \Delta_D.\ u_I^1(x) = t_1 \land \ldots \land u_I^n(x) = t_n \land \langle t_1, \ldots, t_n \rangle \in p^D \}. $$

The semantics of the rest of the $\mathcal{ALC}(D)$ constructors can be found in Table 2.1.

---

**Example 2.5 An $\mathcal{ALC}(D)$-Concept** The concept Elephant $\sqcap \exists$age. $<$10 that we gave earlier is an $\mathcal{ALC}(D)$-concept. Here we show why it is enough to express ‘Elephant younger than 10 years old’.

The predicate exists restriction requires, informally speaking, each instance of the concept must have an *age*, and the *age* should satisfy $<_{10}$. Additionally, *age* is a functional role (feature). Therefore, there is only one *age* for each instance of the concept, and this *age* must be less than 10. This means that if there is an individual Ganesh that is an instance of the above concept, the following assertion

$$\text{Ganesh} : \exists \text{age}. =_{23}$$

will imply a contradiction, since $\langle =_{23} \rangle^D \cap \langle <_{10} \rangle^D = \emptyset$.

---

**Example 2.6 Feature Chains**

With feature chains one can, for example, express the ‘young wife’ concept, i.e., all women whose husbands are even older than their fathers, as follows: $\text{Woman} \sqcap \exists \text{hasFather} \circ \text{age}, \text{hasHusband} \circ \text{age}. <$.

Although $\mathcal{ALC}(D)$ (with a admissible $D$, similarly hereinafter) has been proved decidable (Baader and Hanschke [8]), Lutz [87] proved that reasoning with $\mathcal{ALC}(D)$ and general TBoxes is usually undecidable, even with some simple admissible concrete domains. The undecidability comes from the fact that feature chains are so expressive that one can reduce the general, undecidable PCP (Post Correspondence Problems) to $\mathcal{ALC}(A)$-concept satisfiability, where $A$ is any arithmetic concrete domain (Lutz [88, Sec. 6.1]). The above undecidability result implies that for any DL $\mathcal{L}$ that is expressive enough to internalise a TBox, e.g., all $S$-family languages that provide role inclusion axioms, $\mathcal{L}(D)$ is usually undecidable.
2.3.2 The Type System Approach

In order to integrate expressive DLs with datatypes (unary predicates) defined by external type systems (e.g., the XML Schema type system), Horrocks and Sattler [75] proposed the so called ‘type system approach’, which can be seen as a restricted form of the concrete domain approach, where a universal concrete domain $D$ is treated as a concrete domain\(^{13}\) with datatypes being unary predicates over $D$. The satisfiability problem (2.2) considered in a universal concrete domain is, therefore, much easier than that of the concrete domain approach (2.1).

**Definition 2.12. (Universal Concrete Domain)** A universal concrete domain $D$ consists of a pair $(\Delta_D, \Phi_D)$, where $\Phi_D$ is a set of datatype (unary datatype predicate) names and $\Delta_D$ is the domain of all these datatypes. Each datatype name $d \in \Phi_D$ is associated with a datatype $d^D \subseteq \Delta_D$ defined in the associated type system.

Let $V$ be a set of variables, a datatype conjunction of the form

$$c_1 = \bigwedge_{j=1}^{k} d_j(v^{(j)}), \quad (2.2)$$

where $d_j$ is a (possibly negated) datatype from $\Phi_D$ and the $v^{(j)}$ are variables from $V$, is called *satisfiable* iff there exists a function $\delta$ mapping the variables in $c_1$ to data values in $\Delta_D$ s.t. $\delta(v^{(j)}) \in d_j^D$ for $1 \leq j \leq k$. Such a function $\delta$ is called a *solution* for $c_1$. By $\overline{d}$, we denote the name for the negation of the datatype $d$, and $\overline{d}^D = \Delta_D \setminus d^D$.

A set of datatypes $\Phi_D$ is called *conforming* iff the satisfiability problem for finite (possibly negated) datatype conjunctions over $\Phi_D$ is decidable.

In the type system approach, datatype conjunctions (2.2) allow the use of negated datatypes. On the one hand, it is more user-friendly than the concrete domain approach; on the other hand, however, it can be counter-intuitive.

**Example 2.7 Full Negation Can be Counter-Intuitive** Let us consider the following universal concrete domain $D = (\Delta_D, \Phi_D)$, where $\Delta_D = >_{15}^D \cup \text{string}^D$ and $\Phi_D = \{ >_{15}, \text{string} \}$. ‘Not greater than 15’ can then be represented as $\overline{>_{15}}$, which is interpreted as $\overline{>_{15}}^D = \Delta_D \setminus >_{15}^D = \text{string}^D$. In other words, any string is not greater than 15, which can be counter-intuitive.

\(^{13}\)In [75]’s notation, $D$ refers to $\Phi_D$ the set of datatypes. We call it $\Phi_D$ as it corresponds to $\Phi_D$ in Definition 2.9.
2.3. DESCRIPTION LOGICS AND DATATYPE PREDICATES

In the type system approach, datatypes are considered to be sufficiently structured by type systems, which typically define a set of primitive datatypes and provide derivation mechanisms to define new datatypes from existing ones. [75] does not further explain, however, how the derivation mechanism of a type system affects $\Phi_D$, viz. how one defines the extensions of datatype names in $\Phi_D$ based on the derivation mechanism of a type system.

The $\text{SHOQ}(D)$ DL

Horrocks and Sattler [75] integrated an arbitrary conforming universal concrete domain ($D$) as well as nominals ($O$) into the $\text{SHQ}$ DL to give the $\text{SHOQ}(D)$ DL, where the domain of $D$, i.e., the datatype domain $\Delta_D$, is disjoint with the object domain $\Delta_I$ of an interpretation $I$.

Let us define some notation first. Let $L$ be a Description Logic, $R_A$ and $R_D$ countable sets of abstract role names and concrete role names, such that $R_A$ is disjoint with $R_D$ and $R = R_A \cup R_D$, $\text{Rdsc}_A(L)$ and $\text{Rdsc}_D(L)$ a set of abstract role descriptions and a set of concrete role descriptions (in the following simply called abstract roles and concrete roles) of $L$, respectively.

An $\text{SHOQ}(D)$-role is either an abstract role name, or a datatype role name. Each abstract role $R \in \text{Rdsc}_A(\text{SHOQ}(D))$ is interpreted as a subset of $\Delta_I \times \Delta_I$, and each concrete role $T \in \text{Rdsc}_D(\text{SHOQ}(D))$ is interpreted as a subset of $\Delta_I \times \Delta_D$. The problematic feature chains are not allowed in $\text{SHOQ}(D)$, in order to make $\text{SHOQ}(D)$ decidable. We cannot, thus, use $\text{SHOQ}(D)$ to express the ‘young wife’ concept mentioned in Section 2.3.1 on page 42.

Now we define $\text{SHOQ}(D)$-concepts as follows.

**Definition 2.13. ($\text{SHOQ}(D)$-concepts)** Let $C \in C$ be a concept name, $R \in \text{Rdsc}_A(\text{SHOQ}(D))$ an abstract $\text{SHOQ}(D)$-role, $T \in \text{Rdsc}_D(\text{SHOQ}(D))$ a concrete $\text{SHOQ}(D)$-role, $C, D \in \text{Cdsc}(\text{SHOQ}(D))$ $\text{SHOQ}(D)$-concepts, $o \in I$ an individual, $D$ a universal concrete domain, $d \in \Phi_D$ a datatype, $n, m \in \mathbb{N}$ non-negative integers, valid $\text{SHOQ}(D)$-concepts are defined by the abstract syntax:

$$C := \top | \bot | C \land D | C \lor D | \{o\} | \exists R.C | \forall R.C | \geq nR.C | \leq nR.C | \exists T.d | \forall T.d$$

The semantics of $\text{SHOQ}(D)$-concepts is given in Table 2.2 ($\#$ denotes cardinality).
<p>|</p>
<table>
<thead>
<tr>
<th>Constructor Name</th>
<th>Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>⊤</td>
<td>Δ^I</td>
</tr>
<tr>
<td>bottom</td>
<td>⊥</td>
<td>∅</td>
</tr>
<tr>
<td>concept name</td>
<td>CN</td>
<td>CN^I ⊆ Δ^I</td>
</tr>
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<td>general negation</td>
<td>¬C</td>
<td>Δ^I \ C^I</td>
</tr>
<tr>
<td>conjunction</td>
<td>C \ D</td>
<td>C^I \ D^I</td>
</tr>
<tr>
<td>disjunction</td>
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<td>C^I ∪ D^I</td>
</tr>
<tr>
<td>nominals</td>
<td>{o}</td>
<td>{o}^I ⊆ Δ^I , # {o}^I = 1</td>
</tr>
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<td>∃R.C</td>
<td>{x ∈ Δ^I</td>
</tr>
<tr>
<td>value restriction</td>
<td>∀R.C</td>
<td>{x ∈ Δ^I</td>
</tr>
<tr>
<td>atleast restriction</td>
<td>≧mR.C</td>
<td>{x ∈ Δ^I</td>
</tr>
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<td>≦mR.C</td>
<td>{x ∈ Δ^I</td>
</tr>
<tr>
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<td>∃T.d</td>
<td>{x ∈ Δ^I</td>
</tr>
<tr>
<td>datatype value restriction</td>
<td>∀T.d</td>
<td>{x ∈ Δ^I</td>
</tr>
</tbody>
</table>

Table 2.2: Semantics of SHOQ(D)-concepts

**Example 2.8** An SHOQ(D)-Concept With SHOQ(D), we can construct more expressive concepts than ALC(D), e.g., ‘Items that are cheaper than 100 pounds and have at least two providers from Germany, France, or Italy’:

\[
\text{Item} \sqcap \geq 2\text{providerFrom.} (\{\text{Germany}\} \sqcup \{\text{France}\} \sqcup \{\text{Italy}\}) \sqcap \exists \text{priceInPounds.} <_{100},
\]

where \{\text{Germany}\},\{\text{France}\} and \{\text{Italy}\} are nominals, which are interpreted as singleton sets (see Table 2.2). Note that

\[
\geq 2\text{providerFrom.} (\{\text{Germany}\} \sqcup \{\text{France}\} \sqcup \{\text{Italy}\})
\]

is not a datatype constraint, but a qualified number restriction, which requires that each instance of the above concept should be related by the abstract role \text{providerFrom} to at least two out of the following three individuals: Germany, France, Italy.

\[
\exists \text{priceInPounds.} <_{100}
\]

is a datatype exists restriction, where \text{<}_{100} is a datatype defined in a type system. Note that \text{priceInPounds} is a role instead of a feature, so an instance of the above concept can be related to more than one value by \text{priceInPounds}; in other words, some items may have two prices. This may look a bit odd in some contexts. ◊

The above example indicates that the lack of functional concrete role axioms, or in general, the lack of datatype qualified number restrictions makes SHOQ(D) less useful in some situations. Extensions of the SHOQ(D) DL with datatype qualified number restrictions will be discussed in Chapter 6; such extensions are required in
order to provide DL reasoning support for the Web ontology languages DAML+OIL and OWL (cf. Chapter 3).

Formally, an $SHOQ(D)$-RBox is defined as follows.

**Definition 2.14. ($SHOQ(D)$ RBox)** Let $R_1, R_2 \in Rdsc_A(SHOQ(D))$, $T_1, T_2 \in Rdsc_D(SHOQ(D))$, $SN, RN \in R_A$, an $SHOQ(D)$ RBox $\mathcal{R}$ is a finite, possibly empty, set of role axioms:

- functional role axioms $Func(SN)$;
- transitive role axioms $Trans(RN)$;
- abstract role inclusion axioms $R_1 \sqsubseteq R_2$;
- concrete role inclusion axioms $T_1 \sqsubseteq T_2$.

For a set of role inclusion axioms $\mathcal{R}^{\text{inc}}$, $\mathcal{R}^{+}_{\text{inc}} := (\mathcal{R}^{\text{inc}}, \sqsubseteq)$ is called a role hierarchy, where $\sqsubseteq$ is the transitive-reflexive closure of $\sqsubseteq$ over $\mathcal{R}^{\text{inc}}$. Given $\sqsubseteq$, the set of roles $R^{\downarrow} = \{ S \in R | S \sqsubseteq R \}$ defines the sub-roles of a role $R$. A role $R$ is called a super-role of a role $S$ if $S \in R^{\downarrow}$. A role $R$ is called simple if, for each role $S$, $S \sqsubseteq R$ implies $Trans(S) \notin \mathcal{R}$.

There are two remarks on Definition 2.14. Firstly, only simple roles, viz. roles such that none of their sub-roles are transitive, can be used in number restrictions, since allowing transitive roles in number restrictions leads to undecidability [66]. Secondly, a concrete role $T$ can not be a transitive role; thus, it is always a simple role.

The $SHOQ(D)$-concept satisfiability and subsumption problems w.r.t. a TBox are proved decidable [75]. As nominals and datatypes are widely used in ontologies, and ontologies play an important role in the Semantic Web, $SHOQ(D)$ is expected to be very useful in the Semantic Web.

### 2.3.3 Limitations of Existing Approaches

There are several limitations of the existing approaches that we introduced in this section:

**Datatypes and Data Values** Strictly speaking, the existing approaches mainly focus on datatype predicates,\(^{14}\) but do not adequately represent datatypes and data values.

\(^{14}\)In the type system approach, the datatypes are actually treated as unary predicates.
For example, if one wants to represent ‘Balavan is 1 year old’, \(\langle \text{Balavan}, 1 \rangle : \text{age} \) is not enough, as ‘1’ is a symbol that can represent different data values of different datatypes (e.g., ‘1’ can be used to represent integer 1 and boolean value \text{true}). A more proper way to represent the above assertion is \(\langle \text{Balavan}, L2V(\text{integer})(\text{“1”}) \rangle : \text{age}, \) where \(L2V(\text{integer})\) is the lexical-to-value mapping for the \text{integer} datatype. Note that the type system approach treats datatypes as unary predicates, and it does not consider the lexical spaces and lexical-to-value mappings of datatypes (cf. Section 3.3.2).

**Predicates with Unfixed Arities** It is not user-friendly, for the concrete domain approach, to disallow the use of predicates with unfixed arities and to force users to represent the constraint \(x_1 + x_2 + x_3 = x_4\) with the help of concept languages as follows

\[
\exists x_1, x_2, x_12, + \cap \exists x_12, x_3, x_4, +,
\]

instead of using the more natural predicate ‘\(+\)’ (cf. Example 2.4 on page 41).

**Negation** It is not user-friendly to disallow the use of negated predicates in predicate conjunctions (2.1) in the concrete domain approach (cf. Example 2.3 on page 40). Although the type system approach allows the use of (full) negation in datatype conjunctions (2.2), it can be counter-intuitive (cf. Example 2.7 on page 44).

**Customised Datatype Predicates** Neither of the two approaches provide any formalism to construct customised predicates, such as ‘sum no greater than 15’ presented in Example 1.5 on page 22. This is because the concrete domain approach assumes that datatype constraints are sufficiently structured by \(\Phi_D\). Baader and Hanschke [8] claim that they do not want to define new classes of elements of \(\Delta_D\) (customised datatypes) or new relations between elements of \(\Delta_D\) (customised predicates) with the help of the concept language.
<table>
<thead>
<tr>
<th>Chapter Achievements</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Description Logics are characterised by the use of various constructors to build concepts and roles, an emphasis on the decidability of key reasoning problems and by the provision of (empirically) tractable reasoning services.</td>
</tr>
<tr>
<td>• The concrete domain approach provides a rigorous treatment about datatype predicates, while the type system approach provides the use of unary datatype predicates defined by external type systems. Both approaches have some limitations, e.g., they do not provide any formalism to construct customised datatype predicates.</td>
</tr>
</tbody>
</table>
Chapter 3

The Semantic Web

Chapter Aims

- To explain why DL-based ontologies are important in the Semantic Web.
- To introduce RDF(S), OWL and their datatype formalisms.
- To clarify the two issues that the thesis is going to tackle.

Chapter Plan

3.1 Annotations and Meaning (50)
3.2 Web Ontology Languages (56)
3.3 Web Datatype Formalisms (69)
3.4 Outlook for the Two Issues (78)

3.1 Annotations and Meaning

As described in Chapter 1, the vision of the Semantic Web is to make Web resources (not just HTML pages, but a wide range of Web accessible data and services) more understandable to machines. Machine-understandable annotations are, therefore, introduced to describe the content and functions of Web resources.

3.1.1 RDF

Resource Description Framework (RDF) [81] as a W3C recommendation provides a standard syntax to create, exchange and use annotations in the Semantic Web. It is
Figure 3.1: RDF statements

built upon earlier developments such as the Dublin Core (see Section 3.1.2) and the Platform for Internet Content Selectivity (PICS) [121] content rating initiative.

An RDF statement (or RDF triple) is of the form:

\[
\text{[subject property object .]}
\]  

(3.1)

RDF-annotated resources (i.e., subjects) are usually named by Uniform Resource Identifier references. Uniform Resource Identifiers (URIs) are short strings that identify Web resources [45]. Uniform Resource Locators (URLs) are a particular type of URIs, i.e., those have network locations. A URI reference (or URIref) is a URI, together with an optional fragment identifier at the end. For example, the URI reference http://www.example.org/Elephant#Ganesh consists of the URI http://www.example.org/Elephant and (separated by the # character) the fragment identifier Ganesh. As a convention, name spaces, which are sources where multiple resources are from, are (usually) URIs with the # character. For example, http://www.example.org/Elephant# is a name space. Resources without URIrefs are called blank nodes; a blank node indicates the existence of a resource, without explicitly mentioning the URIref of that resource. A blank node identifier, which is a local identifier, can be used to allow several RDF statements to reference the same blank node. RDF annotates Web resources in terms of named properties. Values of named properties (i.e., objects) can be URIrefs of Web resources or literals, viz. representations of data values (such as integers and strings). A set of RDF statements is call an RDF graph.

To represent RDF statements in a machine-processable way, RDF defines a specific eXtensible Markup Language (XML) syntax, referred to as RDF/XML [94]. As RDF/XML is verbose, in this thesis, we use the Notation 3 (or N3) syntax of RDF,
where each RDF statement is of the form (3.1). Figure 3.1 shows an RDF graph in N3 syntax, where the ‘@prefix’ introduces shorthand identifications (such as ‘ex:’) of XML namespaces and a semicolon ‘;’ introduces another property of the same subject. In these statements, the annotated resource is elp:Ganesh, which is annotated with three properties ex:mytitle, ex:mycreator and ex:mypublisher. Note that _:b1 is a blank node identifier.

Given that RDF provides only a standard syntax for annotations, how do we provide meaning to Web resources through annotations? The meaning comes either from some external agreements, e.g., Dublin Core, or from ontologies.

### 3.1.2 Dublin Core

One way of giving meaning to annotations is to provide some external agreement on the meaning of a set of information properties. For example, the Dublin Core Metadata Element Set [32] provides 15 ‘core’ information properties, such as ‘Title’, ‘Creator’, ‘Date’, with descriptive semantic definitions (in natural language). One can use these information properties in, e.g., RDF or META tags of HTML.

```
@prefix rdf: <http://www.w3.org/1999/02/22-rdf-syntax-ns#>
@prefix dc: <http://purl.org/dc/elements/1.1/>
@prefix elp: <http://example.org/Animal#>

elp:Ganesh dc:title "A resource called Ganesh" ;
    dc:creator "Pat Gregory" ;
    dc:publisher _:b1 .
_:b1 elp:name "Elephant United" .
```

Figure 3.2: Dublin Core properties in RDF statements

If we replace the properties ex:mytitle, ex:mycreator and ex:mypublisher used in Figure 3.1 with dc:title, dc:creator and dc:publisher as shown in Figure 3.2, Dublin Core compatible intelligent agents can then understand that the title of the Web resource is ‘A resource called Ganesh’, and the creator is Pat Gregory. This is not possible for the RDF statements in Figure 3.1 because, in general, users may use arbitrary names for the title, creator and publisher properties, etc.

The limitation of the ‘external agreement’ approach is its inflexibility, i.e., only a limited range of pre-defined information properties can be expressed.
3.1.3 Ontology

An alternative approach is to use ontologies to specify the meaning of Web resources. *Ontology* is a term borrowed from philosophy that refers to the science of describing the kinds of entities in the world and how they are related. In computer science, ontology is, in general, a ‘representation of a shared conceptualisation’ of a specific domain [46, 142]. It provides a shared and common *vocabulary*, including important concepts, properties and their definitions, and *constraints*, sometimes referred to as background assumptions regarding the intended meaning of the vocabulary, used in a domain that can be communicated between people and heterogeneous, distributed application systems.

The ontology approach is more flexible than the external agreement approach because users can customise vocabulary and constraints in ontologies. For example, applications in different domains can use different ontologies. Typically, ontologies can be used to specify the meaning of Web resources (through annotations) by asserting resources as instances of some important concepts and/or asserting resources relating to resources by some important properties defined in ontologies.

Ontologies can be expressed in Description Logics. An ontology usually corresponds to a TBox and an RBox (see Section 2.1.2) in Description Logics. Vocabulary in an ontology can be expressed by named concepts and roles, and concept definitions can be expressed by equivalence introductions. Background assumptions can be represented by general concept and role axioms. Sometimes, an ontology corresponds to a DL knowledge base. For example, in the OWL Web ontology language to be introduced in Section 3.2.2, an ontology also contains instances of important concepts and relationships among these instances, which can be represented by DL assertions.

The following example shows an ontology written in \( SHOQ(D) \) axioms and assertions (cf. Section 2.3.2 on page 45).

**Example 3.1 A DL-based Ontology**

A simple animal ontology may consist of three distinct parts. The first part is a set of important concepts and properties, which may include:\(^1\)

- concepts Animal, Plant, Cow, Sheep and Elephant;
- properties eat, partOf, age, liveIn and weight;

\(^1\)As mentioned earlier, this thesis adopts the notations (i.e., font styles) of individual names (a), class names (CN) and role names (RN) introduced in Section 2.1 on page 26.
3.1. ANNOTATIONS AND MEANING

- a defined concept Herbivore, whose members are exactly those Animals s.t. everything they eat is either a Plant or is a partOf a Plant: \( \text{Herbivore} \equiv \text{Animal} \sqcap \forall \text{eat.}(\text{Plant} \sqcup \exists \text{partOf. Plant}) \);

- a defined concept AdultElephant, whose members are exactly those Elephants whose ages are greater than 20 years: \( \text{AdultElephant} \equiv \text{Elephant} \sqcap \exists \text{age.} > 20 \).

The second part of the elephant ontology is composed of background assumptions of the domain and may include:

- **Cow, Sheep and Elephant are Animals**: \( \text{Cow} \sqsubseteq \text{Animal}, \text{Sheep} \sqsubseteq \text{Animal}, \text{Elephant} \sqsubseteq \text{Animal} \);

- **Cows are Herbivores**: \( \text{Cow} \sqsubseteq \text{Herbivore} \);

- **Elephants liveIn some Habitat**: \( \text{Elephant} \sqsubseteq \exists \text{liveIn. Habitat} \);

- **AdultElephants weigh at least 2000 kg**: \( \text{AdultElephant} \sqsubseteq \exists \text{weigh.} > 2000 \);

- **no individual can be both a Herbivore and a Carnivore**: \( \text{Herbivore} \sqcap \neg \text{Carnivore} \);

- **the property partOf is transitive**: \( \text{Trans}(\text{partOf}) \).

The third part of the elephant ontology is about instances and their inter-relationships, and may include:

- **Ganesh is an Elephant**: \( \text{Ganesh} : \text{Elephant} \);

- **south-sahara is a Habitat**: \( \text{south-sahara} : \text{Habitat} \);

- **Ganesh the Elephant liveIn south-sahara**: \( \langle \text{Ganesh, south-sahara} \rangle : \text{liveIn} \).

DL-based Ontologies can exploit powerful Description Logic reasoning tools, so as to facilitate machine understanding of Web resources. DL reasoning support can be very helpful to ensure the quality of ontologies, which is pivotal to the Semantic Web, in different development phases:

- **Ontology design**: DL Reasoning can be used to test whether concepts are non-contradictory and to derive implied relations. In particular, they can test whether the concepts in the ontology have their intended meaning or consequences. For example, in the animal ontology discussed in Example 3.1, one might want to
test whether elephants can be carnivores by adding a new background assumption Elephant \sqsubseteq \text{Carnivore} (Elephants are Carnivores). A DL reasoner should report that the concept Elephant is satisfiable (could have instances), because no relationship between Elephants and Herbivores has been specified; in other words, the report suggests that this important background knowledge is missing in the ontology.

- Ontology integration: Since it is not reasonable to assume that there will be a single ontology for the whole Web, interoperability and integration of different ontologies are also important issues. Integration can, for example, be supported by asserting inter-ontology relationships and testing for consistency and computing the integrated concept hierarchy. For example, one might want to integrate the above animal ontology with a mad cow ontology, asserting that the concepts Cow in the two ontologies are the same concept. If the following background assumptions are in the mad cow ontology:

- MadCows are Cows that eat SheepBrains: MadCow \sqsubseteq \text{Cow} \sqcap \exists \text{eat. SheepBrain},
- SheepBrains are partOf Sheep: SheepBrain \sqsubseteq \exists \text{partOf. Sheep},

a DL reasoner should report a contradiction because on the one hand, MadCows eat partOf Sheep, thus MadCows are Carnivores, while on the other hand, MadCows are Cows, hence they cannot be Carnivores.

- Ontology deployment: Reasoning may also be used when an ontology is deployed, i.e., when a Web page is already annotated with its concepts and properties. For example, consider a Web page about Ganesh the elephant, with the following annotation [Ganesh age 23 ]\(^2\) (or ⟨Ganesh, 23⟩:age in DL syntax) According to the above animal ontology, a DL reasoner should conclude that Ganesh is an AdultElephant. If one asks for all the Web pages about AdultElephants (as defined by the above elephant ontology), the Web page about Ganesh should be one of the (possibly huge number of) pages returned. In the deployment phase, as one would expect, requirements on the efficiency of reasoning are much more stringent than in the design and integration phases (Baader et al. [9]).

\(^2\)Strictly speaking, we should use the symbol for 23 here, i.e., the typed literal “23”\(^\text{xs:integer}\), instead of 23 itself. For typed literals, cf. Definition 3.7 on page 71.
The importance of DL-based ontologies in the Semantic Web has inspired the development of several DL-based ontology languages specifically designed for this purpose. These include OIL, DAML+OIL and OWL [42, 68, 12], all of which provide a DL-style model theoretic semantics. The design of these languages suggests that Description Logics are an important component of knowledge representation in the Semantic Web context.

We end this section by a brief comparison of the two approaches to providing meaning to Web resources through annotations. The external agreement approach is simple and easy to use. The ontology approach is more expressive and flexible, in that users can define their vocabulary and constraints for their application domains, but requires that ontology languages should have a well-defined semantics and should have powerful reasoning support.

Note that it is not proper to use the information properties defined in Dublin Core as abstract properties (i.e., abstract roles, see Section 2.3.2) in ontologies. Otherwise, there can be unexpected restrictions or implications on the information properties. For example, if one uses dc:author as an abstract property in an ontology and there is a constraint in the ontology that an author should be a person, then it disallows anything but persons, such as organisations, to be authors. In Chapter 4, we suggest that information properties can be used as annotation properties in an ontology.

3.2 Web Ontology Languages

The main Web ontology language considered in this thesis is OWL [12], which is a W3C recommendation for expressing ontologies in the Semantic Web. The design of OWL was mainly motivated by OIL and DAML+OIL, as well as several other pre-existing languages including RDFS, SHOE and DAML-ONT.

The W3C recommendation RDFS (RDF Schema) [22] can be recognised as a simple SW ontology language. RDF and RDFS, also referred to as RDF(S), are expected to be the foundation of the SW languages. There are, however, semantic problems layering OWL on top of RDF(S). In this section, we introduce RDFS and its model theory first (Section 3.1.1) and then describe the layering problems when we introduce OWL (Section 3.2.2). We will discuss datatype components of RDF(S) and OWL in Section 3.3.
3.2. WEB ONTOLOGY LANGUAGES

3.2.1 RDFS

Syntax

Following W3C’s ‘one small step at a time’ strategy, RDFS can be seen as a first try to support expressing simple ontologies with RDF syntax. In RDFS, predefined Web resources rdfs:Class, rdfs:Resource and rdf:Property can be used to define classes (concepts), resources and properties (roles), respectively.

Unlike Dublin Core, RDFS does not predefine information properties but a set of meta-properties that can be used to represent background assumptions in ontologies:

- rdf:type: the instance-of relationship,
- rdfs:subClassOf: the property that models the subsumption hierarchy between classes,
- rdfs:subPropertyOf: the property that models the subsumption hierarchy between properties,
- rdfs:domain: the property that constrains all instances of a particular property to describe instances of a particular class,
- rdfs:range: the property that constrains all instances of a particular property to have values that are instances of a particular class.

```xml
@prefix rdf: <http://www.w3.org/1999/02/22-rdf-syntax-ns#>  
@prefix rdfs: <http://www.w3.org/2000/01/rdf-schema#>  
@prefix elp: <http://example.org/Animal#>

elp:Animal rdf:type rdfs:Class .
elp:Habitat rdf:type rdfs:Class .
elp:Elephant rdf:type rdfs:Class ; rdfs:subClassOf elp:Animal .
elp:liveIn rdf:type rdf:Property ;
    rdfs:domain elp:Animal ; rdfs:range elp:Habitat .
elp:south-sahara rdf:type elp:Habitat .
```

Figure 3.3: An RDFS ontology
RDFS statements are simply RDF triples; viz. RDFS provides no syntactic restrictions on RDF triples. Figure 3.3 shows a fragment of an animal ontology (cf. Example 3.1 on page 53) in RDFS. This fragment defines three classes, i.e., elp:Animal, elp:Habitat and elp:Elephant (which is rdfs:subClassOf elp:Animal), and a property elp:liveIn, the rdfs:domain and rdfs:range of which are elp:Animal and elp:Habitat, respectively. In addition, it states that the resource elp:Ganesh is an instance of elp:Elephant, and that it elp:liveIn a an elp:Habitat called elp:south-sahara.

At a glance, RDFS is a simple ontology language that supports only class and property hierarchies, as well as domain and range constraints for properties. According to the RDF Model Theory, however, it is more complicated than that.

RDF Model Theory

Initially RDF and RDFS [82] had neither a formal model theory nor any formal meaning at all. This led to disagreements about the meaning of parts of RDFS; e.g., whether multiple range and domain constraints on a single property should be interpreted conjunctively or disjunctively. Recently, however, a RDF model theory (RDF MT) [53] has been proposed, which provides semantics not only for RDFS ontologies, but also for RDF triples. It became a W3C recommendation in February 2004 and has since been called RDF Semantics.

Firstly, the semantics of triples (such as [s p o .]) is given in terms of simple interpretations. To simplify presentation, in this thesis we do not cover blank nodes, which are identified by local identifiers instead of URIrefs.

**Definition 3.1.** (Simple Interpretation) Given a set of URI references V, a *simple interpretation* I of V in the RDF model theory is defined by

- an non-empty set IR of resources, called the *domain (or universe) of* I,
- a set IP, called the *set of properties in* I,
- a mapping IEXT, called the *extension function*, from IP to the powerset of IR × IR,
- a mapping IS from URIrefs in V to IR ∪ IP.

Given a triple [s p o .], I([s p o .]) = true if s,p,o ∈ V, IS(p) ∈ IP, and (IS(s), IS(o)) ∈ IEXT(IS(p)); otherwise, I([s p o .]) = false.

Given a set of triples S, I(S) = false if I([s p o .]) = false for some triple [s p o .] in S, otherwise I(S) = true. I *satisfies* S, written as I |= S if I(S) = true; in this case, we say I is a simple interpretation of S.
3.2. WEB ONTOLOGY LANGUAGES

Figure 3.4: A simple interpretation of $V = \{a, b, c\}$ (from [53])

Note that Definition 3.1 does not specify the relationship between $IP$ and $IR$, i.e., $IP$ may or may not be disjoint with $IR$. Figure 3.4 presents a simple interpretation $I$ of $V = \{a, b, c\}$, where the URIref $b$ is simply interpreted as a property because $IS(b) = 1 \in IP$, and $IEXT(IS(b))$, the extension of $IS(b)$, is a set of pairs of resources that are in $IR$, i.e., $\{(1, 2), (2, 1)\}$. Since $\langle IS(a), IS(c) \rangle \in IEXT(IS(b))$, $I([a b c .]) = \text{true}$; hence, we can conclude that $I$ satisfies $[a b c .]$.

Secondly, the semantics of RDF triples is given in terms of RDF-Interpretations.

**Definition 3.2.** (RDF-Interpretation) Given a set of URI references $V$ and the set $rdfV$, called the RDF vocabulary, of URI references in the rdf: namespace, an RDF-interpretation of $V$ is a simple interpretation $I$ of $V \cup rdfV$ that satisfies:

1. for $p \in V \cup rdfV$, $IS(p) \in IP$ iff $\langle IS(p), IS(rdf:Property) \rangle \in IEXT(IS(rdf:type))$,

2. all the RDF axiomatic statements.\(^3\)

Condition 1 of Definition 3.2 implies that each member of $IP$ is a resource in $IR$, due to the definition of $IEXT$ in Definition 3.1; in other words, RDF-interpretations require $IP$ to be a subset of $IR$. RDF axiomatic statements mentioned in Condition 2 are RDF statements about RDF built-in vocabularies in $rdfV$; e.g., $[\text{rdf:type rdf:type}$

\(^3\)Readers are referred to [53] for the list of the RDF axiomatic statements.
3.2. WEB ONTOLOGY LANGUAGES

rdf:Property] is an RDF axiomatic statement. According to Definition 3.2, any RDF-interpretation I should satisfy [rdf:type rdf:type rdf:Property .], viz. IS(rdf:type) should be in IP.

Finally, the semantics of RDFS statements written in RDF triples is given in terms of RDFS-Interpretations.

Definition 3.3. (RDFS-Interpretation) Given rdfV, a set of URI references V and the set rdfsV, called the RDFS vocabulary, of URI references in the rdfs: namespace, an RDFS-interpretation I of V is an RDF-interpretation of V ∪ rdfV ∪ rdfsV which introduces

- a set IC, called the set of classes in I, and
- a mapping IEXT (called the class extension function) from IC to the set of subsets of IR,

and satisfies the following conditions (let x, y, u, v be URIrefs in V ∪ rdfV ∪ rdfsV)\(^4\)

1. IS(x) ∈ IEXT(IS(y)) iff ⟨IS(x), IS(y)⟩ ∈ IEXT(IS(rdf:type)),
2. IC = IEXT(IS(rdfs:Class)) and IR = IEXT(IS(rdfs:Resource)),
3. if ⟨IS(x), IS(y)⟩ ∈ IEXT(IS(rdfs:domain)) and ⟨IS(u), IS(v)⟩ ∈ IEXT(IS(x)), then IS(u) ∈ IEXT(IS(y)),
4. if ⟨IS(x), IS(y)⟩ ∈ IEXT(IS(rdfs:range)) and ⟨IS(u), IS(v)⟩ ∈ IEXT(IS(x)), then IS(v) ∈ IEXT(IS(y)),
5. IEXT(IS(rdfs:subPropertyOf)) is transitive and reflexive on IP,
6. if ⟨IS(x), IS(y)⟩ ∈ IEXT(IS(rdfs:subPropertyOf)), then IS(x), IS(y) ∈ IP and IEXT(IS(x)) ⊆ IEXT(IS(y)),
7. IEXT(IS(rdfs:subClassOf)) is transitive and reflexive on IC,
8. if ⟨IS(x), IS(y)⟩ ∈ IEXT(IS(rdfs:subClassOf)), then IS(x), IS(y) ∈ IC and IEXT(IS(x)) ⊆ IEXT(IS(y)),
9. if IS(x) ∈ IC, then ⟨IS(x), IS(rdfs:Resource)⟩ ∈ IEXT(IS(rdfs:subClassOf)),

\(^4\)We only focus on the core RDFS primitives, i.e., the RDFS predefined meta-properties introduced on page 57.
and satisfies all the RDFS axiomatic statements.\footnote{Again, readers are referred to \cite{53} for a list of the RDFS axiomatic statements, which includes, e.g., \texttt{[rdf:type rdf:range rdf:Class]}.}

Condition 1 indicates that a ‘class’ is not a strictly necessary but convenient semantic construct \cite{53} because the class extension function $IC_{EXT}$ is simply a ‘syntactic sugar’ and is defined in terms of $IEXT$. Handling classes in this way can be counterintuitive (cf. Proposition 3.4). Condition 2 to 8 are about RDFS meta-properties $rdfs:domain$, $rdfs:range$, $rdfs:subPropertyOf$ and $rdfs:subClassOf$. Condition 9 ensures that all classes are sub-classes of $rdfs:Resource$.

**Proposition 3.4.** The RDFS statements $\texttt{[rdfs:Resource rdf:type rdfs:Class]}$ and $\texttt{[rdfs:Class rdfs:subClassOf rdfs:Resource]}$ are always true in all RDFS-interpretations.

**Proof.** For $\texttt{[rdfs:Resource rdf:type rdfs:Class]}$:

1. According to the definition of $IS$ and Definition 3.2, for any resource $x$, we have $IS(x) \in IR$. Due to $IR = IC\texttt{EXT}(IS(rdfs:Resource))$ and Condition 1 in Definition 3.3, $\langle IS(x), IS(rdfs:Resource) \rangle \in IEXT(IS(rdf:type))$. Since $\texttt{rdf:Property}$ is a built-in resource, we have $\langle IS(\texttt{rdf:Property}), IS(rdfs:Resource) \rangle \in IEXT(IS(rdf:type))$.

2. Due to $\texttt{[rdf:type rdfs:range rdfs:Class]}$ (an RDFS axiomatic statement), $\langle IS(\texttt{rdf:Property}), IS(rdfs:Resource) \rangle \in IEXT(IS(rdf:type))$ and Condition 4 in Definition 3.3, we have $IS(rdfs:Resource) \in IC\texttt{EXT}(IS(rdfs:Class))$. Therefore, for any RDFS-interpretation $I$, we have $I \models \texttt{[rdfs:Resource rdf:type rdfs:Class]}$.

For $\texttt{[rdfs:Class rdfs:subClassOf rdfs:Resource]}$: According to the definition of $IC$, every class is its member, including $IS(rdfs:Class)$, viz. $IS(rdfs:Class) \in IC$. Due to Condition 9 of Definition 3.3, $\langle IS(rdfs:Class), IS(rdfs:Resource) \rangle \in IEXT(IS(rdfs:subClassOf))$; hence, for any RDFS-interpretation $I$, we have $I \models \texttt{[rdfs:Class rdfs:subClassOf rdfs:Resource]}$. $\square$.

The two RDFS statements in Proposition 3.4 suggest a strange situation for $rdfs:Class$ and $rdfs:Resource$ as discussed in \cite{108}: on the one hand, $rdfs:Resource$ is an instance of $rdfs:Class$; on the other hand, $rdfs:Class$ is a sub-class of $rdfs:Resource$. Hence is $rdfs:Resource$ an instance of its sub-class? Users find this counter-intuitive and thus
3.2. WEB ONTOLOGY LANGUAGES

hard to understand — this is why we say that RDF(S) is more complicated than it appears.

Having described the semantics, we now briefly discuss reasoning in RDF(S). Since RDF(S) does not support the negation constructor, it is impossible, if we do not consider datatypes, to express contradictions. Consistency checking, therefore, is not needed for RDF(S). Instead, entailment is the key inference problem in RDF(S), which can be defined on the basis of interpretations.

**Definition 3.5. (RDF Entailments)** Given two sets of RDF statements $S_1$ and $S_2$, $S_1$ simply entails (RDF-entails, RDFS-entails) $S_2$ if all the simple interpretations (RDF-interpretations, RDFS-interpretations, respectively) of $S_1$ also satisfy $S_2$. ◦

**Limitations of RDF(S)**

RDF(S) has the following limitations:

1. The expressiveness of RDF(S) is limited [73]; e.g., we cannot express ‘An Elephant is different from a Cow’, because negation is not expressible in RDF(S).

2. Some valid RDF(S) statements can be counter-intuitive and hard to understand (such as the two statements presented in Proposition 3.4 on page 61). [104, 23, 108] argue that this is because vocabulary can play dual (or multiple) roles in RDFS-interpretations; e.g., rdfs:Resource is a super-class and an instance of rdfs:Class.

3. RDF(S) provides very few restrictions on its syntax, and no restrictions at all on the use of URIs in rdfsV ∪ rdfsV [113]. For example, RDF(S) does not disallow statements like:

```
(a rdfs:Class b).
```

```
(rdfs:Resource rdfs:subClassOf rdfs:Type).
```

where in the first statement rdfs:Class is used as a property, and in the second statement rdfs:Resource is asserted as a sub-class of rdf:type.

4. RDF statements are not only standard syntax for annotations, but have built-in semantics [74], i.e., rdf-interpretations; therefore, any language extensions of RDF(S) must be compatible with such built-in semantics.
5. RDF(S) does not distinguish URI references of classes and properties, including the built-in ones of RDF(S) and its subsequent languages, both of which are interpreted as resources in the domain of interpretations [74].

Limitation 1 suggests it is both necessary and desirable to layer more expressive ontology languages on top of RDF(S). Limitation 2 indicates that it is desirable to have a more intuitive sub-language of RDF(S). Due to Limitations 3-5, there are problems layering OWL on top of RDF(S) (see Section 3.2.2), hence it is crucial to have an OWL-compatible sub-language of RDF(S).

### 3.2.2 OWL

The OWL language facilitates greater machine understandability of Web resources than that supported by RDFS by providing more expressive vocabulary (classes and properties) and constraints (axioms) along with a formal semantics.

OWL has three increasingly expressive sub-languages: OWL Lite, OWL DL and OWL Full. **OWL Lite** and **OWL DL** are, like DAML+OIL (which is equivalent to the \(SHOIQ(D^+)_DL\)), basically very expressive description logics; they are almost equivalent to the \(SHIF(D^+)_DL\) and \(SHOIN(D^+)_DL\). Therefore, they can exploit existing DL research, e.g., to have well-defined semantics and well studied formal properties, in particular the decidability and complexity of key reasoning services: OWL Lite and OWL DL are both decidable, and the complexity of the ontology entailment problems of OWL Lite and OWL DL is \(EXPTIME\)-complete and \(NEXPTIME\)-complete, respectively [69]. **OWL Full** is clearly undecidable, as it does not impose restrictions on the use of transitive properties, but presents an attempt at complete integration with RDF(S).

In this section, we will first introduce the syntax and semantics of OWL DL (and therefore OWL Lite), and then briefly describe the problems we encounter when we extend RDF MT to support OWL constructors.

#### Syntax

OWL DL provides an abstract syntax and an RDF/XML syntax, as well as a mapping from the abstract syntax to the RDF/XML syntax [118].

---

6 ‘DL’ for Description Logic.

7 They also support annotation properties, which Description Logics do not.
The abstract syntax is heavily influenced by frames in general and by the design of OIL in particular. An OWL ontology in the abstract syntax may contain a sequence of annotations, axioms and facts (i.e., individual axioms). The abstract syntax for OWL class descriptions, property descriptions and axioms (and the corresponding DL syntax) are listed in Tables 3.1, 3.2 (on page 65) and 3.3 (on page 66). Readers are referred to [118] for full details on the abstract syntax. Figure 3.5 on page 64 presents a fragment of the animal ontology (cf. Example 3.1) in the OWL abstract syntax. Note that OWL does not provide customised datatypes (such as >20), so AdultElephant is not expressible in OWL.

```
Namespace(elp=<http://example.org/Animal#>)
Ontology(elp:ontology
   Class(elp:Animal)
   Class(elp:Plant)
   Class(elp:Cow partial exp:Animal elp:Herbivore)
   Class(elp:Sheep partial exp:Animal elp:Herbivore)
   Class(elp:Elephant partial exp:Animal elp:Herbivore
      restriction(elp:liveIn someValueFrom(elp:Habitat)))
   Class(elp:Habitat)
   Class(elp:Carnivore)
   Class(elp:Herbivore complete elp:Animal
      restriction(elp:eat allValuesFrom(
         unionOf(elp:Plant restriction(elp:partOf
            someValueFrom(elp:Plant))))))
   ObjectProperty(elp:partOf Transitive)
   ObjectProperty(elp:eat)
   ObjectProperty(elp:liveIn)
   DisjointClasses(elp:Herbivore elp:Carnivore)
   Individual(elp:Ganesh type(elp:Elephant)
      value(elp:liveIn
         Individual(elp:south-sahara type(elp:Habitat))))
)
```

Figure 3.5: An example ontology in the OWL abstract syntax

The OWL RDF/XML syntax is the exchange syntax for OWL DL. RDF is known (cf. the ‘Limitation of RDF(S)’ section on page 3.2.1) to have too few restrictions to provide a well-formed syntax for OWL. Therefore, the RDF/XML syntax form of an OWL DL ontology is valid, iff it can be translated (according to the mapping rules
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### Abstract Syntax | DL Syntax | Semantics
--- | --- | ---
ObjectProperty($R$) | $R$ | $R^I \subseteq \Delta^I \times \Delta^I$
ObjectProperty($S$ inverseOf($R$)) | $R^-$ | $(R^-)^I \subseteq \Delta^I \times \Delta^I$

Table 3.1: OWL object property descriptions

Table 3.2: OWL class descriptions

**Direct Semantics**

As OWL DL is basically a Description Logic, its model-theoretic semantics is very similar to the semantics provided for Description Logics (see Section 2.1), except that, in OWL DL, symbols for classes and properties, etc. are URI references instead of the usual names (strings), and that the model-theoretic semantics includes semantics for annotation properties and ontology properties (cf. [118]). Tables 3.1, 3.2 and 3.3 present the semantics of object property descriptions, concept descriptions, as well as OWL DL axioms and facts, respectively.

**OWL Full and OWL DL**

OWL Full is an attempt at complete integration with RDF(S). Syntactically, it differs from OWL DL in that it allows arbitrary RDF/XML forms of OWL ontologies, not only those translated from the abstract syntax form of the ontology.

OWL provides an RDF MT-compatible semantics, which is an extension of RDF MT with additional vocabulary. In fact, two version of this semantics are provided
### 3.2. WEB ONTOLOGY LANGUAGES

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class($A$, partial $C_1 \ldots C_n$)</td>
<td>$A \subseteq C_1 \cap \ldots \cap C_n$</td>
<td>$A^x \subseteq C_1^x \cap \ldots \cap C_n^x$</td>
</tr>
<tr>
<td>Class($A$, complete $C_1 \ldots C_n$)</td>
<td>$A \equiv C_1 \cap \ldots \cap C_n$</td>
<td>$A^x = C_1^x \cap \ldots \cap C_n^x$</td>
</tr>
<tr>
<td>EnumeratedClass($A$, ${o_1 \ldots o_n}$)</td>
<td>$A \equiv {o_1} \cup \ldots \cup {o_n}$</td>
<td>$A^x = {o_1^x \ldots o_n^x}$</td>
</tr>
<tr>
<td>SubClassOf($C_1$, $C_2$)</td>
<td>$C_1 \subseteq C_2$</td>
<td>$C_1^x \subseteq C_2^x$</td>
</tr>
<tr>
<td>DisjointClasses($C_1 \ldots C_n$)</td>
<td>$C_1 \equiv \ldots \equiv C_n$</td>
<td>$C_1^x = \ldots = C_n^x$</td>
</tr>
<tr>
<td>$\forall o \in C_1$</td>
<td>$C_1 \subseteq -C_2$</td>
<td>$C_1^x \cap C_2^x = \emptyset$</td>
</tr>
<tr>
<td>($1 \leq i &lt; j \leq n$)</td>
<td></td>
<td>($1 \leq i &lt; j \leq n$)</td>
</tr>
<tr>
<td>SubPropertyOf($R_1$, $R_2$)</td>
<td>$R_1 \subseteq R_2$</td>
<td>$R_1^x \subseteq R_2^x$</td>
</tr>
<tr>
<td>EquivalentProperties($R_1 \ldots R_n$)</td>
<td>$R_1 \equiv \ldots \equiv R_n$</td>
<td>$R_1^x = \ldots = R_n^x$</td>
</tr>
<tr>
<td>ObjectProperty($R$, super($R_1$) $\ldots$ super($R_n$)) with</td>
<td>$R \subseteq R_1$</td>
<td>$R^x \subseteq R_1^x$</td>
</tr>
<tr>
<td>domain($C_1$) $\ldots$ domain($C_n$)</td>
<td>$1 \leq R \subseteq C_1$</td>
<td>$R^x \subseteq C_1^x \cap \Delta_1^x$</td>
</tr>
<tr>
<td>range($C_1$) $\ldots$ range($C_n$)</td>
<td>$\top \subseteq \forall R.C_i$</td>
<td>$R^x \subseteq \Delta_i^x \cap C_i^x$</td>
</tr>
<tr>
<td>[Symmetric]</td>
<td>$R \equiv R^\top$</td>
<td>$R^x = (R^\top)^x$</td>
</tr>
<tr>
<td>[Functional]</td>
<td>Func($R$)</td>
<td>${(x, y) \mid \exists y. (x, y) \in R^x \leq 1}$</td>
</tr>
<tr>
<td>[InverseFunctional]</td>
<td>Func($R^\top$)</td>
<td>${(x, y) \mid \exists y. (x, y) \in (R^\top)^x \leq 1}$</td>
</tr>
<tr>
<td>[Transitive]</td>
<td>Trans($R$)</td>
<td>$R^x = (R^x)^x$</td>
</tr>
</tbody>
</table>

| AnnotationProperty($R$) | | |

| Individual($o$, type($C_1$) $\ldots$ type($C_n$)) | $o : C_i$, $1 \leq i \leq n$ | $o^x \in C_i^x$, $1 \leq i \leq n$ |
| value($R_1$, $o_1$) $\ldots$ value($R_n$, $o_n$) | $\langle o_1, o_1 \rangle : R_1$, $1 \leq i \leq n$ | $\langle o_1^x, o_1^x \rangle \in R_1^x$, $1 \leq i \leq n$ |
| SameIndividual($o_1 \ldots o_n$) | $o_1 = \ldots = o_n$ | $o_1^x = \ldots = o_n^x$ |
| DifferentIndividuals($o_1 \ldots o_n$) | $o_i \neq o_j$, $1 \leq i < j \leq n$ | $o_i^x \neq o_j^x$, $1 \leq i < j \leq n$ |

Table 3.3: OWL axioms and facts

In [118]: one for OWL DL and the other for OWL Full. In order to (try to) make the RDF MT-compatible semantics for OWL DL equivalent to the direct semantics, the domain of discourse is divided into several disjoint parts. In particular, the interpretations of classes, properties, individuals and OWL/RDF vocabulary are strictly separated. Given such a separation, there is a direct correspondence between RDF MT models and standard first-order models. Note that classes and properties, unsurprisingly, cannot be treated as ordinary resources as they are in RDF MT.

