Positive extensions of Schur multipliers

Ying-Fen Lin

Queen's University Belfast
(joint work with Rupert Levene and Ivan Todorov)

SOAR
November 13, 2015

An $n \times n$ matrix is called *partially defined* if only some of its entries are specified with the unspecified entries treated as complex variables.

A *completion* of a partially defined matrix is simply a specification of the unspecified entries.

Question: determine whether or not a completion of a partially defined matrix exists which has some property, for example, contraction, positive...

An $n \times n$ matrix is called *partially defined* if only some of its entries are specified with the unspecified entries treated as complex variables.
A *completion* of a partially defined matrix is simply a specification of the unspecified entries.

Question: determine whether or not a completion of a partially defined matrix exists which has some property, for example, contraction, positive...

An $n \times n$ matrix is called *partially defined* if only some of its entries are specified with the unspecified entries treated as complex variables.
A *completion* of a partially defined matrix is simply a specification of the unspecified entries.

Question: determine whether or not a completion of a partially defined matrix exists which has some property, for example, contraction, positive...

Known results

Dym and Gohberg (1981) If $T = (t_{ij})$ is a partially defined $n \times n$ matrix with t_{ij} defined only for $|i - j| \leq k$, $0 < k < n - 1$, which has the property that all its fully defined $k \times k$ principal submatrices are positive semi-definite, then T can be completed to a positive semi-definite matrix.

Grone, Johnson, Sa and Wolkowitz (1984) A characterisation is given of those symmetric patterns J such that every partially positive matrix with pattern J has a positive completion.

Known results

Dym and Gohberg (1981) If $T = (t_{ij})$ is a partially defined $n \times n$ matrix with t_{ij} defined only for $|i - j| \leq k$, $0 < k < n - 1$, which has the property that all its fully defined $k \times k$ principal submatrices are positive semi-definite, then T can be completed to a positive semi-definite matrix.

Grone, Johnson, Sa and Wolkowitz (1984) A characterisation is given of those symmetric patterns J such that every partially positive matrix with pattern J has a positive completion.

Some definitions

- A subset $J \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is called a *pattern*.
- A partially defined $n \times n$ matrix $T = (t_{ij})$ is said to have pattern J if t_{ij} is specified if and only if $(i, j) \in J$.
- If $(i, i) \in J$ for all i and if $(i, j) \in J$ then $(j, i) \in J$, we call J a *symmetric pattern*.

To each pattern J, we can associate a subspace S_J of M_n by

$$S_J = \{(a_{ij}) \in M_n : a_{ij} = 0 \text{ if } (i, j) \notin J\}.$$

Note that S_J is an operator system if and only if J is symmetric.

Some definitions

- A subset $J \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is called a *pattern*.
- A partially defined $n \times n$ matrix $T = (t_{ij})$ is said to have pattern J if t_{ij} is specified if and only if $(i, j) \in J$.
- If $(i, i) \in J$ for all i and if $(i, j) \in J$ then $(j, i) \in J$, we call J a *symmetric pattern*.

To each pattern J, we can associate a subspace S_J of M_n by

$$S_J = \{(a_{ij}) \in M_n : a_{ij} = 0 \text{ if } (i, j) \notin J\}.$$

Note that S_J is an operator system if and only if J is symmetric.

Some definitions

- A subset $J \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is called a *pattern*.
- A partially defined $n \times n$ matrix $T = (t_{ij})$ is said to have pattern J if t_{ij} is specified if and only if $(i, j) \in J$.
- If $(i, i) \in J$ for all i and if $(i, j) \in J$ then $(j, i) \in J$, we call J a *symmetric pattern*.

To each pattern J, we can associate a subspace S_J of M_n by

$$S_J = \{(a_{ij}) \in M_n : a_{ij} = 0 \text{ if } (i, j) \notin J\}.$$

Note that S_J is an operator system if and only if J is symmetric.

Some definitions

- A subset $J \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is called a *pattern*.
- A partially defined $n \times n$ matrix $T = (t_{ij})$ is said to have pattern J if t_{ij} is specified if and only if $(i, j) \in J$.
- If $(i, i) \in J$ for all i and if $(i, j) \in J$ then $(j, i) \in J$, we call J a *symmetric pattern*.

To each pattern J, we can associate a subspace S_J of M_n by

$$S_J = \{(a_{ij}) \in M_n : a_{ij} = 0 \text{ if } (i, j) \notin J\}.$$

Note that S_J is an operator system if and only if J is symmetric.

Some definitions

- A subset $J \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is called a *pattern*.
- A partially defined $n \times n$ matrix $T = (t_{ij})$ is said to have pattern J if t_{ij} is specified if and only if $(i, j) \in J$.
- If $(i, i) \in J$ for all i and if $(i, j) \in J$ then $(j, i) \in J$, we call J a *symmetric pattern*.