As far as the RDF MT-compatible semantics for OWL Full is concerned, the above disjointness restriction is not required. It has yet to be proved, however, that this semantics gives a coherent meaning to OWL Full. There exist at least three known problems that the RDF-compatible semantics for OWL Full needs to solve, but a proven solution has yet to be given.

- **Too Few Entailments** [116] Consider the following question: does the following individual axiom

\[
\text{Individual(ex:John} \\
\text{ type(intersectionOf(ex:Student ex:Employee ex:European)))}
\]

entail the individual axiom

\[
\text{Individual(ex:John} \\
\text{ type(intersectionOf(ex:Student ex:European)))}
\]
In OWL DL, the answer is simply ‘yes’, since intersectionOf(ex: Student ex: Employee ex: European) is a sub-class of intersectionOf(ex: Student ex: European). Since in RDF(S) every class is a resource, OWL Full needs to make sure of the existence of the resource intersectionOf(ex: Student ex: European) in every possible interpretation; otherwise, the answer will be ‘no’ which leads to a disagreement between OWL DL and OWL Full.

In general, OWL Full introduces so called comprehension principles to add all the missing resources into the domain for all the OWL class descriptions (see Table 3.2 on page 65). This is surely a very difficult task. It has yet to be proved that the proper resources are all added into the universe, no more and no less, and that the added resources will not bring any side-effects.

- **Contradiction Classes [116, 117, 74]** In OWL Full, it is possible to construct a class the instances of which have no rdf:type relationship linked to:

\[
\_ : c ~ \text{owl:onProperty} \quad \text{rdf:type} ; \quad \text{owl:allValuesFrom} \quad \_ : d . \\
\_ : d ~ \text{owl:complementOf} \quad \_ : e . \\
\_ : e ~ \text{owl:oneOf} \quad \_ : l \\
\_ : l \text{rdf:first} \quad \_ : c ; \quad \text{rdf:rest} \quad \text{rdf:nil}.
\]

The above triples require that rdf:type relates member of the class \_ : c to anything but \_ : c. It is impossible for one to determine the membership of \_ : c. If an object is an instance of \_ : c, then it is not; but if it is not then it is — this is a contradiction class. Note that it is not a valid OWL DL class, as OWL DL disallows using rdf:type as an object property.

With naive comprehension principles, resources of contradiction classes would be added to all possible OWL Full interpretations, which thus have ill-defined class memberships. To avoid the problem, the comprehension principles must also consider avoiding contradiction classes. Unsurprisingly, devising such comprehension principles took a considerable amount of effort [74], and no proof has ever shown that all possible contradiction classes are excluded in the comprehension principles of OWL Full.

- **Size of the Universe [70]** Consider the following question: is it possible that there is only one object in an interpretation of the following OWL ontology?

\[
\text{Individual(elp:Ganesh type(elp:Elephant))} \\
\text{DisjointClasses(elp:Elephant elp:Plant)}
\]
In OWL DL, classes are not objects, so the answer is ‘yes’: The only object in the domain is the interpretation of `elp:Ganesh`, the `elp:Elephant` class thus has one instance, i.e., the interpretation of `elp:Ganesh`, and the `elp:Plant` class has no instances.

In OWL Full, since classes are also objects, besides `elp:Ganesh`, the classes `elp:Elephant` and `elp:Plant` should both be mapped to the only one object in the universe. This is not possible because the interpretation of `elp:Ganesh` is an instance of `elp:Elephant`, but not an instance of `elp:Plant`; hence, `elp:Elephant` and `elp:Plant` should be different, i.e., there should be at least two objects in the universe. As the above axioms are valid OWL DL axioms, this example shows that OWL Full disagrees with OWL DL on valid OWL DL ontologies.

Furthermore, this example shows that the interpretation of OWL Full has different features than the interpretation of standard First Order Logic (FOL) model theoretic semantics. This raises the question as to whether it is possible to layer FOL languages on top of RDF(S).

Consequently, there is a serious mismatch between the two versions of RDF MT-compatible semantics. Even for two OWL DL ontologies $\mathcal{O}_1$ and $\mathcal{O}_2$, $\mathcal{O}_1$ OWL Full entails $\mathcal{O}_2$ does not imply that $\mathcal{O}_1$ OWL DL entails $\mathcal{O}_2$ [118]. Therefore, the semantic connection (at least in terms of entailment) between OWL DL and OWL Full seems rather weak.

**OWL and RDF(S)**

To sum up, OWL DL and RDF(S) are not compatible. OWL DL heavily restricts the syntax of RDF(S), viz. some RDF(S) annotations are not recognisable by OWL DL-compatible agents. Actually, it is not an easy task to check if an RDF graph is an OWL DL ontology. In addition, OWL DL imposes a strong condition on the RDF MT: that the interpretations of classes, properties and individuals be pairwise disjoint. This condition makes the semantics of RDF triples in OWL DL very different from their semantics given by RDF MT. This means that RDF(S) annotations that are valid OWL DL statements do not share the same meaning in OWL DL and RDF(S). As a result, the RDF(S)-compatible intelligent agents and OWL DL-compatible intelligent agents seem unlikely to be able to share their understandings.

The introduction of OWL Full does not make the situation any better. Firstly, due to the three problems discussed earlier, it is unknown whether the RDF-compatible
semantics for OWL Full could give a coherent meaning to OWL Full ontologies. Secondly, as OWL Full is undecidable, it provides no computation guarantee. Furthermore, the descriptions that OWL Full supports are so complex that it is difficult to have tool support for them. In short, it is rather unlikely that intelligent agents could use OWL Full seamlessly to support SW applications.

Therefore, OWL Full can simply be seen as an unsuccessful attempt at integrating OWL and RDF(S). In the rest of the thesis, when we mention ‘OWL’ we mean ‘OWL DL’.

3.3 Web Datatype Formalisms

A major concern when we provide DL reasoning services for OWL DL is the support for OWL datatyping. On the one hand, OWL DL provides some new datatype features that existing DL approaches (cf. Section 2.3) do not support. On the other hand, there have already been complaints about the shortcomings of OWL datatyping. The objective of this section, therefore, is to investigate the new features as well as the limitations of OWL datatyping. The strategy of this thesis is to provide DL reasoning services to support not only OWL DL, but also a more comprehensive extension of OWL DL that overcomes the possibly fatal limitations of OWL datatyping.

OWL datatyping is closely related to two existing Web datatype formalisms. OWL uses many of the built-in XML Schema datatypes as its built-in datatypes, and it adopts the RDF(S) specification of datatypes and data values. In this section, accordingly, we will first introduce XML Schema datatypes, RDF(S) datatyping and then present the OWL datatype formalism. Finally, we will summarise the new features and limitations of OWL datatyping.

3.3.1 XML Schema Datatypes

W3C XML Schema Part 2 [17] defines facilities for defining datatypes (or simple types) to be used in XML Schema as well as other XML specifications.

An XML Schema datatype is a triple, consisting of a value space, a lexical space, and a set of facets that characterize properties of the value space, individual values or lexical items:

- A value space $V(d)$ is the set of values for a given datatype $d$. For example, the value space for the decimal datatype is $V(\text{decimal}) = \{i \times 10^{-n} \mid i, n \in$
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\[ V(\text{integer}) \land n \geq 0 \}\).

- Each value in the value space of a datatype is denoted by one or more literals, or strings, in its lexical space. For example, “100” and “1.0E2” are two different literals from the lexical space of float which both denote the same value. A lexical space is the set of valid literals for a datatype.

- A facet is a single defining aspect of a value space. For example, the minExclusive facet is the exclusive lower bound of the value space for an ordered datatype. The value of minExclusive must be in the value space of the datatype (called a base datatype).

XML Schema datatypes are divided into disjoint built-in datatypes and user-derived datatypes. Derived datatypes can be defined by derivation from primitive or existing derived datatypes by the following three means:

- Derivation by restriction, i.e., by using facets on an existing type, so as to limit the number of possible values of the derived type.

- Derivation by union, i.e., to allow value from a list of datatypes.

- Derivation by list, i.e., to define the list type of an existing datatype.

Note that derivation by restriction is most widely used, while derivation by list is supported by neither RDF(S) nor OWL.

**Example 3.2 An XML Schema User-Derived Datatype** The following is the definition of a user-derived datatype (of the base datatype integer) which restricts values to integers greater than 20, using minExclusive.

```xml
<simpleType name="greaterThan20">
  <restriction base="xsd:integer">
    <minExclusive value="20"/>
  </restriction>
</simpleType>
```

There are two main limitations of the XML Schema type system:

- It does not provide a standard way to access the URI references of user-derived (customised) datatypes. This drawback makes XML Schema user-derived datatypes not accessible by RDF(S) and OWL.

---

8Readers are referred to [17] for the complete list of XML Schema built-in datatypes.
• It does not support n-ary datatype predicates, let alone providing facilities to define customised datatype predicates. This, to some extent, limits the kinds of datatype constraints that RDF(S) and OWL could provide.

### 3.3.2 RDF(S) Datatyping

RDF(S) provides a specification of datatypes and data values; accordingly, it allows the use of datatypes defined by any external type systems, e.g., the XML Schema type system, which conform to this specification.

**Definition 3.6. (Datatype)** A datatype \( d \) is characterised by a lexical space, \( L(d) \), which is an non-empty set of Unicode strings; a value space, \( V(d) \), which is an non-empty set, and a total mapping \( L2V(d) \) from the lexical space to the value space.

For example, boolean is a datatype with value space \{true, false\}, lexical space \{“T”, “F”, “1”, “0”\} and lexical-to-value mapping \{“T” \mapsto true, “F” \mapsto false, “1” \mapsto true, “0” \mapsto false\}.

**Definition 3.7. (Typed and Plain Literals)** Typed literals are of the form \( "v"^{u} \), where \( v \) is a Unicode string, called the lexical form of the typed literal, and \( u \) is a URI reference of a datatype. Plain literals have a lexical form and optionally a language tag as defined by [1], normalised to lowercase.

The denotation of a typed literal is the value mapped from its enclosed Unicode string by the lexical-to-value mapping of the datatype associated with its enclosed datatype URIref. For example, \( "1"^{xsd:boolean} \) is a typed literal that represents the boolean value true, while \( "1"^{xsd:integer} \) represents the integer 1. Plain literals, e.g., “1”, are considered to denote themselves [53].

The associations between datatype URI references (e.g., xsd:boolean) and datatypes (e.g., boolean) can be provided by datatype maps defined as follows.

**Definition 3.8. (Datatype Map)** We consider a datatype map \( M_{d} \) that is a partial mapping from datatype URI references to datatypes.

**Example 3.3 Datatype Map** \( M_{d1} = \{(xsd:string, string), (xsd:integer, integer)\} \) is a datatype map, where xsd:string and xsd:integer are datatype URI references, and string and integer are datatypes.
A datatype map may include some built-in XML Schema datatypes (as seen in Example 3.3), while other built-in XML Schema datatypes are problematic and thus unsuitable for various reasons. For example, xsd:ENTITIES is a list-value datatype that does not fit the RDF datatype model.\textsuperscript{9} Please note that derived XML Schema datatypes are not RDF(S) datatypes, because there is no standard way to access a derived XML Schema datatype through a URI reference. Therefore, there is no way to include a derived XML Schema datatype in a datatype map.

An RDFS-interpretation w.r.t. a datatype map $M_d$ can be defined as follows.

\textbf{Definition 3.9. (RDFS $M_d$-Interpretation)} Given a datatype map $M_d$, an \textit{RDFS $M_d$-interpretation} $I$ of a vocabulary $V$ is any RDFS-interpretation of $V \cup \{ u \mid \exists d. \langle u, d \rangle \in M_d \}$ which introduces

- A distinguished subset $LV$ of $IR$, called the \textit{set of literal values}, which contains all the plain literals in $V$,

- a mapping $IL$ from typed literals in $V$ into $IR$,

and satisfies the following extra conditions:

1. $LV = ICEXT(IS(rdfs:Literal))$,

2. for each pair $\langle u, d \rangle \in M_d$,
   \begin{enumerate}
   \item $ICEXT(d) = V(d) \subseteq LV$,
   \item there exist $d \in IR$ s.t. $IS(u) = d$,
   \item $IS(u) \in ICEXT(IS(rdfs:Datatype))$,
   \item for “$s”\textsuperscript{""}u\textsuperscript{′} $\in V, IS(u\textsuperscript{′}) = d$, if $s \in L(d)$, then $IL(“s”\textsuperscript{""}u\textsuperscript{′}) = L2S(d)(s)$, otherwise, $IL(“s”\textsuperscript{""}u\textsuperscript{′}) \notin LV$,
   \end{enumerate}

3. if $d \in ICEXT(IS(rdfs:Datatype))$, then $\langle d, IS(rdfs:Literal) \rangle \in IEXT(rdfs:subClassOf)$. \hfill $\Diamond$

According to Definition 3.9, $LV$ is a subset of $IR$, i.e., literal values are resources. Condition (1) ensures that the class extension of $rdfs$:Literal is $LV$. Condition (2a) asserts that RDF(S) datatypes are classes, condition (2b) ensures that there is a resource $d$ for datatype $d$ in $M_d$, condition (2c) ensures that the class $rdfs$:Datatype contains

\textsuperscript{9}See the RDF semantics document http://www.w3.org/TR/rdf-mt/#dtype_interp for the complete list of RDF(S) built-in datatypes.
the datatypes used in any satisfying \( M_d \)-interpretation, and condition (2d) explains why the range of \( IL \) is \( IR \) rather than \( LV \) (because, for \( "s" \uparrow \uparrow u \), if \( s \notin L(IS(u)) \), then \( IL("s" \uparrow \uparrow u) \notin LV \)). Condition (3) requires that RDF(S) datatypes are sub-classes of rdfs:Literal.

If the datatypes in the datatype map \( M_d \) impose disjointness conditions on their value spaces, it is possible for an RDF graph to have no RDFS \( M_d \)-interpretation which satisfies it, i.e., there exists a data type clash. For example,

\[
\_ : x \ rdf:type \ xsd:string
\]
\[
\_ : x \ rdf:type \ xsd:decimal
\]

would constitute a data type clash because the value spaces of \( xsd:string \) and \( xsd:decimal \) are disjoint. In RDF(S), an ill-typed literal does not in itself constitute a data type clash (cf. Condition(2d) in Definition 3.9), but a graph which entails that an ill-typed literal has rdf:type rdfs:Literal would be inconsistent.

Let \( S, E \) be two sets of RDFS statements. \( S \) RDFS-\( M_d \)-entails \( E \) if every RDFS \( M_d \)-interpretation of \( S \) also satisfies \( E \).

### 3.3.3 OWL Datatyping

**Datatypes**

OWL uses many of the built-in XML Schema datatypes, and it adopts the RDF(S) specification of datatypes and data values. Some built-in XML Schema datatypes are problematic for OWL, as they do not fit the RDF(S) datatype model. Readers are referred to [119] for the complete list of OWL built-in datatypes. Note that the built-in RDF datatype, rdfs:XMLLiteral, is also an OWL built-in datatype.

The fundamental difference between RDF(S) datatyping and OWL datatyping is the relationship between datatypes and classes. In OWL DL, datatypes are not classes, and object and datatype domains are disjoint with each other. Such disjointness is motivated by both philosophical and pragmatic considerations (Horrocks et al. [73]):

- Datatypes (e.g., the \( >_{20} \) datatype) are different from classes (e.g., Tree) in that datatypes and the predicates (such as \( =, <, + \)) defined over them have fixed extension (e.g., the extension of \( >_{20} \) is all the integers that are greater than 20), while classes could have different interpretations in different models.

- The simplicity, compactness and the semantic integrity of the ontology language are not compromised. We do not have to provide a logical theory for each
datatype, nor do we have to worry whether each of them is still correct whenever we extend the ontology language.

- The ‘implementability’ of the language is not compromised. A hybrid reasoner can easily be implemented by combining a reasoner for the ‘object language’ with one (or possibly more) datatype reasoner(s) that can decide the satisfiability problem of datatype constraints.

OWL datatyping further distinguishes supported from unsupported datatype URI references w.r.t. a datatype map $M_d$ and allows both of them to be used in an OWL DL ontology. The motivation is that different OWL DL reasoners could provide different supported datatype URIrefs, but will not simply disallow the use of datatype URIrefs that they do not support.

**Definition 3.10. (Supported and Unsupported Datatype URIrefs)** Given a datatype map $M_d$, a datatype URI reference $u$ is called a supported datatype URI reference w.r.t. $M_d$ (or simply a supported datatype URIref) if there exists a datatype $d$ s.t. $M_d(u) = d$ (in this case, $d$ is called a supported datatype w.r.t. $M_d$); otherwise, $u$ is called an unsupported datatype URI reference w.r.t. $M_d$ (or simply an unsupported datatype URIref).

For example, xsd:integer is a supported datatype URIref w.r.t. $M_{d1}$ given in Example 3.3, while xsd:boolean is an unsupported datatype URIref w.r.t. $M_{d1}$. OWL DL requires at least xsd:integer and xsd:string be supported datatype URIrefs. Other built-in OWL datatype URIrefs can be either supported or unsupported.

As built-in XML Schema datatypes are usually not enough in many SW and ontology applications, OWL provides the use of so called enumerated datatypes, which are built using literals.

**Definition 3.11. (Enumerated Datatypes)** Let $y_1, \ldots, y_n$ be typed literals. An enumerated datatype is of the form oneOf($y_1, \ldots, y_n$).

For example, oneOf("0"^^xsd:integer "15"^^xsd:integer "30"^^xsd:integer "40"^^xsd:integer) is an enumerated datatype, which is interpreted as $\{0, 15, 30, 40\}$. The above enumerated datatype, e.g., can be used as the range of a datatype property tennisScore.

The semantics of OWL datatypes is defined w.r.t. a datatype map.
Definition 3.12. (Datatype Interpretation) An OWL datatype interpretation w.r.t. to a datatype map $M_d$ is a pair $(\Delta_D, \cdot^D)$, where the datatype domain $\Delta_D = \mathbb{PL} \cup \bigcup$ for each supported datatype URIref $u$ w.r.t. $M_p V(M_p(u))$ ($\mathbb{PL}$ is the value space for plain literals, i.e., the union of the set of Unicode strings and the set of pairs of Unicode strings and language tags) and $\cdot^D$ is a datatype interpretation function, which has to satisfy the following conditions:

1. $\text{rdfs:Literal}^D = \Delta_D$;
2. for each plain literal $l$, $l^D = l \in \mathbb{PL}$;
3. for each supported datatype URIref $u$ (let $d = M_d(u)$):
   (a) $u^D = V(d) \subseteq \Delta_D$,
   (b) if $s \in L(d)$, then $\langle \text{"s"}^{\cdot^D} \rangle u = L2V(d)(s)$,
   (c) if $s \notin L(d)$, then $\langle \text{"s"}^{\cdot^D} \rangle u$ is not defined;
4. for each unsupported datatype URIref $u$, $u^D \subseteq \Delta_D$, and $\langle \text{"s"}^{\cdot^D} \rangle u \in u^D$.
5. each enumerated datatype oneOf($y_1 \ldots y_n$) is interpreted as $y_1^D \cup \ldots \cup y_n^D$.

Condition 1 of Definition 3.12 asserts that $\text{rdfs:Literal}$ is interpreted as the datatype domain of a datatype interpretation. Condition 2 ensures that plain literals are interpreted as themselves. Condition 3 ensures that supported datatype URIrefs are interpreted as the value spaces of the datatypes they represent, and it also ensures the valid interpretations of typed literals with supported datatype URIrefs. Note that, in OWL, ill-typed literals have no interpretations, which is different from RDF(S) datatyping (cf. Definition 3.9 on page 72). Condition 4 states that unsupported datatype URIrefs are interpreted as any subsets of the datatype domain, which implies that typed literals associated with unsupported datatype URIrefs are interpreted as some members of the datatype domain, although we do not know exactly which members (of the datatype domain) they are interpreted as. Condition 5 gives the interpretation of enumerated datatypes.

Object Language and Datatypes

OWL datatype properties relate objects to data values in data ranges (see Table 3.4); they are disjoint with object properties, which relate objects to objects. OWL provides various datatype property axioms: besides the usual sub-property, domain and
3.3. WEB DATATYPE FORMALISMS

### Table 3.4: OWL data ranges

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>rdfs:Literal $u$ a datatype URRef</td>
<td>$\top_D$</td>
<td>$\Delta_D$</td>
</tr>
<tr>
<td>oneOf(&quot;s_1&quot;^^u_1 ... &quot;s_n&quot;^^u_n)</td>
<td>${&quot;s_1&quot;^D, ... , &quot;s_n&quot;^D} \subseteq \Delta_D$</td>
<td>${t_1 \cup \ldots \cup t_n} \subseteq \Delta_D$</td>
</tr>
<tr>
<td>$t_i = L2V(u_i)(s_i)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 3.5: OWL datatype property axioms

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>(SubPropertyOf $T_1, T_2$)</td>
<td>$T_1 \subseteq T_2$</td>
<td>$T^I_1 \subseteq T^I_2$</td>
</tr>
<tr>
<td>(EquivalentProperties $T_1, \ldots, T_n$)</td>
<td>$T_1 = \ldots = T_n$</td>
<td>$T^I_1 = \ldots = T^I_n$</td>
</tr>
<tr>
<td>DatatypeProperty($T \supert T_1 \ldots \supert T_n$)</td>
<td>$T \subseteq T_i$</td>
<td>$T^I \subseteq T^I_i$</td>
</tr>
<tr>
<td>domain($C_1$) ... domain($C_k$)</td>
<td>$\geq 1T \subseteq C_i$</td>
<td>$T^I \subseteq C^I_i \times \Delta_D$</td>
</tr>
<tr>
<td>range($d_1$) ... range($d_h$)</td>
<td>$\top \subseteq \forall T.d_i$</td>
<td>$T^I \subseteq \Delta^I \times d^D$</td>
</tr>
<tr>
<td>[Functional])</td>
<td>$\text{Func}(T)$</td>
<td>$\forall x \in \Delta^I. \sharp{t \mid \langle x, t \rangle \in T^I} \leq 1$</td>
</tr>
</tbody>
</table>

In addition, OWL DL provides several datatype-related concept descriptions (see Table 3.6): datatype exists, datatype value restrictions, and datatype (unqualified) number restrictions, which specify the set of objects related by a datatype property to more (or less) than a certain number of data values.

By default, datatype properties are not functional; e.g., a Person can have multiple telephoneNumbers.

### Table 3.6: OWL datatype-related concept descriptions

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>restriction($T$ someValuesFrom($d$))</td>
<td>$\exists T.d$</td>
<td>${x \mid \exists t. \langle x, t \rangle \in T^I \wedge t \in d^D}$</td>
</tr>
<tr>
<td>restriction($T$ allValuesFrom($d$))</td>
<td>$\forall T.d$</td>
<td>${x \mid \exists t. \langle x, t \rangle \in T^I \rightarrow t \in d^D}$</td>
</tr>
<tr>
<td>restriction($T$ minCardinality($m$))</td>
<td>$\geq mT$</td>
<td>${x \mid #{t \mid \langle x, t \rangle \in T^I} \geq m}$</td>
</tr>
<tr>
<td>restriction($T$ maxCardinality($m$))</td>
<td>$\leq mT$</td>
<td>${x \mid #{t \mid \langle x, t \rangle \in T^I} \leq m}$</td>
</tr>
</tbody>
</table>
New Requirements

To summarise, OWL datatyping presents new requirements to DL research:

- It adopts the RDF(S) specification of datatypes and data values, while existing DL approaches mainly consider predicates instead of datatypes and data values.
- It provides enumerated datatypes and datatype-related number restrictions, which are beyond the expressive power of existing DL languages.

Existing DL approaches should, therefore, be extended to support these new requirements.

Limitations of OWL Datatyping

There have already been complaints [29, 91] from users about two serious limitations of OWL datatyping, which discourage potential users from adopting OWL DL in their SW and ontology applications.

1. Customised Datatypes Many SW and ontology applications need to define customised datatypes for their own purposes. For example, to define the class AdultElephant in the animal ontology presented in Example 3.1 (page 53) requires the customised datatype $>_{20}$. OWL datatyping does not allow the use of user-derived XML Schema datatypes, since the XML Schema type system provides no standard way to access URIrefs of user-derived datatypes. Enumerated datatypes are not expressive enough to be an alternative to user-derived XML Schema datatypes, since they cannot represent infinite datatypes, such as ‘GreaterThan20’ presented in Example 3.2 (page 70). Furthermore, they are clumsy to represent big ranges, e.g., the integer range [1, 1000000].

2. Customised Datatype Predicates Many SW and ontology applications need to represent constraints over multiple datatype properties, such as the sum of height, length and width of SmallItems is no greater than 15cm, the pricing-Pounds and priceInDollars should conform to a certain exchange rate, etc. (see Section 1.3). OWL does not support the use of $n$-ary datatype predicates, not to mention customised datatype predicates.

In addition, OWL datatyping has several other limitations:
1. OWL does not support negated datatypes. For example, ‘all integers but 0’, which is the relativised negation of the enumerated datatype oneOf("0"^^xsd:integer), is not expressible in OWL. Moreover, negated datatypes are necessary in the negated normal form (NNF)\(^{11}\) of datatype-related class descriptions in, e.g., DL tableaux algorithms.

2. An OWL DL datatype domain seriously restricts the interpretations of typed literals with unsupported datatype URIrefs. For example, given the datatype map \(M_{d1}\) (cf. Example 3.3), “1.278e-3”^^xsd:float will have to be interpreted as either an integer, a string or a string with a language tag, which is counter-intuitive.

3. Users cannot define names for enumerated datatypes; viz. users have to construct the enumerated datatypes whenever they want to use them. This is very inconvenient if they need to use enumerated datatypes with large numbers of typed literals multiple times in their ontologies.

### 3.4 Outlook for the Two Issues

**Issue 1: What is the connection between RDF(S) and OWL?** As explained in this chapter, the intended foundation of the Semantic Web RDF(S) and the SW standard ontology language OWL are not compatible with each other, because of RDF(S)’s non-standard semantics. In Chapter 4 we will present a modified semantics for RDF(S) that addresses this problem and restores the desired connection between RDF(S) and OWL.

**Issue 2: How to extend and provide reasoning support for OWL Datatyping?**

In this chapter, we have shown that OWL has a very serious limitation; i.e., it does not support customised datatypes and datatype predicates. This limitation hinders the wider acceptance of OWL DL in SW and ontology applications [126].

We will provide a solution in Chapter 5 to 8. Firstly, we will propose a comprehensive formalism to unify datatypes and predicates and to identify a family of decidable DLs that provide customised datatypes and datatype predicates (Chapter 5). Secondly, we propose two extension of OWL DL, i.e., OWL-E and OWL-Eu, both of which are members of the above identified family of the decidable combined DLs. Thirdly, we

\(^{11}\)A concept is in negation normal form iff negation is applied only to atomic concept names, nominals or datatype ranges/expressions; see page 133.
provide practical decision procedures for a family of DLs that are closely related to OWL DL, OWL-Eu and OWL-E (Chapter 6). Last but not least, we will present the framework architecture and discuss the flexibility of our framework (Chapter 7 and 8).

**Chapter Achievements**

- The construction, integration and deployment of DL-based ontologies benefit greatly from a well-defined DL semantics and powerful DL reasoning tools.

- OWL Lite and OWL DL are DL-based Web ontology languages, which can thus exploit existing DL research, e.g., to have well-defined semantics, well known formal properties, in particular the decidability and complexity of key reasoning services.

- The semantic problems of layering OWL on top of RDF(S) stem from the following characteristics of RDF(S):
  
  - RDF triples have built-in semantics.
  
  - Classes and properties, including built-in classes and properties of RDF(S) and its subsequent languages such as OWL, are treated as objects (or resources) in the domain.

  - There are no restrictions on the use of built-in vocabularies.

- Although OWL DL is very expressive and provides some new requirements about datatypes to DL research, OWL datatyping has many limitations. A very serious limitation that hinders its wider acceptance is the lack of support for customised datatypes and datatype predicates.
Chapter 4

An Important Connection

Chapter Aims:

- To investigate a novel modification of RDF(S) that keeps the main features of RDFS and is able to serve as a firm semantic foundation for the latest DL-based SW ontology languages.

Chapter Plan

4.1 RDFS(FA): A DL-ised Sub-language of RDFS (80)
4.2 RDFS(FA) and OWL (95)
4.3 A Clarified Vision of the Semantic Web (98)

As shown in Chapter 3, there are problems layering OWL on top of RDF(S). In this chapter we propose RDFS(FA) (RDFS with Fixed Architecture), as an alternative to RDFS, to restore the desired connection between RDF(S) and OWL DL.

4.1 RDFS(FA): A DL-ised Sub-language of RDFS

4.1.1 Introduction

Let us first consider design requirements of RDFS(FA) from the following two aspects: (i) Semantics Web language layering and (ii) SW and ontology applications.
4.1. RDFS(FA): A DL-ISED SUB-LANGUAGE OF RDFS

Language Layering Requirements

To restore the connection between RDF(S) and OWL, we shall consider a sub-language of RDFS with a semantics that is compatible with the direct semantics of OWL DL. Ontologies in this sub-language are still valid RDFS ontologies.

From the lessons we learnt in the ‘OWL Full and OWL DL’ Section on page 65, RDFS(FA) should address the following problems of RDF(S):

- RDF triples have built-in semantics.
- Classes and properties, including built-in classes and properties of RDF(S) and its subsequent languages such as OWL, are treated as objects (or resources) in the domain.
- There are no restrictions on the use of built-in vocabularies.

Application Requirements

The main challenge to RDFS(FA) is that it should also provide the main features of RDFS that OWL does not provide, viz., it should provide meta-classes, meta-properties and enable the use of class symbols (URIrefs) as property values.

Example 4.1 RDFS: Meta-classes and Meta-properties

Applications using WordNet [98] to annotate resources, such as images [146], require the use of meta-classes and meta-properties.

```@prefix wnc: <http://www.cogsci.princeton.edu/~wn/concept#>
@prefix wns: <http://www.cogsci.princeton.edu/~wn/schema#>

wns:LexicalConcept rdfs:subClassOf rdfs:Class.
wns:hyponymOf  rdfs:subPropertyOf rdfs:subClassOf;
   rdfs:domain  wns:LexicalConcept ;
   rdfs:range   wns:LexicalConcept .

wnc:100002086  wns:hyponymOf  wnc:100001740 .
```

where wnc:100002086 and wnc:100001740 are WordNet synsets (i.e., concepts like ‘Elephant’ and ‘Animal’). The first statement specifies that the class LexicalConcept is a subclass of the built-in RDFS meta-class rdfs:Class, the instances of which are classes. This means that now all instances of LexicalConcept are also classes. In a similar vein, the second statement defines that the WordNet property hyponymOf is a
sub-property of the built-in RDFS meta-property rdfs:subClassOf. This enables us to interpret the instances of *hyponymOf* as subclass links.

We invite the reader to note that the metamodeling architecture of RDFS may not be what one expects, as it is valid to add RDFS triples such as

```r
rdfs:Class rdfs:subClassOf wnc:100002086 .
```

into the above WordNet ontology, which makes the relationship between wnc:100002086 and wns:LexicalConcept rather confusing: wnc:100002086 is an instance of wns:LexicalConcept, which is an instance of an instance of wnc:100002086; nevertheless, wnc:100002086 and wns:LexicalConcept are not necessarily equivalent to each other.

It has been argued [105] that it is often useful to use classes as values for properties. Users want to use whatever they believe to be intuitive as values of properties, including classes, properties, etc.

**Example 4.2 RDFS: Classes as Property Values**

This example is from [105]. Suppose we have a set of Books about Animals and want to annotate each Book with its *subject*, which is a particular species or class of Animals that it talks about (cf. Figure 4.1). Further, when retrieving all Books about Lions from a repository, we want Books that are annotated as books about AfricanLions to be included in the results.

```r
@prefix bk: <http://protege.stanford.edu/swbp/books#>

bk:AfricanLion rdf:type rdfs:Class;
    rdfs:subClassOf bk:Lion .

bk:LionsLifeInThePrideBook rdf:type bk:Book ;
```
As classes are objects in RDF(S), classes \( bk:\text{Lion} \) and \( bk:\text{AfricanLion} \) can be used as values of properties.

We invite the reader to note that it is not proper to use information properties \((dc:\text{subject})\) as properties in ontologies (except annotation properties, cf. Example 4.4), since properties in ontologies (except annotation properties) could have extra implications, due to various constraints in an ontology (cf. comparison between Dublin Core and ontologies on page 56).

**Overview of RDFS(FA)**

RDFS(FA) has a First Order/Description Logic style semantics, and introduces a Fixed layered metamodeling Architecture to RDFS.

One of the interesting features of RDFS(FA) is its metamodeling architecture. Let us recall that RDFS has a non-layered metamodeling architecture; resources in RDFS can be classes, objects and properties at the same time, viz. classes and their instances (as well as relationships between the instances) are the same layer. RDFS(FA), instead, provides a layered metamodeling architecture that is very similar to that of the widely used Unified Modelling Language (UML) [31]. Like UML, RDFS(FA) divides up the universe of discourse into a series of strata (or layers). The built-in modelling primitives of RDFS are separated into different strata of RDFS(FA), and the semantics of modelling primitives depend on the stratum they belong to. Theoretically there can be a (countably) infinite number of strata in the metamodeling architecture; in practice, four strata (as shown in Figure 4.2) are usually enough. The UML-like meta-modeling architecture makes it easier for users who are familiar with UML to understand and use RDFS(FA).

Figure 4.2 shows strata 0 to strata 3, i.e., the lowest four layers of the Metamodeling Architecture of RDFS(FA); they are also called the Instance Layer, the Ontology Layer, the Language Layer and the Meta-Language Layer, respectively:

- URI references in the Instance Layer (e.g., \( bk:\text{TheAfricanLionBook} \)) are interpreted as individual objects.
4.1. RDFS(FA): A DL-ISED SUB-LANGUAGE OF RDFS

Figure 4.2: (The lowest four layers of) The metamodeling architecture of RDFS(FA)

- URI references in the Ontology Layer (e.g., wnc:100002086, bk:Lion, bk:bookTitle) are interpreted as ontology classes and ontology properties.

- URI references in the Language Layer (e.g., fa:Class2 and fa:Property2) are used to define and describe elements in the Ontology Layer. Note that, besides some built-in language classes and properties, URI references of meta-classes and meta-properties (e.g., wns:LexicalConcept and wns:hyponymOf) can be used in the Language Layer (and the layers above).

- URI references in the Meta-Language Layer (e.g, fa:Class3 and fa:Resource3) are used to define and describe elements in the Language Layer.

As seen in Figure 4.2, rdfs:Resource is stratified into three layers, i.e., fa:Resource1 in the Ontology Layer, fa:Resource2 in the Language Layer and fa:Resource3 in the Meta-Language Layer. Similarly, rdfs:Class and rdfs:Property are stratified into the Language Layer and the Meta-Language Layer.

In RDFS(FA), classes cannot be objects and vice versa; in RDFS, Web resources can be classes, properties, objects or even datatypes all at once. We argue that RDFS(FA) is more intuitive than RDFS based on the following observation: when users design their ontologies, a common concern is to decide whether to model something in the domain as a class or as an object. This concern suggests that users intuitively tend to assume that classes and objects should be different from each other. Therefore, layered meta-models are more intuitive than non-layered meta-models.
4.1. RDFS(FA): A DL-ISED SUB-LANGUAGE OF RDFS

Each RDFS(FA) built-in vocabulary (except fa:AnnotationProperty, fa:Literal, and built-in annotation properties) has a number to indicate the stratum it belongs to. Here are two rules of thumb to get these numbers of strata right: for an RDFS(FA) triple [s p o .], (i) if p is not an instance-of relationship, then s and o should be in the same stratum, and p should be one stratum higher than s and o, e.g., both wns:LexicalConcept and fa:Class_2 are in stratum 2 and fa:subClassOf_3 is in stratum 3 (cf. Example 4.3); (ii) if p is an instance-of relationship, then o is one stratum higher than s, and p is in the same stratum as o (cf. Example 4.4). Developers can instrument tools that would maintain the numbers of strata automatically.

Example 4.3 RDFS(FA): Meta-classes and Meta-properties

We now revisit Example 4.1 and use RDFS(FA) to represent the meta-class wns:LexicalConcept and the meta-property wns:hyponymOf in WordNet.

\[
\text{wns:LexicalConcept} \text{ fa:subClassOf}_3 \text{ fa:Class}_2 . \\
\text{wns:hyponymOf} \text{ fa:type}_3 \text{ fa:AbstractProperty}_3 ; \\
\text{fa:subPropertyOf}_3 \text{ fa:subClassOf}_2 ; \\
\text{fa:domain}_3 \text{ wns:LexicalConcept} ; \\
\text{fa:range}_3 \text{ wns:LexicalConcept} . \\
\text{wnc:100002086} \text{ fa:type}_2 \text{ fa:Class}_2 ; \\
\text{wns:hyponymOf} \text{ wnc:100001740} .
\]

Now the layering of the model is much clearer: wnc:100002086 and wnc:100001740 are in stratum 1, wns:LexicalConcept and wns:hyponymOf are in stratum 2. The rules of thumb presented above disallow asserting that fa:Class_2 is an instance of wnc:100002086. Hence, there is no confusion here.

In RDFS(FA), all Web resources can have data-value annotation properties (cf. Section 4.1.2), i.e., the values of annotation properties can be either typed literals or plain literals. For example, typed literals of the built-in XML Schema datatype xsd:anyURI can be used as values of annotation properties. Their interpretations (see Definition 4.2 on page 89) are URI references; this thus allows one to refer to URI references of any ontology elements through annotation properties.

Example 4.4 RDFS(FA): Class URIrefs as Values of Annotation Properties

We now revisit Example 4.2 and use the annotation property dc:subject to refer to class URIrefs (cf. Figure 4.3). The approach we presents here is slightly different from Approach 5 in [105] in that annotations are class URIrefs instead of classes.

@prefix bk: <http://protege.stanford.edu/swbp/books#>
Figure 4.3: RDFS(FA): class URIs as annotation property values

```
bk:bookTitle rdf:type fa:AnnotationProperty.
dc:subject rdf:type fa:AnnotationProperty.
bk:AfricanLion fa:type2 fa:Class2;
   fa:subClassOf2 bk:Lion .
bk:LionsLifeInThePrideBook fa:type1 bk:Lion ;
   bk:bookTitle "Lions: Life in the Pride" ;
   dc:subject "bk:Lion"^^xsd:anyURI .
bk:LionsLifeInThePrideBook fa:type1 bk:Lion ;
   bk:bookTitle "The African Lion" ;
   dc:subject "bk:AfricanLion"^^xsd:anyURI .
```

We recall that in Example 4.2 dc:subject is used as an object property; its values include class URIs bk:Lion and bk:AfricanLion, which are interpreted as classes (arbitrary sets of resources). Here the values of dc:subject are interpretations of typed literals “bk:Lion”^^xsd:anyURI and “bk:AfricanLion”^^xsd:anyURI, viz. class URIs bk:Lion and bk:AfricanLion. The latter interpretation is more appropriate. As values of information properties, typed or plain literals can properly represent external agreements because they have fixed interpretations, which classes do not have.

Since the result of classification of such an RDFS(FA) ontology can be represented as partial orderings of class URIs (such as bk:AfricanLion < bk:Lion < bk:Animal), we can make use of such result when retrieving all books about bk:Lion from a repository, i.e., by retrieving books that are annotated (through dc:subject) with bk:Lion and books annotated with bk:AfricanLion.

◊
4.1.2 Semantics

Let us introduce the design philosophy of RDFS(FA), before moving on to the formal semantics of RDFS(FA).

Design Philosophy

The design of RDFS(FA) embodies two main principles:

1. In RDFS(FA), RDF is used (only) as standard syntax for annotations, i.e., the built-in semantics for RDF triples are disregarded, and new semantics is given to RDFS(FA) triples.

2. RDFS(FA) provides various Web resources with DL-style semantics.

In RDFS(FA), we distinguish seven groups of Web resources: object resources (or objects for short), class resources (or classes), datatype resources (or datatypes), abstract property resources (or abstract properties), datatype property resources (or datatype properties), annotation property resources (or annotation properties) and instance-of property resources (or instance-of properties). The URIrefs for the six groups of Web resources are pairwise disjoint.

Objects Informally speaking, an object URIref is interpreted as an object; e.g., bk: TheAfricanLionBook is interpreted as an object.

Classes Informally speaking, a class URIref is interpreted as a set of objects or sets in the adjacent lower layer. These objects (or sets) are called the instances of the class. In the Ontology Layer, a class URIref (such as bk:Book) is interpreted as a set of objects in the Instance Layer (including, e.g., the interpretation of the object URIref bk: TheAfricanLionBook). In the Language Layer, a class URIref (such as wns:LexicalConcept) is interpreted as a set of sets that are in the Ontology Layer (such as the interpretation of the class URIrefs bk: Book).

Datatypes The semantics of a datatype in RDFS(FA) is similar to that in RDF(S), except that a datatype is not a class.
4.1. RDFS(FA): A DL-ISED SUB-LANGUAGE OF RDFS

Abstract Properties  Informally speaking, an abstract property URIref is interpreted as a set of binary relationships (or pairs) between instances of two classes that are in the same stratum as the property. In the Ontology Layer, an abstract property URIref (such as elp:liveIn) is interpreted as a set of binary relationships between instances of the interpretations of the class URIrefs (such as elp:Elephant and elp:Habitat) in the Ontology Layer. In the Language Layer, an abstract property URIref (such as wns:hyponymOf) is interpreted as a set of binary relationships between instances of the interpretations of the class URIrefs (such as wns:LexicalConcept) in the Language Layer.

Datatype Properties  Informally speaking, a datatype property URIref (such as elp:age) is interpreted as a set of binary relationships (or pairs) between instances of a class in the Ontology layer (such as elp:Animal) and data values (such as integers).

Annotation Properties  Annotation properties are similar to datatype properties in that their values are data values, but they are not bound to any strata. An annotation property URIref (such as dc:creator) is interpreted as a set of binary relationships between some resources and data values. In RDFS(FA), not only objects, but classes and properties can have (even share) annotation properties (such as dc:creator).

Instance-Of Properties  Informally speaking, an instance-of property is a built-in RDFS(FA) property that connects resources in two adjacent strata. For example, the instance-of property fa:type, which is used between stratum 0 and 1, is interpreted as a set of binary relationships between resources in stratum 0 (e.g., the interpretation of the object URIref bk:TheAfricanLionBook) and resources in stratum 1 (e.g., the interpretation of the class URIref bk:Book).

Interpretations

The semantics of RDFS(FA) starts with the notation of vocabulary. Instead of having a mixed vocabulary like that of RDF(S), RDFS(FA) provides a separated vocabulary as follows. For ease of presentation, this thesis does not cover blank nodes, which can be handled similar to the way that URI references are handled.

Definition 4.1. (RDFS(FA) Vocabulary) An RDFS(FA) vocabulary \( V \) consists of a set of literals \( V_L \), and seven sets of pairwise disjoint URI references, which are \( V_C \) (class URIrefs), \( V_D \) (datatype URIrefs), \( V_{AP} \) (abstract property URIrefs), \( V_{DP} \) (datatype
4.1. RDFS(FA): A DL-ISED SUB-LANGUAGE OF RDFS

property URIrefs), \( V_{ANP} \) (annotation property URIrefs), \( V_I \) (individual URIrefs) and \( V_S = \{ fa:\text{Literal}, fa:\text{type}_1, fa:\text{type}_2, \ldots \} \). \( V_C \) (\( V_{AP} \)) is divided into disjoint stratified subsets \( V_{C_1}, V_{C_2}, \ldots (V_{AP_1}, V_{AP_2}, \ldots) \) of class (abstract property) URIrefs in strata 1, 2, . . . , where we use a subscript \( i \) to indicate URI references in the stratum \( i \).

Let \( i \geq 0 \); the built-in class URIrefs of RDFS(FA) are \( fa:\text{Resource}_{i+1}, fa:\text{Class}_{i+2}, fa:\text{Property}_{i+2}, fa:\text{AbstractProperty}_{i+2}, fa:\text{DatatypeProperty} \) and \( fa:\text{AnnotationProperty} \); the built-in abstract property URIrefs of RDFS(FA) are \( fa:\text{subClassOf}_{i+2}, fa:\text{subPropertyOf}_{i+2}, fa:\text{domain}_{i+2} \) and \( fa:\text{range}_{i+2} \); the built-in annotation property URIrefs of RDFS(FA) are \( fa:\text{label}, fa:\text{comment}, fa:\text{seeAlso} \) and \( fa:\text{isDefinedBy} \); other built-in URIrefs of RDFS(FA) are those in \( V_S \). We use a superscript \( b (u) \) together with \( V_C, V_{AP} \) and their stratified subsets, to indicate the corresponding subsets of built-in (user-defined) URI references.

Formally, the semantics of RDFS(FA) individuals, classes, datatypes, abstract properties, datatype properties and typed literals is defined in terms of an interpretation as follows.

**Definition 4.2. (RDFS(FA) Interpretation)** Given an RDFS(FA) vocabulary \( V \), an RDFS(FA) interpretation w.r.t. a datatype map \( M_d \) is a tuple of the form \( \mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J}) \), where \( \Delta^\mathcal{J} \) is the domain (a non-empty set) and \( \cdot^\mathcal{J} \) is the interpretation function. Let \( \Delta^\mathcal{J}_A \) be the abstract domain (a non-empty set), \( i \) a non-negative integer, \( \Delta^\mathcal{J}_{A_i} \) the abstract domain in stratum \( i \) and \( \Delta^\mathcal{J}_D \) the domain (a non-empty set) for datatypes in a datatype map \( M_d \), \( \mathcal{J} \) satisfies the following conditions:

1. \( \Delta^\mathcal{J}_A = \bigcup_{i \geq 0} \Delta^\mathcal{J}_{A_i} \),
2. \( \Delta^\mathcal{J}_{A_i} \cap \Delta^\mathcal{J}_{A_j} = \emptyset \) \( (0 \leq i < j) \),
3. \( \Delta^\mathcal{J}_D \cap \Delta^\mathcal{J}_A = \emptyset \),
4. \( \bigcup_{u \in M_d(u)} V(d) \subseteq \Delta^\mathcal{J}_D \),
5. \( \Delta^\mathcal{J} = \Delta^\mathcal{J}_A \cup \Delta^\mathcal{J}_D \),
6. \( \forall a \in V_I : a^\mathcal{J} \in \Delta^\mathcal{J}_{A_0} \),
7. \( \forall C \in V_{C_{i+1}} : C^\mathcal{J} \subseteq \Delta^\mathcal{J}_{A_i} \),
8. \( \forall p \in V_{AP_{i+1}} : p^\mathcal{J} \subseteq \Delta^\mathcal{J}_{A_i} \times \Delta^\mathcal{J}_{A_i} \),
9. \( \forall n \in V_{ANP} : \langle x, y \rangle \in n^\mathcal{J} \rightarrow y \in \Delta^\mathcal{J}_D \),
10. \( \forall r \in V_{DP} : r^J \subseteq \Delta_{A_0}^J \times \Delta_D^J \).

11. \( \text{fa:type}_{i+1}^J \subseteq \Delta_{A_i}^J \times \text{fa:Class}_{i+2}^J \).

12. \( \text{fa:Literal}^J = \Delta_D^J \).

13. \( \text{fa:Resource}_{i+1}^J = \Delta_{A_i}^J \).

14. \( \forall C \in V_{C_{i+1}} : C^J \in \text{fa:Class}_{i+2}^J \).

15. \( \forall p \in V_{AP_{i+1}} : p^J \in \text{fa:AbstractProperty}_{i+2}^J \).

16. \( \forall r \in V_{DP} : r^J \in \text{fa:DatatypeProperty}^J \).

17. \( \forall n \in V_{ANP} : n^J \in \text{fa:AnnotationProperty}^J \).

18. \( \text{fa:Class}_{i+2}^J \subseteq \text{fa:Resource}_{i+2}^J \) and \( \text{fa:Property}_{i+2}^J \subseteq \text{fa:Resource}_{i+2}^J \).

19. \( \text{fa:AbstractProperty}_{i+2}^J \subseteq \text{fa:Property}_{i+2}^J \) and \( \text{fa:DatatypeProperty}^J \subseteq \text{fa:Property}_{2}^J \).

20. \( \forall u \in V_D, \text{if } M_d(u) = d, \text{then} \)

   (a) \( u^J = V(d) \),

   (b) \( \text{if } v \in L(d), \text{then } ("v"^u)^J = L2V(d)(v) \),

   (c) \( \text{if } v \notin L(d), \text{then } ("v"^u)^J \text{ is undefined;}^1 \)

   otherwise, \( u^J \subseteq \Delta_D \) and \( "v"^u \in \Delta_D \). \( \diamond \)

There are some remark on Definition 4.2:

1. Conditions 1-5 require that the domain (of universe) \( \Delta^J \) in RDFS(FA) is disjunctively divided into the abstract domain \( \Delta_A^J \) and the datatype domain \( \Delta_D^J \) (cf. Figure 4.4 on page 92), where \( \Delta_A^J \) is further disjunctively divided into sub-abstract domains \( \Delta_{A_i}^J \) in different strata (layers) and \( \Delta_D^J \) is a super-set of the union of the value spaces of all the datatypes in \( M_d \).

\(^1\)The reader is invited to note that there is a tiny difference between OWL and RDF datatyping in handling typed literals with invalid lexical forms. Like RDFS(FA), OWL datatyping treats them as contradictions; RDF datatyping does not, but interprets them as some non-data-valued objects; cf. Section 3.3.
2. Condition 6-11 describe the interpretations of individuals, classes, abstract properties, annotation properties, datatype properties and instance-of properties, respectively. For example, Condition 6 ensures that each individual URIref \( a \in V_1 \) are interpreted as a member of \( \Delta_{A_0} \). We invite the reader to refer to the ‘Design Philosophy’ section (on page 87) for informal descriptions of these interpretations.

3. Conditions 12-19 are extra semantic constraints on the built-in URIrefs in \( V_S \) and \( V_C \). Condition 12 ensures that \( \text{fa:Literal} \) is interpreted as the datatype domain \( \Delta_D \), while condition 13 ensures that \( \text{fa:Resource}_{i+1} \) is interpreted as the abstract domain \( \Delta_{A_i} \). Conditions 14-17 ensures that the interpretations of \( \text{fa:Class}_{i+2} \), \( \text{fa:AbstractProperty}_{i+2} \), \( \text{fa:DatatypeProperty}_{i+2} \) and \( \text{fa:AnnotationProperty} \) should contain the interpretations of corresponding URI references. Condition 18 ensure that classes and properties are resources in corresponding strata; condition 19 ensures that abstract properties and properties in corresponding strata, and that datatype properties are in stratum 2.

4. Condition 20 ensures the valid interpretations of datatypes and typed literals. (20a) ensures that a datatype URIref in the datatype map \( M_d \) is interpreted as the value space of the corresponding datatype. (20b)-(20c) ensure that valid typed literals are interpreted as members of the value space of corresponding datatypes, and ill-typed literals cause contradictions. Note that the interpretations of datatype URIrefs that are not in \( M_d \) are interpreted as unknown subsets of \( \Delta_D \).

It is straightforward, but important, to observe that the above interpretation of RDFS(FA) is similar to that of DLs (cf. Section 2.1.1), except that it also provides interpretations for meta-classes, meta-properties and annotation properties, which are useful in SW applications (cf. Example 4.3 and Example 4.4).

Figure 4.4 illustrates the interpretation of RDFS(FA):

- Typed literals (such as “30”^^xsd:integer) are interpreted as values in the value space corresponding datatypes (such as \( V(\text{integer}) \)). All value spaces of datatypes in \( M_d \) are subset of \( \Delta_D \).

- The datatype domain is disjoint with the abstract domain, which is stratified into sub-abstract domains (\( \Delta_{A_0} \), \( \Delta_{A_1} \), etc.).
In stratum 0 (the Instance Layer), object URIrefs (e.g., `elp:Ganesh` and `elp:south-sahara`) are interpreted as objects (i.e., resources in \( \Delta^J_{A_0} \)).

In stratum 1 (the Ontology Layer), class URIrefs (such as `elp:Elephant` and `elp:Habitat`) are interpreted as sets of objects. Abstract property URIrefs (such as `elp:liveIn`) are interpreted as sets of pairs of objects. Datatype property URIrefs (such as `elp:age`) are interpreted as a set of pairs where the first resource (e.g., `elp:Ganesh`) is an object, and the second resource is a datatyped value (e.g., the integer 30).

In stratum 2 (the Language Layer), fa:Class\(_2\) is interpreted as a set of sets of objects, and fa:AbstractProperty\(_2\) is interpreted as a set of sets of pairs of objects.

### 4.1.3 RDFS(FA) Ontologies

Informally speaking, an RDFS(FA) ontology is a set of RDFS(FA) axioms, which are basically RDF triples (in N3 syntax)\(^2\) with extra syntactic rules to disallow arbitrary use of its built-in vocabulary and to allow the use of meta-classes and meta-properties.

\(^2\)Here we use the N3 syntax, instead of the RDF/XML syntax, as it is more compact.
in specified layers and the use of annotation properties.

**Definition 4.3. (RDFS(FA) Ontologies)** Given an RDFS(FA) vocabulary $V$, let $i$ be a non-negative integer, $a, b \in V_I$, $D_i \in V_{C_1}$, $C \in V_{C_{i+1}}$, $D \in V_{C_{i+1}}$, $H \in V_{C_{i+2}}$, $p_1 \in V_{AP_1}^u$, $p \in V_{AP_{i+1}}^u$, $q \in V_{AP_{i+1}}$, $r, s \in V_{DP}$, $q' \in V_{AP_{i+2}}^u$, $u \in V_D$, $X, Y \in V_{C_{i+1}} \cup V_{AP_{i+1}}$, $n \in V_{ANP}$ and $w \in V \setminus V_L$.

An RDFS(FA) ontology is a finite, possibly empty, set of axioms of the form:

1. $[C \text{fa:subClassOf}_{i+2} D.]$, called class inclusions,
2. $[p \text{fa:subPropertyOf}_{i+2} q.]$, called abstract property inclusions;
3. $[r \text{fa:subPropertyOf}_2 s.]$, called datatype property inclusions;
4. $[p \text{fa:domain}_{i+2} D.]$, called abstract property domain restrictions;
5. $[r \text{fa:domain}_2 D_1.]$, called datatype property domain restrictions;
6. $[p \text{fa:range}_{i+2} D.]$, called abstract property range restrictions;
7. $[r \text{fa:range}_2 u.]$, called datatype property range restrictions;
8. $[a \text{fa:type}_1 D_1.]$, called class assertions,
9. $[a p_1 b.]$, called abstract property assertions,
10. $[a r "v" ^u.]$, called datatype property assertions,
11. $[X \text{fa:type}_{i+2} H.]$, called meta class assertions,
12. $[X q' Y.]$, called meta abstract property assertions,
13. $[w n "v" ^u.]$, called annotation property assertions,
14. $[n \text{rdf:type fa:AnnotationProperty}.]$, called annotation property declarations.

An interpretation $\mathcal{J}$ satisfies an RDFS(FA) axiom $\varphi$, written as $\mathcal{J} \models \varphi$, if $\mathcal{J}$ meets certain semantic conditions:

1. $\mathcal{J} \models [C \text{fa:subClassOf}_{i+2} D.]$ if $C^\mathcal{J} \subseteq D^\mathcal{J}$;
2. $\mathcal{J} \models [p \text{fa:subPropertyOf}_{i+2} q.]$ if $p^\mathcal{J} \subseteq q^\mathcal{J}$;
3. $\mathcal{J} \models [r \text{fa:subPropertyOf}_2 s.]$ if $r^\mathcal{J} \subseteq s^\mathcal{J}$;
4.1. RDFS(FA): A DL-ISED SUB-LANGUAGE OF RDFS

4. \( \mathcal{J} \models [p \ fa:domain_{i+2} D] \) if \( \forall x. \langle x, y \rangle \in p^\mathcal{J} \rightarrow x^\mathcal{J} \in D^\mathcal{J} \);

5. \( \mathcal{J} \models [r \ fa:domain_2 D_1] \) if \( \forall x. \langle x, t \rangle \in r^\mathcal{J} \rightarrow x^\mathcal{J} \in D_1^\mathcal{J} \);

6. \( \mathcal{J} \models [p \ fa:range_{i+2} D] \) if \( \forall y. \langle x, y \rangle \in p^\mathcal{J} \rightarrow y^\mathcal{J} \in D^\mathcal{J} \);

7. \( \mathcal{J} \models [r \ fa:range_2 u] \) if \( \forall t. \langle x, t \rangle \in r^\mathcal{J} \rightarrow t^\mathcal{J} \in u^\mathcal{J} \);

8. \( \mathcal{J} \models [a \ fa:type_1 C_1] \) if \( a^\mathcal{J} \in C_1^\mathcal{J} \);

9. \( \mathcal{J} \models [a \ p_1 \ b] \) if \( \langle a^\mathcal{J}, b^\mathcal{J} \rangle \in p_1^\mathcal{J} \);

10. \( \mathcal{J} \models [a \ r \ "v"^u u] \) if \( \langle a^\mathcal{J}, ("v"^u)^\mathcal{J} \rangle \in r^\mathcal{J} \);

11. \( \mathcal{J} \models [X \ fa:type_{i+2} H] \) if \( X^\mathcal{J} \in H^\mathcal{J} \);

12. \( \mathcal{J} \models [X \ q' Y] \) if \( \langle X^\mathcal{J}, Y^\mathcal{J} \rangle \in q'^\mathcal{J} \);

13. \( \mathcal{J} \models [w \ n \ "v"^u u] \) if \( ("v"^u)^\mathcal{J} \in \Delta_D \).

14. \( \mathcal{J} \models [n \ rdf:type \ fa:AnnotationProperty.] \) if \( n^\mathcal{J} \in fa:AnnotationProperty^\mathcal{J} \).

An interpretation \( \mathcal{J} \) satisfies an ontology \( \mathcal{O} \), written as \( \mathcal{J} \models \mathcal{O} \), if and only if it satisfies all the axioms in \( \mathcal{O} \); \( \mathcal{O} \) is satisfiable (unsatisfiable), written as \( \mathcal{O} \models \top \) (\( \mathcal{O} \models \bot \)), if there exists (does not exist) such an interpretation \( \mathcal{J} \).

Given an RDFS(FA) axiom \( \varphi \), \( \mathcal{O} \) entails \( \varphi \), written as \( \mathcal{O} \models \varphi \), if for all models \( \mathcal{J} \) of \( \mathcal{O} \) we have \( \mathcal{J} \models \varphi \). An ontology \( \mathcal{O} \) entails an ontology \( \mathcal{O}' \), written as \( \mathcal{O} \models \mathcal{O}' \), if for all models \( \mathcal{J} \) of \( \mathcal{O} \) we have \( \mathcal{J} \models \mathcal{O}' \). Two ontologies \( \mathcal{O} \) and \( \mathcal{O}' \) are equivalent, written as \( \mathcal{O} \equiv \mathcal{O}' \), if \( \mathcal{O} \models \mathcal{O}' \) and \( \mathcal{O}' \models \mathcal{O} \).

We invite the reader to note that RDFS(FA) axioms of the form 1-8 and 11 are RDFS statements with extra (subscript) information specifying the strata that the related resources belong to. For example, \( [C \ fa:subClassOf_{i+2} D.] \) requires that the concepts \( C \) and \( D \) should be on stratum \( i+1 \). Furthermore, RDFS(FA) provides the use of three kinds of properties: abstract properties, datatype properties and annotation properties (cf. RDFS(FA) axioms of the form 9,12, 10 and 13). Last but not least, let us point out that rdf:type is used in annotation property declarations because annotation property are not bound to any stratum.

The interpretation of class inclusions, property inclusions in stratum 1 as well as class assertions and property assertions are exactly the same as the corresponding DL axioms that we have seen in Chapter 2. RDFS(FA) meta-axioms are very similar to the
4.2. RDFS(FA) AND OWL

In this section, we show that it is much easier to layer OWL DL, syntactically and semantically, on top of RDFS(FA) than on top of RDF(S).

Figure 4.5: An RDFS(FA) ontology

above, except that they apply on classes and properties that are higher than stratum 1. RDFS(FA) annotation property assertions require that values of annotation properties should be data values in the datatype domain.

Figure 4.5 shows an example RDFS(FA) ontology (cf. Figure 3.3 for the RDFS version). Firstly, the layering structure is clear. `elp:Animal`, `elp:Habitat`, `elp:Elephant` and `elp:liveIn` are in stratum 1 (the Ontology layer), while `elp:Ganesh` and `elp:south-sahara` are in stratum 0 (the Instance Layer). Secondly, RDFS(FA) disallows arbitrary use of its built-in vocabulary. For example, in class inclusion axioms, the subjects can only be only user-defined class URIrefs (such as `elp:Animal`), which could disallow triples like

```
@prefix fa: <http://dl-web.man.ac.uk/rdfsfa/ns#>
@prefix elp: <http://example.org/Animal#>

elp:Animal fa:type2 fa:Class2 .
elp:Habitat fa:type2 fa:Class2 .
elp:Elephant fa:type2 fa:Class2 ; fa:subClassOf2 elp:Animal .
elp:liveIn fa:type2 fa:AbstractProperty2 ;
    fa:domain2 elp:Animal ; fa:range2 elp:Habitat .
elp:south-sahara fa:type1 elp:Habitat .
elp:Ganesh fa:type1 elp:Elephant ; elp:liveIn elp:south-sahara .
```

Furthermore, RDFS(FA) allows users to specify classes and properties in specified strata. For example, the class inclusion axiom

```
elp:Elephant fa:subClassOf2 elp:Animal .
```

requires that both `elp:Elephant` and `elp:Animal` are class URIrefs in stratum 1. More about the support of meta-classes and the use of class URIrefs as values of annotation properties can be found in Example 4.3 and Example 4.4, respectively.

4.2 RDFS(FA) and OWL

In this section, we show that it is much easier to layer OWL DL, syntactically and semantically, on top of RDFS(FA) than on top of RDF(S).
There is a one-to-one bidirectional mapping (as shown in Table 4.1) between the RDFS(FA) axioms in strata 0-2 and OWL DL axioms in OWL abstract syntax. For example, the RDFS(FA) class inclusion axiom \([\text{C}_1 \text{ fa:subClassOf} \text{ D}_1 .] \) can be mapped to the OWL class axiom (SubClassOf \( \text{C}_1 \text{ D}_1 \)) and vice versa.

It is easier, therefore, to syntactically layer OWL DL on top of RDFS(FA) than on top of RDF(S), due to the above bidirectional mapping. According to the OWL Semantics and Abstract Syntax document [119], the mapping between OWL DL axioms, or OWL axioms for short, and RDF(S) statements is only unidirectional, i.e., from OWL axioms to RDF(S) statements. For example, we can map the following OWL axiom (SubClassOf \( \text{C}_1 \text{ D}_1 \)) to the RDF(S) statement \([\text{C}_1 \text{ rdfs:subClassOf} \text{ D}_1 .] \), with an implicit OWL constraint, viz., \( \text{C}_1 \) and \( \text{D}_1 \) can only be class URIs, but not URIs for properties or individuals, etc. An RDF(S) statement \([\text{C}_1 \text{ rdfs:subClassOf} \text{ D}_1 .] \) without such constraint, however, cannot be correctly mapped to the OWL axiom (SubClassOf \( \text{C}_1 \text{ D}_1 \)). Interestingly, in the corresponding RDFS(FA) axioms these kinds of implicit constraints are made explicit via the syntactic constraints of the RDFS(FA) class axioms (cf. Definition 4.3). For example, the RDFS(FA) class inclusion axiom \([\text{C}_1 \text{ fa:subClassOf} \text{ D}_1 .] \) (in place of \([\text{C}_1 \text{ rdfs:subClassOf} \text{ D}_1 .] \)) requires that both \( \text{C}_1 \) and \( \text{D}_1 \) are class URIs in stratum 1. This explains why the above bidirectional mapping (listed in Table 4.1) is possible.

Furthermore, it can be shown (by the following theorem) that the above bidirectional mapping is a semantics-preserved mapping.

**Theorem 4.4.** The bidirectional mapping, shown in Table 4.1, between the RDFS(FA) axioms in strata 0-2 and the corresponding OWL axioms in the OWL abstract syntax

<table>
<thead>
<tr>
<th>RDFS(FA) Axioms</th>
<th>OWL Axioms (Abstract Syntax)</th>
<th>OWL Axioms (RDF Syntax)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [\text{C}_1 \text{ fa:subClassOf} \text{ D}_1 .] )</td>
<td>SubClassOf((\text{C}_1 \text{ D}_1) )</td>
<td>( [\text{C}_1 \text{ rdfs:subClassOf} \text{ D}_1 .] )</td>
</tr>
<tr>
<td>( [\text{p}_1 \text{ fa:subPropertyOf} \text{ q}_1 .] )</td>
<td>SubPropertyOf((\text{p}_1 \text{ q}_1) )</td>
<td>( [\text{p}_1 \text{ rdfs:subPropertyOf} \text{ q}_1 .] )</td>
</tr>
<tr>
<td>( [\text{r}_1 \text{ fa:domain} \text{ D}_1 .] )</td>
<td>ObjectProperty((\text{r}_1 \text{ domain(D}_1)) )</td>
<td>( [\text{r}_1 \text{ rdfs:domain} \text{ D}_1 .] )</td>
</tr>
<tr>
<td>( [\text{r}_1 \text{ fa:range} \text{ D}_1 .] )</td>
<td>DatatypeProperty((\text{r}_1 \text{ range(D}_1)) )</td>
<td>( [\text{r}_1 \text{ rdfs:range} \text{ D}_1 .] )</td>
</tr>
<tr>
<td>( [\text{r}_1 \text{ fa:range} \text{ u .}] )</td>
<td>Individual((a \text{ value(r}_1 \text{ u.})) )</td>
<td>( [\text{r}_1 \text{ rdfs:range} \text{ u .}] )</td>
</tr>
<tr>
<td>( [\text{fa:subClassOf} \text{ C}_1 .] )</td>
<td>Individual((a \text{ type(C}_1)) )</td>
<td>( [\text{a rdfs:subType} \text{ C}_1 .] )</td>
</tr>
<tr>
<td>( [\text{fa:subResource} \text{ C}_1 .] )</td>
<td>Individual((a \text{ value(C}_1)) )</td>
<td>( [\text{a rdfs:subResource} \text{ C}_1 .] )</td>
</tr>
<tr>
<td>( [\text{fa:Class} \text{ C}_1 .] )</td>
<td>Class((\text{C}_1) )</td>
<td>( [\text{C}_1 \text{ rdfType} \text{ rdf:Class .}] )</td>
</tr>
<tr>
<td>( [\text{fa:AbstractProperty} \text{ p}_1 .] )</td>
<td>ObjectProperty((\text{p}_1) )</td>
<td>( [\text{p}_1 \text{ rdfType} \text{ rdf:ObjectProperty .}] )</td>
</tr>
<tr>
<td>( [\text{fa:DatatypeProperty} \text{ p}_1 .] )</td>
<td>DatatypeProperty((\text{p}_1) )</td>
<td>( [\text{p}_1 \text{ rdfType} \text{ rdf:DatatypeProperty .}] )</td>
</tr>
</tbody>
</table>
is a satisfiability-preserved map.

**Proof:** Given a datatype map \( M_d \), we only need to show that there exists an interpretation \( J \) satisfying all the listed RDFS(FA) axioms iff there exists an interpretation \( I \) satisfying all the corresponding OWL DL axioms.

For the *only-if* direction, given an RDFS(FA) interpretation \( J = (\Delta^J, \cdot^J) \) for \( V \) w.r.t. \( M_d \), we can construct an OWL DL interpretation \( I = (\Delta^I, \cdot^I) \) as follows:

- \( \Delta^I = \Delta^J_{A_0} \) and \( \Delta^D_{owl} = \Delta^D_{fa} \); for each class URIref (in stratum 1) \( C, C^I = C^J \); for each datatype URIref (in stratum 1) \( u, u^I = u^J \); for each abstract (object) property URIref \( p \) (in stratum 1), \( p^I = p^J \); for each datatype property URIref \( r \), \( r^I = r^J \).

Now we only need to show that if \( J \) satisfies an RDFS(FA) axiom \( \phi_1 \) in the first column of Table 4.1, we have \( I \) satisfies the corresponding OWL DL axiom \( \phi_2 \) in the second column of Table 4.1. According to the semantics of RDFS(FA) (Definition 4.2 on page 89) and RDFS(FA) axioms (Definition 4.3 on page 93), the semantics of OWL constructs (Tables 3.1, datatypes (Definition 3.12 on page 75) 3.2 and 3.6) and axioms (Tables 3.3 and 3.5), this is trivially true. Therefore, we only give the proof for the class inclusion axiom to illustrate the proofs for the rest: if \( J \models [C_1 fa:subClassOf D_1 ] \), according to Definition 4.3, we have \( C_1^J \subseteq D_1^J \), hence \( C_1^I \subseteq D_1^I \). Thus, \( I \models \text{SubClassOf}(C_1 D_1) \).

Similarly, the *if* direction is trivially true, we only need to show that, in an RDFS(FA) interpretation \( J \), we can construct abstract domains for strata higher than stratum 0. Let \( i \geq 0 \). According to the semantics conditions 7, 8, 13 to 19 in Definition 4.2, we have \( fa:Class_{i+2}^J = 2^{\Delta^J_{A_{i+2}}} \), \( fa:Property_{2}^J = 2^{\Delta^J_{A_0}} \times \Delta^J_{A_0} \cup 2^{\Delta^J_{A_0}} \times \Delta^D_{fa} \), \( fa:Property_{i+3}^J = 2^{\Delta^J_{A_{i+2}}} \times \Delta^J_{A_{i+1}} \) and \( \Delta^J_{A_{i+1}} = fa:Resource_{i+2} = fa:Class_{i+2}^J \cup fa:Property_{i+2}^J \). Hence we have \( \Delta^J_{A_1} = 2^{\Delta^J_{A_0}} \cup 2^{\Delta^J_{A_0}} \times \Delta^J_{A_0} \cup 2^{\Delta^J_{A_0}} \times \Delta^D \) and \( \Delta^J_{A_{i+2}} = 2^{\Delta^J_{A_{i+1}}} \cup 2^{\Delta^J_{A_{i+1}}} \times \Delta^J_{A_{i+1}} \).

We claim that OWL DL can be semantically layered on top of RDFS(FA). Firstly, [113] shows that RDFS(FA) does not have the semantic problems [113, 116, 117, 69] that RDF(S) has, when we layer OWL on top of it. Secondly, OWL DL reserves the semantics of RDFS(FA) built-in primitives; e.g., Table 4.2 shows that owl:Thing is equivalent to \( fa:Resource_1 \). Table 4.3 shows that OWL DL uses some RDFS modelling primitives with RDFS(FA) semantics, instead of RDFS semantics. Furthermore, OWL DL extends RDFS(FA) in strata 0-2 by introducing new class descriptions (such as class intersections), new property descriptions (such as inverse properties) and new axioms (such as functional axioms for properties). Most importantly, Theorem 4.4 shows that OWL DL preserves the meaning of the RDFS(FA) axioms in strata 0-2.
4.3 A CLARIFIED VISION OF THE SEMANTIC WEB

Table 4.2: OWL DL preserves the semantics of built-in RDFS(FA) primitives

<table>
<thead>
<tr>
<th>OWL Modelling Primitives</th>
<th>RDFS(FA) Modelling Primitives</th>
</tr>
</thead>
<tbody>
<tr>
<td>owl:Thing</td>
<td>fa:Resource&lt;sub&gt;1&lt;/sub&gt;</td>
</tr>
<tr>
<td>owl:Class</td>
<td>fa:Class&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>owl:ObjectProperty</td>
<td>fa:AbstractProperty&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>owl:DatatypeProperty</td>
<td>fa:DatatypeProperty</td>
</tr>
</tbody>
</table>

Table 4.3: OWL DL uses RDFS primitives with RDFS(FA) semantics

<table>
<thead>
<tr>
<th>RDFS Modelling Primitives</th>
<th>RDFS(FA) Modelling Primitives</th>
</tr>
</thead>
<tbody>
<tr>
<td>rdfs:subClassOf</td>
<td>fa:subClassOf&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>rdfs:subPropertyOf</td>
<td>fa:subPropertyOf&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>rdfs:domain</td>
<td>fa:domain&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td>rdfs:range</td>
<td>fa:range&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
</tbody>
</table>

shown in Table 4.1. Last but not least, annotation properties and ontology properties in OWL DL extends RDFS(FA) annotation properties, relating resources with not only data literal but also URI references. It is possible to have a new sub-language of OWL — OWL FA, which extends OWL DL with meta classes/properties and provide better support for annotation properties.

Although Theorem 4.4 indicates that we can use DL reasoners to reason with RDFS(FA) axioms in strata 0-2, or possibly in every three adjacent strata, providing reasoning support for RDFS(FA) is beyond the scope of this thesis.

4.3 A Clarified Vision of the Semantic Web

In this chapter we have presented RDFS(FA), an alternative to RDFS with a DL-style semantics, so as to repair the broken link between RDF(S) and OWL.

RDFS(FA), consequently, provides a clarified vision of the Semantic Web: RDF is only a standard syntax for SW annotations and languages (i.e., the built-in semantics of RDF triples is disregarded), and the meaning of annotations comes from either external agreements (such as Dublin Core) or ontologies (which are more flexible), both of which are supported by RDFS(FA).

On the one hand, RDFS(FA) is an ontology language that provides a UML-like layered style for using RDFS; it is more intuitive, and easier to understand and use by

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<sup>3</sup>We are not convinced, however, that it is proper to use annotation properties to relate resources with URI references.
users. Most importantly, strata 0-2 have a standard model theoretic semantics, s.t. more expressive FOL ontology languages, such as the W3C standard OWL, can be layered on top of them and are compatible with RDFS(FA)’s metamodeling architecture.

On the other hand, RDFS(FA) allows the use of Dublin Core information properties as annotation properties. In RDFS(FA), all resources can have annotation properties, such that ‘anyone can say anything about anything’. Typed literals are used to precisely represent values of annotation properties, such as “1999-05-31”^^xsd:date for the dc:date property and “bk:Lion”^^xsd:anyURI for the dc:subject property. In particular, the use of URIrefs as values of annotation properties can enable SW applications to make use of URIrefs of ontology elements, such as classes, in the results of various ontology inferences (cf. Example 4.4 on page 85).

In general, introducing RDFS(FA) as a sub-language of RDF(S) makes it more flexible to layer languages on top of RDF(S), which surely solidifies RDF(S)’s proposed role as the base of the Semantic Web. The Semantic Web tower has become clearer, easier to understand and formalise. The remaining layers of the Semantic Web and the extensions of OWL, such as the Semantic Web Rule Language (SWRL), can be more easily defined in the metamodeling architecture of RDFS(FA).

Having addressed the layering problem and clarified the vision of the Semantic Web, can we now use the DL reasoning services to completely support OWL and DAML+OIL? Not yet — we still need to solve the second problem mentioned in Section 3.4, i.e., the problem of how to extend DL datatype reasoning to support OWL, DAML+OIL and their extensions that provide customised datatypes and datatype predicates. We are going to investigate this problem and provide a solution in the coming chapters.
Chapter Achievements

- RDFS(FA) is an ontology language that has well-formed syntax, DL-style model theoretic semantics and provides a layered style of using RDFS. It provides a firm semantic foundation for DL-based Web ontology languages.

- The bidirectional one-to-one mapping between RDFS(FA) axioms in strata 0-2 and OWL DL axioms enables RDFS(FA)-agents and OWL DL-agents to more easily communicate with each other.

- Annotation properties in RDFS(FA) enable SW applications to make use of URIrefs of ontology elements, such as classes, in results of various ontology inferences.

- Introducing RDFS(FA) as a sub-language of RDFS clarifies the vision of the Semantic Web and solidifies RDF(S)’s proposed role as the base of the Semantic Web.
Chapter 5

A Unified Formalism for Datatypes and Predicates

Chapter Aims:

- To propose a general formalism to unify existing ontology-related datatype and predicate formalisms and to overcome their limitations.
- To investigate, in our formalism, a family of decidable Description Logics that provide customised datatypes and datatype predicates.

Chapter Plan

5.1 Datatype Groups (101)
5.2 Integrating DLs with Datatype Groups (117)
5.3 Related Work (124)

This chapter introduces the datatype group approach, which provides a unified formalism for datatypes and predicates, so as to support all the new datatype features of OWL and to enable combining DLs with customised datatypes and predicates.

5.1 Datatype Groups

Existing ontology-related formalisms either focus on either datatypes (such as RDF(S) and OWL datatyping) or predicates (such as the concrete domain and the type system approach). The datatype group approach to be presented in this chapter provides a
unified formalism (Section 5.1.1) for datatypes and datatype predicates. In this formalism, datatype expressions (Section 5.1.2) can be constructed to represent customised datatypes and datatype predicates, which are very useful in SW and ontology applications (cf. Section 1.3).

5.1.1 Unifying Datatypes and Predicates

Datatypes and datatype predicates are closely related to each other, but they serve different purposes. A datatype $d$ is characterised by its lexical space $L(d)$, value space $V(d)$ and lexical-to-value mapping $L2V(d)$ (cf. Definition 3.6 on page 71). It can be used to represent its member values through typed literals (cf. Definition 3.7 on page 71). A datatype predicate, however, is defined otherwise.

**Definition 5.1. (Datatype Predicate)** A *datatype predicate* (or simply *predicate*) $p$ is characterised by an arity $a(p)$, or a minimum arity $a_{\text{min}}(p)$ if $p$ can have multiple arities, and a predicate extension (or simply *extension*) $E(p)$.

Predicates are mainly used to represent constraints over values of datatypes which they are defined over. Here are some examples:

**Example 5.1 Datatype Predicates**

1. $>^{\text{int}}_{[20]}$ is a (unary) predicate with $a(>^{\text{int}}_{[20]}) = 1$ and $E(>^{\text{int}}_{[20]}) = \{ i \in V(\text{integer}) \mid i > L2V(\text{integer})("20") \}$. This example shows that predicates are defined based on datatypes (e.g., $\text{integer}$) and their values (e.g., the integer $L2V(\text{integer})("20")$, i.e., 20).

2. $=^{\text{int}}$ is a (binary) predicate with arity $a(=^{\text{int}}) = 2$ and extension $E(=^{\text{int}}) = \{ \langle i_1, i_2 \rangle \in V(\text{integer})^2 \mid i_1 = i_2 \}$.  

3. $+^{\text{int}}$ is a predicate with minimum arity $a_{\text{min}}(+^{\text{int}}) = 3$ and extension $E(+^{\text{int}}) = \{ \langle i_0, \ldots, i_n \rangle \in V(\text{integer})^n \mid i_0 = i_1 + \ldots + i_n \}$.  

Datatypes can be regarded as *special* predicates with arity 1 and predicate extensions equal to their value spaces; e.g., the datatype $\text{integer}$ can be seen as a predicate with arity $a(\text{integer}) = 1$ and predicate extension $E(\text{integer}) = V(\text{integer})$. They

\[1\text{Note that some XML Schema user-derived datatypes (such as ‘GreaterThan20’) defined in Example 3.2 on page 70) can represent unary predicates (such as } >^{\text{int}}_{[20]} \text{), but they can not represent predicates with arities greater than 1, such as } =^{\text{int}} \text{ and } +^{\text{int}}.\]
are special because they have lexical spaces and lexical-to-value mappings that ordinary predicates do not have.

Based on the above connection between datatypes and predicates, we can extend the definitions of datatype maps (cf. Section 3.3.2), supported/unsupported datatype URIrefs to predicate maps and supported/unsupported predicate URIrefs as follows.

**Definition 5.2. (Predicate Map)** We consider a predicate map $M_p$ that is a partial mapping from predicate URI references to predicates.

**Example 5.2 Predicate Map** $M_{p_1} = \{ \langle \text{xsd:string}, \text{string} \rangle, \langle \text{xsd:integer}, \text{integer} \rangle, \langle \text{owlx:integerEquality}, \text{=}^{\text{int}} \rangle, \langle \text{owlx:integerGreaterThanx=n}, >^{\text{int}}[^{\text{int}}]n \rangle \}$, where xsd:string, xsd:integer, owlx:integerEquality and owlx:integerGreaterThanx=n are predicate URI references, string, integer and $>^{\text{int}}[^{\text{int}}]n$ are unary predicates, and $=^{\text{int}}$ is a binary predicate. Note that, by `$>^{\text{int}}[^{\text{int}}]n$’, we mean there exist a predicate $>^{\text{int}}[^{\text{int}}]n$ for each integer $L2V(\text{integer})("n")$, which is represented by the predicate URIref owlx:integerGreaterThanOrx=n.

In general, datatype predicates like $>^{\text{int}}[^{\text{int}}]n$ can be seen as parameterised predicates, where $n$ can be seen as a parameter. In this thesis, we use the “=" character to separate the lexical forms of a parameter from the rest of a URIref of a parameterised predicate. For example, in owlx:integerGreaterThanx=20, the lexical form of the parameter is “20”.

**Definition 5.3. (Supported and Unsupported Predicates)** Given a predicate map $M_p$, a predicate URIref $u$ is called a supported predicate URIref w.r.t. $M_p$ (or simply supported predicate URIref), if there exists a predicate $p$ s.t. $M_p(u) = p$ (in this case, $p$ is called a supported predicate w.r.t. $M_p$); otherwise, $u$ is called an unsupported predicate URIref w.r.t. $M_p$ (or simply unsupported predicate URIref).

For example, owlx:integerEquality is a supported predicate URIref w.r.t. $M_{p_1}$ presented in Example 5.2, while owlx:integerInequality is an unsupported predicate URIref w.r.t. $M_{p_1}$. Therefore, according to $M_{p_1}$, we know neither the arity nor the extension of the predicate that owlx:integerInequality represents. Note that we make as few assumptions as possible about unsupported predicates; e.g., we do not even assume that they have a fixed arity.

Now we provide the definition of datatype groups which unifies datatypes and predicates. Informally speaking, a datatype group is a group of supported predicate URIrefs.
‘wrapped’ around a set of base datatype URIs. It can potentially be divided into several sub-groups, so that all predicates in a sub-group are defined over the unique base datatype of the sub-group. This allows us to make use of known decidability results about the satisfiability problems of predicate conjunctions of, e.g., those exploited by the admissible (or computable) concrete domains presented in Section 2.4 of [88]. Formally, a datatype group is defined as follows, and sub-groups will be defined in Definition 5.7.

**Definition 5.4. (Datatype Group)** A datatype group \( \mathcal{G} \) is a tuple \( (\mathcal{M}_p, \mathcal{D}_G, \text{dom}) \), where \( \mathcal{M}_p \) is the predicate map of \( \mathcal{G} \), \( \mathcal{D}_G \) is the set of base datatype URI references of \( \mathcal{G} \), and \( \text{dom} \) is the declared domain function of \( \mathcal{G} \).

We call \( \Phi_G \) the set of supported predicate URI references of \( \mathcal{G} \), i.e., for each \( u \in \Phi_G \), \( \mathcal{M}_p(u) \) is defined; we require \( \mathcal{D}_G \subseteq \Phi_G \). We assume that there exists a unary predicate URI reference \( \text{owlx:DatatypeBottom} \notin \Phi_G \).

The declared domain function \( \text{dom} \) is a mapping defined as follows:

\[
\text{dom}(u) = \begin{cases} 
  u & \text{if } u \in \mathcal{D}_G \\
  (d_1, \ldots, d_n), & \text{where } d_1, \ldots, d_n \in \mathcal{D}_G \\
  \{ (d, \ldots, d) \mid i \geq n \}, & \text{where } d \in \mathcal{D}_G \\
  \{ (d, \ldots, d) \} & \text{i times} \\
  a_{\min}(\mathcal{M}_p(u)) = n & \text{if } u \in \Phi_G \setminus \mathcal{D}_G \text{ and } a(\mathcal{M}_p(u)) = n \\
  \end{cases}
\]

Definition 5.4 ensures that base datatype URIs are among the supported predicate URIs (\( \mathcal{D}_G \subseteq \Phi_G \)), and supported predicate URIs relate to base datatypes via the declared domain function \( \text{dom} \). In this formalism, we assume that if a supported predicate does not have a fixed arity (e.g., \( \text{+int} \)), then it relates to only one base datatype (e.g., \( \text{integer} \)).

**Example 5.3 Datatype Group** \( \mathcal{G}_1 = (\mathcal{M}_{p_1}, \mathcal{D}_{\mathcal{G}_1}, \text{dom}_1) \), where \( \mathcal{M}_{p_1} \) is defined in Example 5.2, \( \mathcal{D}_{\mathcal{G}_1} = \{ \text{xsd:string}, \text{xsd:integer} \} \), and \( \text{dom}_1 = \{ (\text{xsd:string}, \text{xsd:string}), (\text{xsd:integer}, \text{xsd:integer}), (\text{owlx:integerEquality}, (\text{xsd:integer}, \text{xsd:integer})), (\text{owlx:integerGreaterThanx=n}, \text{xsd:integer}) \} \).

According to \( \mathcal{M}_{p_1} \), we have \( \Phi_{\mathcal{G}_1} = \{ \text{xsd:string}, \text{xsd:integer}, \text{owlx:integerEquality}, \text{owlx:integerGreaterThanx=n} \} \), hence \( \mathcal{D}_{\mathcal{G}_1} \subseteq \Phi_{\mathcal{G}_1} \).

**Definition 5.5. (Interpretation of Datatype Groups)** A datatype interpretation \( \mathcal{I}_D \) of a datatype group \( \mathcal{G} = (\mathcal{M}_p, \mathcal{D}_G, \text{dom}) \) is a pair \( (\Delta_D, \cdot^D) \), where \( \Delta_D \) (the datatype
domain) is a non-empty set and \( \cdot^D \) is a datatype interpretation function, which has to satisfy the following conditions:

1. rdfs:Literal\(^D = \Delta_D \) and owlx:DatatypeBottom\(^D = \emptyset;\)
2. for each plain literal \( l, l^D = l \in PL \) and \( PL \subseteq \Delta_D;\)
3. for each supported datatype URIref \( u \in D_G \) (let \( d = M_p(u) \)):
   - (a) \( u^D = E(d) = V(d) \subseteq \Delta_D, \)
   - (b) if \( s \in L(d) \), then \( (\text{"s"}^\cdot u)^D = L2V(d)(s), \)
   - (c) if \( s \not\in L(d) \), then \( (\text{"s"}^\cdot u)^D \) is not defined;
4. for any two \( u_1, u_2 \in D_G \): \( u_1^D \cap u_2^D = \emptyset; \)
5. \( \forall u \in \Phi_G, u^D = E(M_p(u)); \)
6. \( \forall u \in \Phi_G, u^D \subseteq (\text{dom}(u))^D; \)
   - (a) if \( \text{dom}(u) = (d_1, \ldots, d_n) \) and \( a(M_p(u)) = n, \) then \( (\text{dom}(u))^D = d_1^D \times \) \( \ldots \times d_n^D, \)
   - (b) if \( \text{dom}(u) = \{(d, \ldots, d) \mid i \geq n \} \) and \( a_{\min}(M_p(u)) = n, \) then \( (\text{dom}(u))^D = \)
     \( \bigcup_{i \geq n} (d^D)^i; \)
7. \( \forall u \not\in \Phi_G, u^D \subseteq \bigcup_{n \geq 1} (\Delta_D)^n, \) and \( \text{"u"}^\cdot u \in \Delta_D. \)

Moreover, we extend \( \cdot^D \) to (relativised) negated predicate URI references \( \overline{u} \) as follows:

\[
(\overline{u})^D = \begin{cases} 
\Delta_D \setminus u^D & \text{if } u \in D_G \\
(\text{dom}(u))^D \setminus u^D & \text{if } u \in \Phi_G \setminus D_G \\
\bigcup_{n \geq 1} (\Delta_D)^n \setminus u^D & \text{if } u \not\in \Phi_G.
\end{cases}
\]

There are some remarks about Definition 5.5:

1. According to Condition 2 and (3a), the datatype domain \( \Delta_D \) is a superset of value spaces of any base datatypes and \( PL \). As a result, unsupported predicate URIrefs can be interpreted intuitively; e.g., given \( M_{p1}, \) \( \text{"1.278e-3"}^\cdot \text{xsd:float} \) does not have to be interpreted as either an integer, a string or a string with

\(^2PL\) is the value space for plain literals; cf. Definition 3.12 on page 75.
5.1. DATATYPE GROUPS

a language tag. In OWL datatyping, given \( M_{d_1}, \ "1.278e-3" \textsuperscript{\text{\textasciitilde \textasciitilde}} \text{xsd:float} \) has to be interpreted as either an integer, a string or a string with a language tag (cf. Section 3.3.3 on page 77).

2. Base datatype URIrefs and typed literals with base datatype URIrefs are interpreted in the same way as OWL datatyping (cf. Condition 3 and its counterpart in Definition 3.12 on page 75). Value spaces of the base datatypes are disjoint (Condition 4), which is essential to dividing \( \Phi_G \) into sub-groups. Note that, in case the value space of a base datatype \( d_1 \) is a subset of another base datatype \( d_2, d_1 \) can be seen as a unary predicate of \( d_2 \); cf. Example 2.4 on page 41 for the ‘integer’ predicate of the rational datatype.

3. Supported predicate URIrefs are interpreted as the extensions of the predicates they represent (Condition 5), which are subsets of the corresponding declared domains (Condition 6). Extensions of base datatypes are equal to the value spaces of the base datatypes (Condition (3a)).

4. Unsupported predicate URIrefs are not restricted to any fixed arity, and typed literals with unsupported predicates are interpreted as some member of the datatype domain (Condition 7).

5. Supported predicate URIrefs \( u \in \Phi_G \setminus D_G \) have relativised negations (w.r.t. their declared domains). For example, \( \text{owlx:integerGreaterThan}x=15 \), the negated predicate URIref for \( \text{owlx:integerGreaterThan}x=15 \), is interpreted as \( V(\text{integer}) \setminus (\text{owlx:integerGreaterThan}x=15)^D \); therefore, its interpretation includes the integer 5, but not the string “Fred”, no matter if \textit{string} is a base datatype in \( D_G \) or not.

Now we introduce the kind of basic reasoning mechanisms required for a datatype group.

**Definition 5.6. (Predicate Conjunction)** Let \( V \) be a set of variables, \( G = (M_p, D_G, \text{dom}) \) a datatype group, we consider predicate conjunctions of \( G \) of the form

\[
\mathcal{C} = \bigwedge_{j=1}^{k} w_j(v_1^{(j)}, \ldots, v_n^{(j)}),
\]

where the \( v_i^{(j)} \) are variables from \( V \), \( w_j \) are (possibly negated) predicate URI references of the form \( u_j \) or \( \overline{u_j} \), and if \( u_j \in \Phi_G, a(M_p(u_j)) = n_j \) or \( a_{\min}(M_p(u_j)) \leq \)
A predicate conjunction \( C \) is called satisfiable iff there exist an interpretation \((\Delta_D, \cdot^D)\) of \( G \) and a function \( \delta \) mapping the variables in \( C \) to data values in \( \Delta_D \) s.t. \( \langle \delta(v_1^{(j)}), \ldots, \delta(v_n^{(j)}) \rangle \in w_j^D \) for all \( 1 \leq j \leq k \). Such a function \( \delta \) is called a solution for \( C \) (w.r.t. \((\Delta_D, \cdot^D)) \).

Note that we may have negated predicate URIrefs in a predicate conjunction of datatype groups. For example,

\[
C_1 = \text{owlx:integerGreaterThan}x=38(v_1) \land \\
\text{owlx:integerGreaterThan}x=12(v_2) \land \\
\text{owlx:integerEquality}(v_1, v_2)
\]

is a predicate conjunction of \( G_1 \) (which is presented in Example 5.3 on page 104). For any interpretation \((\Delta_D, \cdot^D)\) of \( G_1 \), we have \( 26 \in (\text{owlx:integerGreaterThan}x=12)^D = E(>_{\text{int}}^{12}) \subseteq \Delta_D \) and \( 26 \in (\text{owlx:integerGreaterThan}x=38)^D = V(\text{integer}) \setminus E(>_{\text{int}}^{38}) \subseteq \Delta_D \). Therefore, the function \( \delta = \{ v_1 \mapsto 26, v_2 \mapsto 26 \} \) is a solution for \( C_1 \); in other words, \( C_1 \) is satisfiable.

The predicate conjunction over a datatype group \( G \) can possibly be divided into independent sub-conjunctions of sub-groups of \( G \). Informally speaking, a sub-group includes a base datatype URI reference and the set of supported predicate URIrefs defined over it.

**Definition 5.7. (Sub-Group)** Given a datatype group \( G = (M_p, D_G, \text{dom}) \) and a base datatype URI reference \( w \in D_G \), the sub-group of \( w \) in \( G \), abbreviated as \( \text{sub-group}(w, G) \), is defined as:

\[
\text{sub-group}(w, G) = \{ u \in \Phi_G | \text{dom}(u) = \underbrace{w, \ldots, w}_{a(M_p(u)) \text{ times}} \text{ or } \underbrace{w, \ldots, w}_{i \text{ times}} \} \\
\text{dom}(u) = \{ (w, \ldots, w) | i \geq a_{\text{min}}(M_p(u)) \} \}
\]

**Example 5.4 The Sub-Group of** \( \text{xsd:integer} \) \( \text{in} \) \( G_1 \) \( \text{(presented in Example 5.3)} \) \is \( \text{sub-group}(\text{xsd:integer}, G_1) = \{ \text{xsd:integer, owlx:integerEquality, owlx:integerGreaterThan}x=n \} \). According to Definition 5.7 and Condition 4 of Definition 5.5, predicate conjunctions over \( \text{sub-group}(\text{xsd:integer}, G_1) \) and \( \text{sub-group}(\text{xsd:string}, G_1) \) can be divided into two sub-conjunctions: one over \( \text{sub-group}(\text{xsd:integer}, G_1) \) and the other over \( \text{sub-group}(\text{xsd:string}, G_1) \). These sub-conjunctions can be handled
5.1. DATATYPE GROUPS

separately if there are no common variables; if there are common variables, there exist contradictions, due to the disjointness of the value spaces \( V(\text{integer}) \) and \( V(\text{string}) \).

There is a strong connection between a sub-group and a concrete domain.

**Definition 5.8. (Corresponding Concrete Domain)** Let \( \mathcal{G} = (\mathcal{M}_p, \mathcal{D}_\mathcal{G}, \text{dom}) \) be a datatype group, \( w \in \mathcal{D}_\mathcal{G} \) a base datatype URI reference and \( \mathcal{M}_p(w) = \mathcal{D} \). If all the supported predicates in sub-group \((w, \mathcal{G})\) have fixed arities, the corresponding concrete domain of sub-group \((w, \mathcal{G})\) is \((\Delta_D, \Phi_D)\), where \(\Delta_D := V(\mathcal{D})\) and \(\Phi_D := \{ \bot \} \cup \{ \mathcal{M}_p(u)|u \in \text{sub-group}(w, \mathcal{G}) \}\), where \( \bot \) corresponds to \( \pi \).

Note that concrete domains do not support predicates that can have multiple arities, and sub-groups including such predicates do not have corresponding concrete domains.

**Example 5.5 The Corresponding Concrete Domain of sub-group** \((\text{xsd:integer}, \mathcal{G}_1)\) is \((\Delta_{\text{integer}}, \Phi_{\text{integer}})\), where \(\Delta_{\text{integer}} := V(\text{integer})\) and \(\Phi_{\text{integer}} := \{ \bot_{\text{integer}}, \text{integer}, \text{int} = \text{int}, \text{int} > \text{int} \}\). Note that the predicate \( \bot_{\text{integer}} \) corresponds to \( \text{xsd:integer} \), the negated form of \( \text{xsd:integer} \).

One of the benefits of introducing the corresponding concrete domain for a sub-group is that if the corresponding concrete domain is admissible, the sub-group is computable.

**Lemma 5.9.** Let \( \mathcal{G} = (\mathcal{M}_p, \mathcal{D}_\mathcal{G}, \text{dom}) \) be a datatype group, \( w \in \mathcal{D}_\mathcal{G} \) a base datatype URI reference and \( \mathcal{D} = \mathcal{M}_p(w) \) a base datatype. If the corresponding concrete domain of \( w \), \((\Delta_\mathcal{D}, \Phi_\mathcal{D})\), is admissible, then the satisfiability problem for finite predicate conjunctions \( \xi_w \) of the sub-group \((w, \mathcal{G})\) is decidable.