To each pattern J, we can associate a subspace S_J of M_n by

$$S_J = \{(a_{ij}) \in M_n : a_{ij} = 0 \text{ if } (i, j) \notin J\}.$$

Note that S_J is an operator system if and only if J is symmetric.

A generalisation of previous results in $n \times n$ matrices

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern and let $T = (t_{ij})$ be a partially defined matrix with pattern J. Then the following are equivalent:

1. *T has a positive completion,*
2. *$\phi_T : S_J \to M_n$ defined by $\phi_T((a_{ij})) = (a_{ij}t_{ij})$ is positive,*
3. *$\Psi_T : S_J \to \mathbb{C}$ defined by $\Psi_T((a_{ij})) = \sum_{ij} a_{ij}t_{ij}$ is positive.*

A generalisation of previous results in $n \times n$ matrices

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern and let $T = (t_{ij})$ be a partially defined matrix with pattern J. Then the following are equivalent:

1. *T has a positive completion,*
2. *$\phi_T : S_J \to M_n$ defined by $\phi_T((a_{ij})) = (a_{ij} t_{ij})$ is positive,*
3. *$\Psi_T : S_J \to \mathbb{C}$ defined by $\Psi_T((a_{ij})) = \sum_{ij} a_{ij} t_{ij}$ is positive.*

A generalisation of previous results in $n \times n$ matrices

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern and let $T = (t_{ij})$ be a partially defined matrix with pattern J. Then the following are equivalent:

1. *T has a positive completion,*
2. *$\phi_T : S_J \to M_n$ defined by $\phi_T((a_{ij})) = (a_{ij}t_{ij})$ is positive,*
3. *$\Psi_T : S_J \to \mathbb{C}$ defined by $\Psi_T((a_{ij})) = \sum_{ij} a_{ij}t_{ij}$ is positive.*

Let J be a symmetric pattern and $T = (t_{ij})$ be a partially defined matrix with pattern J. The matrix T is called *partially positive* if it is symmetric and every $m \times m$ submatrix $B = (b_{k,l})$ of T with $b_{k,l} = t_{i_k,i_l}$, where $(i_k, i_l) \in J$ for $1 \leq k, l \leq m$, is positive.

Note: T is partially positive if and only if $\phi_T(P)$ is positive for every rank one positive P in S_J.

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern, then the following are equivalent:

1. *every partially positive matrix with pattern J has a positive completion,*
2. *every positive $P \in S_J$ is a sum of rank one positives in S_J,*
3. *the graph G_J is chordal.*

Let J be a symmetric pattern and $T = (t_{ij})$ be a partially defined matrix with pattern J. The matrix T is called *partially positive* if it is symmetric and every $m \times m$ submatrix $B = (b_{k,l})$ of T with $b_{k,l} = t_{i_k, i_l}$, where $(i_k, i_l) \in J$ for $1 \leq k, l \leq m$, is positive.

Note: T is partially positive if and only if $\phi_T(P)$ is positive for every rank one positive P in S_J.

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern, then the following are equivalent:

1. *every partially positive matrix with pattern J has a positive completion,*
2. *every positive $P \in S_J$ is a sum of rank one positives in S_J,*
3. *the graph G_J is chordal.*

Let J be a symmetric pattern and $T = (t_{ij})$ be a partially defined matrix with pattern J. The matrix T is called *partially positive* if it is symmetric and every $m \times m$ submatrix $B = (b_{k,l})$ of T with $b_{k,l} = t_{i_k,i_l}$, where $(i_k, i_l) \in J$ for $1 \leq k, l \leq m$, is positive.

Note: T is partially positive if and only if $\phi_T(P)$ is positive for every rank one positive P in S_J.

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern, then the following are equivalent:

1. *every partially positive matrix with pattern J has a positive completion,*
2. *every positive $P \in S_J$ is a sum of rank one positives in S_J,*
3. *the graph G_J is chordal.*

Let J be a symmetric pattern and $T = (t_{ij})$ be a partially defined matrix with pattern J. The matrix T is called *partially positive* if it is symmetric and every $m \times m$ submatrix $B = (b_{k,l})$ of T with $b_{k,l} = t_{i_k,i_l}$, where $(i_k, i_l) \in J$ for $1 \leq k, l \leq m$, is positive.

Note: T is partially positive if and only if $\phi_T(P)$ is positive for every rank one positive P in S_J.