**Proof.** Direct consequence of Definition 5.8 and Definition 2.8 on page 28 of [88]: If \((\Delta_\mathcal{D}, \Phi_\mathcal{D})\) is admissible, \( \Phi_\mathcal{D} \) is closed under negation and predicate conjunctions over \((\Delta_\mathcal{D}, \Phi_\mathcal{D})\) are decidable. Hence \( \forall u \in \text{sub-group}(w, \mathcal{G}) \setminus \{w\} \), there exists \( u' \in \text{sub-group}(w, \mathcal{G}) \), such that \( \pi_\mathcal{D} = u'^\mathcal{D} \). Accordingly, negations in \( \xi_w \) can be eliminated, and \( \xi_w \) can be equivalently transformed into predicate conjunctions of \((\Delta_\mathcal{D}, \Phi_\mathcal{D})\). \( \xi_w \) is therefore decidable.

Note that the use of predicates with multiple arities in a sub-group does not mean that the predicate conjunction of the sub-group is not decidable. For example, if we consider the concrete domain \( \mathcal{Q} \) (cf. Example 2.4 on page 41) and check carefully with
its decidability proof presented in [90, p.29-30], it is easy to show that if we replace the ternary predicate

\[ E(+) = \{ \langle t_1, t_2, t_3 \rangle \in V(\text{rational})^3 \mid t_1 = t_2 + t_3 \} \]

with the predicate

\[ E(+) = \{ \langle t_1, \ldots, t_n \rangle \in V(\text{rational})^n \mid t_1 = t_2 + \ldots + t_n \}, \]

the predicate conjunction is still decidable. This is because the original proof is based on a reduction to mixed integer programming (MIP), where a mixed integer programming problem has the form \( Ax = b \), where \( A \) is an \( m \times n \)-matrix of rational numbers, \( x \) is an \( n \)-vector of variables (there is no restriction that \( n \) must be 3), each of them being an integer variable or a rational variable, and \( b \) is an \( m \)-vector of rational numbers (see, e.g. [132]). In the original proof, the construct \(+\langle x_1, x_2, x_3 \rangle\) is transformed into an equation \(-x_1 + x_2 + x_3 = 0\). This can be easily extended to the \( +' \) predicate, where the construct \(+'\langle x_1, \ldots, x_n \rangle\) can be transformed into an equation \(-x_1 + x_2 + \ldots + x_n = 0\).

In general, the computability of a sub-group depends on the decidability of the satisfiability of its predicate conjunctions, instead of the existence of a corresponding concrete domain. Lemma 5.9 simply shows a way to reuse existing results.

We end this section by elaborating the conditions that computable datatype groups require.

**Definition 5.10. (Conforming Datatype Group)** A datatype group \( G \) is conforming iff

1. for any \( u \in \Phi_G \setminus D_G \) with \( a(M_p(u)) = n \geq 2 \): \( \text{dom}(u) = (w, \ldots, w) \) for some \( w \in D_G \), and

2. for any \( u \in \Phi_G \setminus D_G \): there exist \( u' \in \Phi_G \setminus D_G \) such that \( u'^D = \overline{u}^D \), and

3. the satisfiability problems for finite predicate conjunctions of each sub-group of \( G \) is decidable, and

4. for each datatype \( u_i \in D_G \), there exists \( w_i \in \Phi_G \), s.t. \( M_p(w_i) = \neq u_i \), where \( \neq u_i \) is the binary inequality predicate for \( M_p(u_i) \).
In Definition 5.10, Condition 1 ensures that \( \Phi_G \) can be completely divided into sub-groups. Condition 2 and 3 ensure that all the sub-groups are computable, and Condition 4 ensures that number restrictions can be handled (cf. Section 5.1.3).

**Example 5.6 A Conforming Datatype Group**

\( \mathcal{G}_1 \) (presented in Example 5.3) is not conforming because it does not satisfy Condition 2 and 4 of Definition 5.10. To make it conforming, we should extend \( M_{p1} \) as follows:

\[
M_{p1} = \{ \langle \text{xsd:string}, \text{string} \rangle, \langle \text{owlx:stringEquality}, =^{\text{str}} \rangle, \langle \text{owlx:stringInequality}, \neq^{\text{str}} \rangle, \langle \text{xsd:integer}, \text{integer} \rangle, \langle \text{owlx:integerEquality}, =^{\text{int}} \rangle, \langle \text{owlx:integerInequality}, \neq^{\text{int}} \rangle, \langle \text{owlx:integerGreaterThan}, >^{\text{int}} \rangle, \langle \text{owlx:integerLessThanOrEqualTo}, \leq^{\text{int}} \rangle \}.
\]

It is possible to devise a polynomial time algorithm to decide satisfiability predicate conjunction over sub-group(xsd:integer, \( \mathcal{G}_1 \)) ([88, Sec. 2.4.1]) and sub-group(xsd:string, \( \mathcal{G}_1 \)).

**Theorem 5.11.** If \( \mathcal{G} = (M_p, D_G, \text{dom}) \) is a conforming datatype group, then the satisfiability problem for finite predicate conjunctions of \( \mathcal{G} \) is decidable.

**Proof.** Let the predicate conjunction be \( C = C_{w_1} \land \ldots \land C_{w_k} \land C_U \), where \( D_G = \{ w_1, \ldots, w_k \} \) and \( C_{w_i} \) is the predicate conjunction for sub-group(\( w_i, \mathcal{G} \)) and \( C_U \) the sub-conjunction of \( C \) where only unsupported predicate appear.

According to Definition 5.10, the set of \( C_S = C_{w_1} \land \ldots \land C_{w_k} \) is decidable. According to Definition 5.4, \( C_U \) is unsatisfiable iff there exist \( u(v_1, \ldots, v_n) \) and \( \overline{u}(v_1, \ldots, v_n) \) for some \( u \notin \Phi_G \) appearing in \( C_U \), which is clear decidable. \( C \) is satisfiable iff both \( C_S \) and \( C_U \) are satisfiable.

### 5.1.2 Datatype Expressions

In the last section, we have shown that datatype groups provide an OWL-compatible unified formalism for datatypes and predicates. In this section, we further describe how to construct datatype expressions in the unified formalism, so as to represent customised datatypes and predicates.

**Definition 5.12.** (\( \mathcal{G} \)-Datatype Expressions) Let \( \mathcal{G} = (M_p, D_G, \text{dom}) \) be a datatype group, the set of \( \mathcal{G} \)-datatype expressions (or simply datatype expressions), abbreviated \( \text{Dexp}(\mathcal{G}) \), is inductively defined as follows:
Table 5.1: Semantics of datatype expressions

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>oneOf(l₁, ..., lₙ)</td>
<td>{l₁, ..., lₙ}</td>
<td>{l₁^D} ∪ ... ∪ {lₙ^D}</td>
</tr>
<tr>
<td>domain(u₁, ..., uₙ)</td>
<td>[u₁, ..., uₙ]</td>
<td>u₁^D × ... × uₙ^D</td>
</tr>
<tr>
<td>and(P, Q)</td>
<td>P ∧ Q</td>
<td>P^D ∩ Q^D</td>
</tr>
<tr>
<td>or(P, Q)</td>
<td>P ∨ Q</td>
<td>P^D ∪ Q^D</td>
</tr>
</tbody>
</table>

1. for any predicate URIrefs \( u : u, \overline{u} \in \text{Dexp}(\mathcal{G}) \);

2. let \( l₁, \ldots, lₙ \) be typed literals, \( \{l₁, \ldots, lₙ\} \in \text{Dexp}(\mathcal{G}) \) with arity 1, where \{\} is called the oneOf constructor;

3. let \( u_i' \) be either rdfs:Literal, \( w, u_i \), or \( w_i \) (for all \( 1 \leq i \leq n \)), where \( w \in \text{D}_\mathcal{G} \), \( u_i \in \text{sub-group}(w, \mathcal{G}) \setminus \{w\} \) and \( a(\text{M}_p(u_i)) = 1 \), \( \overline{u_1', \ldots, u_n'} \in \text{Dexp}(\mathcal{G}) \) with arity \( n \);

4. \( \forall P, Q \in \text{Dexp}(\mathcal{G}) \), \( P \land Q, P \lor Q \in \text{Dexp}(\mathcal{G}) \) if:
   - (a) both \( P \) and \( Q \) have arity \( n \) (in this case, \( P \land Q, P \lor Q \) have arity \( n \)); or
   - (b) one of them has arity \( n \), the other has minimal arity \( n_{\text{min}} \) and \( n \geq n_{\text{min}} \) (in this case \( P \land Q, P \lor Q \) have arity \( n \)); or
   - (c) one of them has minimum arity \( n_{\text{min}} \), the other has minimum arity \( n'_{\text{min}} \) and \( n_{\text{min}} \geq n'_{\text{min}} \) (in this case \( P \land Q, P \lor Q \) have minimum arity \( n_{\text{min}} \)).

\( \land, \lor \) are called the and, or constructors, respectively.

The semantics of datatype expressions of the above form 1 is already defined in Definition 5.4. The abstract syntax and semantics of the rest forms of \( \mathcal{G} \)-datatype expressions are given in Table 5.1.

Let \( P \) be a \( \mathcal{G} \)-datatype expression. The negation of \( P \) is of the form \( \neg P \): if the arity of \( P \) is \( n \) (\( n > 0 \)), \( \neg P \) is interpreted as \( (\Delta_D)^n \setminus P^D \); if the minimum arity of \( P \) is \( n \) (\( n > 0 \)), then \( \neg P \) is interpreted as \( \bigcup_{m \geq n}(\Delta_D)^m \setminus P^D \).

Note that, similar to predicates, a datatype expression either has a fixed arity or has a minimum arity.

The following examples show how to use datatype expressions to represent customised datatypes and predicates.
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Example 5.7 How to Use Datatype Expressions to Represent Customised Datatypes and Predicates

Customised Datatypes: $G$-datatype expressions can be used to represent two kinds of user-derived XML Schema datatypes: derivation by restriction and derivation by union.\(^3\)

- The customised XML Schema datatype ‘`greaterThan20`’ (derivation by restriction)

  ```xml
  <simpleType name = "greaterThan20">
  <restriction base = "xsd:integer">
    <minExclusive value = "20"/>
  </restriction>
  </simpleType>
  ```

  can be represented as

  ```xml
  owlx:integerGreaterThan#20
  ```

  which is a unary predicate URIRef.

- The customised XML Schema datatype ‘`cameraPrice`’, by which one might want to express camera prices as a float number between 8.00 and 100000.00, or with one of the strings “low”, “medium” or “expensive” (derivation by union and restriction)

  ```xml
  <simpleType name = "cameraPrice">
  <union>
    <simpleType>
      <restriction base = "xsd:float">
        <minInclusive value = "8.00"/>
        <maxInclusive value = "100000.00"/>
      </restriction>
    </simpleType>
    <simpleType>
      <restriction base = "xsd:string">
        <enumeration value = "low"/>
        <enumeration value = "medium"/>
        <enumeration value = "expensive"/>
      </restriction>
    </simpleType>
  </union>
  </simpleType>
  ```

\(^3\)List-value datatypes do not fit the RDF datatype model.
can be represented by
\[(\text{owlx:floatGreaterThan}x=8.00 \land \text{owlx:floatLessThanOrEqualTo}x=100000.00)\]
\[\lor\{\text{“low”}^\text{xsd:string}, \text{“medium”}^\text{xsd:string}, \text{“expensive”}^\text{xsd:string}\},\]
which is a disjunction expression with arity 1, where the first disjunct is a conjunction expression and the second disjunct is an oneOf expression.4

**Customised Predicates:** G-datatype expressions can be used to represent customised predicates, e.g., those we have seen in Section 1.3.

- The customised predicate ‘sumNoGreaterThan15’, with extension \(E(\text{sumNoGreaterThan15}) = \{\langle i_0, i_1, i_2, i_3 \rangle \in V(\text{integer})^4 \mid i_0 = i_1 + i_2 + i_3 \text{ and } \neg(i_0 > 15)\}\) and arity \(a(\text{sumNoGreaterThan15}) = 4\), can be represented by
  \[\text{owlx:integerAddition} \land \]
  \[\{\text{owlx:integerGreaterThan}x=15, \text{xsd:integer}, \text{xsd:integer}, \text{xsd:integer}\},\]
  which is a conjunction expression, where the first conjunct is a predicate URIref and the second conjunct is a domain expression. Note that the predicate owlx:integerAddition (i.e., \(+^\text{int}\)) has a minimum arity 3, while the arity of the domain expression is 4. According to Definition 5.12, the arity of the conjunction expression is 4.

- The customised predicate ‘multiplyBy1.6’, with extension \(E(\text{multiplyBy1.6}) = \{\langle b_0, b_1, b_2 \rangle \in V(\text{double})^3 \mid b_0 = b_1 \ast b_2 \text{ and } b_1 = 1.6\}\) and arity \(a(\text{multiplyBy1.6}) = 3\), can be represented by
  \[\text{owlx:doubleMultiplication} \land \]
  \[\{\text{xsd:double}, \text{owlx:doubleEqualTo}x=1.6, \text{xsd:double}\},\]
  which is a conjunction expression with arity 3, where owlx:doubleMultiplication is the predicate URIref for \(*^\text{double}\).\5

- The customised predicate ‘multiplyBy1.6v2’, which is a variation of ‘multiplyBy1.6’, with extension \(E(\text{multiplyBy1.6v2}) = \{\langle b_0, b_1 \rangle \in V(\text{double})^2 \mid b_0 = 1.6 \ast b_1\}\) and arity \(a(\text{multiplyBy1.6v2}) = 2\), can be represented by
  \[\text{owlx:doubleMultiplicationx}=1.6 \land \{\text{xsd:double}, \text{xsd:double}\},\]
  which is a conjunction expression with arity 2, where owlx:doubleMultiplicationx=y is the predicate URIref for \(*^\text{double}_{[y]}\), where \(E(\ast^\text{double}_{[y]}) = \{\langle d_0, \ldots, d_n \rangle \mid\]

---

4Note that the oneOf constructor is redundant (but sometimes convenient) and can be simulated by the or constructor and unary equality predicates.
Let us now consider a more expressive reasoning mechanism than predicate conjunctions that we introduced in Definition 5.6 (page 106).

**Definition 5.13. (Datatype Expression Conjunction)** Let $G$ be a conforming datatype group, $V$ a set of variables, we consider $G$-datatype expression conjunctions of the form

$$C_E = \bigwedge_{j=1}^{k} P_j(v_1^{(j)}, \ldots, v_n^{(j)}), \tag{5.2}$$

where $P_j$ is a (possibly negated) $G$-datatype expression, with its arity equal to $n_j$ or its minimum arity less than or equal to $n_j$, and the $v_i^{(j)}$ are variables from $V$. A $G$-datatype expression conjunction $C_E$ is called *satisfiable* iff there exist an interpretation $(\Delta_D, \cdot^D)$ of $G$ and a function $\delta$ mapping the variables in $C_E$ to data values in $\Delta_D$ s.t. \langle $\delta(v_1^{(j)}), \ldots, \delta(v_n^{(j)})$$\rangle \in P^D_j$ for $1 \leq j \leq k$. Such a function $\delta$ is called a *solution* for $C_E$ (w.r.t. $(\Delta_D, \cdot^D)$).

**Lemma 5.14.** Let $G$ be a conforming datatype group; $G$-datatype expression conjunctions of the form (5.2) are decidable.

**Proof.** It is trivial to reduce the satisfiability problem for $G$-datatype expression conjunctions to the satisfiability problem for predicate conjunctions over $G$:  

1. Due to Condition 2 of a conforming datatype group (cf. Definition 5.10 on page 109), we can eliminate relativised negations in a domain expression. Hence, the domain constructor simply introduces predicate conjunctions because $\text{domain}(u_1, \ldots, u_n)(v_1, \ldots, v_n) \equiv u_1(v_1) \land \ldots \land u_n(v_n)$. Similarly, its negation introduces disjunctions, $\neg \text{domain}(u_1, \ldots, u_n)(v_1, \ldots, v_n) \equiv \neg u_1(v_1) \lor \ldots \lor \neg u_n(v_n)$, where

$$\neg u_i(v_i) \equiv \begin{cases} \overline{u_i}(v_i) \lor \overline{\text{dom}(u_i)}(v_i) & \text{if } u_i \in \Phi_G \setminus \text{D}_G, \\ \overline{u_i}(v_i) & \text{otherwise,} \end{cases}$$

according to Definition 5.5.

---

$^5$See the discussion about parameterised predicates on page 103.
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2. The \texttt{and} and \texttt{or} constructors simply introduce disjunctions of predicate conjunctions of \( \mathcal{G} \). According to Lemma 5.11, the satisfiability problem of predicate conjunctions of \( \mathcal{G} \) is decidable; therefore, a \( \mathcal{G} \)-datatype expression conjunction is satisfiable iff one of its disjuncts is satisfiable.

We end this section by a brief explanation of the several kinds of conjunctions we have seen so far in this Chapter. From the viewpoint of users points of view, \( \mathcal{G} \)-datatype expressions allows users to use conjunctions to construct customised datatypes and predicates (see Definition 5.12). From the viewpoint of datatype reasoners, they are used to solve the satisfiability problems of predicate conjunctions (see Definition 5.6) and datatype expression conjunctions (see Definition 5.13).

5.1.3 Relations with Existing Formalisms

In this section, we briefly discuss the relations between our formalism and existing ontology-related formalisms.

Relation with Concrete Domains

The concrete domain approach provides a rigorous treatment of datatype predicates within concrete domains. The basic reasoning task is to compute the satisfiability of finite predicate conjunctions, which is required to be decidable in admissible concrete domains. In our formalism, a conforming datatype group requires that the satisfiability of finite predicate conjunctions of all its sub-groups should be decidable (cf. Condition 3 of Definition 5.10 on page 109). Sub-groups of a datatype group can, but not necessarily, correspond to concrete domains (cf. Definition 5.8 on page 108).

A concrete domain is similar to a datatype in that its domain corresponds to the value space of a datatype. The concrete domain approach, however, does not consider the lexical space and lexical-to-value mapping of a datatype. Accordingly, it does not provide a formal way to represent data values of a datatype (with the help of the lexical-to-value mapping of the datatype). This could introduce confusions in the ABox when data values of different concrete domains (e.g., \texttt{boolean} and \texttt{integer}) share the same lexical forms (e.g., “1” and “0”). In addition, this could also affect definitions of some predicate extensions. For example, the extension of \( >_{[n]}^{int} \) is defined as follows in the concrete domain approach:

\[
E(>_{[n]}^{int}) = \{ i \in E(int) \mid i > n \}.
\]
Please note that the “n” in the predicate name \( >_{[n]}^{int} \) is a string, while the \( n \) in the semantic condition ‘\( i > n \)’ is an integer. So strictly speaking they are not the same thing. In the datatype group approach, on the contrary, the extension of \( >_{[n]}^{int} \) is defined as follows:

\[
E(>_{[n]}^{int}) = \{ i \in E(integer) \mid i > L2V(integer)("n") \},
\]

where the lexical-to-value mapping of integer, viz. \( L2V(integer) \), maps “n”, the lexical form, to its corresponding integer value.

The concrete domain approach is rather compact, or simplified, and thus not very user-friendly. Our formalism overcomes various limitations of concrete domains discussed in Section 2.3.3:

1. It supports predicates with non-fixed arities (cf. Definition 5.1 on page 102).

2. It provides relativised negations for supported predicate URIrefs in predicate conjunctions (5.1) of a datatype group.

3. Most importantly, it provides datatype expressions (cf. Definition 5.12 on page 110) to represent customised datatypes and customised predicates, which are very useful in SW and ontology applications.

Furthermore, our formalism provides the use of unsupported predicate URI references, (cf. Definition 5.3 on page 103). This again makes the formalism more user-friendly, as datatype reasoners supporting different datatype groups will not simply reject the predicate URI references that are not in their predicate maps.

**Relation with OWL Datatyping**

OWL datatyping, on the other hand, is focused on datatypes instead of predicates. It uses the RDF(S) specification of datatypes and data values, but it is different from RDF(S) in that datatypes are not classes in its concept language and the datatype domain (w.r.t. a datatype map) is disjoint with the object domain (of an interpretation). It uses datatype maps to distinguish supported datatype URIrefs from unsupported ones, and it also provides enumerated datatypes. The very serious limitation of OWL datatyping is that it does not support customised datatypes and predicates.

Our formalism is compatible with OWL datatyping. Furthermore, it overcome all the limitations of OWL datatyping discussed in the ‘Limitations of OWL Datatyping’ section on page 77:
• It extends OWL datatyping to support predicates.

• Most importantly, it introduces datatype expressions to provide the use of customised datatypes and predicates.

• It provides (relativised) negations for predicate URIrefs in a datatype group (cf. Definition 5.4 on page 104).

• The datatype domain in an interpretation of a datatype group is a superset of (instead of equivalent to) the value spaces of base datatypes and plain literals, so typed literals with unsupported predicates are interpreted more intuitively (cf. Definition 5.5 on page 104).

5.2 Integrating DLs with Datatype Groups

We shall now present our scheme for integrating an arbitrary datatype group into so called $\mathcal{G}$-combinable Description Logics, a family of Description Logics that satisfy some weak conditions. We will show that members of the family of integrated DLs are decidable.

5.2.1 $\mathcal{G}$-combinable Description Logics

Formally, a $\mathcal{G}$-combinable Description Logics is defined as follows.

**Definition 5.15. ($\mathcal{G}$-combinable Description Logics)** Given a Description Logic $\mathcal{L}$, $\mathcal{L}$ is called a $\mathcal{G}$-combinable Description Logic if (i) $\mathcal{L}$ provides the conjunction ($\sqcap$) and bottom ($\bot$) constructors and (ii) $\mathcal{L}$-concept satisfiability w.r.t. TBoxes and RBoxes is decidable. $\mathcal{L}(\mathcal{G})$ is called a $\mathcal{G}$-combined Description Logic.

Many well-known DLs, such as $\mathcal{SHIQ}$, $\mathcal{SHOIQ}$ and their sub-languages (e.g., $\mathcal{ALC}$), are $\mathcal{G}$-combinable Description Logics. We can integrate an arbitrary datatype group $\mathcal{G}$ into a $\mathcal{G}$-combinable Description Logic $\mathcal{L}$. The result of this integration is called $\mathcal{L}(\mathcal{G})$. For $\mathcal{L}(\mathcal{G})$, an interpretation is of the form $(\Delta^I, \cdot^I, \Delta_D, \cdot^D)$, where $(\Delta^I, \cdot^I)$ is an interpretation of the object domain, $(\Delta_D, \cdot^D)$ is an interpretation of the datatype group $\mathcal{G}$ and we require $\Delta^I$ and $\Delta_D$ are disjoint with each other. In addition to the language constructors of $\mathcal{L}$, $\mathcal{L}(\mathcal{G})$ allows datatype group-based concept descriptions shown in Table 5.2.
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<table>
<thead>
<tr>
<th>Constructor</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>expressive predicate</td>
<td>( \exists T_1, \ldots, T_n. E )</td>
<td>( { x \in \Delta^T</td>
</tr>
<tr>
<td>exists restriction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>expressive predicate value</td>
<td>( \forall T_1, \ldots, T_n. E )</td>
<td>( { x \in \Delta^T</td>
</tr>
<tr>
<td>value restriction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>expressive predicate atleast</td>
<td>( \geq m T_1, \ldots, T_n. E )</td>
<td>( { x \in \Delta^T</td>
</tr>
<tr>
<td>restriction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>expressive predicate atmost</td>
<td>( \leq m T_1, \ldots, T_n. E )</td>
<td>( { x \in \Delta^T</td>
</tr>
</tbody>
</table>

Table 5.2: Datatype group-based concept descriptions

**Definition 5.16.** \((\mathcal{L}(\mathcal{G})\text{-concepts})\) Let \( \mathcal{G} \) be a datatype group, \( \mathcal{L} \) a \( \mathcal{G} \)-combinable Description Logic, \( \mathcal{C}N \in \mathcal{C} \) be an atomic concept name, \( T_1, \ldots, T_n \in \mathsf{Rdsc}(\mathcal{L}) \) concrete roles (where \( T_i \nsubseteq T_j, T_j \nsubseteq T_i \) for all \( 1 \leq i < j \leq n \), \(^6\) if \( \mathcal{L} \) provides role inclusion axioms) and \( E \) a \( \mathcal{G} \)-datatype expression. Valid \( \mathcal{L}(\mathcal{G}) \)-concepts include valid \( \mathcal{L} \)-concepts as well as the following concept descriptions defined by the abstract syntax:

\[
\mathcal{C} ::= \exists T_1, \ldots, T_n. E \text{ (expressive predicate exists restriction) } \mid \\
\forall T_1, \ldots, T_n. E \text{ (expressive predicate value restriction) } \mid \\
\geq T_1, \ldots, T_n E. \text{ (expressive predicate atleast restriction) } \mid \\
\leq T_1, \ldots, T_n E. \text{ (expressive predicate atmost restriction) }
\]

The semantics of the above \( \mathcal{L}(\mathcal{G}) \)-concepts is given in Table 5.2

\(^6\) \( \equiv \) is the transitive reflexive closure of \( \sqsubseteq \).

5.2.2 Datatype Queries

To decide \( \mathcal{L}(\mathcal{G}) \)-concept satisfiability and subsumption problem w.r.t. TBoxes and RBoxes, a DL reasoner can use a datatype reasoner to answer so called \( \mathcal{G} \)-datatype
queries.

**Definition 5.17. (G-Datatype Query)** For a datatype group \(G\), a G-datatype query is of the form 
\[
Q := \bigvee_{j=1}^{k} CE_j \land \bigwedge_{j_1=1}^{k_1} \neq (\vec{v}(j_1,1); \vec{v}(j_1,2)) \land \bigvee_{j_2=1}^{k_2} = (\vec{v}(j_2,1); \ldots; \vec{v}(j_2,m_{j_2})),
\]  
(5.3)

where \(CE_j\) is a (possibly negated) \(G\)-datatype expression conjunction, \(\vec{v}(s)\) are tuples of variables appearing in \(CE_1, \ldots, CE_k\), and \(\neq\) and \(=\) are called the value inequality predicate and value equality predicate, respectively. A G-datatype query is satisfiable iff there exists an interpretation \((\Delta_D, \cdot_D)\) of \(G\) and a function \(\delta\) mapping the variables in \(CE_1, \ldots, CE_k\) to data values in \(\Delta_D\) s.t.

- \(\delta\) is a solution for one of \(CE_1, \ldots, CE_k\) w.r.t. \((\Delta_D, \cdot_D)\) and,
- \(\delta(\vec{v}(j_1,1)) \neq \delta(\vec{v}(j_1,2))\) for all \(1 \leq j_1 \leq k_1,\)
- there exist some \(j_2 (1 \leq j_2 \leq k_2)\) s.t. \(\delta(\vec{v}(j_2,1)) = \ldots = \delta(\vec{v}(j_2,m_{j_2})).\)

Such a function \(\delta\) is called a solution for \(Q\) w.r.t. \((\Delta_D, \cdot_D)\).

**Lemma 5.18.** For \(G\) a conforming datatype group, G-datatype queries of the form (5.3) are decidable.

**Proof.** Since the satisfiability problem of (possibly negated) \(G\)-datatype expression conjunctions is decidable (cf. Lemma 5.14 on page 114), we only need to show how to handle the extra constraints introduced by the value inequality predicate and value equality predicate. We can transform the tuple-level equality and inequality constraints into \(\forall\), a disjunction of conjunctions of variable-level constraints, which are of the forms \(= (v_i, v_j)\) or \(\neq (v_i, v_{j'}).\) For each satisfiable \(G\)-datatype expression conjunction \(CE_j\), we can further extend \(CE_j\) to \(CE_j'\) by adding new conjuncts \(=u (v_i, v_j)\) and/or \(\neq u (v_i, v_{j'})\) into \(CE\). \(Q\) is unsatisfiable if all \(CE_j'\) are unsatisfiable; otherwise, \(Q\) is satisfiable. \(\square\)

### 5.2.3 Decidability

Now we show that \(L(G)\) is decidable. The proof is inspired by the proof (Lutz [88, p.32-33]) of the decidability of \(ALCF(D)\)-concept satisfiability w.r.t. to general TBoxes,

\[\text{Note that, if } \vec{v} = \langle v_1, \ldots, v_n \rangle, \delta(\vec{v}) \text{ is an abbreviation for } \langle \delta(v_1), \ldots, \delta(v_n) \rangle.\]
where $\mathcal{ALC}_f(D)$ is obtained from $\mathcal{ALC}(D)$ by restricting the concrete domain constructor to concrete features in place of feature chains. The $\mathcal{ALC}_f(D)$ DL can be seen a restricted form of $\mathcal{ALC}(\mathcal{G})$, where, roughly speaking, all concrete roles are functional, $\mathcal{D}$ is the only base datatype in $\mathcal{G}$, unsupported predicates are disallowed and the only kind of datatype expression are predicates of $\mathcal{D}$.

We will show the decidability of $\mathcal{L}(\mathcal{G})$-concept satisfiability w.r.t. TBoxes and RBoxes by reducing it to the $\mathcal{L}$-concept satisfiability w.r.t. TBoxes and RBoxes. The basic idea behind the reduction is that we can replace each datatype group-based concept $C$ in $T$ with a new atomic primitive concept $A_C$ in $T'$. We then compute the satisfiability problem for all possible conjunctions of datatype group-based concepts in $T$ (of which there are only a finite number), and in case a conjunction $C_1 \sqcap \ldots \sqcap C_n$ is unsatisfiable, we add an axiom $A_{C_1} \sqcap \ldots \sqcap A_{C_n} \sqsubseteq \bot$ to $T'$.

For example, datatype group-based concepts $\exists T. > 1$ and $\forall T. \leq 0$ occurring in $T$ would be replaced with $A_{\exists T. > 1}$ and $A_{\forall T. \leq 0}$ in $T'$, and $A_{\exists T. > 1} \sqcap A_{\forall T. \leq 0} \sqsubseteq \bot$ would be added to $T'$ because $\exists T. > 1 \sqcap \forall T. \leq 0$ is unsatisfiable (i.e., there is no solution for the predicate conjunction $> 1(v) \land \leq 0(v)$).

**Theorem 5.19.** Given a conforming datatype group $\mathcal{G}$, the $\mathcal{L}(\mathcal{G})$-concept satisfiability problem w.r.t. TBoxes and RBoxes is decidable.

**Proof:** We prove the theorem by reducing $\mathcal{L}(\mathcal{G})$-concept satisfiability w.r.t. TBoxes and RBoxes to the $\mathcal{L}$-concept satisfiability w.r.t. TBoxes and RBoxes. Let $D$ be an $\mathcal{L}(\mathcal{G})$-concept for satisfiability checking, $\text{cl}_T(D)$ the set of all the sub-concepts of concepts in $\{D\} \cup \{D_1, D_2 \mid D_1 \sqsubseteq D_2 \in T \text{ or } D_1 = D_2 \in T\}$, and $\{C_1, \ldots, C_k\} \subseteq \text{cl}_T(D)$ the set of all the datatype group-based concepts in $\text{cl}_T(D)$, i.e., each $C_i$ ($1 \leq i \leq k$) is of one of the four forms shown in the ‘DL Syntax’ column of Table 5.2 on page 118. Let $T$ be a concrete role name. We assume that all the functional concrete role axioms in $R$ of the form $\text{Func}(T)$ are encoded into concept inclusion axioms of the form $\top \sqsubseteq \leq_T \sqsubseteq T_D$ in $T$.

We define a mapping $\pi$ that maps datatype group-based concept conjunctions of the form $S = B_1 \sqcap \ldots \sqcap B_h$, where $\{B_1, \ldots, B_h\} \subseteq \{C_1, \ldots, C_k\}$, to a corresponding $\mathcal{G}$-datatype query (cf. equation (5.3) on page 119) $\pi(S)$, according to the semantics of the datatype group-based concepts given in Table 5.2:

**(Step 1)** For each $B_j$ of the form $\exists T_1, \ldots, T_{n_j}. E$, $\pi(S)$ contains a conjunct $E(v_{j1}^{T_1}, \ldots, v_{jn_j}^{T_{n_j}})$, where each $v_{ji}^{T_i}$ ($1 \leq i \leq n_j$) is a variable, with the corresponding concrete role name $T_i$ as its superscript.
(Step 2) For each \( B_j \) of the form \( \geq mT_1, \ldots, T_{n_j}. E \), \( \pi(S) \) contains a conjunct

\[
\bigwedge_{a=1}^{m} E(v_{j1}^{T_1}, \ldots, v_{ja_{n_j}}^{T_{n_j}}) \land \bigwedge_{1 \leq a < b \leq m} \neq (v_{j1}^{T_1}, \ldots, v_{ja_{n_j}}^{T_{n_j}}; v_{jb1}^{T_1}, \ldots, v_{ja_{n_j}}^{T_{n_j}})
\]

where the inequality constraints are used to make sure all the tuples \( \langle v_{j1}^{T_1}, \ldots, v_{ja_{n_j}}^{T_{n_j}} \rangle, \ldots, \langle v_{jm1}^{T_1}, \ldots, v_{ja_{n_j}}^{T_{n_j}} \rangle \) are different from each other. We will not introduce any more new variables (with superscriptions) in the following steps.

(Step 3) For each \( B_j \) of the form \( \forall T_1, \ldots, T_{n_j}. E \), let \( A_j \) be the set of all tuples \( \vec{v} \) of variables that were introduced in (Step 1) and (Step 2), of the form \( \langle v_{1}^{T'_1}, \ldots, v_{n_j}^{T'_{n_j}} \rangle \), where the superscript of each \( v_{1}^{T'_1} (1 \leq i \leq n_j) \) matches the corresponding concrete role name \( T_i \) in \( \forall T_1, \ldots, T_{n_j}. E \). A variable \( v^{T'} \) matches a concrete role \( T \) if \( T' \sqsubseteq T \). Then \( \pi(S) \) contains a conjunct

\[
\bigwedge_{\vec{v} \in A_j} E(\vec{v}).
\]

(Step 4) For each \( B_j \) of the form \( \leq mT_1, \ldots, T_{n_j}. E \), similarly to (Step 3), we can define a set \( A_j \) for \( B_j \). Let \( |A_j| = m' \). If \( m' \leq m \), then \( B_j \) will not introduce any new datatype constraints. Otherwise, let \( P(A, x) \) be the function that maps a set \( A \) to the set of all the partitions of \( A \) with size \( x \); i.e., for each partition \( Q = \{q_1, \ldots, q_x\} \in P(A, x), q_1, \ldots, q_n \) are non-empty sets, \( q_1 \cap \ldots \cap q_x = \emptyset \) and \( A = q_1 \cup \ldots \cup q_x \). Then \( \pi(S) \) contains a conjunct

\[
\bigvee_{Q \in P(A_j, m)} \bigwedge_{q \in Q} \bigwedge_{\vec{v}_1, \vec{v}_2 \in q} E(\vec{v}_1) \land E(\vec{v}_2) \rightarrow = (\vec{v}_1; \vec{v}_2),
\]

we can apply the “\( x \Rightarrow y \equiv \neg x \lor y \)” equivalence and DeMorgan’s law to this conjunct to give

\[
\bigvee_{Q \in P(A_j, m)} \bigwedge_{q \in Q} \bigwedge_{\vec{v}_1, \vec{v}_2 \in q} \neg E(\vec{v}_1) \lor \neg E(\vec{v}_2) \lor = (\vec{v}_1; \vec{v}_2).
\]

Since the satisfiability problem for a datatype query is decidable, for each possible \( S = B_1 \cap \ldots \cap B_h \), where \( \{B_1, \ldots, B_h\} \subseteq \{C_1, \ldots, C_k\} \), we can decide if \( \pi(S) \) is satisfiable or not.
5.2. INTEGRATING DLS WITH DATATYPE GROUPS

Now we can reduce the \( \mathcal{L}(\mathcal{G}) \)-concept satisfiability problem to the \( \mathcal{L} \)-concept satisfiability problem, by introducing some new atomic primitive concepts (to represent \( C_i \), for each \( 1 \leq i \leq k \)) and some concept inclusion axioms about these atomic primitive concepts (to capture all the possible contradictions caused by \( S \)) as follows:

1) We create an atomic primitive concept \( A_{C_i} \) for each \( C_i \in \{ C_1, \ldots, C_k \} \), and transform \( T \) and \( D \) into \( T' \) and \( D' \) by replacing all \( C_i \) with \( A_{C_i} \) in \( T \) and \( D \). We transform \( \mathcal{R} \) into \( \mathcal{R}' \) by removing all the concrete role inclusion axioms.

2) For each \( S = B_1 \sqcap \ldots \sqcap B_h \), where \( \{ B_1, \ldots, B_h \} \subseteq \{ C_1, \ldots, C_k \} \):

- If \( \pi(S) \) is satisfiable, we do nothing.
- If \( \pi(S) \) is unsatisfiable, we add the following concept inclusion axiom into \( T' \):

\[
A_{B_1} \sqcap \ldots \sqcap A_{B_h} \sqsubseteq \bot,
\]

Claim: (i) For any \( S = B_1 \sqcap \ldots \sqcap B_h \), where \( \{ B_1, \ldots, B_h \} \subseteq \{ C_1, \ldots, C_k \} \), \( S \) is satisfiable iff \( \pi(S) \) is satisfiable. (ii) All the possible contradictions caused by possible datatype group-based sub-concept conjunctions in \( cl_T(D) \) have been encoded in the TBox \( T' \). (iii) \( D \) is satisfiable w.r.t. \( T \) and \( \mathcal{R} \) iff \( D' \) is satisfiable w.r.t. \( T' \) and \( \mathcal{R}' \).

Claim (i) is true because the mappings in (Step1) - (Step4) exactly generate the needed \( \mathcal{G} \)-datatype queries \( \pi(S) \) according to the semantics of datatype group-based concepts.

- (Step1): For each \( B_j \) of the form \( \exists T_1, \ldots, T_{n_j} \cdot E \), \( \pi(S) \) contains a conjunct \( E(v_{j1}^{T_1}, \ldots, v_{jn_j}^{T_{n_j}}) \). If \( (\Delta_D, \cdot_D) \) is an interpretation of \( \mathcal{G} \) and \( \delta \) is a solution of \( E(v_{j1}^{T_1}, \ldots, v_{jn_j}^{T_{n_j}}) \) w.r.t. \( (\Delta_D, \cdot_D) \), we have \( (\delta(v_{j1}^{T_1}), \ldots, \delta(v_{jn_j}^{T_{n_j}})) \in E^D \). Furthermore, the concrete role names \( T_i \) are used in superscripts of the corresponding variables, so as to assure that further constraints from datatype expression value and atmost restrictions can be properly added to these variables.

- (Step2): For each \( B_j \) of the form \( \geq m T_1, \ldots, T_{n_j} \cdot E \), \( \pi(S) \) contains a conjunct \( \bigwedge_{a=1}^{m} E(v_{ja1}^{T_1}, \ldots, v_{jan_j}^{T_{n_j}}) \land \bigwedge_{1 \leq a < b \leq m} \neq (v_{ja1}^{T_1}, \ldots, v_{jan_j}^{T_{n_j}}; v_{ja1}^{T_1}, \ldots, v_{jan_j}^{T_{n_j}}) \). If \( (\Delta_D, \cdot_D) \) is an interpretation of \( \mathcal{G} \) and \( \delta \) is a solution w.r.t. \( (\Delta_D, \cdot_D) \) of this conjunct, we have \( (\delta(v_{ja1}^{T_1}), \ldots, \delta(v_{jan_j}^{T_{n_j}})) \in E^D \) for all \( 1 \leq a \leq m \) and \( (\delta(v_{ja1}^{T_1}), \ldots, \delta(v_{jan_j}^{T_{n_j}})) \neq (\delta(v_{ja1}^{T_1}), \ldots, \delta(v_{ja1}^{T_{n_j}})) \) for all \( 1 \leq a < b \leq m \); viz. there are at least \( m \) tuples of data values that satisfy the datatype expression \( E \). The purpose of using superscripts in variables is the same as (Step1).
5.2. INTEGRATING DLS WITH DATATYPE GROUPS

- (Step3): For each \( B_j \) of the form \( \forall T_1, \ldots, T_{n_j}, E, \pi(S) \) contains a conjunct \( \bigwedge_{v \in A_j} E(\vec{v}) \). Since in (Step1) and (Step2) we have generated all the needed variables, the set \( A_j \) includes all the tuples of variables, the superscripts of which match \( T_1, \ldots, T_{n_j} \). If \( (\Delta_D, \pi^D) \) is an interpretation of \( \mathcal{G} \) and \( \delta \) is a solution w.r.t. \( (\Delta_D, \pi^D) \) of the above conjunct, we have \( \delta(\vec{v}) \in E^D \), for all \( \vec{v} \in A_j \).

- (Step4): For each \( B_j \) of the form \( \leq m T_1, \ldots, T_{n_j}, E, \pi(S) \) contains a conjunct

\[
\bigvee_{Q \in P(A_j, m)} \bigwedge_{q \in Q} \bigwedge_{v_1, v_2 \in q} E(\vec{v}_1) \land E(\vec{v}_2) \rightarrow (\vec{v}_1; \vec{v}_2),
\]

if \( m < |A_j| \). The set \( A_j \) is constructed as that in (Step3), and \( P(A_j, m) \) is the set of all the partitions of \( A_j \) with size \( m \). If \( (\Delta_D, \pi^D) \) is an interpretation of \( \mathcal{G} \) and \( \delta \) is a solution w.r.t. \( (\Delta_D, \pi^D) \) of this conjunct, there exists a partition \( Q \), s.t. for all \( q_i \in Q \) \( (1 \leq i \leq m) \), any pairs of variable tuples \( \vec{v}_1, \vec{v}_2 \) must satisfy that if both \( \delta(\vec{v}_1) \in E^D \) and \( \delta(\vec{v}_2) \in E^D \) are true, then \( \delta(\vec{v}_1) = \delta(\vec{v}_2) \). In other words, there are at most \( m \) different tuples of data values that are linked through the concrete roles \( T_1, \ldots, T_{n_j} \) and satisfy \( E \).

For claim (ii). Firstly, due to the 1), it is obvious that \( D' \) is an \( \mathcal{L} \)-concept and \( T' \) contains no datatype group-based concepts, and there are no concrete roles in \( \mathcal{R}' \). Secondly, due to 2), claim (i) and that \( \mathcal{G} \)-datatype queries are decidable, for any possible datatype group-based concept conjunction \( S = B_1 \sqcap \ldots \sqcap B_h \) and if \( \pi(S) \) is unsatisfiable, there is an axiom \( A_{B_1} \sqcap \ldots \sqcap A_{B_h} \sqsubseteq \bot \) in \( T' \). Therefore, all the possible contradictions caused by possible datatype group-based sub-concept conjunctions in \( cl_T(D) \) have been encoded in the TBox \( T' \).

For claim (iii). If \( D \) is satisfiable w.r.t. \( T \) and \( \mathcal{R} \), then there is a model \( \mathcal{I} \), s.t. \( \mathcal{I} \models D, \mathcal{I} \models T \) and \( \mathcal{I} \models \mathcal{R} \). We show how to construct a model \( \mathcal{I}' \) of \( D' \) w.r.t. \( T' \) and \( \mathcal{R}' \) from \( \mathcal{I} \). \( \mathcal{I}' \) will be identical to \( \mathcal{I} \) in every respect except for concrete roles (there are no concrete roles in \( \mathcal{I}' \)) and the atomic primitive concepts \( A_{C_i} \) for each \( C_i \in \{C_1, \ldots, C_k\} \) (there are no \( A_{C_i} \) in \( \mathcal{I} \)). So we only need to construct \( A_{C_i}^{T'} : A_{C_i}^{T'} = C_i^{T'} \). Due to the constructions of \( D', T', \mathcal{R}' \), we have \( D^{T'} \neq \emptyset \), \( T' \models T' \) and \( T' \models \mathcal{R}' \).

For the converse direction, let \( \mathcal{I}' \) be a model of \( D' \) w.r.t. \( T' \) and \( \mathcal{R}' \). \( \mathcal{I} \) will be identical to \( \mathcal{I}' \) in every respect except for concrete roles and datatype group-based concepts \( C_1, \ldots, C_k \). We can construct \( C_i^{T'} \) \( (1 \leq i \leq k) \) as \( C_i^{T'} = A_{C_i}^{T'} \) and the interpretations of concrete roles as follows: Let \( C = C_1^{T'} \cup \ldots \cup C_k^{T'} \). For each \( x_j \in C \),
there exists a set \( \{C_{j_1}, \ldots, C_{j_{nx}}\} \) s.t. for each \( C_{j_h} \in \{C_{j_1}, \ldots, C_{j_{nx}}\} \), \( x_j \in C^T_{j_h} \). Let \( S_j = C_{j_1} \cap \ldots \cap C_{j_{nx}} \). Obviously, \( \mathcal{I} \models S_j \). Due to claim (i), the datatype query \( \pi(S_j) \) is decidable; therefore, there exists a datatype interpretation \((\Delta_D, \cdot_D)\) and a solution \( \delta \) of \( \pi(S_j) \) w.r.t. \((\Delta_D, \cdot_D)\). Let \( T \) be a concrete role, \( V_{T}(j) \) the set of variables in \( \pi(S_j) \) that match \( T \), \( \delta(V_{T}(j)) \) the set of data values to which \( \delta \) maps the set of variables in \( V_{T}(j) \). Initially, we set all \( T^I = \emptyset \), then for each \( T \) used in each \( S_j \), we have \( T^I = T^I \cup \{S_j^I \times \delta(V_{T}(j))\} \). Obviously, we have \( \mathcal{I} \models D \). Due to claim (ii) and the construction of \( T' \), we have \( \mathcal{I} \models T \). Due to the definition of match, the constructions of \( \mathcal{R}' \) and the interpretations of concrete roles, we have \( \mathcal{I} \models \mathcal{R} \). \qed

## 5.3 Related Work

The story started when Lutz [85] proved that the \( \mathcal{ALC}(\mathcal{D}) \)-concept satisfiability problem w.r.t. TBoxes is usually undecidable even when we use simple concrete domains, such as some arithmetic concrete domains. In order to retain decidability, an obvious way out is to restrict the powerful concrete domain constructor, viz. feature chains (cf. Definition 2.10), to concrete features.

In this direction, Haarslev et al. [50] proved that \( \mathcal{SHN}(\mathcal{D})^- \)-concept satisfiability w.r.t. TBoxes and RBoxes is decidable, where \( \mathcal{D}^- \) means disallowing feature chains. Horrocks and Sattler [75] showed that the \( \mathcal{SHOQ}(\mathcal{D}) \)-concept satisfiability w.r.t. TBoxes and RBoxes is decidable, where \( \mathcal{D} \) is a universal concrete domain (cf. Definition 2.12). In \( \mathcal{SHOQ}(\mathcal{D}) \) (cf. Definition 2.3.2), feature chains are disallowed and only unary datatype predicates are considered. This result is strengthened by Pan and Horrocks [109], who show that \( \mathcal{SHOQ}(\mathcal{D}) \) can be extended to allow arbitrary arity datatype predicates and predicate qualified number restriction constructors to give \( \mathcal{SHOQ}(\mathcal{D}_n) \), and \( \mathcal{SHOQ}(\mathcal{D}_n) \)-concept satisfiability w.r.t. TBoxes and RBoxes is still decidable.

Interestingly, Baader et al. [5] shows that we can combine an arbitrary DL \( \mathcal{L}' \) with \( \mathcal{D}^- \) to give \( \mathcal{L}'(\mathcal{D})^- \), and \( \mathcal{L}'(\mathcal{D})^- \)-concept satisfiability w.r.t. TBoxes and RBoxes is decidable if:

1. \( \mathcal{L}' \) provides the conjunction (\( \cap \)) and full concept negation (\( \neg \)) constructors;
2. \( \mathcal{L}' \) does not provide any of the following constructors: nominals (\( \mathcal{O} \)), role complement (\( \mathcal{\overline{\cdot}} \)), universal role (\( \mathcal{U} \));
3. \( \mathcal{L}' \)-concept satisfiability w.r.t. TBoxes and RBoxes is decidable.
5.3. RELATED WORK

Obviously, Baader et al. [5]’s result covers $\mathcal{SHN}(\mathcal{D})^-$ and many other DLs integrated with $(\mathcal{D})^-$, but not $\mathcal{SHOQ}(\mathcal{D})$ or $\mathcal{SHOQ}(\mathcal{D}_n)$.

As far as the Description Logics of practical interest (e.g., the $\mathcal{AL}$-family DLs) are concerned, we have further generalised Baader et al. [5]’s result with Theorem 5.19: $\mathcal{L}(\mathcal{G})$-concept satisfiability w.r.t. TBoxes and RBoxes is decidable if

1. $\mathcal{L}$ provides the conjunction ($\sqcap$) and bottom ($\bot$) constructors;
2. $\mathcal{L}$-concept satisfiability w.r.t. TBoxes and RBoxes is decidable.

$\mathcal{SHN}(\mathcal{D})^-$, $\mathcal{SHOQ}(\mathcal{D})$ and $\mathcal{SHOQ}(\mathcal{D}_n)$ are all $\mathcal{G}$-combinable Description Logics.

In Chapter 6, we shall provide two extensions of OWL DL, which use the $\mathcal{SHOIQ}(\mathcal{G})$ and $\mathcal{SHOIN}(\mathcal{G}_1)$ $\mathcal{G}$-combined Description Logics as their underpinnings.

**Chapter Achievements**

- The datatype group approach provides a general formalism to unify existing ontology-related datatype and predicate formalisms and to overcome their limitations.
- In the unified formalism, $\mathcal{G}$-datatype expressions can be used to represent customised datatypes and predicates.
- The family of $\mathcal{G}$-combinable Description Logics are identified, in the formalism, to integrate with conforming datatype groups, so as to give decidable Description Logics that can provide customised datatypes and datatype predicates.
- Theorem 5.19 generalises (almost) all the known decidability results, about concept satisfiability w.r.t. TBoxes and RBoxes, of feature-chain-free Description Logics combined with admissible concrete domains.
Chapter 6
OWL-E: OWL Extended with Datatype Expressions

Chapter Aims

- To provide decidable extensions of OWL DL that support customised datatypes and datatype predicates.
- To design practical decision procedures for a family of Description Logics that are closely related to OWL DL and its above extensions.

Chapter Plan

6.1 The Design of OWL-E (126)
6.2 Reasoning with OWL-E (132)
6.3 OWL-Eu: A Smaller Extension of OWL (161)

In Chapter 5, the datatype group-based framework extends OWL datatyping to support customised datatypes and predicates. In this chapter, accordingly, we will propose OWL-E and OWL-Eu, which are decidable extensions of OWL DL with datatype groups, and present practical decision procedures for a family of Description Logics that are related to OWL DL, OWL-Eu and OWL-E.

6.1 The Design of OWL-E

A serious limitation of OWL DL is that its datatype support is too limited, i.e., it does not provide customised datatypes and predicates. Therefore, we have designed
an extension of OWL DL that overcomes this limitation.

### 6.1.1 Meeting the Requirements

Four main requirements have been considered when OWL-E is designed:

1. OWL-E should provide customised datatypes and predicates; therefore, it should be based on a datatype formalism which is compatible with OWL datatyping and provides facilities to construct customised datatypes and predicates.

2. OWL-E should be a decidable extension of OWL DL.

3. It is desirable that OWL-E is also an extension of DAML+OIL.

4. It is desirable for OWL-E to overcome all the other limitations mentioned in the ‘Limitations of OWL Datatyping’ section on page 77 (i.e., OWL-E should provide negations of datatypes, the datatype domain should properly accommodate interpretations of typed literals with unsupported predicate URIrefs, and OWL-E should provide names for enumerated datatypes).

According to Chapter 5, the datatype group approach matches the datatype formalism needed in Requirement 1, and we can integrate an arbitrary datatype group into a $\mathcal{G}$-combinable Description Logic without losing decidability of concept satisfiability w.r.t. TBoxes and RBoxes. As OWL-E is designed to be an extension of OWL DL, and it is desirable also to be an extension of DAML+OIL (Requirement 3), we should check whether the concept languages of OWL DL (i.e., $\mathcal{SHOIN}$) and DAML+OIL (i.e., $\mathcal{SHOIQ}$) are $\mathcal{G}$-combinable Description Logics. Since both $\mathcal{SHOIN}$-concept and $\mathcal{SHOIQ}$-concept satisfiability problems w.r.t. TBoxes and RBoxes are decidable (Tobies [139, Corollary 6.31]), according to Definition 5.15, both of them are $\mathcal{G}$-combinable Description Logics. Given that both $\mathcal{SHOIN}(\mathcal{G})$ and $\mathcal{SHOIQ}(\mathcal{G})$ are closed under negation, Theorem 5.19 now yields the following decidability result.

**Corollary 6.1.** The $\mathcal{SHOIN}(\mathcal{G})$- and $\mathcal{SHOIQ}(\mathcal{G})$-concept satisfiability and subsumption problems w.r.t. TBoxes and RBoxes are decidable.

Combining this result with Theorem 2.7, we obtain the following theorem.

**Theorem 6.2.** The knowledge base satisfiability problems of $\mathcal{SHOIN}(\mathcal{G})$ and $\mathcal{SHOIQ}(\mathcal{G})$ are decidable.
According to Schaerf [128], if a Description Logic is closed under negation, then all the basic reasoning services (described in Section 2.1.3) are reducible to knowledge base satisfiability. Therefore, Theorem 6.2 indicates the following theorem.

**Theorem 6.3.** All the basic reasoning services of $\mathit{SHOIN}(\mathcal{G})$ and $\mathit{SHOIQ}(\mathcal{G})$ are decidable.

Since $\mathit{SHOIN}(\mathcal{G})$ is a special case of $\mathit{SHOIQ}(\mathcal{G})$, if we choose $\mathit{SHOIQ}$ as the underpinning of OWL-E, then we can satisfy both Requirement 2 and 3.

Finally, we consider the minor limitations of OWL DL mentioned in the ‘Limitations of OWL Datatyping’ (Requirement 4). The first two limitations, i.e., those about negations and the datatype domain, have been considered in the datatype group approach (cf. Section 5.1.3), and we only need to take care of the last limitation, i.e., names for enumerated datatypes. In response to this requirement, OWL-E allows users to define names (in fact, URIrefs) for $\mathcal{G}$-datatype expressions.

### 6.1.2 From OWL DL to OWL-E

As discussed in the last section, OWL-E extends OWL DL as follows.

1. OWL-E contains an arbitrary conforming datatype group. Informally speaking, a datatype group contains a set of datatypes and the predicates defined over them (cf. Definition 5.4 on page 104).

2. OWL-E provides $\mathcal{G}$-datatype expressions. Given a datatype group $\mathcal{G} = (\mathcal{M}_p, \mathcal{D}_\mathcal{G}, \text{dom})$, Table 6.1 shows the abstract syntax and DL syntax of datatype expressions, where $u$ is a datatype predicate URIref, \$s_i\text{" }d_i\$ are typed literals, $v_1, \ldots, v_n$ are (possibly negated) unary supported predicate URIrefs, $P, Q$ are datatype expressions and $\Phi_\mathcal{G}$ is the set of supported predicate URIrefs in $\mathcal{G}$. Note that there are some extra syntactic conditions on the conjunctive and disjunctive expressions: if both $P$ and $Q$ have fixed arity, viz. $a(P)$ and $a(Q)$, then we require $a(P) = a(Q)$; or if one of them has minimum arity (suppose it is $P$), then we require $a_{\text{min}}(P) \leq a(Q)$ (cf. Definition 5.12 on page 110). Unary datatype expressions can be used as data ranges in datatype range axioms (cf. Table 3.5), while arbitrary datatype expressions can be used in datatype group-based class descriptions (cf. Table 6.3).
3. OWL-E provides *datatype expression axioms*, which introduce URIrefs for datatype expressions. Note that the set of URIrefs of datatype expressions should be disjoint from the URIrefs of classes, properties, individuals datatypes and datatype predicates, etc. Table 6.2 shows the abstract syntax of OWL-E datatype expression axioms, where $u$ is a datatype expression URIref and $E$ is a datatype expression as defined in Table 6.1.

4. OWL-E provides some new classes descriptions, which are listed in Table 6.3, where $T, T_1, \ldots, T_n$ are datatype properties (where $T_i \notin T_j, T_j \notin T_i$ for all $1 \leq i < j \leq n$), $R$ is an object property, $C$ is a class, $E$ is a datatype expression or a

---

1. $\bowtie$ is the transitive reflexive closure of $\sqsubseteq$. 

### Table 6.1: OWL-E datatype expressions

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>rdfs:Literal</td>
<td>$\top_D$</td>
<td>$\Delta_D$</td>
</tr>
<tr>
<td>owlx:DatatypeBottom</td>
<td>$\bot_D$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$u$ a predicate URIref</td>
<td>$u$</td>
<td>$u^D$</td>
</tr>
<tr>
<td>$\neg(u)$</td>
<td>$\pi$</td>
<td>if $u \in D_G$, $\Delta_D \setminus u^D$</td>
</tr>
<tr>
<td>oneOf($&quot;s_1&quot; \ldots &quot;s_n&quot;$)</td>
<td>$(\mathcal{U}D \cup \ldots \cup {&quot;s_1&quot; \ldots &quot;s_n&quot;})$</td>
<td>if $u \notin \Phi_G$, $(\mathcal{U}D \cup \ldots \cup {&quot;s_1&quot; \ldots &quot;s_n&quot;}) \setminus u^D$</td>
</tr>
<tr>
<td>domain($v_1, \ldots, v_n$)</td>
<td>$[v_1, \ldots, v_n]$</td>
<td>$\forall$</td>
</tr>
<tr>
<td>and($P, Q$)</td>
<td>$P \land Q$</td>
<td>$P \land Q$</td>
</tr>
<tr>
<td>or($P, Q$)</td>
<td>$P \lor Q$</td>
<td>$P \lor Q$</td>
</tr>
</tbody>
</table>

### Table 6.2: OWL-E datatype expression axiom

<table>
<thead>
<tr>
<th>Abstract Syntax</th>
<th>DL Syntax</th>
<th>Semantics</th>
</tr>
</thead>
<tbody>
<tr>
<td>restriction(${T}$ someTuplesSatisfy($E$))</td>
<td>$\exists T_1, \ldots, T_n. E$</td>
<td>${x \in \Delta^T</td>
</tr>
<tr>
<td>restriction(${T}$ allTuplesSatisfy($E$))</td>
<td>$\forall T_1, \ldots, T_n. E$</td>
<td>${x \in \Delta^T</td>
</tr>
<tr>
<td>restriction(${T}$ minCardinality(m) someTuplesSatisfy($E$))</td>
<td>$\exists m. T_1, \ldots, T_n. E$</td>
<td>${x \in \Delta^T</td>
</tr>
<tr>
<td>restriction(${T}$ maxCardinality(m) someTuplesSatisfy($E$))</td>
<td>$\exists m. T_1, \ldots, T_n. E$</td>
<td>${x \in \Delta^T</td>
</tr>
<tr>
<td>restriction($R$ minCardinality(m) someValuesFrom($C$))</td>
<td>$\exists m. R.C$</td>
<td>${x \in \Delta^T</td>
</tr>
<tr>
<td>restriction($R$ maxCardinality(m) someValuesFrom($C$))</td>
<td>$\exists m. R.C$</td>
<td>${x \in \Delta^T</td>
</tr>
</tbody>
</table>
6.1. THE DESIGN OF OWL-E

datatype expression URIref, and $\#$ denotes cardinality. Note that the first four are datatype group-based class descriptions, and the last two are qualified number restrictions.

6.1.3 How OWL-E Helps

Now we revisit the examples presented in Section 1.3 to show that how OWL-E supports the use of customised datatypes and predicates in various SW and ontology applications.

Example 6.1 OWL-E: Matchmaking

A service requirement may ask for a PC with memory size greater than 512Mb, unit price less than 700 pounds and delivery date earlier than 15/03/2004. The set of PCs that satisfy this required capability can be expressed by the following OWL-E class description.

```owl
Class(RequiredPC complete PC
    restriction(memorySizeInMb
        maxCardinality(1) minCardinality(1)
        someTuplesSatisfy(owlx:integerGreatThanx=512))
    restriction(priceInPound
        maxCardinality(1) minCardinality(1)
        someTuplesSatisfy(owlx:integerLessThanx=700))
    restriction(deliveryDate
        maxCardinality(1) minCardinality(1)
        someTuplesSatisfy(owlx:dateEarlierThanx=2004-03-15)))
```

This class description says, informally speaking, a RequiredPC is a PC which relates to (at most) three values, through the datatype properties memorySizeInMb, pricingPound and deliveryDate, that satisfy the unary predicate URIrefs owlx:integerGreatThanx=512, owlx:integerLessThanx=700 and owlx:dateEarlierThanx=2004-03-15, respectively.

Example 6.2 OWL-E: Unit Mapping

Now we revisit Example 1.4 on page 21. Assume we need to ensure that the distance in miles (in ontology A) should be equal to 1.6 times the data value of distance in kilometres (in the current ontology B) between the same pairs of locations. We can use the following OWL-E class inclusion axioms to represent this constraint:
SubClassOf(owl:Thing

    restriction(a:distanceInMile distanceInKilometre
    allTuplesSatisfy(and(owlx:doubleMultiplicationx=1.6
domain(xsd:double xsd:double)))))

where owlx:doubleMultiplicationx=1.6 is the predicate URIref for \(s^\text{double}\) (cf. Example 5.7 on page 112). The axiom ensures that if an object has all the two datatype properties, then the value it relates to through the distanceInKilometre property should be 1.6 times of that through the distanceInMile property. ◊

Example 6.3 OWL-E: Small Items

Assume that electronic-shops want to define small items as items of which the sum of height, length and width is no greater than 15cm. SmallItem can be represented by the following class description:

Class(SmallItem complete Item

    restriction(hlwSumInCm heightInCm lengthInCm widthInCm
    maxCardinality(1) minCardinality(1)
someTuplesSatisfy(sumNoGreaterThan15)))

DatatypeExpression(sumNoGreaterThan15 and(

    owlx:integerAddition
domain(not(owlx:integerGreaterThanx=15) xsd:integer
xsd:integer xsd:integer))))

The class description says a SmallItem is an Item which relates through the group of datatype properties hlwSumInCm, heightInCm, lengthInCm and widthInCm to exactly one tuple of values that satisfy the datatype expression sumNoGreaterThan15. The datatype expression axiom defines sumNoGreaterThan as a conjunction, where the first conjunct is a predicate URIref owlx:integer-Addition and the second conjunct is a domain expression, which requires the first argument of owlx:integerAddition should be no greater than 15 (cf. Example 5.7). Note that datatype expression axioms are very useful when datatype expressions (e.g., sumNoGreaterThan15) need to be used multiple times in the ontology. ◊
6.2 Reasoning with OWL-E

In this section, we will investigate how to provide practical decision procedures for OWL-E. To the best of our knowledge, there exist no published tableaux algorithms to handle the SHOIQ DL. Therefore, in what follows, we use the ‘divide and conquer’ strategy. That is, instead of providing a decision procedure for the SHOIQ(\mathcal{G}) DL, we provide decision procedures for the SHOQ(\mathcal{G}), SHIQ(\mathcal{G}) and SHIO(\mathcal{G}) DLs, where the SHOQ(\mathcal{D}) DL has been argued to be useful in the context of the Semantic Web \cite{75},\footnote{SHOQ(\mathcal{D}) is a special case of SHOQ(\mathcal{G}), where datatype expressions are restricted to simply datatype URIrefs and datatype-related number restrictions are not allowed; cf. Table 2.2 on page 46 and Table 5.2 on page 118.} and the SHIQ DL is the underpinning of the OIL Web ontology language and is implemented in popular DL systems like FaCT \cite{59} and RACER \cite{51}.

Most importantly, the decision procedures we will present for the above DLs call for a datatype reasoner as a sub-procedure for checking the satisfiability of some datatype queries via a well-defined ‘interface’. Therefore, once we have a decision procedure for SHOIQ, we can easily upgrade it to one for SHOIQ(\mathcal{G}) (cf. Theorem 6.14 on page 156).

Before introducing the above DLs, we define some notation. Let \(R_A\) and \(R_D\) be a countable set of abstract role names and datatype role names, such that \(R = R_A \cup R_D\); accordingly, for a Description Logic \(L\), let \(\text{Rdsc}_A(L)\) and \(\text{Rdsc}_D(L)\) be a set of abstract role expressions and a set of concrete role expressions (or simply called abstract roles and concrete roles) of \(L\), respectively. For a datatype group \(\mathcal{G}\), let \(\text{DPexp}(\mathcal{G})^{(1)} \subseteq \text{Dexp}(\mathcal{G})\) be a set of unary \(\mathcal{G}\)-predicate expressions. We use \(\top_D\) and \(\bot_D\) to represent rdfs:Literal and owlx:DatatypeBottom, respectively, in the DL syntax.

6.2.1 The SHOQ(\mathcal{G}) DL

Syntax and Semantics

The SHOQ(\mathcal{G}) DL extends the SHOQ(\mathcal{D}) DL (cf. Section 2.3.2) by replacing the two datatype related constructors (cf. Table 2.2 on page 46) with the four datatype group-based concept constructors given in Table 5.2 on page 118. I.e., SHOQ(\mathcal{G})-roles and SHOQ(\mathcal{G})-RBoxes are the same as SHOQ(\mathcal{D})-roles and SHOQ(\mathcal{D})-RBoxes (cf. Definition 2.14 on page 47).

Formally, SHOQ(\mathcal{G})-concepts are defined as followed.
Definition 6.4. \((S\text{HO}Q(G))-\text{concepts}\) Let \(CN \in C, R \in \text{Rdsc}_A(S\text{HO}Q(G)), T_1, \ldots, T_n \in \text{Rdsc}_D(S\text{HO}Q(G))\) and \(T_i \not\sqsubseteq T_j, T_j \not\sqsubseteq T_i\) for all \(1 \leq i < j \leq n\), \(C, D \in \text{Cdsc}(S\text{HO}Q(G)), o \in I, E \in \text{Dexp}(G), n, m \in \mathbb{N}, n \geq 1\). Valid \(S\text{HO}Q(G)\)-concepts are defined by the abstract syntax:

\[
C ::= \top | \bot | CN | \neg C | C \sqcap D | C \sqcup D | \{o\} \\
| \exists R.C | \forall R.C | \geq m R.C | \leq m R.C \\
| \exists T_1, \ldots, T_n.E | \forall T_1, \ldots, T_n.E | \geq m T_1, \ldots, T_n.E | \leq m T_1, \ldots, T_n.E.
\]

The semantics of datatype group-based \(S\text{HO}Q(G)\)-concepts is given in Table 5.2 on page 118; the semantics of other \(S\text{HO}Q(G)\)-concepts is given in Table 2.2 on page 46.

\(\diamond\)

A \(S\text{HO}Q(G)\)-Tableau

Since the \(S\text{HO}Q(G_1)\)-concept satisfiability and subsumption problem w.r.t. to a knowledge base can be reduced to the \(S\text{HO}Q(G_1)\)-concept satisfiability problem w.r.t. an RBox (cf. Section 2.1.3), we only discuss a tableaux algorithm for the latter problem.

For ease of presentation, we assume all concepts to be in negation normal form (NNF): a concept is in negation normal form iff negation is applied only to atomic concept names, nominals or datatype expressions. Each \(S\text{HO}Q(G)\)-concept can be transformed into an equivalent one in NNF by pushing negation inwards, making use of DeMorgan’s laws and the following equivalences:

\[
\neg \exists R.C \equiv \forall R.(\neg C) \\
\neg \forall R.C \equiv \exists R.(\neg C) \\
\neg \leq n R.C \equiv \geq (n + 1) R.C \\
\neg \geq (n + 1) R.C \equiv \leq n R.C \\
\neg \geq 0 R.C \equiv \bot \\
\neg \leq 0 R.C \equiv \top
\]

We use \(\sim C\) to denote the NNF of \(\neg C\).

Like other tableaux algorithms, the tableaux algorithm for \(S\text{HO}Q(G)\) tries to prove the satisfiability of a concept expression \(D\) by constructing a model of \(D\). The model is represented by a so-called completion forest (cf. Section 2.2), nodes of which correspond to either individuals (labelled nodes) or variables (non-labelled nodes), each labelled node being labelled with a set of \(S\text{HO}Q(G)\)-concepts. When testing the satisfiability of a \(S\text{HO}Q(G)\)-concept \(D\), these sets are restricted to subsets of \(\text{cl}(D)\),
which denotes the set of all sub-concepts of \( D \) and the NNF of their negations. We use \( \text{cl}_{d_h}(D) \) to denote the set of all the \( G \)-datatype expressions and their negations occurring in these (NNFs of) sub-concepts.

The soundness and completeness of the algorithm will be proved by showing that it creates a tableau for \( D \).

**Definition 6.5.** Let \( D \) be a \( SHOQ(G) \)-concept in NNF, \( R \) a \( SHOQ(G) \) RBox, \( R^D_A \) and \( R^D_D \) the sets of abstract and concrete roles, respectively, occurring in \( D \) or \( R \), \( G \) a datatype group, \( P \in \text{cl}_{d_h}(D) \) a (possibly negated) datatype predicate expression. A tableau \( \mathcal{T} = (S_A, S_D, \mathcal{L}, DC^T, \mathcal{E}_A, \mathcal{E}_D) \) for \( D \) w.r.t. \( R \) is defined as follows:

- \( S_A \) is a set of individuals,
- \( S_D \) is a set of variables,
- \( \mathcal{L} : S_A \to 2^{\text{cl}(D)} \) maps each individual to a set of concepts which is a sub-set of \( \text{cl}(D) \),
- \( DC^T \) is a set of datatype constraints of the form \( P(v_1, \ldots, v_n) \), where \( P \in \text{cl}_{d_h}(D) \) and \( v_1, \ldots, v_n \in S_D \),
- \( \mathcal{E}_A : R^D_A \to 2^{S_A \times S_A} \) maps each abstract role in \( R^D_A \) to a set of pairs of individuals,
- \( \mathcal{E}_D : R^D_D \to 2^{S_A \times S_D} \) maps each concrete role in \( R^D_D \) to a set of pairs of individuals and variables.

There is some individual \( s_0 \in S_A \) such that \( D \in \mathcal{L}(s_0) \). For all \( s, x \in S_A, v \in S_D, C, C_1, C_2 \in \text{cl}(D), R, S \in R^D_A, T, T' \in R^D_D, d \in \text{cl}_{d_1}(D) \),

\[
S^T(s, C) := \{x | \langle s, x \rangle \in \mathcal{E}_A(S) \land C \in \mathcal{L}(x)\},
\]

and it holds that:

**\( P0 \)** there exists a datatype interpretation \( \mathcal{I}_D = (\Delta_D, \text{D}) \) and a mapping \( \delta \) from \( S_D \) to \( \Delta_D \) s.t. for each \( P(v_1, \ldots, v_n) \in DC^T \), there exist \( t_i = \delta(v_i) \) for all \( 1 \leq i \leq n \), and \( \langle t_1, \ldots, t_n \rangle \in P^D \); based on \( \mathcal{I}_D \) and \( \delta \), we define \( T_1, \ldots, T_n^T(s, P) \) as follows

\[
T_1, \ldots, T_n^T(s, P) := \{\langle \delta(v_1), \ldots, \delta(v_n) \rangle | \langle s, v_1 \rangle \in \mathcal{E}_D(T_1) \land \ldots \land \langle s, v_n \rangle \in \mathcal{E}_D(T_n) \land \langle \delta(v_1), \ldots, \delta(v_n) \rangle \in P^D\},
\]
(P1) if \( C \in \mathcal{L}(s) \), then \( \sim C \notin \mathcal{L}(s) \).

(P2) if \( C_1 \cap C_2 \in \mathcal{L}(s) \), then \( C_1 \in \mathcal{L}(s) \) and \( C_2 \in \mathcal{L}(s) \).

(P3) if \( C_1 \cup C_2 \in \mathcal{L}(s) \), then \( C_1 \in \mathcal{L}(s) \) or \( C_2 \in \mathcal{L}(s) \).

(P4) if \( \langle s, x \rangle \in \mathcal{E}_A(R) \) and \( R \sqsubseteq S \), then \( \langle s, x \rangle \in \mathcal{E}_A(S) \), and if \( \langle s, t \rangle \in \mathcal{E}_D(T) \) and \( T \sqsubseteq T' \), then \( \langle s, t \rangle \in \mathcal{E}_D(T') \).

(P5) if \( \forall R.C \in \mathcal{L}(s) \) and \( \langle s, x \rangle \in \mathcal{E}_A(R) \), then \( C \in \mathcal{L}(x) \).

(P6) if \( \exists R.C \in \mathcal{L}(s) \), then there is some \( x \in S \) such that \( \langle s, x \rangle \in \mathcal{E}_A(R) \) and \( C \in \mathcal{L}(x) \).

(P7) if \( \forall S.C \in \mathcal{L}(s) \) and \( \langle s, x \rangle \in \mathcal{E}_A(R) \) for some \( R \sqsubseteq S \) with \( \text{Trans}(R) \), then \( \forall R.C \in \mathcal{L}(x) \).

(P8) if \( \geq n S.C \in \mathcal{L}(s) \), then \( \sharp S^T(s, C) \geq n \).

(P9) if \( \leq n S.C \in \mathcal{L}(s) \), then \( \sharp S^T(s, C) \leq n \).

(P10) if \( \{ \leq n S.C, \geq n S.C \} \cap \mathcal{L}(s) \neq \emptyset \) and \( \langle s, x \rangle \in \mathcal{E}_A(S) \), then \( \{ C, \sim C \} \cap \mathcal{L}(x) \neq \emptyset \).

(P11) if \( \{ \circ \} \in \mathcal{L}(s) \cap \mathcal{L}(x) \), then \( s = x \).

(P12) if \( \forall T_1, \ldots, T_n.P \in \mathcal{L}(s) \) and \( \langle s, v_i \rangle \in \mathcal{E}_D(T_i) \) for all \( 1 \leq i \leq n \), then \( P(v_1, \ldots, v_n) \in DC^T \).

(P13) if \( \exists T_1, \ldots, T_n.P \in \mathcal{L}(s) \), then there is some \( v_1, \ldots, v_n \in S_D \) such that \( \langle s, v_i \rangle \in \mathcal{E}_D(T_i) \) for all \( 1 \leq i \leq n \), and \( P(v_1, \ldots, v_n) \in DC^T \).

(P14) if \( \geq m T_1, \ldots, T_n.P \in \mathcal{L}(s) \), then \( \sharp T^T(s, P) \geq m \).

(P15) if \( \leq m T_1, \ldots, T_n.P \in \mathcal{L}(s) \), then \( \sharp T^T(s, P) \leq m \).

(P16) if \( \{ \leq m T_1, \ldots, T_n.P, \geq m T_1, \ldots, T_n.P \} \cap \mathcal{L}(s) \neq \emptyset \) and \( \langle s, v_i \rangle \in \mathcal{E}_D(T_i) \) for all \( 1 \leq i \leq n \), then we have either \( P(v_1, \ldots, v_n) \in DC^T \) or \( \neg P(v_1, \ldots, v_n) \in DC^T \). \( \Diamond \)

**Lemma 6.6.** A \( SHOQ(G) \)-concept \( D \) in NNF is satisfiable w.r.t. an RBox \( \mathcal{R} \) iff \( D \) has a tableau w.r.t. \( \mathcal{R} \).
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**Proof:** For the *if* direction, given a datatype group $G$, if $\mathcal{I} = (S_A, S_D, \mathcal{L}, DC^T, \mathcal{E}_A, \mathcal{E}_D)$ is a tableau for $D$, a model $\mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I}, \Delta_D, \cdot^\mathcal{D})$ of $D$ w.r.t. $G$ can be defined as:

$$\Delta^\mathcal{I} = S_A$$

$$CN^\mathcal{I} = \{s \in S_A | \text{CN} \in \mathcal{L}(s)\} \text{ for all concept names CN in}\ cl(D)$$

$$R^\mathcal{I} = \begin{cases} \mathcal{E}_A(R)^+ & \text{if Trans}(R) \in \mathcal{R} \\ \mathcal{E}_A(R) \cup \bigcup_{P \in R, P \neq R} P^\mathcal{I} & \text{otherwise} \end{cases}$$

$$T^\mathcal{I} = \{\langle s, \delta(v) \rangle | \langle s, v \rangle \in \mathcal{E}_D(T)\}$$

where $\mathcal{E}_A(R)^+$ denotes the transitive closure of $\mathcal{E}_A(R)$, and according to (P0), there exist a datatype interpretation $(\Delta_D, \cdot^\mathcal{D})$ of $G$.

The interpretation of non-transitive roles is recursive in order to correctly interpret those non-transitive roles that have a transitive sub-role. From the definition of $R^\mathcal{I}$ and (P4) of a tableau, it follows that, if $\langle s, x \rangle \in S^\mathcal{I}$, then either $\langle s, x \rangle \in \mathcal{E}_A(S)$ directly, or there exists a path $\langle s, s_1 \rangle, \langle s_1, s_2 \rangle, \ldots, \langle s_n, x \rangle \in \mathcal{E}_A(R)$ for some $R$ with Trans$(R)$ and $R \sqsubseteq S$. From the definition of $T^\mathcal{I}$ and (P4) of a tableau, it follows that, if $\langle s, v \rangle \in \mathcal{E}_D(T)$ and $T \sqsubseteq T'$, then $\langle s, \delta(v) \rangle \in T^\mathcal{I}$.

To show that $\mathcal{I}$ is a model of $D$ w.r.t. $\mathcal{R}$, we have to prove (1) $\mathcal{I} \models \mathcal{R}$ and (2) $D^\mathcal{I} \neq \emptyset$. The first one is obvious due to (P4) and the construction of the model. The second one is shown by proving $\mathcal{I} \models \mathcal{R}$ and $C \in \mathcal{L}(s) \Rightarrow s \in C^\mathcal{I}$ for any $s \in S_A$. This implies $D^\mathcal{I} \neq \emptyset$ since $T$ is a tableau for $D$ and hence there must be some $s_0 \in S_A$ with $D \in \mathcal{L}(s_0)$.

We will use an induction on the structure of concept to show that $C \in \mathcal{L}(s)$ implies $s \in C^\mathcal{I}$ for any $C$. The two base cases of the induction are $C = \text{CN}$ or $C = \neg \text{CN}$. If $\text{CN} \in \mathcal{L}(s)$, then by definition of $CN^\mathcal{I}$, $s \in CN^\mathcal{I}$. If $\neg \text{CN} \in \mathcal{L}(s)$, then by (P1), $\text{CN} \notin \mathcal{L}(s)$ and hence $s \notin CN^\mathcal{I}$. For the induction step we have to distinguish several cases:

- $C = C_1 \cap C_2$. Since $T$ is a tableau, $C \in \mathcal{L}(s)$ implies $C_1 \in \mathcal{L}(s)$ and $C_2 \in \mathcal{L}(s)$. Hence, by induction, we have $s \in C_1^\mathcal{I}$ and $s \in C_2^\mathcal{I}$, which yields $s \in (C_1 \cap C_2)^\mathcal{I}$.

- $C = C_1 \cup C_2$. Similar to the previous case.

- $C = \forall S.E$. Let $s \in S_A$ with $C \in \mathcal{L}(s)$ and $x \in S_A$ an arbitrary individual such that $\langle s, x \rangle \in \mathcal{E}_A(S)$. Then there are two possibilities:
  - $\langle s, x \rangle \in \mathcal{E}_A(S)$. Then (P5) implies that $E \in \mathcal{L}(x)$ and by induction $x \in \mathcal{L}(x)$. Therefore, $s \in C^\mathcal{I}$.
  - $\langle s, x \rangle \notin \mathcal{E}_A(S)$. Then $\langle s, x \rangle \in \mathcal{E}_A(S)$, which implies $s \in C^\mathcal{I}$.

- $C = \exists S.E$. Let $s \in S_A$ with $C \in \mathcal{L}(s)$ and $x \in S_A$ an arbitrary individual such that $\langle s, x \rangle \in \mathcal{E}_A(S)$. Then there are two possibilities:
  - $\langle s, x \rangle \in \mathcal{E}_A(S)$. Then (P5) implies that $E \in \mathcal{L}(x)$ and by induction $x \in \mathcal{L}(x)$.
  - $\langle s, x \rangle \notin \mathcal{E}_A(S)$. Then $\langle s, x \rangle \notin \mathcal{E}_A(S)$, which implies $s \notin C^\mathcal{I}$.

- $C = \exists S.E$. Let $s \in S_A$ with $C \in \mathcal{L}(s)$ and $x \in S_A$ an arbitrary individual such that $\langle s, x \rangle \in \mathcal{E}_A(S)$. Then there are two possibilities:
  - $\langle s, x \rangle \in \mathcal{E}_A(S)$. Then (P5) implies that $E \in \mathcal{L}(x)$ and by induction $x \in \mathcal{L}(x)$.
  - $\langle s, x \rangle \notin \mathcal{E}_A(S)$. Then $\langle s, x \rangle \notin \mathcal{E}_A(S)$, which implies $s \notin C^\mathcal{I}$.
\[ E^T. \]

- \( \langle s, x \rangle \not\in E_A(S) \). Then there exists a path \( \langle s, s_1 \rangle, \langle s_1, s_2 \rangle, \ldots, \langle s_n, x \rangle \in E_A(R) \) for some \( R \) with Trans\((R)\) and \( R \subseteq S \). Then (P7) implies \( \forall R.E \in L(s_i) \) for all \( 1 \leq i \leq n \), and \( \forall R.E \in L(x) \). From (P5), we have \( E \in L(x) \).

By induction, this implies \( x \in E^T \).

- \( C = \exists S.E \). Since \( T \) is a tableau, \( C \in L(s) \) implies the existence of an individual \( x \in S_A \) such that \( \langle s, x \rangle \in E_A(S) \) and \( x \in L(E) \). By induction, we have \( x \in E^T \), and from the definition of \( S^T \) it follows that \( \langle s, x \rangle \in S^T \) and hence \( s \in C^T \).

- \( C = (\geq n S.E) \). For an \( s \in S_A \) with \( C \in L(s) \), we have \( \#S^T(s,E) \geq n \). Hence there are \( n \) individuals \( x_1, \ldots, x_n \) such that \( x_i \neq x_j \) for \( i \neq j \), \( \langle s, x_i \rangle \in E_A(S) \), and \( E \in L(x_i) \) for all \( i \). By induction, we have \( x_i \in E^T \) and, since \( E_A(S) \subseteq S^T \), \( s \in C \) also holds.

- \( C = (\leq n S.E) \). For this case, we need that \( S \) is a simple role, which implies that \( S^T = E_A(S) \). Let \( s \in S_A \) with \( C \in L(s) \). Due to (P10) we have \( E \in L(x) \) or \( \sim E \in L(x) \) for each \( x \) with \( \langle s, x \rangle \in E_A(S) \). Moreover, \( \#S^T(s,E) \leq n \) holds due to (P9). We define

\[ S^T(s,E) := \{ x \in \Delta^T \mid \langle s, x \rangle \in S^T \land x \in E^T \} \]

and can show that \( \#S^T(s,E) \leq \#S^T(s,E) \): assume \( \#S^T(s,E) > \#S^T(s,E) \), this implies the existence of some \( x \) with \( \langle s, x \rangle \in S^T \) with \( x \in E^T \) but \( E \not\in L(x) \) (because \( S^T = E_A(S) \)). By (P10) this implies \( \sim E \in L(x) \), which, by induction, yields \( x \in (\sim E)^T \) in contradiction to \( x \in E^T \). Therefore \( \#S^T(s,E) \leq n \), and \( s \in C^T \) also holds.

- \( C = \{o\} \). Let \( s, x \in S_A \), \( \{o\} \in L(s) \cap L(x) \) implies \( \{o\} \in L(s) \) and \( \{o\} \in L(x) \). By induction, we have \( s \in \{o\} \) and \( x \in \{o\} \) and hence \( s \in C^T \). By (P11) we have \( s = x \), which ensures that nominals are interpreted as singletons.

- \( C = \forall T_1, \ldots, T_n.P \). Let \( s \in S_A \) with \( C \in L(s) \), and \( v_1, \ldots, v_n \in S_D \) with \( \langle s, v_i \rangle \in E_D(T_i) \) for all \( 1 \leq i \leq n \). By the definition of \( T^T \), we have \( \langle s, \delta(v_i) \rangle \in T^T_i \) for all \( 1 \leq i \leq n \). (P12') implies that \( P(v_1, \ldots, v_n) \in DC^T \). By (P0'), we have \( \langle \delta(v_1), \ldots, \delta(v_n) \rangle \in PD \), hence \( s \in C^T \).

- \( C = \exists T_1, \ldots, T_n.P \). Since \( T \) is a tableau, \( C \in L(s) \) implies the existence of variables \( v_1, \ldots, v_n \in S_D \) such that \( \langle s, v_1 \rangle \in E_D(T_i) \) for all \( 1 \leq i \leq n \), and
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\[ P(v_1, \ldots, v_n) \in DC^\mathcal{T}. \] By the definition of \( T^\mathcal{T} \), we have \( \langle s, \delta(v_i) \rangle \in T^\mathcal{T}_i \) for all \( 1 \leq i \leq n \). By \((P0')\), we have \( \langle \delta(v_1), \ldots, \delta(v_n) \rangle \in P^D \), hence \( s \in C^\mathcal{T} \).

• \( C = (\geq m T_1, \ldots, T_n P) \). For an \( s \) with \( C \in L(s) \), we have \( \sharp T^\mathcal{T}(s, P) \geq m \).

Hence there are \( m \cdot n \) variables \( v_{11}, \ldots, v_{mn} \in S_D \) such that \( \langle s, v_{ij} \rangle \in E_D(T_j) \), \( \langle \delta(v_{i1}), \ldots, \delta(v_{in}) \rangle \in P^D \) and \( \langle \langle \delta(v_{i1}), \ldots, \delta(v_{in}) \rangle, \langle \delta(v_{j1}), \ldots, \delta(v_{jn}) \rangle \rangle \notin P^D \) for all \( 1 \leq i < k \leq m, 1 \leq j \leq n \). By the definition of \( T^\mathcal{T} \), we have \( \langle s, \delta(v_{ij}) \rangle \in T^\mathcal{T}_j \) for all \( 1 \leq i \leq m, 1 \leq j \leq n \). Hence \( s \in C^\mathcal{T} \).

• \( C = (\leq m T_1, \ldots, T_n P) \). Let \( s \in S_A \) with \( C \in L(s) \). Due to \((P16')\) we have \( P(v_1, \ldots, v_n) \in DC^\mathcal{T} \) or \( \neg P(v_1, \ldots, v_n) \in DC^\mathcal{T} \) for each tuple \( \langle v_1, \ldots, v_n \rangle \) with \( \langle s, v_i \rangle \in E_D(T_i) \) for all \( 1 \leq i \leq n \). Moreover, \( \sharp T^\mathcal{T}(s, P) \leq m \) holds due to \((P15')\). We define

\[ T^\mathcal{T}(s, P) = \{ \langle t_1, \ldots, t_n \rangle \mid \langle s, t_i \rangle \in T^\mathcal{T}_i \text{ (for all } 1 \leq i \leq n) \}
\]

We can show that \( \sharp T^\mathcal{T}(s, P) \leq \sharp T^\mathcal{T}(s, P) \): assume \( \sharp T^\mathcal{T}(s, P) > \sharp T^\mathcal{T}(s, P) \). This implies the existence of some tuple \( \langle t_1, \ldots, t_n \rangle \) with \( \langle s, t_i \rangle \in T^\mathcal{T}_i \) for all \( 1 \leq i \leq n \) and \( \langle t_1, \ldots, t_n \rangle \in P^D \), but there is no \( v_1, \ldots, v_n \) such that \( \delta(v_i) = t_i \) for all \( 1 \leq i \leq n \), which is in contradiction with the definition of \( T^\mathcal{T} \).

For the only-if direction, we have to show that satisfiability of \( D \) w.r.t. \( \mathcal{R} \) implies the existence of a tableau \( \mathcal{T} \) for \( D \) w.r.t. \( \mathcal{R} \).

Let \( \mathcal{T} = \langle \Delta^\mathcal{T}, \mathcal{I}, \Delta_D, P^D \rangle \) be a model of \( D \) w.r.t. \( \mathcal{G} \) and \( \mathcal{T} \models \mathcal{R} \). A tableau \( \mathcal{T} = \langle S_A, S_D, L, DC^\mathcal{T}, E_A, E_D \rangle \) for \( D \) can be defined as:

• \( S_A = \Delta^\mathcal{T} \);

• \( L(s) = \{ C \in \text{cl}(D) \mid s \in C^\mathcal{T} \} \);

• \( E_A(R) = R^\mathcal{T} \), where \( R \) is an abstract role;

• we initialise \( S_D \) as an empty set, \( \delta \) and \( E_D(T) \) as empty relations, and construct them as follows: for all concrete role \( T \) and for each \( \langle s, t \rangle \in T^\mathcal{T} \),

1. we generate a new variable \( v \),

2. \( S_D := S_D \cup \{ v \} \).

\( \neq \) is the value inequality predicate; cf. Definition 5.17 on page 119.
3. \( \delta := \delta \cup \{ \langle v, t \rangle \} \),
4. \( \mathcal{E}_D(T) := \mathcal{E}_D(T) \cup \{ \langle s, v \rangle \} \);

- \( \mathcal{D}^T := \{ P(v_1, \ldots, v_n) | \exists v_1, \ldots, v_n \in S_D. P \in \text{cl}_D(D), \delta(v_i) = t_i \text{ (for all } 1 \leq i \leq n) \text{ and } \langle t_1, \ldots, t_n \rangle \in P^D \} \).

It remains to demonstrate that \( \mathcal{T} \) is a tableau for \( D \):

- (P0): Since \( \mathcal{T} \) is an interpretation w.r.t. to \( \mathcal{G}, \mathcal{T}_D = (\Delta_D, \cdot^D) \) is a datatype interpretation of \( \mathcal{G} \). By definition of \( \mathcal{D}^T \), for each \( P(v_1, \ldots, v_n) \in \mathcal{D}^T \), there exist \( t_1, \ldots, t_n \) with \( t_i = \delta(v_i) \) for all \( 1 \leq i \leq n \) and \( \langle t_1, \ldots, t_n \rangle \in P^D \).

- (P1) - (P3), (P5) - (P6), and (P8) - (P11) are satisfied as a direct consequence of the definition of the semantics of \( S\mathcal{HOQ}(\mathcal{G}_1) \)-concepts.

- (P4) is satisfied because \( \mathcal{T} \models \mathcal{R} \).

- (P7): If \( s \in (\forall S.C)^T \) and \( \langle s, x \rangle \in R^T \) with \( \text{Trans}(R) \) and \( R \subseteq S \), then \( x \in (\forall R.C)^T \) unless there is some \( y \) such that \( \langle x, y \rangle \in R^T \) and \( y \notin C^T \). In this case, if \( \langle s, x \rangle \in R^T, \langle x, y \rangle \in R^T \) and \( \text{Trans}(R) \), then \( \langle s, y \rangle \in R^T \). Hence \( \langle s, y \rangle \in S^T \) and \( s \notin (\forall S.C)^T \) in contradiction to the assumption. \( \mathcal{T} \) therefore satisfies (P7).

- (P12): Assume \( \forall T_1, \ldots, T_n. P \in \mathcal{L}(s) \) and \( \langle s, v_j \rangle \in \mathcal{E}_D(T_j) \) for all \( 1 \leq j \leq n \). According to the definition of \( \mathcal{E}_D(T) \) and \( \delta \), there must be \( n \) data values \( t_1, \ldots, t_n \in \Delta_D \) such that \( \langle s, t_j \rangle \in T_j^T \) and \( \delta(v_j) = t_j \) for all \( 1 \leq j \leq n \). By the definition of \( \mathcal{L}(s) \), we have \( s \in \forall T_1, \ldots, T_n. P \), hence \( \langle t_1, \ldots, t_n \rangle \in P^D \). By the definition of \( \mathcal{D}^T \), we have \( P(v_1, \ldots, v_n) \in \mathcal{D}^T \).

- (P13): Assume \( \exists T_1, \ldots, T_n. P \in \mathcal{L}(s) \). By the definition of \( \mathcal{L}(s) \), we have \( s \in \exists T_1, \ldots, T_n. P \), then there must be \( n \) data values \( t_1, \ldots, t_n \in \Delta_D \) such that \( \langle s, t_j \rangle \in T_j^T \) for all \( 1 \leq j \leq n \) and \( \langle t_1, \ldots, t_n \rangle \in P^D \). By the definition of \( \mathcal{S}_D \) and \( \delta \), there are \( n \) variables \( v_1, \ldots, v_n \in \mathcal{S}_D \) s.t. \( t_j = \delta(v_j) \) for all \( 1 \leq j \leq n \). By the definition of \( \mathcal{D}^T \), we have \( P(v_1, \ldots, v_n) \in \mathcal{D}^T \).

- (P14) and (P15): Assume \( \geq m T_1, \ldots, T_n. P \in \mathcal{L}(s) \). By the definition of \( \mathcal{L}(s) \), we have \( s \in \geq m T_1, \ldots, T_n. P \). This implies that there are \( m \cdot n \) data values \( t_{11}, \ldots, t_{mn} \) such that \( \langle s, t_{ij} \rangle \in T_j^T \) and \( \langle \langle \delta(v_{i1}), \ldots, \delta(v_{in}) \rangle, \langle \delta(v_{k1}), \ldots, \delta(v_{kn}) \rangle \in \neq^D \text{ for } 1 \leq i < k \leq m, 1 \leq j \leq n \). By the definition of \( \mathcal{E}_D(T) \), there are \( m \cdot n \) variables \( v_{11}, \ldots, v_{mn} \in \mathcal{S}_D \) s.t. \( t_{ij} = \delta(v_{ij}) \) and \( \langle s, v_{ij} \rangle \in \mathcal{E}_D(T_j) \) for
1 \leq i \leq m, 1 \leq j \leq n. By the definition of \( T^\exists(s, P) \), we have \( \#T^\exists(s, P) \geq m \). Similarly, (P15) is satisfied.

- (P16): Assume \( \{ \geq mT_1, \ldots, T_n.P, \leq mT_1, \ldots, T_n.P \} \cap \mathcal{L}(s) \neq \emptyset \) and \( \langle s, v_j \rangle \in E_D(T_j) \) for \( 1 \leq j \leq n \). By the definition of \( \mathcal{L}(s) \), we have \( s \in \geq mT_1, \ldots, T_n.P \), or \( s \in \leq mT_1, \ldots, T_n.P \). According to the definition of \( E_D(T) \), there must be \( n \) data values \( t_1, \ldots, t_n \in \Delta_D \) such that \( \langle s, t_j \rangle \in T_j^\exists \) and \( t_j = \delta(v_j) \) for all \( 1 \leq j \leq n \). Since either \( \langle t_1, \ldots, t_n \rangle \in P^D \) or \( \langle t_1, \ldots, t_n \rangle \in (\neg P)^D \), by the definition of \( DC^\exists \), we have either \( P(v_1, \ldots, v_n) \in DC^\exists \) or \( \neg P(v_1, \ldots, v_n) \in DC^\exists \).

**Constructing a \( SHOQ(\mathcal{G}) \)-Tableau**

Now we consider a tableaux algorithm for \( SHOQ(\mathcal{G}) \). Generally speaking, given a \( \mathcal{G} \)-combinable Description Logic \( \mathcal{L} \), if we have a tableaux algorithm for \( \mathcal{L} \) (we call its completion rules \( \mathcal{L} \)-rules), then the set of the datatype group-related completion rules, or \( \mathcal{G} \)-rules (cf. Figure 6.2 on page 143) and the set of clash conditions about datatype constraints, or \( \mathcal{G} \)-clash conditions (cf. Figure 6.4 on page 144) can be seen as plug-ins to the tableaux algorithm of \( \mathcal{L}(\mathcal{G}) \). I.e., the set of completion rules for \( \mathcal{L}(\mathcal{G}) \) can be constructed as the union of the sets of \( \mathcal{L} \)-rules and \( \mathcal{G} \)-rules, and the set of clash conditions for \( \mathcal{L}(\mathcal{G}) \) can be constructed as the union of the sets of \( \mathcal{L} \)-clash conditions (those for \( \mathcal{L} \)) and \( \mathcal{G} \)-clash conditions.

In what follows, we will demonstrate that we can extend the known decidable tableaux algorithm for \( SHOQ \), which is provided by Horrocks and Sattler [75], in the above way and give a tableaux algorithm for \( SHOQ(\mathcal{G}) \). From Lemma 6.6, an algorithm which constructs a tableau for a \( SHOQ(\mathcal{G}) \)-concept \( D \) can be used as a decision procedure for the satisfiability of \( D \) w.r.t. a role box \( \mathcal{R} \).

**Definition 6.7. (A Tableaux Algorithm for \( SHOQ(\mathcal{G}) \))** Let \( \mathcal{R} \) be an \( \mathcal{R} \)Box, \( D \) a \( SHOQ(\mathcal{G}) \)-concept in NNF, \( R^D_A \) and \( R^D_B \) the sets of abstract roles and concrete roles occurring in \( D \) or \( \mathcal{R} \), \( I^D \) the set of nominals occurring in \( D \), and \( \mathcal{G} \) a datatype group.

A tableaux algorithm works on a completion forest \( F \) for \( D \) w.r.t. \( \mathcal{R} \). Labels on nodes and edges are defined as usual, except that there are two kinds of nodes in the completion forest: abstract nodes (the normal labelled nodes, by default we simply call them nodes) and concrete nodes (non-labelled leaves of \( F \)). Each (abstract) node

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4For a brief introduction of tableaux algorithms, see Section 2.2.
Let $x$ be labelled with a set

$$L(x) \subseteq \text{cl}(D) \cup \{\uparrow (R, \{o\}) \mid R \in R^D_A \text{ and } \{o\} \in I^D\},$$

and each edge $\langle x, y \rangle$ is labelled with a set of role names $L(\langle x, y \rangle)$ containing either abstract roles occurring in $R^D_A$, or concrete roles in $R^D$: in the first case, $y$ is a node and called an abstract successor (or simply successor) of $x$; in the second case, $y$ is a concrete node, and called a concrete successor of $x$. Predecessors, ancestors, and roots are defined as usual. Additionally, we keep track of inequalities between (abstract) nodes of the forest with a symmetric binary relation $\neq$. For each $\{\circ\} \in I^D$, there is a distinguished node $z_{\{\circ\}}$ in $F$ such that $\{\circ\} \in L(z)$. We use $\uparrow (R, \{o\}) \in L(x)$ to represent an $R$ labelled edge from $x$ to $z_{\{o\}}$.