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern, then the following are equivalent:

1. *every partially positive matrix with pattern J has a positive completion,*
2. *every positive $P \in S_J$ is a sum of rank one positives in S_J,*
3. *the graph G_J is chordal.*

Let J be a symmetric pattern and $T = (t_{ij})$ be a partially defined matrix with pattern J. The matrix T is called *partially positive* if it is symmetric and every $m \times m$ submatrix $B = (b_{k,l})$ of T with $b_{k,l} = t_{i_k,i_l}$, where $(i_k, i_l) \in J$ for $1 \leq k, l \leq m$, is positive.

Note: T is partially positive if and only if $\phi_T(P)$ is positive for every rank one positive P in S_J.

Theorem (Paulsen, Power and Smith (1989))

Let J be a symmetric pattern, then the following are equivalent:

1. *every partially positive matrix with pattern J has a positive completion,*
2. *every positive $P \in S_J$ is a sum of rank one positives in S_J,*
3. *the graph G_J is chordal.*

On the setting of $\ell^2(X)$

Let X be a set and $H = \ell^2(X)$ with the canonical orthonormal basis $(e_x)_{x \in X}$. For $x, y \in X$, denote by $E_{x,y}$ the corresponding matrix unit in $B(H)$. For $\kappa \subseteq X \times X$, define

$$\mathcal{S}(\kappa) := \overline{\text{span}\{E_{x,y} : (x,y) \in \kappa\}}^{\text{w}^*}$$

Note:

- $T \in B(H)$ is in $\mathcal{S}(\kappa)$ if and only if the matrix $(t_{x,y})$, $t_{x,y} = (Te_y, e_x)$, has $t_{x,y} = 0$ whenever $(x,y) \in \kappa^c$.
- the subspace $\mathcal{S}(\kappa)$ is an operator system if and only if
    1. κ is symmetric,
    2. κ contains the diagonal of $X \times X$.

Note that each such κ gives rise to a graph on X.

On the setting of $\ell^2(X)$

Let X be a set and $H = \ell^2(X)$ with the canonical orthonormal basis $(e_x)_{x \in X}$. For $x, y \in X$, denote by $E_{x,y}$ the corresponding matrix unit in $B(H)$. For $\kappa \subseteq X \times X$, define

$$\mathcal{S}(\kappa) := \overline{\operatorname{span}\{E_{x,y} : (x,y) \in \kappa\}}^{\mathrm{w}^*}$$

Note:

- $T \in B(H)$ is in $\mathcal{S}(\kappa)$ if and only if the matrix $(t_{x,y})$, $t_{x,y} = (Te_y, e_x)$, has $t_{x,y} = 0$ whenever $(x,y) \in \kappa^c$.
- the subspace $\mathcal{S}(\kappa)$ is an operator system if and only if
  1. κ is symmetric,
  2. κ contains the diagonal of $X \times X$.

Note that each such κ gives rise to a graph on X.

On the setting of $\ell^2(X)$

Let X be a set and $H = \ell^2(X)$ with the canonical orthonormal basis $(e_x)_{x \in X}$. For $x, y \in X$, denote by $E_{x,y}$ the corresponding matrix unit in $B(H)$. For $\kappa \subseteq X \times X$, define

$$\mathcal{S}(\kappa) := \overline{\text{span}\{E_{x,y} : (x,y) \in \kappa\}}^{w^*}$$

Note:

- $T \in B(H)$ is in $\mathcal{S}(\kappa)$ if and only if the matrix $(t_{x,y})$, $t_{x,y} = (Te_y, e_x)$, has $t_{x,y} = 0$ whenever $(x,y) \in \kappa^c$.
- the subspace $\mathcal{S}(\kappa)$ is an operator system if and only if
    1. κ is symmetric,
    2. κ contains the diagonal of $X \times X$.

Note that each such κ gives rise to a graph on X.

On the setting of $\ell^2(X)$

Let X be a set and $H = \ell^2(X)$ with the canonical orthonormal basis $(e_x)_{x \in X}$. For $x, y \in X$, denote by $E_{x,y}$ the corresponding matrix unit in $B(H)$. For $\kappa \subseteq X \times X$, define

$$\mathcal{S}(\kappa) := \overline{\text{span}\{E_{x,y} : (x,y) \in \kappa\}}^{\mathrm{w}^*}$$

Note:

- $T \in B(H)$ is in $\mathcal{S}(\kappa)$ if and only if the matrix $(t_{x,y})$, $t_{x,y} = (Te_y, e_x)$, has $t_{x,y} = 0$ whenever $(x,y) \in \kappa^c$.
- the subspace $\mathcal{S}(\kappa)$ is an operator system if and only if
  1. κ is symmetric,
  2. κ contains the diagonal of $X \times X$.

Note that each such κ gives rise to a graph on X.