Given a completion forest, a node $y$ is called an $R$-successor of a node $x$ if, for some $R'$ with $R' \sqsubseteq R$, either $y$ is a successor of $x$ and $R' \in L(\langle x, y \rangle)$, or $\uparrow (R', \{o\}) \in L(x)$ and $y = z_{\{o\}}$. For an abstract role $S$ and a node $x$ in $F$, we define $S^F(x, C)$ as

$$S^F(x, C) := \{y \mid y \text{ is an abstract } S \text{-successor of } x \text{ and } C \in L(y)\}.$$

Given a completion forest, a concrete node $v$ is called a concrete $T$-successor of a node $x$ if, for some concrete role $T'$ with $T' \sqsubseteq T$, $v$ is a concrete successor of $x$ and $T' \in L(\langle x, v \rangle)$. A tuple of concrete nodes $\langle v_1, \ldots, v_n \rangle$ is called a $\langle T_1 \ldots T_n \rangle$-successor of a node $x$ if, for all $1 \leq j \leq n$, $v_j$ is a concrete $T_j$-successor of $x$.

We will use a set $DC(x)$ to store the datatype expressions that must hold w.r.t. concrete successors of a node $x$ in $F$. Each element of $DC(x)$ is either of the form $\langle (v_{k1}, \ldots, v_{kn}), P \rangle$, or of the form $\langle (v_{l1}, \ldots, v_{ln}), (v_{j1}, \ldots, v_{jn}), \neq \rangle$, where $\langle v_{k1}, \ldots, v_{kn} \rangle$, $\langle v_{l1}, \ldots, v_{ln} \rangle$, and $\langle v_{j1}, \ldots, v_{jn} \rangle$ are tuples of concrete successors of $x$, $P \in \text{cl}(D)$ is a (possibly negated) datatype predicate expression, and $\neq$ is the value inequality predicate. The tableau algorithm calls a datatype reasoner as a sub-procedure for the satisfiability of $DC(x)$. We say that $DC(x)$ is satisfiable if the datatype query

$$\bigwedge_{\langle(v_{k1}, \ldots, v_{kn}), P \rangle \in DC(x)} P(v_{k1}, \ldots, v_{kn}) \land$$

$$\bigwedge_{\langle(v_{l1}, \ldots, v_{ln}), (v_{j1}, \ldots, v_{jn}), \neq \rangle \in DC(x)} \neq (v_{l1}, \ldots, v_{ln}; v_{j1}, \ldots, v_{jn}) \tag{6.1}$$

is satisfiable. For $n$ concrete role $T_1, \ldots, T_n$ and a node $x$ in $F$, we define $T_1, \ldots, T_n^F(x,$

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5cf. page 134 for the definition of $\text{cl}(D)$. 


\[ T_1 \ldots T_n(x, P) := \{ \langle v_1, \ldots, v_n \rangle \mid \langle v_1, \ldots, v_n \rangle \text{ is a } (T_1 \ldots T_n)\text{-successor of } x \text{ and } (\langle v_1, \ldots, v_n \rangle, P) \in DC(x) \}. \]
A node $x$ is *directly blocked* if none of its ancestors are blocked, and it has an ancestor $x'$ that is not distinguished such that $L(x) \subseteq L(x')$. We say $x'$ blocks $x$. A node is *blocked* if it is directly blocked or if its predecessor is blocked.

If $\{o_1, \ldots, o_l\} = I^D$, the algorithm initialises the completion forest $F$ to contain $l + 1$ root nodes $x_0, z_{\{o_1\}}, \ldots, z_{\{o_l\}}$ with $L(x_0) = \{D\}$ and $L(z_{\{o_i\}}) = \{\{o_i\}\}$. The inequality relation $\neq$ is initialised with the empty relation and the $DC(x)$ of any

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| $\forall$-rule: | if $1.\forall T_1, \ldots, T_n, P \in L(x)$, $x$ is not blocked, and  
2. there is a $\langle T_1 \ldots T_n \rangle$-successor $\langle v_1, \ldots, v_n \rangle$ of $x$  
with $(\langle v_1, \ldots, v_n \rangle, P) \notin DC(x)$,  
then $DC(x) \rightarrow DC(x) \cup \{\langle v_1, \ldots, v_n \rangle, P\}$. |
| $\exists$-rule: | if $1.\exists T_1, \ldots, T_n, P \in L(x)$, $x$ is not blocked, and  
2. $x$ has no $\langle T_1 \ldots T_n \rangle$-successor $\langle v_1, \ldots, v_n \rangle$ with  
$(\langle v_1, \ldots, v_n \rangle, P) \in DC(x)$,  
then 1. create a $\langle T_1 \ldots T_n \rangle$-successor $\langle v_1, \ldots, v_n \rangle$ of $x$ with  
$L(\langle x, v_j \rangle) = \{T_j\}$ for all $1 \leq j \leq n$, and  
2. $DC(x) \rightarrow DC(x) \cup \{\langle v_1, \ldots, v_n \rangle, P\}$. |
| $\geq$-rule: | if $1.\geq m T_1, \ldots, T_n, P \in L(x)$, $x$ is not blocked, and  
2. there are no $m$ $\langle T_1 \ldots T_n \rangle$-successors $\langle v_1, \ldots, v_{1n} \rangle$,  
$s.t. \langle v_{1m} \ldots v_{nm} \rangle$ of $x$ with $(\langle v_1, \ldots, v_{1n} \rangle, P) \in DC(x)$ and  
$(\langle v_1, \ldots, v_{1n}, \langle v_1, \ldots, v_{1n}, P \rangle \notin DC(x) \forall 1 \leq i < j \leq m$. |
| $\leq$-rule: | if $1.\leq m T_1, \ldots, T_n, P \in L(x)$, $x$ is not blocked, and  
2. $x$ has $m + 1 \langle T_1 \ldots T_n \rangle$-successors $\langle v_1, \ldots, v_{1n} \rangle$,  
$s.t. \langle v_{m+1,1} \ldots v_{m+1,n} \rangle$ of $x$, with $(\langle v_1, \ldots, v_{1n} \rangle, P) \in DC(x)$  
for $1 \leq i < j \leq m + 1$, and  
3. among them, there exist two different $\langle T_1 \ldots T_n \rangle$-successors  
$\langle v_1, \ldots, v_{in} \rangle$ and $\langle v_{j1}, \ldots, v_{jn} \rangle \equiv j \neq i$, s.t.  
$(\langle v_1, \ldots, v_{in}, \langle v_{j1}, \ldots, v_{jn}, \rangle \neq DC(x)$. |
| choose $p$-rule: | if $1.\{\leq m T_1, \ldots, T_n, P, \geq m T_1, \ldots, T_n P\} \cap L(x) \neq \emptyset$, $x$ is not blocked,  
2. $(\langle v_1, \ldots, v_{1n} \rangle, P)$ is a $\langle T_1 \ldots T_n \rangle$-successor of $x$,  
$s.t. (\langle v_{1m} \ldots v_{nm}, P \rangle \notin DC(x)$ and $(\langle v_1, \ldots, v_{1n}, \neg P \rangle \notin DC(x)$. |

Figure 6.2: The $G$-rules
A node \( x \) of a completion forest \( F \) contains a clash if (at least) one of the following conditions holds:

1. \( \bot \in L(x) \);
2. for some \( A \in C \), \( \{A, \neg A\} \subseteq L(x) \);
3. for some abstract role \( S \), \( S.C \in L(x) \) and there are \( n+1 \) \( S \)-successors \( y_1, \ldots, y_{n+1} \) of \( x \) with \( C \in L(y_i) \) for each \( 1 \leq i \leq n+1 \) and \( y_i \neq y_j \) for each \( 1 \leq i < j \leq n+1 \);
4. for some \( \{o\} \in L(x) \), \( x \neq z_{\{o\}} \).

Figure 6.3: The \( SHOQ \)-clash conditions

(abstract) node \( x \) is initialised to the empty set. \( F \) is then expanded by repeatedly applying the completion rules, listed in Figure 6.1 and Figure 6.2, until \( F \) is complete, or some \( L(x) \) contains a clash (see Figure 6.3 and Figure 6.4 for clash conditions for \( SHOQ(G) \)).

A node \( x \) of a completion forest \( F \) contains a clash if (at least) one of the following conditions holds:

\( (G1) \) for some concrete roles \( T_1, \ldots, T_n, \leq m T_1, \ldots, T_n, P \in L(x) \) and there are \( m+1 \) \( \langle T_1 \ldots T_n \rangle \)-successors \( \langle v_{11}, \ldots, v_{1n} \rangle, \ldots, \langle v_{m+1,1}, \ldots, v_{m+1,n} \rangle \) of \( x \) with \( \langle \langle v_{i1}, \ldots, v_{in} \rangle, P \rangle \in DC(x) \) for each \( 1 \leq i \leq m+1 \), and \( \langle \langle v_{i1}, \ldots, v_{in} \rangle, \langle v_{j1}, \ldots, v_{jn} \rangle, \neq \rangle \in DC(x) \) for each \( 1 \leq i < j \leq m+1 \);

\( (G2) \) for some abstract node \( x \), \( DC(x) \) is not satisfiable.

Figure 6.4: The \( G \)-clash conditions

If the completion rules can be applied in such a way that they yield a complete, clash-free completion forest, then the algorithm returns “\( D \) is satisfiable w.r.t. \( R \)”, and “\( D \) is unsatisfiable w.r.t. \( R \)” otherwise.

There are some remarks about \( DC(x) \) in the algorithm:

1. \( DC(x) \) represents a local datatype query for the node \( x \); it is the ‘interface’ between the tableaux algorithm and datatype reasoners.

2. For any \( \langle \langle v_1, \ldots, v_n \rangle, P \rangle \in DC(x) \), the algorithm is not affected by the structure of the datatype expression \( P \). I.e., the kind of datatype expressions and the kind of supported predicates provided by datatype reasoners are independent of the tableaux algorithm.
3. The merging of concrete nodes (variables) is handled by the \( \leq_P \)-rule, instead of by datatype reasoners (through the value equality predicate \( = \)). Therefore, the form of datatype queries (5.3) is simplified to

\[
Q' := C_E \wedge \bigwedge_{j=1}^{k} \neg \left( \vec{v}_{(j,1)} = \vec{v}_{(j,2)} \right),
\]

(6.2)

where \( C_E \) is a datatype expression conjunction of the form (5.14), and \( \neq \) is the value inequality predicate. According to Lemma 5.18, if \( G \) is conforming, a datatype query of the form (6.2) is decidable.

### Sound and Complete

The soundness and completeness of the algorithm will be demonstrated by that, for a \( SHOQ(G) \)-concept \( D \), it always terminates and that it returns \textit{satisfiable} if and only if \( D \) is satisfiable.

**Lemma 6.8. (Termination)** For each \( SHOQ(G) \)-concept \( D \) and \( RBox \ R \), the completion algorithm terminates.

**Proof:** Let \( h = |c_1(D)| \), \( k = |R^D| \), \( m_{\text{max}} \) the maximal number in atleast number restrictions or datatype atleast restrictions in \( D \) and \( \ell = |I^P| \). Note that datatype checkers will be used to test the satisfiability of \( DC(x) \), and that this is known to be decidable (cf. Lemma 5.18 on page 119). Termination is a consequence of the following properties of the expansion rules:

1. All the rules except the \( \leq_-, \leq_P \)- and \( O \)-rules strictly extend the completion forest by extending node labels or adding nodes, while removing neither nodes nor elements from node labels.

2. New nodes are only generated by the \( \exists_- \), \( \exists_P \)-, \( \geq_ \)-rule or the \( \geq_P \)-rule as successors of an abstract node \( x \) for concepts of the form \( \exists R.C, \exists T_1, \ldots, T_n P \), \( \geq S.C \) and \( \geq mT_1, \ldots, T_n P \) in \( L(x) \). For \( x \), each of these concepts can trigger the generation of successors at most once—even if the node(s) generated are later removed by either the \( \leq_-, \leq_P \)- or the \( O \)-rule.

   - For the \( \exists \)-rule, if a successor \( y \) of \( x \) was generated for a concept \( \exists S.C \in L(x) \) and later \( y \) is removed by an application of the \( \leq \)-rule or the \( O \)-rule, then there will always be some \( S \)-successor \( z \) of \( x \) such that \( C \in L(z) \), and hence the \( \exists \)-rule can not be applied again to \( x \) and \( \exists S.C \).
6.2. REASONING WITH OWL-E

For the \( \triangleright \)-rule, if \( y_1, \ldots, y_n \) were generated by an application of the \( \triangleright \)-rule for a concept \( (\triangleright nS.C) \), then \( y_i \neq y_j \) holds for all \( 1 \leq i < j \leq n \).

This implies that there will always be \( n \) \( S \)-successors \( y'_1, \ldots, y'_n \) of \( x \) with \( C \in L(y'_i) \) and \( y'_i \neq y'_j \) holds for all \( 1 \leq i < j \leq n \), since neither the \( \subseteq \)-rule nor the \( O \)-rule can ever merge two nodes \( y'_i, y'_j \) (because \( y'_i \neq y'_j \)).

and, whenever an application of the \( \subseteq \)-rule or the \( O \)-rule removes \( y'_i \), there will be an \( S \)-successor \( z \) of \( x \) that “inherits” all the inequalities from \( y'_i \).

For the \( \exists _p \)-rule, if a concrete \( \langle T_1, \ldots, T_n \rangle \)-successor \( \langle v_1, \ldots, v_n \rangle \) of \( x \) was generated by an application of the \( \exists _p \)-rule for a concept \( \exists T_1, \ldots, T_n.P \), then the \( \subseteq _p \)-rule can only remove some \( v_i \) (\( 1 \leq i \leq n \)) by merging \( L(\langle x, v_i \rangle) \) into \( L(\langle x, v'_i \rangle) \), where \( v' \) is some other concrete \( T_i \)-successor of \( x \), and replacing \( v_i \) with \( v'_i \) in \( DC(x) \) such that \( \exists T^p_\emptyset (x, P \neq 0) \).

For the \( \triangleright _p \)-rule, if \( \langle T_1 \ldots T_n \rangle \)-successors \( \langle v_{i1}, \ldots, v_{in} \rangle, \ldots, \langle v_{m1}, \ldots, v_{mn} \rangle \) were generated by an application of the \( \triangleright _p \)-rule for a concept \( (\triangleright mT_1, \ldots, T_n.P) \), then \( (\langle v_{i1}, \ldots, v_{in} \rangle, \langle v_{j1}, \ldots, v_{jn} \rangle, \neq) \in DC(x) \) holds for all \( 1 \leq i < j \leq m \). This implies that any \( \langle v_{i1}, \ldots, v_{in} \rangle, \langle v_{j1}, \ldots, v_{jn} \rangle \) of these \( \langle T_1 \ldots T_n \rangle \)-successors of \( x \) can never be merged by an application of the \( \subseteq _p \)-rule. Whenever an application of the \( \subseteq _p \)-rule merges \( \langle v_{i1}, \ldots, v_{in} \rangle \) and \( \langle v_{k1}, \ldots, v_{kn} \rangle \), and removes some \( v_{ig} \), where \( 1 \leq g \leq n \), there will be a concrete \( T_g \)-successor \( v_{kg} \) of \( x \) within \( \langle v_{k1}, \ldots, v_{kn} \rangle \) which ‘inherits’ \( L(\langle x, v_{ig} \rangle) \), and the \( \langle T_1 \ldots T_n \rangle \)-successor \( \langle v_{k1}, \ldots, v_{kn} \rangle \) will ‘inherit’ all the datatype constraints (including all the inequalities) in \( DC(x) \) about \( \langle v_{i1}, \ldots, v_{in} \rangle \).

Since \( cl(D) \) contains a total of at most \( h \exists S.C, (\triangleright nS.C), \exists T_1, \ldots, T_n.P \) and \((\triangleright mT_1, \ldots, T_n.P)\) concepts, the out-degree of the forest is bounded by \( h \cdot m_{\text{max}} \).

3. Nodes are labeled with subsets of \( cl(D) \cup \{ \uparrow (R, \{ \circ \}) | R \in R^D_A \} \). Since concrete nodes have no labels and are always leaves, there are at most \( 2^{h+k-\ell} \) different abstract node labellings. Therefore, if a path \( p \) is of length at least \( 2^{h+k-\ell} \), then, from the blocking condition above, there are two nodes \( x, y \) on \( p \) such that \( x \) is directly blocked by \( y \). Hence paths are of length at most \( 2^{h+k-\ell} \).

\[ \square \]

In the following proof of soundness, we will show that we can construct a tableau \( T = (S_A, S_D, L, DC^\tau, E_A, E_D) \) from a complete and clash-free completion forest \( F \).
6.2. REASONING WITH OWL-E

Intuitively, it works as follows: An individual in \( S_A \) corresponds to a \emph{path} in \( F \) from the root node to some (abstract) node that is not blocked. As these paths may be \emph{cyclic}, tableaux can be infinite. Due to qualifying number restrictions, we must distinguish different nodes that are blocked by the same node. A variable in \( S_D \) corresponds to a concrete node in \( F \), which is a concrete successor of the \emph{tail} of a path, and datatype constraints in \( DC^T \) correspond to the elements of the \( DC \) set in the completion forest \( F \).

**Lemma 6.9. (Soundness)** If the expansion rules can be applied to a SHOQ(\( G \))-concept \( D \) in NNF and an RBox \( \mathcal{R} \) such that they yield a complete and clash-free completion forest, then \( D \) has a tableau w.r.t. \( \mathcal{R} \).

**Proof:** Let \( F \) be the complete and clash-free completion forest constructed by the completion algorithm for \( D \). A path is sequence of pairs of (abstract) nodes of \( F \) of the form \( p = [(x_0, x'_0), \ldots, (x_n, x'_n)] \) (we disregard concrete nodes). For such a path, we define \( \text{Tail}(p) = x_n \) and \( \text{Tail}'(p) = x'_n \). With \([p|(x_n, x'_{n+1})] \) we denote the path \([(x_0, x'_0), \ldots, (x_n, x'_n), (x_{n+1}, x'_{n+1})] \). The set \( \text{Paths}(F) \) is defined inductively as follows:

1. For a root node \( x_0 \) in \( F \), \([(x_0, x_0)] \in \text{Paths}(F) \), and

2. For a path \( p \in \text{Paths}(F) \) and a node \( z \) in \( F \):
   
   (a) if \( z \) is a successor of \( \text{Tail}(p) \) and \( z \) is not blocked, then \([p|(z, z)] \in \text{Paths}(F) \), or
   
   (b) if, for some node \( y \) in \( F \), \( y \) is a successor of \( \text{Tail}(p) \) and \( z \) blocks \( y \), then \([p|(z, y)] \in \text{Paths}(F) \).

Please note that, due to the construction of \( \text{Paths} \), for \( p \in \text{Paths}(F) \) with \( p = [p'|(x, x')] \), \( x' \) is blocked iff \( x \neq x' \).

Now we can define a tableau \( \mathcal{T} = (S_A, S_D, \mathcal{L}, DC^T, \mathcal{E}_A, \mathcal{E}_D) \) with:

\[
\begin{align*}
S_A &= \text{Paths}(F) \\
S_D &= \{ v \mid v \text{ is a concrete node in } F \} \\
\mathcal{L}(p) &= \mathcal{L}(\text{Tail}(p)) \\
DC^T &= \{ P(v_1, \ldots, v_n) \mid \exists p \in S_A, (\langle v_1, \ldots, v_n \rangle, P) \in DC(\text{Tail}(p)) \} \\
\mathcal{E}_A(R) &= \{ (p, q) \in S_A \times S_A \mid q = [p|(x, x')] \text{ and } x' \text{ is an } R\text{-successor of } \text{Tail}(p) \} \\
\mathcal{E}_D(T) &= \{ (p, v) \in S_A \times S_D \mid p \in S_A \text{ and } v \text{ is a concrete } T\text{-successor of } \text{Tail}(p) \}
\end{align*}
\]

By default, we simply call abstract successors \emph{successors}; see Definition 6.7 on page 140.
6.2. REASONING WITH OWL-E

Claim: \( \mathcal{T} \) is a tableau for \( D \) w.r.t. \( \mathcal{R} \).

We have to show that \( \mathcal{T} \) satisfies all the properties from Definition 6.5.

- (P0): because \( F \) is clash-free, there exists a datatype interpretation \( I_D = (\Delta_D, \cdot^D) \) and a mapping \( \delta : S_D \to \Delta_D \) s.t. for each \( p \in S_A \) and each \( \langle (v_1, \ldots, v_n), P \rangle \in DC(Tail(p)) \). Thus, according to the definition of \( DC^\mathcal{T} \), for each \( P(v_1, \ldots, v_n) \in DC^\mathcal{T} \), there exists \( t_j = \delta(v_j) \) for all \( 1 \leq j \leq n \) and \( \langle t_1, \ldots, t_n \rangle \in P^D \).

- (P1): If \( C \in \mathcal{L}(p) \), then \( \neg C \notin \mathcal{L}(p) \); otherwise \( F \) is not clash-free.

- (P2) and (P3) hold because \( Tail(p) \) is not blocked and \( F \) is complete.

- (P4) is satisfied due to the definition of \( R \)-successor that takes into account the role hierarchy \( \sqsubseteq \).

- (P5): Assume \( \forall S.C \in \mathcal{L}(p) \) and \( \langle p, q \rangle \in \mathcal{E}_A(S) \). Let \( q = [p|(x, x')] \), \( x' \) is an \( S \)-successor of \( Tail(p) \) and thus \( C \in \mathcal{L}(x') \) must hold (otherwise the \( \forall \)-rule is still applicable). Since \( \mathcal{L}(x') \subseteq \mathcal{L}(x) = \mathcal{L}(q) \), we have \( C \in \mathcal{L}(q) \).

- (P6): Assume \( \exists S.C \in \mathcal{L}(p) \). Define \( x := Tail(p) \). In \( F \) there must be an \( S \)-successor \( y' \) of \( x \) with \( C \in \mathcal{L}(y') \), otherwise the \( \exists \)-rule is still applicable. Let \( q := [p|(y, y')] \in S_A \), since \( \mathcal{L}(y') \subseteq \mathcal{L}(y) = \mathcal{L}(q) \), we have \( C \in \mathcal{L}(q) \).

- (P7): Assume \( \forall S.C \in \mathcal{L}(p), \langle p, q \rangle \in \mathcal{E}_A(R) \) for some \( R \sqsupseteq S \) with \( Trans(R) \). Let \( q := [p|(x, x')] \), then \( x' \) is an \( R \)-successor of \( Tail(p) \) and thus \( \forall R.C \in \mathcal{L}(x') \) must hold, otherwise the \( \forall_+ \)-rule would be applicable. Since \( \mathcal{L}(x') \subseteq \mathcal{L}(x) = \mathcal{L}(q) \), we have \( \forall R.C \in \mathcal{L}(q) \).

- (P8): Assume \( \langle \geq n S.C \rangle \in \mathcal{L}(P) \). This implies that there exist \( n \) individuals \( y_1, \ldots, y_n \) in \( F \) such that each \( y_i \) is an \( S \)-successor of \( Tail(p) \) and \( C \in \mathcal{L}(y_i) \). We claim that for each of these individuals, there is a path \( q_i \) such that \( \langle p, q_i \rangle \in \mathcal{E}_A(S), C \in \mathcal{L}(q_i) \), and \( q_i \neq q_j \) for all \( 1 \leq i < j \leq n \). Obviously, this implies \( \# S^{\mathcal{T}}(p, C) \geq n \). For each \( y_i \) there are two possibilities:

  - \( y_i \) is an \( S \)-successor of \( x \) and \( y_i \) is not blocked in \( F \). Then \( q_i = [p|(y_i, y_i)] \) is a path with the desired properties.

\[ \text{If } x' \text{ is not blocked, } \mathcal{L}(x') = \mathcal{L}(x) \text{; otherwise } \mathcal{L}(x') \subseteq \mathcal{L}(x). \]
\( y_i \) is an \( S \)-successor of \( x \) and \( y_i \) is blocked in \( F \) by some node \( z \). Then \( q_i = [p|(z,y_i)] \) is a path with the desired properties. Since the same \( z \) may block several of the \( y_i \)s, it is indeed necessary to include \( y_i \) explicitly in the path to make them distinct.

- (P9): Assume that there is some \( p \in S_A \) with \( (\leq nS.C) \in L(p) \) and \( S^T(p,C) > n \). We will show that this implies \( S^F(\text{Tail}(p),C) > n \) which is a contradiction to either the clashingness or completeness of \( F \). Define \( x := \text{Tail}(p) \) and \( P := S^T(p,C) \). Due to the assumption, we have \( \sharp P > n \) and \( P \) contains paths of the form \( q = [p|(y,y')] \).

We claim that the function \( \text{Tail}' \) is injective on \( P \). If we assume that there are two paths \( q_1, q_2 \in P \) with \( q_1 \neq q_2 \) and \( \text{Tail}'(q_1) = \text{Tail}'(q_2) = y' \), then this implies that \( q_1 \) is of the form \( q_1 = [p|(y_1,y')] \) and \( q_2 \) is of the form \( q_2 = [p|(y_2,y')] \) with \( y_1 \neq y_2 \). If \( y' \) is not blocked in \( F \), then \( y_1 = y' = y_2 \) holds, contradicting \( y_1 \neq y_2 \). If \( y' \) is blocked in \( F \), then both \( y_1 \) and \( y_2 \) block \( y' \), which implies \( y_1 = y_2 \), again a contradiction.

Since \( \text{Tail}' \) is injective on \( P \), it holds that \( \sharp P = \sharp \text{Tail}'(P) \). Also for each \( y' \in \text{Tail}'(P) \), \( y' \) is an \( S \)-successor of \( x \) and \( C \in L(y') \). This implies \( S^F(x,C) > n \), i.e., \( S^F(\text{Tail}(p),C) > n \).

- (P10): Assume \( \{\exists nS.C, \leq nS.C\} \cap L(p) \neq \emptyset, \langle p,q \rangle \in E_A(S) \). Let \( q = [p|(x,x')] \), \( x' \) is an \( S \)-successor of \( \text{Tail}(p) \) and thus \( \{C,\sim C\} \cap L(x') \neq \emptyset \) must hold (otherwise the choose-rule is still applicable). Since \( L(x') \subseteq L(x) = L(q) \), we have \( \{C,\sim C\} \cap L(q) \neq \emptyset \).

- (P11): Assume \( \{o\} \in L(p) \cap L(q) \). \( \text{Tail}(p) = \text{Tail}(q) = z_{\{o\}} \) must hold, otherwise the O-rule is still applicable. Since distinguished nodes can never be blocked, we have \( p = q \).

- (P12): Assume \( \forall T_1, \ldots, T_n.P \in L(p) \) and \( \langle p,v_j \rangle \in E_D(T_j) \). This implies that \( v_j \) is a concrete \( T_j \)-successor of \( \text{Tail}(p) \), for all \( 1 \leq j \leq n \), and that \( \langle v_1, \ldots, v_n \rangle \) is a \( \langle T_1 \ldots T_n \rangle \)-successor of \( \text{Tail}(p) \). Hence \( \langle \langle v_1, \ldots, v_n \rangle, P \rangle \in DC(\text{Tail}(p)) \) must hold, otherwise the \( \forall P \)-rule is still applicable. Due to the construction of \( DC^\tau \), we have \( P(v_1, \ldots, v_n) \in DC^\tau \).

- (P13): Assume \( \exists T_1, \ldots, T_n.P \in L(p) \). In \( F \) there must be a \( \langle T_1 \ldots T_n \rangle \)-successor \( \langle v_1, \ldots, v_n \rangle \) of \( \text{Tail}(p) \) s.t. \( \langle \langle v_1, \ldots, v_n \rangle, P \rangle \in DC(\text{Tail}(p)) \), otherwise the
\( \exists_p \)-rule is still applicable. Due to the construction of \( DC^\tau \), we have \( P(v_1, \ldots, v_n) \in DC^\tau \).

- (P14): Assume \( \geq mT_1, \ldots, T_n.P \in \mathcal{L}(p) \). This implies \( \geq mT_1, \ldots, T_n.P \in \mathcal{L}(\text{Tail}(p)) \), and hence there are \( m \langle T_1 \ldots T_n \rangle \)-successors \( \langle v_{i_1}, \ldots, v_{i_{n}} \rangle, \ldots, \langle v_{m_1}, \ldots, v_{m_n} \rangle \) of \( \text{Tail}(p) \) in \( F \) such that \( (\langle v_{i_1}, \ldots, v_{i_{n}} \rangle, P) \in DC(\text{Tail}(p)) \) and \( (\langle v_{j_1}, \ldots, v_{j_{n}} \rangle, P) \notin DC(\text{Tail}(p)) \) for all \( 1 \leq i < j \leq m \) must hold (otherwise the \( \geq_p \)-rule is still applicable). Due to the completeness of \( F \), we have \( \langle \delta(v_{i_1}), \ldots, \delta(v_{m_n}) \rangle \in P^D \) and \( (\langle \delta(v_{j_1}), \ldots, \delta(v_{j_{n}}) \rangle, P) \notin P^D \) for all \( 1 \leq i < j \leq m \). Hence \( T^\tau(p, P) \geq m \).

- (P15): Assume that there is some \( p \in S_A \) such that \( \leq mT_1, \ldots, T_n.P \in \mathcal{L}(p) \) and \( T^\tau(p, P) > m \). Define \( x := \text{Tail}(p) \). Due to the assumption, we have at least \( m + 1 \langle T_1 \ldots T_n \rangle \)-successors \( \langle v_{i_1}, \ldots, v_{i_{n}} \rangle, \ldots, \langle v_{m_1}, \ldots, v_{m_n} \rangle \) of \( x \), such that \( \langle v_{i_1}, \ldots, v_{i_{n}} \rangle \in P^D \) and \( (\langle \delta(v_{i_1}), \ldots, \delta(v_{i_{n}}) \rangle, P) \notin P^D \) for all \( 1 \leq i < j \leq m \). This implies that we have \( (\langle v_{i_1}, v_{j_1} \rangle, P) \notin DC(x) \) for all \( 1 \leq i < j \leq m + 1 \), which means there is a clash of form (G1), contradicting the clash-freeness of \( F \).

- (P16): Assume \( \{ \leq mT_1, \ldots, T_n.P, \geq mT_1, \ldots, T_n.P \} \cap \mathcal{L}(p) \neq \emptyset \), and there exists a \( \langle T_1 \ldots T_n \rangle \)-successor \( \langle v_{1}, \ldots, v_{n} \rangle \) of \( \text{Tail}(p) \), such that \( (p, v_j) \in E_D(T_j) \) for all \( 1 \leq j \leq n \). Thus \( (\langle v_{1}, \ldots, v_{n} \rangle, P) \in DC(x) \) or \( (\langle v_{1}, \ldots, v_{n} \rangle, \neg P) \in DC(x) \) must hold (otherwise the \( \text{choose}_p \)-rule is still applicable). Due to the definition of \( DC^\tau \), we have either \( P(v_1, \ldots, v_n) \in DC^\tau \) or \( \neg P(v_1, \ldots, v_n) \in DC^\tau \).

**Lemma 6.10. (Completeness)** If a SHOQ(\( G \))-concept \( D \) in NNF has a tableau w.r.t. an RBox \( \mathcal{R} \), then the expansion rules can be applied to \( D \) such that they yield a complete, clash-free completion forest w.r.t. \( \mathcal{R} \).

**Proof:** Let \( \mathcal{T} = (S_A, S_D, \mathcal{L}, DC^\tau, E_A, E_D) \) be a tableau for \( D \) w.r.t. an RBox \( \mathcal{R} \). Using \( \mathcal{T} \), we trigger the application of the completion rules such that they yield a completion forest \( F \) that is both complete and clash-free.

We start with \( F \) consisting of \( l + 1 \) root nodes \( x_0, z_{o_1}, \ldots, z_{o_l} \) with \( \mathcal{L}(x_0) = \{ D \}, \mathcal{L}(z_{o_i}) = \emptyset \), and \( DC(x) = \emptyset \). Since \( \mathcal{T} \) is a tableau, there is some \( s_0 \in S_A \) with \( D \in \mathcal{L}(s_0) \), for all abstract nodes \( x \). Due to (P0), there exists a datatype interpretation \( I_D = (\Delta_D, \cdot^D) \) and a mapping \( \delta \) that maps the variables in \( S_D \) to data values in \( \Delta_D \).

When applying the completion rules to \( F \), the application of the non-deterministic rules are guided by the labelling in the tableau \( \mathcal{T} \). To this purpose, we define a mapping
\(\pi\) which maps the abstract nodes of \(F\) to elements of \(S_A\) and maps concrete nodes of \(F\) to elements of \(S_D\). We steer the non-deterministic rules such that \(\mathcal{L}(x) \subseteq \mathcal{L}(\pi(x))\) holds for all abstract nodes \(x\) of the completion forest. More precisely, we define \(\pi\) inductively as follows, for each \(x, y, v, v_1, \ldots, v_n, w_1, \ldots, w_n\) in \(F\):

\[
\begin{align*}
\mathcal{L}(x) &\subseteq \mathcal{L}(\pi(x)) \\
\langle \pi(x), \pi(y) \rangle &\in \mathcal{E}_A(S) \quad \text{if } y \text{ is an } S\text{-successor of } x, \\
\langle \pi(x), \pi(v_j) \rangle &\in \mathcal{E}_D(T_j) \quad (\text{for all } 1 \leq j \leq n) \quad \text{if } \langle v_1, \ldots, v_n \rangle \text{ is a } \langle T_1 \ldots T_n \rangle\text{-successor of } x \\
\pi(x) &\neq \pi(y) \quad \text{if } x \text{ and } y \text{ are abstract nodes and } x \neq y \\
\langle \langle v_1, \ldots, v_n \rangle, P \rangle &\in DC(x) \quad \text{if } v_1, \ldots, v_n \text{ are a concrete successors of } x \text{ and } P(\pi(v_1), \ldots, \pi(v_n)) \in DC^T(x) \\
\langle \langle \delta(\pi(v_1)), \ldots, \delta(\pi(v_n)) \rangle, \langle \delta(\pi(v_1)), \ldots, \delta(\pi(v_n)) \rangle \rangle &\in \mathcal{D}_P \quad \text{if } v_1, \ldots, v_n, w_1, \ldots, w_n \text{ are concrete successors of } x \text{ and } \\
\langle \langle v_1, \ldots, v_n \rangle, \langle w_1, \ldots, w_n \rangle, \neq P \rangle &\in DC(x)
\end{align*}
\]

\(\textbf{Claim:}\) Let \(F\) be a completion forest and \(\pi\) a function that satisfies \((*)\). If a rule is applicable to \(F\) then the rule is applicable to \(F\) in a way that yields a completion forest \(F'\) and a function \(\pi'\) that satisfy \((*)\).

Let \(F\) to be a completion forest and \(\pi\) a function that satisfies \((*)\). We have to consider the various rules.

- The \(\sqcap\)-rule: If \(C_1 \sqcap C_2 \in \mathcal{L}(x)\), then \(C_1 \sqcap C_2 \in \mathcal{L}(\pi(x))\). This implies \(C_1 \in \mathcal{L}(\pi(x))\) and \(C_2 \in \mathcal{L}(\pi(x))\) due to \((P2)\) from Definition 6.5, and hence the rule can be applied without violating \((*)\).

- The \(\sqcup\)-rule: If \(C_1 \sqcup C_2 \in \mathcal{L}(x)\), then \(C_1 \sqcup C_2 \in \mathcal{L}(\pi(x))\). Since \(T\) is a tableau, 
\((P3)\) from Definition 6.5 implies \(\{C_1, C_2\} \cap \mathcal{L}(\pi(x)) \neq \emptyset\). Hence the \(\sqcup\)-rule can add a concept \(E \in \{C_1, C_2\}\) to \(\mathcal{L}(x)\) such that \(\mathcal{L}(x) \subseteq \mathcal{L}(\pi(x))\) holds.

- The \(\exists\)-rule: If \(\exists \mathcal{S}.C \in \mathcal{L}(x)\), then \(\exists \mathcal{S}.C \in \mathcal{L}(\pi(x))\) and, since \(T\) is a tableau, \((P6)\) from Definition 6.5 implies that there is an element \(t \in S_A\) such that \(\langle \pi(x), t \rangle \in \mathcal{E}_A(S)\) and \(C \in \mathcal{L}(t)\). The application of the \(\exists\)-rule generates a new abstract node \(y\) with \(\mathcal{L}(\langle x, y \rangle) = \{S\}\) and \(\mathcal{L}(y) = \{C\}\). Hence we set \(\pi' := \pi[y \mapsto t]\) which yields a function that satisfies \((*)\) for the modified forest.

- The \(\forall\)-rule: If \(\forall \mathcal{S}.C \in \mathcal{L}(x)\), then \(\forall \mathcal{S}.C \in \mathcal{L}(\pi(x))\), and if \(y\) is an \(S\)-successor
of \( x \), then also \( (\pi(x), \pi(y)) \in E_A(S) \) due to (\(*\)). Since \( \mathcal{T} \) is a tableau, (P5) from Definition 6.5 implies \( C \in \mathcal{L}(\pi(y)) \) and hence the \( \forall \)-rule can be applied without violating (\(*\)).

- The \( \forall_+ \)-rule: If \( \forall S. C \in \mathcal{L}(x) \), then \( \forall S. C \in \mathcal{L}(\pi(x)) \), and, if there is some \( R \in S \) with \( \text{Trans}(R) \) and \( y \) is an \( R \)-successor of \( x \), then also \( (\pi(x), \pi(y)) \in E_A(S) \) due to (\(*\)). Since \( \mathcal{T} \) is a tableau, (P7) from Definition 6.5 implies \( \forall R. C \in \mathcal{L}(\pi(y)) \) and hence the \( \forall_+ \)-rule can be applied without violating (\(*\)).

- The \( \geq \)-rule: If \( (\geq n \cdot S. C) \in \mathcal{L}(x) \), then \( (\geq n \cdot S. C) \in \mathcal{L}(\pi(x)) \). Since \( \mathcal{T} \) is a tableau, (P8) from Definition 6.5 implies \( \not\exists^\mathcal{T} S. (\pi(x), C) \geq n \). Hence there are individuals \( t_1, \ldots, t_n \in S_A \) such that \( (\pi(x), t_i) \in E_A(S), C \in \mathcal{L}(t_i) \), and \( t_i \neq t_j \) for \( 1 \leq i < j \leq n \). The \( \geq \)-rule generates \( n \) new abstract nodes \( y_1, \ldots, y_n \). By setting \( \pi' := \pi[y_1 \mapsto t_1, \ldots, y_n \mapsto t_n] \), we have a function \( \pi' \) that satisfies (\(*\)) for the modified forest.

- The \( \leq \)-rule: If \( (\leq n \cdot S. C) \in \mathcal{L}(x) \), then \( (\leq n \cdot S. C) \in \mathcal{L}(\pi(x)) \). Since \( \mathcal{T} \) is a tableau, (P9) from Definition 6.5 implies \( \not\exists^\mathcal{T} S. (\pi(x), C) \leq n \). If the \( \leq \)-rule is applicable, we have \( \not\exists^\mathcal{T} S. (\pi(x), C) = n \), which implies that there are at least \( n + 1 \) \( S \)-successors \( y_1, \ldots, y_{n+1} \) of \( x \) such that \( C \in \mathcal{L}(y_1) \). Thus, there must be two nodes \( y, z \in \{y_1, \ldots, y_{n+1}\} \) such that \( \pi(y) = \pi(z) \) (because otherwise \( \not\exists^\mathcal{T} S. (\pi(x), C) > n \) would hold). \( \pi(y) = \pi(z) \) implies that \( y \neq z \) cannot hold because of (\(*\)). Hence the \( \leq \)-rule can be applied without violating (\(*\)).

- The \textit{choose}-rule: If \( (\geq n \cdot S. C, \leq n \cdot S. C) \cap \mathcal{L}(x) \neq \emptyset \), then \( (\geq n \cdot S. C, \leq n \cdot S. C) \cap \mathcal{L}(\pi(x)) \neq \emptyset \), and, if there is an \( S \)-successor \( y \) of \( x \), then \( (\pi(x), \pi(y)) \in E_A(S) \) due to (\(*\)). Since \( \mathcal{T} \) is a tableau, (P10) from Definition 6.5 implies \( \{C, \sim C\} \cap \mathcal{L}(\pi(y)) \neq \emptyset \). Hence the \textit{choose}-rule can add an appropriate concept \( E \in \{C, \sim C\} \) to \( \mathcal{L}(y) \) such that \( \mathcal{L}(y) \subseteq \mathcal{L}(\pi(y)) \) holds.

- The \textit{O}-rule: If \( \{\odot\} \in \mathcal{L}(x) \), then \( \{\odot\} \in \mathcal{L}(\pi(x)) \), and if \( z \) is distinguished with \( \{\odot\} \in \mathcal{L}(z) \), then \( \{\odot\} \in \mathcal{L}(\pi(z)) \) due to (\(*\)). Since \( \mathcal{T} \) is a tableau, (P11) from Definition 6.5 implies \( \pi(x) = \pi(z) \); hence \( \mathcal{L}(\pi(x)) = \mathcal{L}(\pi(z)) = \mathcal{L}(\pi(x)) \cup \mathcal{L}(\pi(z)) \). Hence the \textit{O}-rule can be applied without violating (\(*\)).

- The \( \forall_P \)-rule: If \( \forall T_1, \ldots, T_n. P \in \mathcal{L}(x) \), then \( \forall T_1, \ldots, T_n. P \in \mathcal{L}(\pi(x)) \), and, if \( (v_1, \ldots, v_n) \) is a \( (T_1 \ldots T_n) \)-successor of \( x \), then also \( (\pi(x), \pi(v_j)) \in E_D(T_j) \) for all \( 1 \leq j \leq n \) due to (\(*\)). Since \( \mathcal{T} \) is a tableau, (P12') from Definition 6.5
implies \( P(\pi(v_1), \ldots, \pi(v_n)) \in DC^\mathcal{T}, \) hence \((v_1, \ldots, v_n, P) \in DC(x)\) due to (\(\ast\)). Hence the \(\forall_P\)-rule can be applied without violating (\(\ast\)).

- The \(\exists_P\)-rule: If \(\exists T_1, \ldots, T_n.P \in \mathcal{L}(x),\) then \(\exists T_1, \ldots, T_n.P \in \mathcal{L}(\pi(x))\). Since \(\mathcal{T}\) is a tableau, (P13') from Definition 6.5 implies that there are variables \(w_1, \ldots, w_n \in S_D\) such that \(\langle \pi(x), w_j \rangle \in E_D(T_j)\) for all \(1 \leq j \leq n\), and \(P(w_1, \ldots, w_n) \in DC^\mathcal{T}\). The application of the \(\exists_P\)-rule generates a new \(\langle T_1 \ldots T_n\rangle\)-successor \(\langle v_1, \ldots, v_n \rangle\) with \(\mathcal{L}(\langle x, v_j \rangle) = \{T_j\}\) for all \(1 \leq j \leq n\), and \((\langle v_1, \ldots, v_n \rangle, P) \in DC(x)\). Hence we set \(\pi' := \pi[\pi v_1 \mapsto w_1, \ldots, v_n \mapsto w_n]\) which yields a function that satisfies (\(\ast\)) for the modified forest.

- The \(\geq_P\)-rule: If \(\langle \geq m T_1, \ldots, T_n.P \rangle \in \mathcal{L}(x),\) then \(\langle \geq m T_1, \ldots, T_n.P \rangle \in \mathcal{L}(\pi(x))\).

Since \(\mathcal{T}\) is a tableau, (P14') from Definition 6.5 implies \(\sharp_T^\mathcal{T}(\pi(x), P) \geq m\).

Hence there are \(mn\) variables \(w_1, \ldots, w_{mn} \in S_D\) such that \(\langle \pi(x), w_{ij} \rangle \in E_D(T_j)\), \(\langle \delta(w_{i1}), \ldots, \delta(w_{in}) \rangle \in P^D\) and \((\langle \delta(w_{i1}), \ldots, \delta(w_{in}) \rangle, \langle \delta(w_{k1}), \ldots, \delta(w_{kn}) \rangle) \in \neq D\) for all \(1 \leq i < k \leq m, 1 \leq j \leq n\). The application of the \(\geq_P\)-rule generates \(m\) new \(\langle T_1 \ldots T_n\rangle\)-successors \(\langle v_{11}, \ldots, v_{1n}, \ldots, v_{m1}, \ldots, v_{mn} \rangle\) of \(x\), with \(\mathcal{L}(\langle x, v_{ij} \rangle) = \{T_j\}\), \((\langle v_{11}, \ldots, v_{in} \rangle, P) \in DC(x)\) and \((\langle v_{11}, \ldots, v_{in} \rangle, \langle v_{k1}, \ldots, v_{kn}, \neq \rangle) \notin DC(x)\) for all \(1 \leq i < k \leq m, 1 \leq j \leq n\). By setting \(\pi' := \pi[v_{11} \mapsto w_{11}, \ldots, v_{mn} \mapsto w_{mn}]\), we have a function \(\pi'\) that satisfies (\(\ast\)) for the modified forest.

- The \(\leq_P\)-rule: If \(\langle \leq m T_1, \ldots, T_n.P \rangle \in \mathcal{L}(x),\) then \(\langle \leq m T_1, \ldots, T_n.P \rangle \in \mathcal{L}(\pi(x))\).

Since \(\mathcal{T}\) is a tableau, (P15') from Definition 6.5 implies \(\sharp_T^\mathcal{T}(\pi(x), P) \leq m\).

If the \(\leq_P\)-rule is applicable, we have \(\sharp_T^\mathcal{F}(x, P) > m\), which implies that there are at least \(m + 1\) \(\langle T_1 \ldots T_n\rangle\)-successors \(\langle v_{11}, \ldots, v_{1n} \rangle, \ldots, \langle v_{m+1,1}, \ldots, v_{m+1,n} \rangle\) of \(x\) such that \((\langle v_{i1}, \ldots, v_{in} \rangle, P) \in DC(x)\) for all \(1 \leq i \leq m + 1\). Thus, there must be two \(\langle T_1 \ldots T_n\rangle\)-successors \(\langle v_{11}, \ldots, v_{in} \rangle, \langle v_{j1}, \ldots, v_{jn} \rangle\) among these \(m + 1\) ones, such that \((\langle \delta(\pi(v_{i1})), \ldots, \delta(\pi(v_{in})) \rangle, \langle \delta(\pi(v_{j1})), \ldots, \delta(\pi(v_{jn})) \rangle) \in \neq D\) does not hold (otherwise \(\sharp_T^\mathcal{T}(\pi(x), P) > m\) would hold). This implies that \((\langle v_{11}, \ldots, v_{in} \rangle, \langle v_{j1}, \ldots, v_{jn} \rangle, \neq) \notin DC(x)\). Hence the \(\leq_P\)-rule can be applied without violating (\(\ast\)).

- The \textit{choose}_P-rule: If \(\langle \leq m T_1, \ldots, T_n.P, \geq m T_1, \ldots, T_n.P \rangle \cap \mathcal{L}(x) \neq \emptyset,\) then \(\langle \leq m T_1, \ldots, T_n.P, \geq m T_1, \ldots, T_n.P \rangle \cap \mathcal{L}(\pi(x)) \neq \emptyset,\) and, if \(\langle v_1, \ldots, v_n \rangle\) is a \(\langle T_1 \ldots T_n\rangle\)-successor of \(x\), then \(\langle \pi(x), \pi(v_j) \rangle \in E_D(T_j)\) for all \(1 \leq j \leq n\) due to (\(\ast\)). Since \(\mathcal{T}\) is a tableau, (P16') from Definition 6.5 implies that
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\[ P(\pi(v_1), \ldots, \pi(v_n)) \in DC^T \text{ or } \neg P(\pi(v_1), \ldots, \pi(v_n)) \in DC^T. \]

Hence the \textit{choose}_{P}\text{-rule} can add an appropriate constraint \((\langle v_1, \ldots, v_n \rangle, E)\), where \(E \in \{P, \neg P\}\), to \(DC(x)\) due to (*).

Why does this claim yield the completeness of the completion algorithm? For the initial completion forest including a node \(x_0\) with \(L(x_0) = \{D\}\), we can give a function \(\pi\) that satisfies (*) by setting \(\pi(x_0) := s_0\) for some \(s_0 \in S_A\) with \(D \in L(s_0)\) (such an \(s_0\) exists since \(T\) is a tableau for \(D\)). Whenever a rule is applicable to \(F\), it can be applied in a way that maintains (*) and, from Lemma 6.8, we have that any sequence of rule applications must terminate. Since (*) holds, any forest generated by these rule-applications must be clash-free. This can be seen as follows:

- \(F\) cannot contain a node \(x\) such that \(\bot \in L(x)\) because \(L(x) \subseteq L(\pi(x))\), hence \(\bot \in L(\pi(x))\). This implies \(\pi(x) \in \bot^T\), which contradicts the interpretation of \(\bot\).

- \(F\) cannot contain a node \(x\) such that \(\{C, \sim C\} \in L(x)\) because \(L(x) \subseteq L(\pi(x))\) and hence (P1) of Definition 6.5 would be violated for \(\pi(x)\).

- \(F\) cannot contain a node \(x\) with \((\leq nS.C) \in L(x)\) and \(n + 1 S\)-successors \(y_1, \ldots, y_{n+1}\) of \(x\) with \(C \in L(y_i)\) and \(y_i \neq y_j\) for \(1 \leq i < j \leq n + 1\) because \((\leq nS.C) \in L(\pi(x))\) and \(y_i \neq y_j\) implies \(\pi(y_i) \neq \pi(y_j)\), \#S^T(\pi(x), C) > n would hold which contradicts (P9) of Definition 6.5.

- \(F\) cannot contain a node \(x\) with \((\leq mT_1, \ldots, T_n, P) \in L(x)\) and \(m + 1 \langle T_1 \ldots T_n \rangle\)-successors \(\langle v_1, \ldots, v_{1n} \rangle, \ldots, \langle v_{m+1}, \ldots, v_{m+1,n} \rangle\) of \(x\) with \((\langle v_{i1}, \ldots, v_{in} \rangle, P) \in DC(x)\) for each \(1 \leq i \leq m + 1\), and \((\langle v_{i1}, \ldots, v_{in}, \langle v_{j1}, \ldots, v_{jn} \rangle, \neq \rangle) \in DC(x)\) for all \(1 \leq i < j \leq m + 1\), which implies that \(\langle \delta(\pi(v_{i1})), \ldots, \delta(\pi(v_{in}))\rangle, \langle \delta(\pi(v_{j1})), \ldots, \delta(\pi(v_{jn}))\rangle \notin \neg P\) would hold which contradicts (P15) of Definition 6.5.

- \(F\) cannot contain a node \(x\) where \(DC(x)\) is not satisfied, otherwise \(\delta\) is not a solution for the datatype constraints in \(DC(x)\), which implies that \(\delta\) is not a solution for datatype constraints about \(\pi(x)\) in \(DC^T\), or it is not a solution for the number restrictions of \(T_1, \ldots, T_n^T(\pi(x), P)\) due to (*). This contradicts (P0) of Definition 6.5.

- \(F\) cannot contain a node \(x\) with \(\{o\} \in L(x)\) and \(x \neq z_{\{o\}}\) because \(\{o\} \in L(\pi(x))\) and \(\{o\} \in L(\pi(z_{\{o\}}))\), hence \(\{o\} \in L(\pi(x)) \cap L(\pi(z_{\{o\}}))\), while \(x \neq
As an immediate consequence of Lemmas 6.6, 6.8, 6.9 and 6.10, the completion algorithm always terminates, and answers with “$D$ is satisfiable w.r.t. $\mathcal{R}$” iff $D$ has a tableau $T$. Next, subsumption can be reduced to (un)satisfiability because $\mathcal{SHOQ}(\mathcal{G})$ is closed under negation (cf. Section 2.1.3). Finally, $\mathcal{SHOQ}(\mathcal{G})$ can internalise general concept inclusion axioms (cf. Theorem 2.8 on page 36). In the presence of nominals, however, we must also add $\exists O.o_1 \cap \ldots \cap \exists O.o_l$ to the concept internalising the general concept inclusion axioms to make sure that the universal role $O$ indeed reaches all nominals $o_i$ occurring in the input concept and terminology. Thus, we can decide these inference problems also w.r.t. TBoxes and RBoxes [75].

**Theorem 6.11.** The completion algorithm presented in Definition 6.7 is a decision procedure for satisfiability and subsumption of $\mathcal{SHOQ}(\mathcal{G})$-concepts w.r.t. TBoxes and RBoxes.

Combining this result with Theorem 2.7, we obtain the following theorem.

**Theorem 6.12.** The completion algorithm presented in Definition 6.7 is a decision procedure for $\mathcal{SHOQ}(\mathcal{G})$ knowledge base satisfiability.

In fact, we can generalise the way we extend the tableaux algorithm for $\mathcal{SHOQ}$ to the one for $\mathcal{SHOQ}(\mathcal{G})$ by so call $\mathcal{G}$-augmented tableaux algorithms.

**Definition 6.13.** ($\mathcal{G}$-augmented Tableaux Algorithm) Given a conforming datatype group $\mathcal{G}$ and a $\mathcal{G}$-combinable Description Logic $\mathcal{L}$, if there exists a tableaux algorithm $\text{TA}_\mathcal{L}$ that is a decision procedure for $\mathcal{L}$-concept satisfiability w.r.t. RBoxes, then the $\mathcal{G}$-augmented tableaux algorithm (w.r.t. $\text{TA}_\mathcal{L}$) $\text{TA}_{\mathcal{L}(\mathcal{G})}$ is the same as $\text{TA}_\mathcal{L}$, except that:

- the set of completion rules in $\text{TA}_{\mathcal{L}(\mathcal{G})}$ is the union of the set of completion rules (or $\mathcal{L}$-rules) in $\text{TA}_\mathcal{L}$ and the $\mathcal{G}$-rules listed in Figure 6.2 on page 143;

- the set of clash conditions in $\text{TA}_{\mathcal{L}(\mathcal{G})}$ is the union of the set of clash conditions (or $\mathcal{L}$-clash conditions) in $\text{TA}_\mathcal{L}$ and the $\mathcal{G}$-clash conditions listed in Figure 6.4 on page 144.

The following theorem shows that $\text{TA}_{\mathcal{L}(\mathcal{G})}$ is a decision procedure for $\mathcal{L}(\mathcal{G})$-concept satisfiability w.r.t. RBoxes.
Theorem 6.14. Let $\mathcal{L}$ be a $\mathcal{G}$-combinable Description Logic, $\mathcal{T}_\mathcal{L}$ a tableaux algorithm that is a decision procedure for $\mathcal{L}$-concept satisfiability w.r.t. to $\mathcal{R}$Boxes. The $\mathcal{G}$-augmented tableaux algorithm $\mathcal{T}_\mathcal{L}(\mathcal{G})$ is a decision procedure for $\mathcal{L}(\mathcal{G})$-concept satisfiability w.r.t. to $\mathcal{R}$Boxes.

Proof. This is mainly due to the locality of $\mathcal{G}$-rules and $\mathcal{G}$-clash conditions.

- $\mathcal{G}$-rules are applicable only when there exist datatype group-based concepts in the label of a node $x$, and they only affect the datatype query of $x$, its concrete successors and the edges connecting $x$ and its concrete successors (cf. Definition 6.7). In other words, they will not affect the applicability of $\mathcal{L}$-rules.

- The $\mathcal{G}$-clash conditions will be triggered only when some local datatype constraint is not satisfied. Hence, $\mathcal{G}$-clash conditions and $\mathcal{L}$-clash conditions cover completely different aspects of the completion forest (or tree).

Therefore, $\mathcal{G}$-rules and $\mathcal{G}$-clash conditions can be seen as extra local checking about datatype constraints added into $\mathcal{T}_\mathcal{L}$. As $\mathcal{T}_\mathcal{L}$ is decidable, we only need to show that $\mathcal{G}$-rules and $\mathcal{G}$-clash conditions are sound and complete on checking local datatype constraints, and that $\mathcal{G}$-rules are terminating. All these are witnessed by the fact that the $\mathcal{G}$-augmented tableaux algorithm $\mathcal{T}_{\mathcal{SHOIQ}(\mathcal{G})}$ defined in Definition 6.7 on page 140 is sound, complete and terminating.

Theorem 6.14 suggests that if we have a decidable tableaux algorithm for $\mathcal{SHOIQ}$-concept satisfiability w.r.t. $\mathcal{R}$Boxes, we can construct a decidable $\mathcal{G}$-augmented tableaux algorithm for $\mathcal{SHOIQ}(\mathcal{G})$-concept satisfiability w.r.t. $\mathcal{R}$Boxes.

In the next section, we will provide a $\mathcal{G}$-augmented tableaux algorithm to decide $\mathcal{SHIQ}(\mathcal{G})$-concept satisfiability w.r.t. $\mathcal{R}$Boxes.

6.2.2 The $\mathcal{SHIQ}(\mathcal{G})$ DL

Syntax and Semantics

The $\mathcal{SHIQ}(\mathcal{G})$ DL extends the $\mathcal{SHIQ}$ DL with an arbitrary datatype group. As usual, we define $\mathcal{SHIQ}(\mathcal{G})$-roles, $\mathcal{SHIQ}(\mathcal{G})$-concepts and $\mathcal{SHIQ}(\mathcal{G})$-$\mathcal{R}$Box as follows.

Definition 6.15. ($\mathcal{SHIQ}(\mathcal{G})$-roles) Let $\mathcal{R}_A \subseteq \mathcal{R}_A$, $\mathcal{R} \in \mathcal{R}_{\mathcal{dsc}}$, $\mathcal{T}_D \subseteq \mathcal{R}_D$ and $\mathcal{T} \in \mathcal{R}_{\mathcal{dsc}}$. Valid $\mathcal{SHIQ}(\mathcal{G})$ abstract roles are defined by the abstract syntax:

$$R ::= \mathcal{R} \cup \mathcal{R}_A \cup \mathcal{R}_D \cup \mathcal{R}_{\mathcal{dsc}}$$
where for some \( x, y \in \Delta^I \), \( \langle x, y \rangle \in R^I \) iff \( \langle y, x \rangle \in R^{-I} \). The inverse relation of roles is symmetric, and to avoid considering the roles such as \( R^{-} \), we define a function \( \text{Inv} \) which returns the inverse of a role, more precisely

\[
\text{Inv}(R) := \begin{cases} 
R^{-} & \text{if } R = RN, \\
RN & \text{if } R = RN^{-}.
\end{cases}
\]

Valid \( S\mathcal{HIQ}(G) \) concrete roles are defined by the abstract syntax:

\[
T ::= TN.
\]

**Definition 6.16.** (\( S\mathcal{HIQ}(G) \)-concepts) Let \( CN \in C, R \in \text{Rdsc}_A(S\mathcal{HIQ}(G)), T_1, \ldots, T_n \in \text{Rdsc}_D(S\mathcal{HIQ}(G)) \) and \( T_i \notin T_j, T_j \notin T_i \) for all \( 1 \leq i < j \leq n \), \( C, D \in \text{Cdsc}(S\mathcal{HIQ}(G)) \), \( E \in \text{Dexp}(G) \), \( n, m \in \mathbb{N}, n \geq 1 \). Valid \( S\mathcal{HIQ}(G) \)-concepts are defined by the abstract syntax:

\[
C ::= \top | \bot | CN | \neg C | C \sqcap D | C \sqcup D | \exists R.C | \forall R.C | \geq mR.C | \leq mR.C \\
\exists T_1, \ldots, T_n.E | \forall T_1, \ldots, T_n.E | \geq mT_1, \ldots, T_n.E | \leq mT_1, \ldots, T_n.E.
\]

The semantics of datatype group-based \( S\mathcal{HIQ}(G) \)-concepts is given in Table 5.2 on page 118; the semantics of other \( S\mathcal{HIQ}(G) \)-concepts is given in Table 2.2 on page 46.

As the \( S\mathcal{HIQ}(G) \) DL provides the inverse role constructor, its RBoxes are more expressive than those of the \( S\mathcal{HOQ}(D) \) and \( S\mathcal{HOQ}(G) \) DLs.

**Definition 6.17.** (\( S\mathcal{HIQ}(G) \) RBox) Let \( R_1, R_2 \in \text{Rdsc}_A(S\mathcal{HIQ}(G)), T_1, T_2 \in \text{Rdsc}_D(S\mathcal{HIQ}(G)), SN \in R \), a \( S\mathcal{HIQ}(G) \) RBox \( \mathcal{R} \) is a finite, possibly empty, set of role axioms:

- functional role axioms \( \text{Func}(SN) \);
- transitive role axioms \( \text{Trans}(R_1) \);\(^8\)
- abstract role inclusion axioms \( R_1 \sqsubseteq R_2 \);
- concrete role inclusion axioms \( T_1 \sqsubseteq T_2 \).

\(^8\)Note that a concrete role \( T \) can not be a transitive role.
We extend $\mathcal{R}$ to the role hierarchy of $\mathcal{SHIQ}(\mathcal{G})$ as follows:

$$\mathcal{R}^+ := (\mathcal{R} \cup \{ \text{Inv}(R) \sqsubseteq \text{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R} \}, \sqsubseteq)$$

where $\sqsubseteq$ is the transitive-reflexive closure of $\sqsubseteq$ over $\mathcal{R} \cup \{ \text{Inv}(R) \sqsubseteq \text{Inv}(S) \mid R \sqsubseteq S \in \mathcal{R} \}$.

A $\mathcal{G}$-augmented Tableaux Algorithm for $\mathcal{SHIQ}(\mathcal{G})$

Based on the tableaux algorithm for $\mathcal{SHIQ}$-concept satisfiability w.r.t. role hierarchies presented by Horrocks et al. [65], we can construct a $\mathcal{G}$-augmented tableaux algorithm to decide $\mathcal{L}(\mathcal{G})$-concept satisfiability problem w.r.t. a role hierarchy.

Definition 6.18. (A $\mathcal{G}$-augmented Tableaux Algorithm for $\mathcal{SHIQ}(\mathcal{G})$) Let $\mathcal{R}^+$ be a role hierarchy, $D$ a $\mathcal{SHIQ}(\mathcal{G})$-concept in NNF, $\mathcal{R}_A^D$ the set of abstract roles in $D$ or $\mathcal{R}^+$, together with their inverse, $\mathcal{R}_B^D$ the set of concrete roles occurring in $D$ or $\mathcal{R}^+$ and $\mathcal{G}$ a datatype group.

A tableaux algorithm for $\mathcal{SHIQ}(\mathcal{G})$ is similar to the one for $\mathcal{SHOQ}(\mathcal{G})$ defined in Definition 6.7 except that

- it works on a completion tree (cf. Section 2.2) $T$ for $D$ w.r.t. $\mathcal{R}^+$, and
- for (abstract) nodes $x, y$, the label on the edge $\langle x, y \rangle$, namely $\mathcal{L}(\langle x, y \rangle)$, could possibly contain inverse roles, and
- $S^T(x, C)$ is defined in terms of ‘neighbours’, instead of ‘successors’, and
- this tableaux algorithm uses pair-wise blocking.

Given a completion tree, an (abstract) node $y$ is called an $R$-neighbour of a node $x$ if, for some $R'$ with $R' \sqsubseteq R$, either $y$ is a successor of $x$ and $R' \in \mathcal{L}(\langle x, y \rangle)$, or $y$ is a predecessor of $x$ and $\text{Inv}(R') \in \mathcal{L}(\langle x, y \rangle)$. For an abstract role $S$ and a node $x$ in $T$, we define $S^T(x, C)$ as

$$S^T(x, C) := \{ y \mid y \text{ is an } S\text{-neighbour of } x \text{ and } C \in \mathcal{L}(y) \}.$$ 

A node $x$ is directly blocked if none of its ancestors are blocked, and it has ancestors $x', y$ and $y'$ such that

1. $x$ is a successor of $x'$ and $y$ is a successor of $y'$ and
6.2. REASONING WITH OWL-E

\[ \Box \text{-rule: if } 1.C_1 \cap C_2 \in \mathcal{L}(x), \ x \text{ is not indirectly blocked, and} \]
\[ 2.\{C_1, C_2\} \not\subseteq \mathcal{L}(x), \]
\[ \text{then } \mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{C_1, C_2\} \]

\[ \sqcup \text{-rule: if } 1.C_1 \cup C_2 \in \mathcal{L}(x), \ x \text{ is not indirectly blocked, and} \]
\[ 2.\{C_1, C_2\} \cap \mathcal{L}(x) = \emptyset, \]
\[ \text{then } \mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{C\} \text{ for some } C \in \{C_1, C_2\} \]

\[ \exists \text{-rule: if } 1.\exists R.C \in \mathcal{L}(x), \ x \text{ is not blocked, and} \]
\[ 2.x \text{ has no } R\text{-neighbour } y \text{ with } C \in \mathcal{L}(y), \]
\[ \text{then create a new node } y \text{ with } \mathcal{L}((x, y)) = \{R\} \text{ and } \mathcal{L}(y) = \{C\} \]

\[ \forall \text{-rule: if } 1.\forall R.C \in \mathcal{L}(x), \ x \text{ is not indirectly blocked, and} \]
\[ 2.\text{there is an } R\text{-neighbour } y \text{ of } x \text{ with } C \notin \mathcal{L}(y), \]
\[ \text{then } \mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{C\} \]

\[ \geq \text{-rule: if } 1.\geq n.S.C \in \mathcal{L}(x), \ x \text{ is not blocked, and} \]
\[ 2.\text{there are no } n \text{-S-neighbour } y_1, \ldots, y_n \text{ of } x \text{ with } C \in \mathcal{L}(y_i) \text{ and} \]
\[ 3.y_i \neq y_j \text{ for } 1 \leq i < j \leq n, \]
\[ \text{then create } n \text{ new nodes } y_1, \ldots, y_n \text{ with } \mathcal{L}((x, y_i)) = \{S\}, \]
\[ \mathcal{L}(y_i) = \{C\}, \text{ and } y_i \neq y_j \text{ for } 1 \leq i < j \leq n. \]

\[ \leq \text{-rule: if } 1.\leq n.S.C \in \mathcal{L}(x), \ x \text{ is not indirectly blocked, and} \]
\[ 2.S^T(x, C) > n \text{ and there are two } S\text{-neighbours } y, z \text{ of } x \text{ with} \]
\[ C \in \mathcal{L}(y), C \in \mathcal{L}(z) \text{ y is not an ancestor of } z, \text{ and not } y \neq z \]
\[ \text{then } 1.\mathcal{L}(y_i) \rightarrow \mathcal{L}(y_i) \cup \mathcal{L}(y_j) \text{ and} \]
\[ 2.\text{ add } y \neq y_i \text{ for each } y \text{ with } y \neq y_j \text{, and} \]
\[ 3.\mathcal{L}((x, y_j)) \rightarrow \emptyset \]

\[ \text{choose-rule: if } 1.\{\geq n.S.C, \leq n.S.C\} \cap \mathcal{L}(x) \neq \emptyset, \ x \text{ is not indirectly blocked, and} \]
\[ 2.y \text{ is an } S\text{-neighbour of } x \text{ with } \{C, \sim C\} \cap \mathcal{L}(y) = \emptyset, \]
\[ \text{then } \mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{E\} \text{ for some } E \in \{C, \sim C\} \]

Figure 6.5: The completion rules for $\text{SHIQ}$

A node $x$ of a completion forest $F$ contains a clash if (at least) one of the following conditions holds:

1. $\bot \in \mathcal{L}(x)$;
2. for some $A \in C, \{A, \neg A\} \subset \mathcal{L}(x)$;
3. for some abstract role $S, \leq n.S.C \in \mathcal{L}(x)$ and there are $n+1$ $S$-neighbours $y_1, \ldots, y_{n+1}$ of $x$ with $C \in \mathcal{L}(y_i)$ for each $1 \leq i \leq n+1$ and $y_i \neq y_j$ for each $1 \leq i < j \leq n+1$.

Figure 6.6: The clash conditions of $\text{SHIQ}$

2. $\mathcal{L}(x) = \mathcal{L}(y)$ and $\mathcal{L}(x') = \mathcal{L}(y')$ and

3. $\mathcal{L}((x', x)) = \mathcal{L}((y', y))$. 

In this case we say that \( y \) blocks \( x \).

A node is \( \textit{indirectly blocked} \) if its predecessor is blocked, and in order to avoid wasted expansion after an application of the \( \leq \)-rule, a node \( y \) will also be taken to be indirectly blocked if it is a successor of a node \( x \) and \( \mathcal{L}(\langle x, y \rangle) = \emptyset \).

The algorithm initialises the tree \( T \) the root node \( x_0 \) labeled with \( \mathcal{L}(x_0) = \{ D \} \), the inequality relation \( \neq \) is initialised with the empty relation and the \( DC(x) \) of any (abstract) node \( x \) is initialised to the empty set. \( T \) is then expanded by repeatedly applying the completion rules, listed in Figure 6.5 (page 159) and Figure 6.2 (page 143), until \( T \) is complete, or some \( \mathcal{L}(x) \) contains a clash (cf. Figure 6.6 on page 159 and Figure 6.4 on page 144).

If the completion rules can be applied in such a way that they yield a complete, clash-free completion tree, then the algorithm returns "\( D \) is \textit{satisfiable} w.r.t. \( \mathcal{R}^+ \)" and "\( D \) is \textit{unsatisfiable} w.r.t. \( \mathcal{R}^+ \)" otherwise.

According to Definition 6.17 on page 157, a role hierarchy is simply a syntactic variant of the set of role inclusion axioms in an RBox when inclusion axioms (\( \mathcal{H} \)) is present. According to Theorem 6.14 and the fact that \( SHIQ(\mathcal{G}) \) can internalise TBoxes (cf. Theorem 2.8 on page 36), we have the following theorem.

**Theorem 6.19.** The \( \mathcal{G} \)-augmented tableaux algorithm \( TA_{SHIQ(\mathcal{G})} \) presented in Definition 6.18 is a decision procedure for \( SHIQ(\mathcal{G}) \)-concept satisfiability and subsumption problems w.r.t. TBoxes and RBoxes.

### 6.2.3 The \( SHIO(\mathcal{G}) \) DL

Having introduced \( SHOQ(\mathcal{G}) \) and \( SHIQ(\mathcal{G}) \), now we briefly describe \( SHIO(\mathcal{G}) \), which extends \( SHIO \) with an arbitrary datatype group. \( SHIO \)-roles are the same as \( SHIQ \)-roles, \( SHIO \)-RBoxes are the same as \( SHIQ \)-RBoxes and \( SHIO \)-concepts are defined as follows.

**Definition 6.20.** (\( SHIO(\mathcal{G}) \)-concepts) Let \( CN \in \mathcal{C}, R \in \mathbf{Rdsc}_A(SHIO(\mathcal{G})), T_1, \ldots, T_n \in \mathbf{Rdsc}_D(SHIO(\mathcal{G})) \) and \( T_i \nsubseteq T_j, T_j \nsubseteq T_i \) for all \( 1 \leq i < j \leq n, C, D \in \mathbf{Cdsc}(SHIO(\mathcal{G})), o \in \mathcal{I}, E \in \mathbf{Dexp}(\mathcal{G}), n, m \in \mathbb{N}, n \geq 1 \). Valid \( SHIO(\mathcal{G}) \)-concepts are defined by the abstract syntax:

\[
C ::= \top | \bot | CN | \neg C | C \sqcap D | C \sqcup D | \{ o \} | \exists R.C | \forall R.C | \exists T_1, \ldots, T_n.E | \forall T_1, \ldots, T_n.E | \geq m T_1, \ldots, T_n.E | \leq m T_1, \ldots, T_n.E.
\]
The semantics of datatype group-based $SHIO(G)$-concepts is given in Table 5.2 on page 118; the semantics of other $SHIO(G)$-concepts is given in Table 2.2 on page 46.

According to Theorem 6.14, we can construct a $G$-augmented tableaux algorithm, based on the tableaux algorithm for $SHIO$-concept satisfiability w.r.t. RBoxes presented by Jladik and Model [78], to decide $SHIO(G)$-concept satisfiability w.r.t. RBoxes. Due to Theorem 2.7 (page 35) and Theorem 2.8 (page 36), such an algorithm is also a decision procedure for $SHIO(G)$-knowledge base satisfiability.

We end this section by concluding that we can construct decidable $G$-augmented tableaux algorithms for a family of DLs, of the form $L(G)$, which satisfy the following conditions:

1. $L$ is a $G$-combinable Description Logic,
2. $G$ is a conforming datatype group, and
3. there exists a decidable tableaux algorithm for $L$-concept satisfiability w.r.t. RBoxes (or role hierarchies).

We have investigated several example $G$-augmented tableaux algorithms, viz. those for $SHOQ(G)$, $SHIQ(G)$ and $SHIO(G)$, which can be used to provide reasoning support for OWL-E under the ‘divide and conquer’ strategy.

### 6.3 OWL-Eu: A Smaller Extension of OWL DL

Until now, we have discussed OWL-E as a decidable extension of OWL DL that provides customised datatypes and predicates, but we may also consider the possibility of having a smaller extension of OWL DL. The alternative extension we are going to present in this section is called OWL-Eu, which is a unary restriction of OWL-E.

**The Small Extension Requirement**

When we extend OWL DL to support customised datatypes and predicates, some people prefer the so called small extension requirement, which is two folded: on the one hand, an extension should be a substantial extension to support customised datatypes and predicates; on the other hand, following W3C’s ‘one small step at a time’ strategy, the extension should only be small.
6.3. OWL-EU: A SMALLER EXTENSION OF OWL DL 162

Abstract Syntax  | DL Syntax  | Semantics
---|---|---
rdfs:Literal  | ⊤_D  | Δ_D
owlx:DatatypeBottom  | ⊥_D  | ∅
    | u  | u^D
    | not(u)  | π  | if u ∈ D_G, Δ_D \ u^D
    |  |  | if u ∈ Φ_G, (dom(u))^D \ u^D

<s_1>"^d_1"..."s_n""^d_n>  | \{"s_1""^d_1"..."s_n""^d_n\}  | P \lor q  | P \lor q^D
and(p, q)  | \{"s_1""^d_1"..."s_n""^d_n\}  | P \land q  | P \land q^D
or(p, q)  | \{"s_1""^d_1"..."s_n""^d_n\}  | P \lor q  | P \lor q^D

Table 6.4: OWL-Eu unary datatype expressions

In this section, we provide OWL-Eu, a unary restriction of OWL-E, as a small extension of OWL DL.

From OWL DL to OWL-Eu

OWL-Eu only allows unary datatype groups; i.e., given a datatype group G = (M_p, D_G, dom), M_p contains only pairs of predicate URIrefs and unary datatype predicates and thus Φ_G contains only unary supported predicate URIrefs. Accordingly, OWL-Eu only supports unary G-datatype expressions. Customised datatypes, such as ‘greaterThan20’ and ‘cameraPrice’ presented in Example 5.7 are expressible in OWL-Eu, while customised datatype predicates with arities greater than one, such as ‘sumNoGreaterThan15’ or ‘multiplyBy1.6’ in Example 5.7, are not expressible in OWL-Eu.

The only difference between OWL DL and OWL-Eu is that the latter one extends data ranges in the former one (listed in Table 3.4) to unary datatype expressions listed in Table 6.4, where u is a unary datatype predicate URIref, “s_i”"^d_i" are typed literals, p, q are datatype expressions. Therefore, unary datatype expressions can be used as data ranges in datatype range axioms and datatype-related class descriptions (cf. Table 3.5 and 3.6).

Example 6.4 OWL-Eu: Matchmaking

The PCs with memorySizeInMb greater than 512, unit priceInPound less than 700 and deliveryDate earlier than 15/03/2004 can be expressed by the following OWL-Eu class description

```xml
Class(RequiredPC complete PC
   restriction(memorySizeInMb
      someValuesFrom/owlx:integerGreatThanx=512))
   restriction(priceInPound
```
Note that (i) the key word `someValuesFrom`, instead of `someTuplesSatisfy` is used in the above class description (cf. Example 6.1), and that (ii) functional property axioms are used in place of qualified number restrictions.

### Reasoning with OWL-Eu

The underpinning of OWL-Eu is the $SHOIN(G_1)$ DL, where ‘1’ means it only allows unary datatype groups. To support OWL-Eu, similarly to how we provide reasoning support for OWL-E, we can use the ‘divide and conquer’ strategy and exploit decision procedures for $SHON(G_1)$, $SHIN(G_1)$ and $SHOI(G_1)$, which are just simplified versions of $SHOQ(G)$, $SHIQ(G)$ and $SHIO(G)$, respectively.$^9$ Note that the $SHIN(G_1)$ DL fully covers OWL Lite.

According to Theorem 6.14 (page 156), Theorem 2.7 (page 35) and Theorem 2.8 (page 36) if we have a decidable tableaux algorithm for $SHOIN$-concept satisfiability w.r.t. role hierarchies, we can easily extend it and provide a $G$-augmented tableaux algorithm for $SHOIN(G_1)$-knowledge base satisfiability.

In Chapter 7, we will introduce the architecture of our datatype framework for providing DL inferencing services for OWL and OWL-E (and therefore OWL-Eu).

---

$^9$Note that ‘Q’ stands for qualified number restrictions, while ‘N’ stands for non-qualified number restrictions.
Chapter Achievements

- We provide OWL-E and its unary restriction OWL-Eu, both of which support customised datatypes and datatype predicates, as decidable extensions of OWL DL.

- OWL-E is an extension of both OWL DL and DAML+OIL. It overcomes all the limitations of OWL datatyping discussed in Chapter 3.

- OWL-Eu is only a small extension of OWL DL, it extends OWL data ranges to support unary $G$-datatype expressions.

- Theorem 6.14 shows that if we have a tableaux algorithm for the concept satisfiability problem w.r.t. to RBoxes of a $G$-combinable Description Logic $L$, then we can provide a $G$-augmented tableaux algorithm to decide $L(G)$-concept satisfiability w.r.t. RBoxes.

- We can provide reasoning support for OWL DL, DAML+OIL, OWL-Eu and OWL-E by using the decision procedures for the $SHOQ(G)$, $SHIQ(G)$ and $SHIO(G)$ DLs under the ‘divide and conquer’ strategy.
Chapter 7

Framework Architecture

Chapter Aims

• To investigate a general DL interface for OWL DL and OWL-E.

• To propose a framework architecture to provide flexible and decidable DL reasoning services for DLs integrated with datatype groups.

Chapter Plan

7.1 General DL API (165)
7.2 Architecture (176)
7.3 Datatype Reasoners (178)
7.4 Flexibility (183)

Having provided a framework for combining DLs with customised datatypes and predicates in previous chapters, this chapter further investigates a flexible architecture for our framework.

7.1 General DL API

A key component of our framework architecture is a general DL API for OWL DL and OWL-E. This API is an extension of DIG/1.1 [10], which is a simple API for a general DL system developed by the DL Implementation Group (DIG) [36, 37]. There is a commitment from the implementors of the leading DL reasoners (FaCT [59, 40], FaCT++ [140, 41], Racer [51, 124], Cerebra [27] and Pellet [120]) to provide implementations conforming to the DIG specifications. In the following, Section 7.1.1...
will describe the current version of DIG specification, DIG/1.1, which is not designed for OWL but for DAML+OIL. Section 7.1.2 will present our extension of DIG/1.1.

### 7.1.1 DIG/1.1 Interface

![DIG/1.1 framework architecture](image)

DIG/1.1 is effectively an XML Schema for the $\mathcal{SHOIQ}(\mathcal{D})$ description language (Table 7.1 on page 167)\(^1\) along with tell/ask/response functionality. Figure 7.1 on page 166 presents the DIG/1.1 framework architecture. Applications can take an OWL DL ontology, either in OWL/RDF syntax or in OWL abstract syntax (cf. Section 3.2.2) and use the OWL-API\(^2\) to translate it into an ontology in DIG/1.1 syntax. Then the clients (applications) communicate with a DIG/1.1 server (a Description Logic reasoner) through the use of HTTP POST request, and the DIG server should return 200 OK, unless there is a low-level error.

### DIG/1.1 Communications

There are four kinds of requests from clients and a server:

---

\(^1\) Notations of Table 7.1: $C, C_i$ are concepts, $R$ is a role, $F, F_i$ are features, $A$ is an attribute.

\(^2\) OWL-API can also translate OWL/RDF syntax into OWL abstract syntax, cf. [http://owl.man.ac.uk/api.shtml](http://owl.man.ac.uk/api.shtml).
7.1. GENERAL DL API

| Primitive Concepts | <top/>
|                   | <bottom/>
|                   | <catom name = “CN”/> |
| Boolean Constructors | <and> C₁ ... Cₙ </and>
|                      | <or> C₁ ... Cₙ </or>
|                      | <not> C </not> |
| Abstract Role      | <some> R C </some>
| Restrictions       | <all> R C </all>
|                      | <atmost num = “n”> R C </atmost>
|                      | <atleast num = “n”> R C </atleast>
|                      | <iset> I₁ ... Iₙ </iset> |
| Unary Predicate     | <defined> A </defined>
| Restrictions       | <stringmin val = “s”> A </stringmin>
|                      | <stringmax val = “s”> A </stringmax>
|                      | <stringequals val = “s”> A </stringequals>
|                      | <stringrange min = “s” max = “t”> A </stringrange>
|                      | <intmin val = “i”> A </intmin>
|                      | <intmax val = “i”> A </intmax>
|                      | <intequals val = “i”> A </intequals>
|                      | <intrange min = “i” max = “j”> A </intrange> |
| Role Constructors   | <ratom name = “RN”/> |
|                      | <feature name = “FN”/> |
|                      | <attribute name = “AN”/> |
|                      | <inverse> R </inverse> |
|                      | <chain> F₁ ... Fₙ A </chain> |
| Individuals        | <individual name = “IN”/> |

Table 7.1: DIG/1.1 description language

- Clients can find out which reasoner is actually behind the interface by sending an IDENTIFIER request. The server then returns the name and the version of the reasoner, together with the list of constructors, TELL and ASK operations supported by the reasoner.

- Clients can request to create or release knowledge bases.

- Clients can send TELL requests using the TELL language presented in Table 7.2 on page 168, where I₁, Iᵢ are individual objects and V is either an integer value or a string value. The TELL requests are monotonic, i.e., DIG/1.1 does not support removing the TELL requests that have been sent – the only work-around is to release the knowledge base and start again. A TELL request must be made in the context of a particular knowledge base (by using the knowledge base URI). The server will response using the basic responses presented in Table 7.4 on
• Clients can send ASK requests using the ASK language presented in Table 7.3 on page 169. An ASK request can contain multiple queries (with different IDs) and must be made in the context of a particular knowledge base (by using the knowledge base URI). The server will respond using the RESPONSE language presented in Table 7.4 on page 170.

The following example shows how to use the TELL, ASK and RESPONSE languages of DIG/1.1.

**Example 7.1 DIG/1.1: the TELL, ASK and RESPONSE Statements**

The following simplified³ TELL statements define three classes, Animal, Elephant and AdultElephant that we have seen in Example 3.1 on page 53:

```xml
<tell ...>
  <releaseKB/>
  <defconcept name = "Animal"/>
  <defconcept name = "Elephant"/>
  <equalc>
```

³To save space, we ignore the name space declarations in this example.
Then we use the following ASK statement to define two queries: the first asks about the satisfiability of the `AdultElephant` concept; the second asks for all the concepts subsuming the given description, i.e., Elephants with `age` greater than 30.

```xml
<catom name = “AdultElephant”/>
<and>
    <catom name = “Elephant”/>
    <intmin val = “21”> <attribute name = “age”/> </intmin>
</and>
</equalc>
</tells>
```

Table 7.3: DIG/1.1 ASK language

<table>
<thead>
<tr>
<th>Primitive Queries</th>
<th>&lt;allConceptNames/&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satisfiability Queries</td>
<td>&lt;satisfiable&gt; C &lt;/satisfiable&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;subsumes&gt; C₁ C₂ &lt;/subsumes&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;disjoint&gt; C₁ C₂ &lt;/disjoint&gt;</td>
</tr>
<tr>
<td>Concept Hierarchy Queries</td>
<td>&lt;parents&gt; C &lt;/parents&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;children&gt; C &lt;/children&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;ancestors&gt; C &lt;/ancestors&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;descendants&gt; C &lt;/descendants&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;equivalents&gt; C &lt;/equivalents&gt;</td>
</tr>
<tr>
<td>Role Hierarchy Queries</td>
<td>&lt;rparents&gt; R &lt;/rparents&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;rchildren&gt; R &lt;/rchildren&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;rancestors&gt; R &lt;/rancestors&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;rdescendants&gt; R &lt;/rdescendants&gt;</td>
</tr>
<tr>
<td>Individual Queries</td>
<td>&lt;instances&gt; C &lt;/instances&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;types&gt; I &lt;/types&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;instance&gt; I C &lt;/instance&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;roleFilters&gt; I R &lt;/roleFilters&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;relatedIndividuals&gt; R &lt;/relatedIndividuals&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;toldValues&gt; I A &lt;/toldValues&gt;</td>
</tr>
</tbody>
</table>
The following \texttt{RESPONSE} statements show a possible response to the query above: the answer to the first query is a boolean, while the answer to the second query is a collection of concepts.

```xml
<responses ...>
  <true id = "q1"/>
  <conceptSet id = "q2">
    <synonyms>
      <catom name = "Animal"/>
    </synonyms>
    <synonyms>
      <catom name = "Elephant"/>
    </synonyms>
    <synonyms>
      <catom name = "AdultElephant"/>
    </synonyms>
  </conceptSet>
</responses>
```

Table 7.4: DIG/1.1 \texttt{RESPONSE} language
7.1. GENERAL DL API

Limitations

DIG/1.1 is designed to support DAML+OIL; it has the following limitations w.r.t. OWL DL\(^4\) and OWL-E:

1. The underpinning Description Logic for DIG/1.1 is \(SHOIQ(D_I)\), while OWL DL and OWL-E correspond to the \(SHOIN(D^+)\) and \(SHOIQ(G)\) DLs.

   (a) In DIG/1.1, datatype properties are features (functional roles, cf. Definition 2.10 on page 42) while in OWL DL and OWL-E, they are roles (not necessarily with the functional restriction).

   (b) As far as datatype-related concept descriptions are concerned, the DIG/1.1 description language only supports (very limited) unary predicate exists restrictions,\(^5\) while OWL DL provides datatype exists (\(\exists T.d\)), datatype value (\(\forall T.d\)), datatype atleast (\(\geq nT.d\)) and datatype atmost (\(\leq nT.d\)) restrictions, and OWL-E provides expressive predicate exists (\(\exists T_1,\ldots, T_n.E\)), expressive predicate value (\(\forall T_1,\ldots, T_n.E\)), expressive predicate qualified atleast (\(\geq mT_1,\ldots, T_n.E\)) and expressive predicate qualified atmost (\(\leq mT_1,\ldots, T_n.E\)) restrictions.

   (c) DIG/1.1 does not support the datatype expression axioms that OWL-E provides.

2. DIG/1.1 uses unique name assumption (UNA, cf. Section 2.1.2 on page 33), while neither OWL DL nor OWL-E use this assumption.

3. DIG/1.1 supports only values of the integer and string datatypes, while both OWL DL and OWL-E provide typed literals (cf. Definition 3.7 on page 71).

7.1.2 DIG/OWL-E Interface

To overcome the limitations of DIG/1.1 presented at the end of the last section, we propose DIG/OWL-E, which supports OWL-E (viz. \(SHOIQ(G)\)) and extends the DIG/1.1 description language with the following constructors (see Table 7.5):\(^6\)

\(^4\) Section 3.2 of [35] also provides some discussions on issues arising from translating OWL to DIG.

\(^5\) DIG/1.1 only supports a few integer and string unary predicates; cf. Table 7.1 on page 167.

\(^6\) Notations of Table 7.5: \(V_i\) are typed literals, \(E_i\) are datatype expressions or datatype expression names, \(U_i\) are unary datatype predicates or their relativised negations, and \(T_i\) are concrete roles.
### Table 7.5: New constructors in the DIG/OWL-E description language

<table>
<thead>
<tr>
<th>Typed Literals</th>
<th><code>&lt;typedValue lxform = “L” datatype = “DN”/&gt;</code></th>
</tr>
</thead>
</table>
| Datatype Expressions | `<dttop/>`  
|                  | `<dtbottom/>`  
|                  | `<predicate name = “PN”/>`  
|                  | `<dtnot name = “PN”/>`  
|                  | `<vset> V_1 \ldots V_n </vset>`  
|                  | `<dtand> E_1 \ldots E_n </dtand>`  
|                  | `<dtor> E_1 \ldots E_n </dtor>`  
|                  | `<dtdomain> U_1 \ldots U_n </dtdomain>`  
|                  | `<dtexpression name = “EN”/>` |
| Concrete Roles | `<dtratom name = “TN”/>` |
| Datatype Expression-related Concept Descriptions | `<dtsome> T_1 \ldots T_n E </dtsome>`  
|                  | `<dtall> T_1 \ldots T_n E </dtall>`  
|                  | `<dtatmost num = “n”> T_1 \ldots T_n E </dtatmost>`  
|                  | `<dtatleast num = “n”> T_1 \ldots T_n E </dtatleast>` |

1. **Typed Literals** (cf. Definition 3.7 on page 71): A `<typedValue>` element represents a typed literal; e.g., `<typedValue lxform = “s” datatype = “u”/>` represents the typed literal “s”ˆˆu.

2. **Datatype Expressions** (cf. Definition 5.12 on page 110):
   
   (a) `<dttop/>` and `<dtbottom/>` correspond to rdfs:Literal and owlx:Datatype-Bottom in OWL-E, respectively (cf. Definition 5.4 on page 104).
   
   (b) A `<predicate>` element introduces a predicate URI reference, and a `<dtnot>` element represents a negated predicate URI reference.
   
   (c) A `<vset>` element represents an enumerated datatype (cf. Definition 3.11 on page 74), i.e., a datatype defined by enumerating all its member typed literals.
   
   (d) A `<dtnot>` element represents the relativised negation of a unary supported predicate URIref (cf. Definition 5.12 on page 110).
   
   (e) The `<dtand>`, `<dtor>` and `<dtdomain>` elements corresponds to the and, or and domain constructors defined in Definition 5.12, respectively.

3. **Concrete Roles**:\(^7\) A `<dtratom>` element represents a concrete role. Note that attributes (`<attribute>` elements) are simply functional concrete roles.

\(^7\)We use the term ‘concrete roles’ following [75]. In OWL DL and OWL-E, they are also called ‘datatype properties’; cf. Section 3.3.3 on page 75.
4. **Datatype Expression-related Concept Descriptions** (cf. Table 5.2): \(<}\text{dtsome}\), \(<}\text{dtall}\), \(<}\text{dtatleast}\) and \(<}\text{dtatmost}\) elements corresponds to expressive predicate exists restriction (\(\exists T_1, \ldots, T_n.E\)), expressive predicate value restriction (\(\forall T_1, \ldots, T_n.E\)), expressive predicate qualified atleast restriction (\(\leq mT_1, \ldots, T_n.E\)) and expressive predicate qualified atmost restriction (\(\leq mT_1, \ldots, T_n.E\)), respectively; note that \(E\) can either be a datatype expression or a datatype expression name.

**Example 7.2 A Concept Description in DIG/OWL-E**

The AdultElephant concept can be defined in DIG/OWL-E as follows (cf. the DIG/1.1 version of the AdultElephant concept in Example 7.1):

\[
\begin{align*}
\langle \text{tells} \ldots \rangle \\
\langle \text{equalc} \rangle \\
\langle \text{catom name} = \text{"AdultElephant"} / \rangle \\
\langle \text{and} \rangle \\
\langle \text{catom name} = \text{"Elephant"} / \rangle \\
\langle \text{dtsome} \rangle \\
\langle \text{attribute name} = \text{"age"} / \rangle \\
\langle \text{predicate id} = \text{"owlx:integerGreaterThan20"} / \rangle \\
\langle /\text{dtsome} \rangle \\
\langle /\text{and} \rangle \\
\langle /\text{equalc} \rangle \\
\langle \text{defconcept name} = \text{"Elephant"} / \rangle \\
\langle \text{defattribute name} = \text{"age"} / \rangle \\
\langle /\text{tells} \rangle
\end{align*}
\]

where \(\text{owlx:integerGreaterThan20}\) is the URId for the integer predicate \(\text{int}_{\geq 20}\).

Here we use the \(<}\text{dtsome}\) construct, instead of the \(<}\text{intmin}\) construct; the main benefit is that we can now use arbitrary predicate URIs, or even datatype expressions.

\(\diamond\)

DIG/OWL-E extends the DIG/1.1 TELL language by introducing the following new axioms (see Table 7.6 on page 174):

1. **Datatype Expression Axiom** (cf. Section 6.1): A \(<}\text{defdtexpression}\) element represents a datatype expression axiom, which introduces a name “EN” for a datatype expression \(E\). Therefore, the datatype expression name can be used in datatype expression-related concept descriptions and the concrete role range axioms.

2. **Concrete Role Axioms**: Many of the concrete role axioms are very similar to abstract role axioms. We could have modified the existing abstract role axioms
Datatype Expression Axiom

<defdtexpression name = “EN”>
  E
</defdtexpression>

Concrete Role Axioms

<defcrole name = “RN”/>
<impliescr T₁ T₂ </impliescr>
<equalcr T₁ T₂ </equalcr>
<crdomain> T C </crdomain>
<crrange> T E </crrange>
<crfunctional> T </crfunctional>

Individual Axioms

<sameindividual> I₁ ... Iₙ </sameindividual>
<diffindividual> I₁ ... Iₙ </diffindividual>

<table>
<thead>
<tr>
<th>Datatype Expression Axiom</th>
<th>&lt;defdtexpression name = “EN”&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>E</td>
</tr>
<tr>
<td></td>
<td>&lt;/defdtexpression&gt;</td>
</tr>
<tr>
<td>Concrete Role Axioms</td>
<td>&lt;defcrole name = “RN”/&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;impliescr T₁ T₂ &lt;/impliescr&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;equalcr T₁ T₂ &lt;/equalcr&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;crdomain&gt; T C &lt;/crdomain&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;crrange&gt; T E &lt;/crrange&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;crfunctional&gt; T &lt;/crfunctional&gt;</td>
</tr>
<tr>
<td>Individual Axioms</td>
<td>&lt;sameindividual&gt; I₁ ... Iₙ &lt;/sameindividual&gt;</td>
</tr>
<tr>
<td></td>
<td>&lt;diffindividual&gt; I₁ ... Iₙ &lt;/diffindividual&gt;</td>
</tr>
</tbody>
</table>

Table 7.6: New axioms in the DIG/OWL-E TELL̊ language

to accommodate the concrete role axioms. We propose otherwise in order to maintain backward compatibility. Another advantage is that we can easily disallow asserting that an abstract role is a sub-role of (or equivalent to) a concrete one, or the other way around. Note that the concrete role range axiom (represented by a <crrange> element) is quite different from the abstract role range axiom (represented by a <range> element) in that the range in the former one is a datatype expression or a name of a datatype expression, instead of a concept.

3. Same and Different Individual Axioms: A <sameindividual> element asserts that two individual names are interpreted as the same individual, while a <diffindividual> element asserts that two individual names are interpreted as different individuals. Note that these two individual axioms are convenient, but not necessary, for people to use OWL and OWL-E.¹⁸

Example 7.3 A Datatype Expression Axiom in DIG/OWL-E

The sumLessThan15 customised datatype predicate mentioned in Example 1.5 on page 22 can be defined by the following DIG/OWL-E datatype expression axiom:

<defdtexpression name = “sumNoGreaterThan15”>
  <dtand>
    <predicate id = “owlx:integerAddition”/>
  </dtand>
  <dtdomain>
    <dtnot id = “owlx:integerGreaterThanx=15”/>
    <predicate id = “owlx:integerGreaterThanx=0”/>
    <predicate id = “owlx:integerGreaterThanx=0”/>
    <predicate id = “owlx:integerGreaterThanx=0”/>
  </dtdomain>
</defdtexpression>

¹⁸cf. Section 2.1.2 on page 33.
7.1. GENERAL DL API

| Individual Query | `<relatedIndividualValues>`
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><code>&lt;relatedIndividualValues&gt;</code> T</td>
</tr>
<tr>
<td></td>
<td><code>&lt;/relatedIndividualValues&gt;</code></td>
</tr>
</tbody>
</table>

Table 7.7: New queries in the DIG/OWL-E ASK language

| Individual Value Pair Sets | `<individualValuePairSet>`
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><code>&lt;indvalPair&gt;</code> I_1 V_1</td>
</tr>
<tr>
<td></td>
<td><code>&lt;indvalPair&gt;</code> I_1 V_2</td>
</tr>
<tr>
<td></td>
<td><code>&lt;/indvalPair&gt;</code></td>
</tr>
<tr>
<td></td>
<td><code>&lt;/individualValuePairSet&gt;</code></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Typed Value Sets</th>
<th><code>&lt;typedValueSet&gt;</code> V_1 ... V_n</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><code>&lt;/typedValueSet&gt;</code></td>
</tr>
</tbody>
</table>

Table 7.8: New responses in the DIG/OWL-E RESPONSE language

```xml
</dtdomain>
</dtand>
</defdtexpression>
</tells>
```

where owlx:integerAddition is the URIref of the integer predicate \( +^{\text{int}} \) mentioned in Section 5.1 on page 102. The datatype expression `sumNoGreaterThan15` is a conjunction, where the first conjunct is the predicate URIref owlx:integerAddition, and the second conjunct is a domain constraint, which sets the domains of all its four arguments: the first one (corresponding to the sum of addition) should be integers that are no greater than 15, and the rest are positive integers. Based on `sumNoGreaterThan15`, we can define the SmallItem concept as follows:

```xml
<tells>
  <defconceptname = “Item”/>
  <defconceptname = “SmallItem”/>
  <equalc>
    <catom name = “SmallItem”/>
    <and>
      <catom name = “Item”/>
      <dtsome>
        <dtratom name = “hlwSumInCM”/>
        <dtratom name = “heightInCM”/>
        <dtratom name = “lengthInCM”/>
        <dtratom name = “widthInCM”/>
        <dtexpression name = “sumNoGreaterThan15”/>
      </dtsome>
      </and>
    </equalc>
</tells>
```

◊
DIG/OWL-E extends the DIG/1.1 ASK language by introducing a new form of individual query. With a \(<\text{relatedIndividualValues}>\) element, clients can request the instances (pairs of individual and typed values) of a concrete role. Accordingly, DIG/OWL-E extends the DIG/1.1 RESPONSE language (see Table 7.8) with individual value pair sets (\(<\text{individualValuePairSet}>\) elements) and typed value sets. Furthermore, a response to an identifier request in DIG/OWL-E should include the list of base datatype URIs, supported predicate URIs and the supported datatype expression constructors.

### 7.2 Architecture

We now propose a revised DIG framework architecture, which is designed to be more flexible and compatible with the OWL DL and OWL-E languages. The proposed framework architecture (see Figure 7.2) is different from the existing DIG/1.1 framework architecture we introduced in Section 7.1.1 (see Figure 7.1 on page 166) mainly in the following aspects:

1. We use DIG/OWL-E instead of DIG/1.1 as the general DL reasoner interface in the new architecture.
2. The datatype reasoning is not performed in the DL reasoner, but by one or more datatype reasoners. This architecture makes the framework more flexible, as when new datatypes and predicates are needed, we only need to update our datatype reasoners, but not the DL reasoner; in other words, we have a general-purpose DL reasoner.

There are two kinds of datatype reasoners in the framework. The first kind of datatype reasoner is called a datatype manager, which provides decision procedures to reduce the satisfiability problem of datatype expression conjunctions to the satisfiability problems of various datatype predicate conjunctions. The second kind of datatype reasoner is called a datatype checker. A datatype checker decides the satisfiability problem of datatype predicate conjunctions, where the datatype predicates are defined over a base datatype in a datatype group. More technical details of datatype managers and datatype checkers will be presented in the next section.

As shown in Figure 7.2, when a client sends an identification request to the DIG/OWL-E server, the server returns the names and versions of the DL reasoner, the datatype manager and the datatype checkers. In addition, the server returns the description language that the DL reasoner supports, the datatype expressions that the datatype manager supports and the base datatype URIrefs as well as supported predicate URIrefs that the datatype checkers provide.

The client can then upload the knowledge base using the `TELL` language and query using the `ASK` language. To answer the query, the DL reasoner runs as usual and depends on the datatype manager to decide the satisfiability problem of datatype expression conjunctions whenever necessary. The datatype manager reduces the datatype expression conjunctions to predicate conjunctions and then passes them to the proper datatype checkers to verify their satisfiability. Finally, the DL reasoner uses the `RESPONSE` language to return the answer to the client.

Whenever we need to support a new form of datatype expression, we update our datatype manager; whenever we need to provide some new datatype predicates that are defined over a new base datatype, we add a new datatype checker. In either case, we do not have to update the DL reasoner.
7.3 Datatype Reasoners

The soundness of our proposed framework depends on the proper design of the DL reasoner and datatype reasoners. In Chapter 6, we have presented the tableaux algorithms for the $G$-family DLs, which allows us to make use of datatype reasoners to answer expression conjunctions of the form (6.1). In this section, we present how to design datatype managers and datatype checkers.

7.3.1 Datatype Checkers

We start with datatype checkers. Let $G = (M_p, D_G, \text{dom})$ be a datatype group, $d \in D_G$ a base datatype URIref. According to Lemma 5.9 on page 108, if the corresponding concrete domain of $w$ is admissible, we can use a datatype checker for $d$ to decide the satisfiability of finite predicate conjunctions over $\text{sub-group}(w, G)$. Formally, a datatype checker for $d$ is defined as follows.

**Definition 7.1. (Datatype Checker)** Given a conforming datatype group $G = (M_p, D_G, \text{dom})$ and a base datatype URIref $d \in D_G$, a **datatype checker for $d$**, written as $\text{DC}_d$, is a program that takes as input a finite predicate conjunction $C$,

$$C = \bigwedge_{j=1}^{k} p_j(v_1^{(j)}, \ldots, v_n^{(j)}),$$

where $p_j \in \text{sub-group}(d, G)$ with arity $n_j$ and $v_i^{(j)}$ are variables, and answers **satisfiable** if $C$ is satisfiable and **unsatisfiable** otherwise.

In general, we can reuse existing constraint solvers for the corresponding predicate conjunctions as our datatype checkers.

7.3.2 Datatype Manager

As shown in the framework architecture (see Figure 7.2), a datatype manager works between a DL reasoner and a set of datatype checkers. We assume that datatype managers defined as follows support all forms of OWL-E datatype expressions.

Intuitively, a datatype manager transforms an input datatype expression into a disjunction of predicate conjunctions and then divides each of the predicate conjunction into several sub-conjunctions. Each of these sub-conjunctions contains predicates of certain base datatype. If there exist variables being used across these sub-conjunctions,
then the corresponding disjunct is *unsatisfiable*, as the value spaces of base datatypes are disjoint with each other; otherwise, the datatype manager can send these sub-conjunctions to appropriate datatype checkers to decide their satisfiabilities. If all the sub-conjunctions of a predicate conjunction is *satisfiable*, then it is *satisfiable*, and if any one of the predicate conjunction is *satisfiable*, then the input datatype expression is *satisfiable*.

**Definition 7.2.** *(Datatype Manager)* Given a conforming datatype group \( G = (M_p, D_G, \text{dom}) \), the *datatype manager for \( G \)*, written as \( \text{DM}_G \), is a program that takes as input a datatype query \( \Psi \) of the form of (6.1) and answers *satisfiable* if \( \Psi \) is satisfiable and *unsatisfiable* otherwise.

Let \( p \in \Phi_G, q \notin \Phi_G, d \in D_G, d_1, \ldots, d_n \in \text{sub-group}(d, G) \) and \( a(d_i) = 1 \) for all \( 1 \leq i \leq n \), and \( P, Q \) are \( G \)-datatype expression with the same arity. \( \text{DM}_G \) decides the satisfiability problem of a \( G \)-datatype expression conjunction \( \Psi \) in the following steps:

1. \( \text{DM}_G \) transforms \( \Psi \) into \( \Psi_1 \) by eliminating the negation constructor \( \neg \) for datatype expressions within \( \Psi \) as follows:

\[
\neg p \equiv \overline{p} \sqcup (d, \ldots, d) \text{ (where dom}(p) = (d, \ldots, d)),\]

\[
\neg \overline{p} \equiv p \sqcup (d, \ldots, d) \text{ (where dom}(p) = (d, \ldots, d)),\]

\[
\neg q \equiv \overline{q}\]

\[
\neg \overline{q} \equiv q\]

\[
\neg (d_1 \ldots d_n) \equiv (d_1 \ldots d_n)\]

\[
\neg (d_1 \ldots d_n) \equiv (d_1 \ldots d_n)\]

\[
\neg (P \sqcap Q) \equiv \neg P \sqcap \neg Q\]

\[
\neg (P \sqcup Q) \equiv \neg P \sqcup \neg Q.\]

2. \( \text{DM}_G \) transforms \( \Psi_1 \) into \( \Psi_2 \) by rewriting \( (d_1, \ldots, d_n) \) and \( (d_1, \ldots, d_n) \) in \( \Psi_1 \) as follows:

- \( (d_1, \ldots, d_n)(v_1, \ldots, v_n) \equiv d_1(v_1) \sqcap \ldots \sqcap d_n(v_n); \)
\[ (d_1, \ldots, d_n)(v_1, \ldots, v_n) \equiv C_{d_1}(v_1) \sqcup \ldots \sqcup C_{d_n}(v_n), \text{ and} \]
\[ C_{d_i}(v_i) = \begin{cases} d(v_i) & \text{if } d_i = d, \\ \overline{d_i}(v_i) \sqcup \overline{d}(v_i) & \text{otherwise.} \end{cases} \]

where \( d = \text{dom}(d_i) \) for \( 1 \leq i \leq n \).

3. \( \text{DM}_G \) transforms \( \Psi_2 \) into \( \Psi_3 \) by eliminating the negated predicate names \( \overline{p} \) for \( p \in \Phi_G \setminus D_G \) as follows:
\[ \overline{p} \equiv e \quad \text{(where } E(e) = E(p) \text{).} \]

4. \( \text{DM}_G \) transforms \( \Psi_3 \) into \( \Psi_4 \) by rewriting \( \neq (v_{i_1}, \ldots, v_{j_1}; v_{j_1}, \ldots, v_{j_n}) \) as follows:
\[ \neq (v_{i_1}, \ldots, v_{j_1}; v_{j_1}, \ldots, v_{j_n}) \equiv \neq (v_{i_1}, v_{j_1}) \sqcup \ldots \sqcup \neq (v_{i_n}, v_{j_n}). \]

5. \( \text{DM}_G \) transforms \( \Psi_4 \) into its disjunctive normal form \( \Psi_5 \).

6. \( \text{DM}_G \) checks each disjunct \( M \) of \( \Psi_5 \) as follows:

(a) Let \( v_1, \ldots, v_n \) be a tuple of variables and \( p \) a predicate URIref. If both \( p(v_1, \ldots, v_n) \) and \( \overline{p}(v_1, \ldots, v_n) \) are in \( M \), then \( M \) is unsatisfiable.

(b) Let \( v \) be a variable. If \( v \) occurs as an argument of both \( p \) and \( \overline{d} \) in \( M \), and \( p \in \text{sub-group}(d, G) \), then \( M \) is unsatisfiable.

(c) Let \( v \) be a variable. If \( v \) occurs as an argument of both \( p_1 \) and \( p_2 \) in \( M \), where \( p_1 \in \text{sub-group}(d_1, G), p_2 \in \text{sub-group}(d_2, G) \) and \( d_1 \neq d_2 \), then \( M \) is unsatisfiable.

(d) Otherwise, since \( \Phi_G = \bigcup_{d \in D_G} \text{sub-group}(d, G) \), \( \text{DM}_G \) disregards the unsupported predicate URI references and splits \( M \) into conjunctions \( N \), \( M_{d_1}, \ldots, M_{d_k} \), where \( k \) is the number of sub-groups used in \( M \) and \( \{d_1, \ldots, d_k\} \subseteq D_G \), such that

- \( N \) contains all the value inequality predicate conjunction;
- for each \( p \) in \( M_{d_h} \), \( p \in \text{sub-group}(d_h, G) \).

For each \( \neq (v_{i_1}, v_{j_1}) \), if both the variables \( v_{i_1}, v_{j_1} \) occurs in \( M_{d_h} \), \( \text{DM}_G \) will add \( \neq_{d_h} (v_{i_1}, v_{j_1}) \) as a conjunct into \( M_{d_h} \).
7.3. DATATYPE REASONERS

\( \text{DM}_G \) then uses the corresponding datatype checker \( \text{DC}_{d_h} \) to decide the satisfiability of \( M_{d_h} \). Finally, \( M \) is \textit{satisfiable} if all datatype checkers return \textit{satisfiable}; \( \text{DM}_G \) returns \textit{satisfiable} if one of the disjuncts \( M \) of \( \Psi_5 \) is \textit{satisfiable}; otherwise, \( \text{DM}_G \) returns \textit{unsatisfiable}.

**Lemma 7.3.** Given a conforming datatype group \( G = (M_p, D_G, \text{dom}) \), the type manager \( \text{DM}_G \) decides the satisfiability of datatype queries \( \Psi \) of the form of (6.1).

**Proof:** In Definition 7.2, the first five steps use equivalent transformations to rewrite the input \( G \)-predicate expression conjunction \( \Psi \) into a disjunction of predicate conjunctions \( \Psi_5 \).

In step 1, the equivalent transformations are based on the semantics of negated predicate names and datatype expressions.

- Let \( p \) be a supported predicate URIref and \( p \in \text{sub-group}(d, G) \), we have
  \[
  (\neg p)^D = (\Delta_D)^n \setminus p^D = (\Delta_D)^n \setminus E(p) = ((\Delta_D)^n \setminus (\text{dom}(p))^D) \cup ((\text{dom}(p))^D \setminus E(p)) = \underbrace{(d, \ldots, d)}_{n \text{ times}} \cup \underbrace{\overline{p}}_{n \text{ times}},
  \]
  so we have \( \neg p \equiv \overline{p} \cup \underbrace{(d, \ldots, d)}_{n \text{ times}} \). Similarly, we have \( \neg \overline{p} \equiv p \cup \underbrace{(d, \ldots, d)}_{n \text{ times}} \).

- Let \( q \) be an un-supported predicate URIref, i.e., \( q \not\in \Phi_G \). According to Definition 5.4, we have \( \overline{q}^D = (\Delta_D)^n \setminus q^D \), so \( \overline{q} \equiv \neg q \).

- According to Definition 5.12, we have \( \neg(d_1, \ldots, d_n)^D = (\Delta_D)^n \setminus (d_1, \ldots, d_n)^D = \underbrace{(d_1, \ldots, d_n)}_{n \text{ times}} \), so we have \( \neg(d_1, \ldots, d_n) \equiv (d_1, \ldots, d_n) \).

- Let \( P, Q \) be \( G \)-datatype expressions with the same arity. According to De Morgan’s Law, we have \( \neg(P \sqcap Q) \equiv \neg P \sqcup \neg Q \) and \( \neg(P \sqcup Q) \equiv \neg P \sqcap \neg Q \).

In step 2, let \( d \in D_G \), \( d_1, \ldots, d_n \in \text{sub-group}(d, G) \) and \( a(d_i) = 1 \) for each \( 1 \leq i \leq n \). By definition, we have \( (d_1, \ldots, d_n)(v_1, \ldots, v_n) \equiv d_1(v_1) \sqcap \ldots \sqcap d_n(v_n) \). As for its negation, we have
  \[
  (d_1, \ldots, d_n)(v_1, \ldots, v_n) \equiv \neg(d_1, \ldots, d_n)(v_1, \ldots, v_n) \equiv \neg d_1(v_1) \sqcup \ldots \sqcup \neg d_n(v_n).
  \]
There are two possibilities here:

- if \( d_i = d \), then \((-d_i)^D = \neg d^D = \overline{d} \), hence we have \(-d_i \equiv \overline{d} \);

- otherwise \((-d_i)^D = \Delta_D \setminus d_i^D = (\Delta_D \setminus d^D) \cup (d^D \setminus d_i^D) = \overline{d} \cup \overline{d_i} \); hence we have \(-d_i \equiv \overline{d} \sqcap \overline{d_i} \).

In step 3, the equivalent transformations are guaranteed by the condition 2 of a conforming datatype group. As for step 4 and 5, the equivalent transformations are obvious.

Finally step 6, which checks every disjunct \( M \) of \( \Psi_5 \), where \( M \) is simply a predicate conjunction.

- Sub-step (6a) is obvious, a tuple of variables cannot satisfy both \( p \) and \( \overline{p} \).

- In sub-step (6b), assume we have a solution \( \delta \) for \( M \). Since \( p \in \text{sub-group}(d, G) \), we have \( \delta(v) \in d^D \); due to \( \overline{d}(v) \), we have \( \delta(v) \in \overline{d}^D \). Since \( d^D \cap \overline{d}^D = \emptyset \), we have a contradiction; therefore, we have no solution for \( M \), i.e., \( M \) is unsatisfiable.

- In sub-step (6c), assume we have a solution \( \delta \) for \( M \). Since \( p_1 \in \text{sub-group}(d_1, G) \), we have \( \delta(v) \in d_1^D \); similarly as \( p_2 \in \text{sub-group}(d_2, G) \) we have \( \delta(v) \in d_2^D \). According to Definition 5.5 on page 104, \( d_1^D \cap d_2^D = \emptyset \), hence there is a contradiction; therefore, we have no solution for \( M \), i.e., \( M \) is unsatisfiable.

- In sub-step (6d), \( D_MG \) only concerns the supported predicates because the only possible contradictions caused by unsupported predicate URIs have been considered in sub-step (6a). \( D_MG \) splits \( M \) into \( M_{d_1}, \ldots, M_{d_k} \) and \( N \). For each inequality constraint \( \neq (v_{i_s}, v_{j_s}) \), there are two possibilities:

  - if both \( v_{i_s} \) and \( v_{j_s} \) occur in same \( M_{d_h} \), \( D_MG \) adds \( \neq_{d_h} (v_{i_s}, v_{j_s}) \) into \( M_{d_h} \) to ensure, for any solution \( \delta_h \) of \( M_{d_h} \), \( \delta_h(v_{i_s}) \neq \delta_h(v_{i_s}) \);

  - if \( v_{i_s} \) and \( v_{j_s} \) do not occur in some \( M_{d_h} \), they will not be mapped to the same data value, due to the disjointness of the interpretations of base datatypes.

Finally, \( D_MG \) can then uses the corresponding datatype checker \( DC_{d_h} \) to decide the satisfiability of each \( M_{d_h} \). Since \( M_{d_1}, \ldots, M_{d_k} \) are conjuncts of \( M \), \( M \) is satisfiable iff all the corresponding datatype checkers return satisfiable. Finally, \( D_MG \) returns satisfiable if one of the disjuncts \( M \) of \( \Psi_5 \) is satisfiable. \(\square\)
7.4 Flexibility

We end the chapter by a conclusion of the flexibility of our framework in the following points of view.

**Users** The framework enables the users to use customised datatypes and datatype predicates. In addition, it provides minimum checking for unsupported datatype predicates.

**DIG/OWL-E interface** Our general DL API does not need to be updated when new datatype predicates are supported by datatype reasoners.

**DL reasoner** In the framework, the DL reasoner can implement $G$-augmented tableaux algorithms for a wide range of $G$-combined DLs, including express ones like $\mathcal{SHI}Q(G)$, $\mathcal{SHO}Q(G)$ and $\mathcal{SHIO}(G)$, and less express but more efficient ones like the datatype group extensions of DL-Lite [26] and $\mathcal{ELH}$ [20]. To support new DLs, we only need to upgrade the DL reasoner.

**Datatype Manager** To support new forms of datatype expressions, we only need to upgrade the datatype manager and ensure the datatype query of the form of (6.1) are still decidable.

**Datatype Checkers** To support datatype predicates of some new base datatypes, we only have to add new datatype checkers.

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### Chapter Achievements

- **DIG/OWL-E** extends DIG/1.1 by using $\mathcal{SHO}I\mathcal{Q}(G)$ as the underpinning Description Logic, which is expressive enough to cover DAML+OIL, OWL DL and OWL-E.

- **DIG/OWL-E** supports $G$-datatype expressions and allows the use of arbitrary datatype predicates in $G$-datatype expressions.

- Our proposed framework is highly extendable to support new datatype predicates, new forms of datatype expressions and new decidable Description Logics.
Chapter 8

Implementation

Chapter Aims

• To present a prototype as a test bed for verifying the validity of our approach and for suggesting new directions to investigate.

• To show the flexibility of the framework architecture.

• To describe the matchmaking use case which exploits our prototype.

Chapter Plan

8.1 A Datatype Extension of FaCT (184)

8.2 Case Study: Match Making (188)

This chapter presents an implementation of the tableaux algorithm for $SHIQ(G)$ presented in Chapter 6 and the datatype manager algorithm presented in Chapter 7.

8.1 A Datatype Extension of FaCT

The goal of this section is to describe a datatype extension of the FaCT reasoner [60].

FaCT was the first sound and complete DL system to demonstrate the usefulness of expressive Description Logics for developing practical applications [100]. Our implementation is designed for two purposes. Firstly, it is meant to be a light-weight test bed for the $G$-augmented tableaux algorithm of the $SHIQ(G)$ DL presented in Definition 6.18. Secondly, it aims to show the flexibility of the framework architecture.
presented in Chapter 7. As a concept-proof prototype, it provides neither an implementation of the DIG/OWL-E interface nor any optimisations for datatype reasoning. In other words, performance is not its main concern.

8.1.1 System Overview

As per the framework presented in Chapter 7, our prototype is actually a hybrid reasoner that has three highly independent components: (i) a DL reasoner that is built based on the FaCT reasoner and supports the $SHIQ(\mathcal{G})$ DL (see Section 6.2.2), (ii) a datatype manager which decides the satisfiability of datatype expression conjunctions, and (iii) two simple datatype checkers for integers and strings. The DL reasoner asks the datatype manager to check the satisfiability of datatype queries of the form of (6.1). The datatype manager asks the two datatype checkers to check the satisfiability of predicate conjunctions of the form of (5.6). All the three components of our system is implemented in Lisp, although in principle they do not have to be implemented in same the programming language.

8.1.2 Extended DL Reasoner

The extended FaCT DL reasoner supports datatype group-based concept descriptions. It implements the $\mathcal{G}$-rules described in Figure 6.2 on page 143. We usually apply these rules (in the following order: $\exists_p$-rule, $\geq_p$-rule, $\forall_p$-rule, $\text{choose}_p$-rule and $\leq_p$-rule) on an abstract node just before we further check its abstract successors and query the datatype manager to check resulting datatype constraints. It may need to query the datatype manager again if new datatype constraints are added later, e.g., from some of its abstract successors via some inverse roles.

The DL reasoner is independent of the forms of datatype expressions that the datatype manager supports. It does not have to understand the syntax of datatype expressions, and it simply leaves them untouched and passes them to the datatype manager. Therefore, the DL reasoner does not have to be modified even if the datatype manager is upgraded to support some forms of new datatype expressions.

We extend the FaCT syntax by introducing datatype expressions (see Table 8.1) and datatype expression-related concept descriptions (see Table 8.2), where we use a positive integer $n$ to indicate the number of concrete roles used in the concept descriptions.

The syntax of TBox and RBox axioms remains the same as FaCT. Users can now
8.1. A DATATYPE EXTENSION OF FACT

### Extended FaCT Syntax

| **TOPD** | rdfs:Literal |
| **BOTTOMD** | owlx:DatatypeBottom |
| (and $E_1 \ldots E_n$) | and$(E_1, \ldots, E_n)$ |
| (or $E_1 \ldots E_n$) | or$(E_1, \ldots, E_n)$ |
| (neg p) | neg(p) |
| (domain $d_1 \ldots d_n$) | domain$(d_1, \ldots, d_n)$ |

Table 8.1: FaCT datatype expressions

| **TOP** | ⊤ |
| **BOTTOM** | ⊥ |
| (and $C_1, \ldots, C_n$) | $C_1 \sqcap \ldots \sqcap C_n$ |
| (or $C_1, \ldots, C_n$) | $C_1 \sqcup \ldots \sqcup C_n$ |
| (not C) | ¬C |
| (some $R.C$) | $\exists R.C$ |
| (all $R.C$) | $\forall R.C$ |
| (atleast $m \ R.C$) | $\geq mR.C$ |
| (atmost $m \ R.C$) | $\leq mR.C$ |
| (dt-some $n \ T_1 \ldots T_n \ E$) | $\exists T_1, \ldots, T_n.E$ |
| (dt-all $n \ T_1 \ldots T_n \ E$) | $\forall T_1, \ldots, T_n.E$ |
| (dt-atleast $m \ n \ T_1 \ldots T_n \ E$) | $\geq mT_1, \ldots, T_n.E$ |
| (dt-atmost $m \ n \ T_1 \ldots T_n \ E$) | $\leq mT_1, \ldots, T_n.E$ |

Table 8.2: FaCT concepts

use datatype expression-related concept descriptions in TBox axioms. Note that the set of abstract role names and the set of concrete role names should be disjoint, otherwise the system will report an error. Users can define concrete role inclusion axioms and functional axioms. Note that if users use an abstract role and a concrete role in a role inclusion axiom, the system will report an error. The syntax of TBox queries remains the same as FaCT too; i.e., this datatype extension of FaCT provides concept satisfiability, concept subsumption and classification checking.

8.1.3 **Datatype Reasoning Components**

The datatype reasoning components of the hybrid reasoner include a datatype manager and two simple datatype checkers (concrete domain reasoners). The datatype manager
8.1. A DATATYPE EXTENSION OF FACT

reduces the satisfiability problem of datatype expression conjunctions to the satisfiability problem of predicate conjunctions that the datatype checkers can handle.

The datatype manager is independent of the kinds of datatype predicates that the datatype checkers support. Theoretically it can work with an arbitrary set of datatype checkers, as long as the datatype checkers satisfy the following conditions.

1. the domains of the base datatypes that the datatype checkers support are pairwise disjoint;

2. each datatype checker provides the following registration information:
   (a) its satisfiability checking function,
   (b) its base datatype URIref,
   (c) the inequality predicate URIref for its base datatype,
   (d) the set of supported predicate URIrefs for its base datatype;

3. each registered datatype checker supports the syntax of predicate conjunctions that the datatype manager uses in its queries.

Table 8.3 lists the registration information of the two datatype checkers implemented in our hybrid reasoner.

<table>
<thead>
<tr>
<th>Datatype Checker ( #1 )</th>
<th>Datatype Checker ( #2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sat. Checking Func.</td>
<td>sint-dcf-sat</td>
</tr>
<tr>
<td>Base Datatype URIref</td>
<td>xsd:integer</td>
</tr>
<tr>
<td>Inequality Pred. URIref</td>
<td>owlx:integerInequality</td>
</tr>
<tr>
<td>Other Supported Pred. URIrefs</td>
<td>owlx:integerEquality</td>
</tr>
<tr>
<td></td>
<td>owlx:integerLessThan</td>
</tr>
<tr>
<td></td>
<td>owlx:integerGreaterThan</td>
</tr>
<tr>
<td></td>
<td>owlx:integerLessThanOrEqualTo</td>
</tr>
<tr>
<td></td>
<td>owlx:integerGreaterThanOrEqualTo</td>
</tr>
<tr>
<td></td>
<td>owlx:integerInequalityx=n</td>
</tr>
<tr>
<td></td>
<td>owlx:integerEqualityx=n</td>
</tr>
<tr>
<td></td>
<td>owlx:integerLessThanx=n</td>
</tr>
<tr>
<td></td>
<td>owlx:integerGreaterThanx=n</td>
</tr>
<tr>
<td></td>
<td>owlx:integerLessThanOrEqualTox=n</td>
</tr>
<tr>
<td></td>
<td>owlx:integerGreaterThanOrEqualTox=n</td>
</tr>
</tbody>
</table>

Table 8.3: Registration information of our datatype checkers

---

\(^2\)Here we assume that each datatype checker supports only one base datatype, but it is easy to extend it to the case where a datatype checker supports multiple base datatypes.

\(^3\)In general, it should not be difficult to translate the syntax between the datatype manager and datatype checkers.
8.2. CASE STUDY: MATCHMAKING

It is straightforward, but important, to observe that the system is very flexible. For example, to support new datatypes and predicates, we simply need to add datatype checkers that satisfy the above three conditions.

8.2 Case Study: Matchmaking

We invite the reader to consider a use case of our hybrid reasoner. Section 1.3 has shown an example of using datatype reasoning in matchmaking. This section will further describe how to use our prototype to support matchmaking.

8.2.1 Matchmaking

Let us consider the following scenario: agents advertise services with their capabilities through a registry and query the registry for services with specified capabilities. Matchmaking is a process that takes a query as input and return all advertisements which may potentially satisfy the capabilities specified in the query.

We can use the $SHIQ(G)$ DL to describe service capabilities for both advertisement and query. More precisely, the capability (either in advertisements or queries) of a service can be represented as an OWL-E class or class restriction, e.g. the capability that memory size should be either 256Mb or 512Mb, can be represented the datatype expression-related concept $\exists memoryUnitSizeInMb.(=$256 $\lor=$512).

Usually, we are not only interested in finding the exact match, viz., there could be several degrees of matching. Following [84], we consider five levels of matching:

1. **Exact** If the capabilities of an advertisement $A$ and a request $R$ are equivalent classes, we call it an exact match, noted as $C_A \equiv C_R$.

2. **PlugIn** If the capability of a request $R$ is a sub-class of that of an advertisement $A$, we call it a PlugIn match, noted as $C_R \subseteq C_A$.

3. **Subsume** If the capability of a request $R$ is a super-class of that of an advertisement $A$, we call it a Subsume match, noted as $C_A \subseteq C_R$.

4. **Intersection** If the intersection of the capabilities of an advertisement $A$ and a request $R$ are satisfiable, we call it a Intersection match, noted as $\neg(C_A \cap C_R \sqsubseteq \bot)$.

5. **Disjoint** Otherwise, we call it a Disjoint (failed) match, noted as $C_A \cap C_R \sqsubseteq \bot$. 
8.2. Working Examples

To gain a further insight into the above five levels of matching, it is often helpful to have some working examples. Suppose, in a scenario of computer selling, that an agent would like to buy a PC with the following capabilities:

- the \textit{processor} must be Pentium4;
- the \textit{memoryUnitSizeInMb} must be 128;
- the \textit{priceInPound} must be less than 500.

This can be represented by the following $\text{SHIQ}(G)$ concept:

$$C_{R1} \equiv \text{PC} \sqcap \exists \text{processor.} \text{Pentium4} \sqcap \exists \text{memoryUnitSizeInMb.} \geq [128] \sqcap \exists \text{priceInPound.} < [500]$$

<table>
<thead>
<tr>
<th>Exact match: $C_{A1} \equiv$</th>
<th>$\text{PC} \sqcap \exists \text{processor.} \text{Pentium4} \sqcap \geq \text{memoryUnitSizeInMb.} \geq [128] \sqcap \exists \text{priceInPound.} &lt; [500]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PulgIn match: $C_{A2} \equiv$</td>
<td>$\text{PC} \sqcap \exists \text{processor.} \text{Pentium4} \sqcap \geq \text{memoryUnitSizeInMb.} \geq [128] \lor \geq [256] \sqcap \exists \text{priceInPound.} &lt; [500]$</td>
</tr>
<tr>
<td>Subsume match: $C_{A3} \equiv$</td>
<td>$\text{PC} \sqcap \exists \text{processor.} \text{Pentium4} \sqcap \geq \text{memoryUnitSizeInMb.} \geq [128] \sqcap \exists \text{priceInPound.} &lt; [500]$</td>
</tr>
<tr>
<td>Intrsect. match: $C_{A4} \equiv$</td>
<td>$\text{PC} \sqcap \exists \text{processor.} \text{Pentium4} \sqcap \geq \text{memoryUnitSizeInMb.} \geq [128] \sqcap \exists \text{priceInPound.} &gt; [400]$</td>
</tr>
<tr>
<td>Disjoint match: $C_{A5} \equiv$</td>
<td>$\text{PC} \sqcap \exists \text{processor.} \text{Pentium4} \sqcap \geq 2 \text{memoryUnitSizeInMb.} \geq [256] \sqcap \exists \text{priceInPound.} &lt; [500]$</td>
</tr>
</tbody>
</table>

Figure 8.1: Example matching advertisements

Figure 8.1 presents five example matching advertisements for $C_{R1}$ in five different matching levels. Among them, $C_{A1}$ is the exact match. In realistic situations, however, it is not to easy have an exact match, since advertisements might provide more general
or more specific information. For example, $C_{A2}$ states that $priceInPound$ is only less than 700 and that the $memoryUnitSizeInMb$ can be either 128 or 256 (represented by the datatype expression $=int_{[128]} \lor =int_{[256]}$). $C_{A3}$ adds two restrictions on $orderDates$: firstly, the order date must be in August and September of 2004, which is represented by the datatype expression $(\geq int_{[20040801]} \land \leq int_{[20040831]}) \lor (\geq int_{[20040901]} \land \leq int_{[20040930]})$; secondly, $orderDates$ should be sooner than (represented by the binary predicate $<int$) $DeliverDates$, indicating that PCs will be delivered on some date after orders are made. As a result, $C_{A2}$ and $C_{A3}$ are PlugIn match and Subsume match of $C_{R1}$, respectively. $C_{A4}$ says the $priceInPound$ is greater than 400, and the $CPUFreqInGHz$ of their PCs is 2.8; it is an Intersection match. Finally, $C_{A5}$ advertises that their PCs have exactly two memory chips, with the $memoryUnitSizeInMb$ of each chip is 256, and the $HardDiskBrand$ and $USBKeyBrand$ in their PCs are the same (represented by the binary predicate $=str$); hence it is a disjoint (failed) match.

### 8.2.3 Matching Algorithm

We use the algorithm presented in [84] to detect the above five levels of matching. Firstly, we use our prototype to classify the hierarchy for all the advertised services and then compute the taxonomy position of the capability request $C_{R}$. Advertisements with capability equivalent to $C_{R}$ are considered to be Exact matches, those subsumed by but not equivalent to $C_{R}$ are considered to be PlugIn matches, and those subsuming but not equivalent to $C_{R}$ are considered to be Subsume match. We can then compute the taxonomy position of the negation of the capability request $\neg C_{R}$. Advertisements with capability subsuming but not equivalent to $\neg C_{R}$ are considered to be Intersection matches, while those subsumed by $\neg C_{R}$ are considered to be Disjoint (failed) matches.

We have done some tests with our prototype for the matchmaking use case. In terms of functionality, our implemented matchmaking algorithm has achieved its purpose: it can respond to an input capability request with the results of matched advertisements of all the five degrees. Performance is relatively poor: e.g., classifying a TBox with 20 concepts (each of which has at least two datatype expression-related sub-concepts), a test that requires 235 subsumption tests, takes 34.8 seconds, i.e., 0.148 second per subsumption test.

This is, however, to be expected. Firstly, reasoning with datatypes is generally

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4Note that $=real_{[2.8]}$ is not a supported predicate for our prototype: our prototype will not reject it and even provides minimum checking for it; i.e., if $=real_{[2.8]}$ and $=real_{[2.8]}$ are both in a predicate conjunction, this conjunction is unsatisfiable.
hard when there is more than one base datatype (our prototype supports both integer and string). This is because full negations of datatype expressions can introduce non-determinism. For example, when we check if $C_{A1}$ is subsumed by $C_{R1}$, we actually check if $C_{A1} \sqcap \neg C_{R1}$ is unsatisfiable. The tableaux algorithm we use requires the negated normal form (NNF) of the input concept. Therefore, we transform $\neg C_{R1}$ into its NNF form $\sim C_{R1}$:

$$\sim C_{R1} \equiv \neg P \sqcup \forall \text{processor}. \neg \text{Pentium4} \sqcup$$

$$\forall \text{memoryUnitSizeInMb} (\neg \mathit{int}_{[128]} \sqcup \forall \text{priceInPound} (\neg <\mathit{int}_{[500]})$$

which is further transformed (according to Definition 7.2 on page 179) into

$$\sim C_{R1} \equiv \neg P \sqcup \forall \text{processor}. \neg \text{Pentium4} \sqcup$$

$$\forall \text{memoryUnitSizeInMb} (\neg \mathit{int}_{[128]} \vee \neg \text{integer}) \sqcup$$

$$\forall \text{priceInPound} (\geq \mathit{int}_{[500]} \vee \neg \text{integer})$$

with the result that disjunctions of datatype expressions are introduced.

Secondly, in our prototype, we did not cache the normalised forms of datatype expressions. Therefore, when the datatype manager receives a query from the DL reasoner, it always normalises all the datatype expressions no matter whether they have been normalised before.

### 8.2.4 Speeding Up the Answer

Some optimisation techniques can be used to improve the performance of our prototype. Firstly, we can apply known optimisation techniques for concrete domain reasoning [141, 144, 49], such as incremental concrete domain reasoning and dependency-directed backtracking, in our framework. Secondly, we could devise some new optimisation techniques that are related to the new features of our framework.

**Incremental Datatype Checker** The idea is that a datatype checker should be able to keep track of previous internal states and only check if the additional constraints would introduce contradictions. For example, if a datatype checker has once determined that

$$p_1(v_1) \land p_2(v_2)$$
is satisfiable, when it receives a new query as follows

\[ p_1(\vec{v}_1) \land p_2(\vec{v}_2) \land p_3(\vec{v}_3), \]

it should not check it all over again, but recall the internal state for \( p_1(\vec{v}_1) \land p_2(\vec{v}_2) \) and check if adding \( p_3(\vec{v}_3) \) will cause any contradictions.

A key question here is how to efficiently keep track of previous internal states, since datatype checkers may result in keeping many unused previous states. Proper strategies should be introduced so that datatype checkers know when to keep a copy of an internal state and how long they should keep the state.

Dependency-Directed Backtracking  An adaptation of dependency-directed backtracking (DDB) to support datatype reasoning is described in [141]. The idea is that a DDB-enabled datatype checker should be able to identify all minimal, inconsistent sets of predicates (also referred to as clash culprits). These minimal sets define the necessary dependencies for backtracking. If this is not supported by the datatype checker, or is not computationally feasible, DDB must be disabled after a CD clash.

If a datatype checker is not able to identify all minimal inconsistent sets, [49] proposes that using a technique similar to semantic backtracking can help to keep the recorded predicates and their dependencies as small as possible. The main idea is that when we want to check whether adding \( p(\vec{v}) \) causes a contradiction, we try \( \neg p(\vec{v}) \) first. If \( \neg p(\vec{v}) \) causes a clash, the current internal state already entails \( p(\vec{v}) \); therefore, the backtracking dependencies of \( p(\vec{v}) \) can be safely ignored. If \( \neg p(\vec{v}) \) does not cause a clash, \( p(\vec{v}) \) is added to the current internal state. In case the datatype checker signals an inconsistency, the last predicate added to its current internal state is guaranteed to be a clash culprit. A possible problem here is that, when we have multiple base datatypes in a datatype group, full negation would introduce more disjunctions; hence, it is unclear whether this relaxed version will actually work in this situation.

Other Optimisations  Further optimisation techniques might be useful for handling datatype expressions. For example, we could further improve the performance by caching the normalised datatype expressions in the datatype manager. A key question is how to cache reusable normalised datatype expressions and avoid caching unused ones.

We did not investigate the above ideas yet — they will be the subject of our future work.
Chapter Achievements

- We briefly described our implementation for the SHIQ(DL).
- We demonstrated the usefulness and flexibility of our framework.
- We discussed possible ways to improve the performance of our prototype.
Chapter 9

Discussion

This thesis concludes with a review of the work presented and an assessment of the extent to which the objectives set out in Chapter 1 have been met. The significance of the major results is summarised, and directions for future work are suggested.

9.1 Thesis Overview

This thesis proposes solutions for the two problems that hold back the wider adoption of DL-based SW ontology languages (cf. Chapter 3). To solve the first problem, it proposes RDFS(FA), a novel sub-language of RDF(S), as a firm semantic foundation for the latest DL-based SW ontology languages. To solve the second problem, it proposes two decidable extensions of OWL DL, viz. OWL-E and OWL-Eu, that support customised datatypes and datatype predicates.

Furthermore, it presents a DL reasoning framework to represent customised datatypes and datatype predicates with datatype expressions, and to provide a wide range of $\mathcal{G}$-combined DLs, including the very expressive OWL-E and OWL-Eu DLs, that are decidable and support datatype expressions. The framework can be used to provide decision procedures for $\mathcal{G}$-combined DLs, including those that are closely related to OWL DL, DAML+OIL, OWL-Eu and OWL-E. An important feature of the proposed framework is its flexibility: the hybrid reasoner in the framework is highly extensible to support new datatypes and datatype predicates, new forms of datatype expressions and new decidable Description Logics.

To sum up, the thesis has demonstrated that Description Logics can provide clear semantics, decision procedures and flexible reasoning services for SW ontology languages, including those that provide customised datatype and datatype predicates.
9.2 Significance of Major Contributions

The objectives set out in Section 1.4 have been successfully fulfilled by the following major contributions of the thesis:

- the design of RDFS(FA), a sub-language of RDF(S) with DL-style model theoretic semantics, which provides a firm foundation for using DL reasoning in the Semantic Web and thus solidifies RDF(S)’s proposed role as the foundation of the Semantic Web;

- the datatype group approach, which specifies a formalism to unify datatypes and datatype predicates and to provide a wide range of (including very expressive) decidable Description Logics integrated with customised datatypes and datatype predicates;

- the design of OWL-E, a decidable extension of OWL DL that provides customised datatypes and datatype predicates based on datatype groups, and its unary restriction OWL-Eu, which is a much smaller extension of OWL DL;

- the design of practical tableaux algorithms for a wide range of DLs that are combined with arbitrary conforming datatype groups, including those of a family of DLs that are closely related to OWL DL, DAML+OIL, OWL-Eu and OWL-E;

- a flexible framework architecture to support decidable Description Logic reasoning services for Description Logics integrated with datatype groups.

RDFS(FA)

The first objective of the thesis was to propose a novel modification of RDF(S) as a firm semantic foundation for the latest DL-based SW ontology languages. Such a modification of RDF(S) should satisfy the requirements on language layering and applications presented in Section 4.1.1; i.e., its semantics should be compatible with that of OWL DL, and it should still provide the main features of RDFS.

RDFS(FA) satisfies all these requirements. Firstly, strata 0-2 in RDFS(FA) have a standard model theoretic semantics such that more expressive FOL ontology languages, such as the W3C standard OWL, can be layered on top of them and are compatible with RDFS(FA)’s metamodeling architecture. The bidirectional one-to-one mapping presented in Table 4.1 (page 96) between RDFS(FA) axioms in strata 0-2
9.2. SIGNIFICANCE OF MAJOR CONTRIBUTIONS

and OWL DL axioms enables RDFS(FA)-agents and OWL DL-agents to communicate with each other (cf. Theorem 4.4 on page 96) in a much easier way.

Furthermore, RDFS(FA) clarifies the vision of the Semantic Web: RDF is only a standard syntax for SW annotations and languages; the meaning of annotation comes from either external agreements (such as Dublin Core) or ontologies, both of which are supported by RDFS(FA). As an ontology language, RDFS(FA) provides a UML-like layered style for using RDFS, which makes it more intuitive and easier to understand and use. Data-valued annotation properties in RDFS(FA) enable SW applications to make use of URIs of ontology elements, such as classes, in results of various ontology inferences.

Note that RDFS(FA) does not allow cross strata abstract properties (such as a property relating an individual in stratum 0 and a class in stratum 3); hence, users who prefer this kind of non-layered style can/should use the full RDFS. RDFS(FA), instead, provide annotation properties to allow that ‘anyone can say anything about anything’ (such that users can, e.g., annotate an individual in stratum 0 with a typed literal that represents a class URIref in stratum 3).

Summarily, RDFS(FA) solidifies RDF(S)’s intended role as the foundation of the Semantic Web by restoring the broken link between RDF(S) and OWL DL and by clarifying the vision of the Semantic Web.

OWL-E and OWL-Eu

The second objective of the thesis was to propose some extensions of OWL DL, so as to support customised datatypes and datatype predicates. Such extensions of OWL DL should satisfy the requirements presented in Section 6.1.1. In other words, these extensions should be decidable and should provide customised datatypes and datatype predicates; furthermore, it is desirable that they should also be extensions of DAML+OIL and should overcome the limitations of OWL DL presented in the ‘Limitations of OWL Datatyping’ section on page 77.

OWL-E satisfies all these requirements. Firstly, using $SHOIQ(G)$ as its underpinning, OWL-E is a decidable extension of both OWL DL and DAML+OIL, which provides customised datatypes and predicates; in fact, all the basic reasoning services of OWL-E are decidable (cf. Theorem 6.3). Secondly, as a $G$-combined DL, OWL-E overcomes the limitations of OWL DL with respect to negated datatypes and datatype domain. Finally, OWL-E allows users to define names (in fact, URIrefs) for $G$-datatype expressions, including enumerated datatypes.
9.2. SIGNIFICANCE OF MAJOR CONTRIBUTIONS

OWL-Eu, as a unary restriction of OWL-E, is a small extension of OWL DL.\(^1\) The design of OWL-Eu is motivated by the W3C’s ‘one small step at a time’ strategy — the only extension it presents is to extend OWL data ranges to support unary \(G\)-datatype expressions, which can be used to represent unary customised datatype predicates. OWL-Eu is much easier for existing OWL tools to adapt to, as unary \(G\)-datatype expressions can be treated as unrecognised data ranges (unrecognised in general, but can make sense if they are recognised by certain supporting tools).

Note that OWL-Eu does not provide arbitrary datatype expressions (such as ‘sum no greater than 15’, ‘multiply by 1.6’ or even simple binary comparison predicates such as ‘\(<\)’) and qualified number restrictions. Users who need these features thus can/should use OWL-E instead.

To sum up, as customised datatypes and datatype predicates are necessary in SW and ontology applications, OWL-E and OWL-Eu are two useful and decidable extensions to OWL DL.

A DL Reasoning Framework

The third objective of the thesis was to provide a DL reasoning framework, which (i) supports customised datatypes and datatype predicates, (ii) integrates a family of decidable DLs, including very expressive ones, with customised datatypes and datatype predicates, and (iii) provides decision procedures and flexible reasoning services for some members of this family that are closely related to OWL and the proposed extensions. The DL reasoning framework proposed in the thesis satisfies all these requirements.

Datatype Group

Firstly, the datatype group approach provides a general formalism to unify existing ontology-related datatype and predicate formalisms (such as the concrete domain approach and OWL datatyping) and to overcome their limitations (cf. Section 5.1.3). Most importantly, \(G\)-datatype expressions, in the unified formalism, can be used to represent customised datatypes, including XML Schema user-derived datatypes by restriction and union (cf. Section 3.3.1), and datatype predicates (cf. Example 5.7).

\(G\)-combinable and \(G\)-combined DLs

Secondly, a scheme for integrating an arbitrary

\(^1\)The underpinning of OWL-Eu is \(SHOIN(G_1)\), which does not provide datatype qualified number restrictions; hence, OWL-Eu is not an extension of DAML+OIL.
conforming datatype group into $G$-combinable Description Logics has been provided. An outstanding feature of the $G$-combined DLs is that they provide customised datatypes and datatype predicates. Theorem 5.19 shows that all members of the family of $G$-combined DLs, including $SHOIQ(G)$, are decidable; interestingly, it also generalises (almost) all the known decidability results, on concept satisfiability w.r.t. TBoxes and RBoxes, of feature-chain-free DLs combined with admissible concrete domains.

$G$-augmented Tableaux Algorithm Thirdly, given an arbitrary $G$-combinable DL $L$, a $G$-augmented tableaux algorithm for $L(G)$ can be constructed, based on a tableaux algorithm that is a decision procedure for $L$-concept satisfiability w.r.t. to RBoxes (cf. Theorem 6.14). As there are no published tableaux algorithms for $SHOIQ$, we stick to the ‘divide and conquer’ strategy and use the $G$-augmented tableaux algorithms of the $SHOQ(G)$, $SHIQ(G)$ and $SHIO(G)$ DLs to provide reasoning support for SW ontology languages OWL DL, DAML+OIL, OWL-Eu and OWL-E. Theorem 6.14 ensures that once we have a tableaux algorithm for $SHOIQ$, we can easily upgrade it to one for $SHOIQ(G)$.

Framework Architecture Finally, the proposed framework architecture ensures that the framework can provide flexible reasoning services for $G$-combined DLs $L(G)$ if the corresponding $G$-augmented tableaux algorithms exist. From the viewpoint of users, the DIG/OWL-E interface (cf. Section 7.1.2) allows applications to use arbitrary $G$-datatype expressions, so as to construct customised datatypes and predicates. From the viewpoint of reasoners, the hybrid reasoner is highly extensible to support new datatype predicates, new forms of datatype expressions and new $G$-combined Description Logics.

9.3 Future Work

The proposal of RDFS(FA), OWL-E and OWL-Eu and the study of our DL reasoning framework have suggested several promising paths for further research.

Reasoning Support for RDFS(FA) Although Theorem 4.4 indicates that we can use DL reasoners to reason with RDFS(FA) axioms in strata 0-2, this thesis has not presented a detailed solution for reasoning with RDFS(FA).

\(^2\)Or even in every three adjacent strata.
The Design of OWL FA Although having suggested that it is possible to have a new sub-language of OWL — OWL FA, this thesis has not specified any details of this new sub-language. OWL FA could further enjoy useful features, such as meta-classes, meta-properties and data-value annotation properties, of RDFS(FA).

A Tableaux Algorithm for $SHOIQ$ How to design a tableaux algorithm of the $SHOIQ$ DL has been an open question for quite a while. Once we have it, we can design a $G$-augmented tableaux algorithm for $SHOIQ$, the $G$-combined DL that underpins OWL-E.

Complexity Although it has proved the decidability of the $SHOIQ$ DL, this thesis has not provided any complexity result about it. Nor has the thesis provided any complexity results for the $G$-augmented tableaux algorithms, such as those for $SHOQ$, $SHIQ$ and $SHIO$.

Optimisations As full negations of even datatype predicates can introduce nondeterminism, reasoning with $G$-datatype expressions is generally hard. It could be helpful to investigate if the known optimisation techniques for concrete domains can be applied in $G$-augmented tableaux algorithms and the datatype manager algorithm (cf. Definition 7.2).

More Datatype Predicates More research is needed to investigate some less well understood predicates (such as the string predicates concatenation), predicates that are defined over multiple disjoint base datatypes (such as the string predicate length), predicates that do not fit RDF and OWL datatyping, such as XML Schema user-derived datatypes by list and some datatype-related functions and operators provided in XPath [28] and XQuery [18, 93].

Other future work includes investigations of the application of datatype groups in various extensions (such as the Semantic Web Rule Language [72, 71]) or variants (such as OWL Flight [33]) of OWL DL, so as to facilitate the use of datatypes and datatype predicates in such languages.

Solving the problems that discourages potential users from adopting DL-based Semantic Web ontology languages is an encouraging result for both the DL and SW communities. It is hoped that RDFS(FA), OWL-E, OWL-Eu and the proposed reasoning framework will provide a firm foundation for ongoing research into realising the Semantic Web and, in general, into reasoning support for ontology applications.
Bibliography


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