On the setting of $\ell^2(X)$

Let X be a set and $H = \ell^2(X)$ with the canonical orthonormal basis $(e_x)_{x \in X}$. For $x, y \in X$, denote by $E_{x,y}$ the corresponding matrix unit in $B(H)$. For $\kappa \subseteq X \times X$, define

$$\mathcal{S}(\kappa) := \overline{\text{span}\{E_{x,y} : (x, y) \in \kappa\}}^{\mathrm{w}^*}$$

Note:

- $T \in B(H)$ is in $\mathcal{S}(\kappa)$ if and only if the matrix $(t_{x,y})$, $t_{x,y} = (Te_y, e_x)$, has $t_{x,y} = 0$ whenever $(x, y) \in \kappa^c$.
- the subspace $\mathcal{S}(\kappa)$ is an operator system if and only if
  1. κ is symmetric,
  2. κ contains the diagonal of $X \times X$.

Note that each such κ gives rise to a graph on X.

On the setting of $\ell^2(X)$

Let X be a set and $H = \ell^2(X)$ with the canonical orthonormal basis $(e_x)_{x\in X}$. For $x, y \in X$, denote by $E_{x,y}$ the corresponding matrix unit in $B(H)$. For $\kappa \subseteq X \times X$, define

$$\mathcal{S}(\kappa) := \overline{\text{span}\{E_{x,y} : (x, y) \in \kappa\}}^{\text{w}^*}$$

Note:

- $T \in B(H)$ is in $\mathcal{S}(\kappa)$ if and only if the matrix $(t_{x,y})$, $t_{x,y} = (Te_y, e_x)$, has $t_{x,y} = 0$ whenever $(x, y) \in \kappa^c$.
- the subspace $\mathcal{S}(\kappa)$ is an operator system if and only if
    1. κ is symmetric,
    2. κ contains the diagonal of $X \times X$.

Note that each such κ gives rise to a graph on X.

Schur multipliers and positive extensions

Definition

- A function $\psi : \kappa \to B(K)$ is called an (operator-valued) *Schur multiplier* if

$$S_\psi(t_{x,y}) := (t_{x,y}\psi(x,y))_{x,y\in X} \in B(H \otimes K)$$

for every $(t_{x,y})_{x,y\in X} \in \mathcal{S}(\kappa)$.

- A Schur multiplier $\phi : X \times X \to B(K)$ is called *positive* if S_ϕ is positive, i.e., for every positive $T \in B(H)$, the operator $S_\phi(T) \in B(H \otimes K)$ is positive.
- Let $\psi : \kappa \to B(K)$ be a Schur multiplier. We say that ψ is *partially positive* if for $\alpha \subseteq X$ with $\alpha \times \alpha \subseteq \kappa$, the Schur multiplier $\psi|_{\alpha\times\alpha}$ is positive.

Schur multipliers and positive extensions

Definition

- A function $\psi : \kappa \to B(K)$ is called an (operator-valued) *Schur multiplier* if

$$S_\psi(t_{x,y}) := (t_{x,y}\psi(x,y))_{x,y\in X} \in B(H \otimes K)$$

  for every $(t_{x,y})_{x,y\in X} \in \mathcal{S}(\kappa)$.
- A Schur multiplier $\phi : X \times X \to B(K)$ is called *positive* if S_ϕ is positive, i.e., for every positive $T \in B(H)$, the operator $S_\phi(T) \in B(H \otimes K)$ is positive.
- Let $\psi : \kappa \to B(K)$ be a Schur multiplier. We say that ψ is *partially positive* if for $\alpha \subseteq X$ with $\alpha \times \alpha \subseteq \kappa$, the Schur multiplier $\psi|_{\alpha\times\alpha}$ is positive.

Schur multipliers and positive extensions

Definition

- A function $\psi : \kappa \to B(K)$ is called an (operator-valued) *Schur multiplier* if

$$S_\psi(t_{x,y}) := (t_{x,y}\psi(x,y))_{x,y\in X} \in B(H \otimes K)$$

  for every $(t_{x,y})_{x,y\in X} \in \mathcal{S}(\kappa)$.
- A Schur multiplier $\phi : X \times X \to B(K)$ is called *positive* if S_ϕ is positive, i.e., for every positive $T \in B(H)$, the operator $S_\phi(T) \in B(H \otimes K)$ is positive.
- Let $\psi : \kappa \to B(K)$ be a Schur multiplier. We say that ψ is *partially positive* if for $\alpha \subseteq X$ with $\alpha \times \alpha \subseteq \kappa$, the Schur multiplier $\psi|_{\alpha\times\alpha}$ is positive.

Positive extensions on $\ell^2(X)$

Theorem (Levene, L., Todorov)

Let $\kappa \subseteq X \times X$ be a graph. The following conditions are equivalent:

1. *every partially positive Schur multiplier $\psi : \kappa \to B(K)$ has a positive extension;*
2. *κ is chordal;*
3. *every positive operator in $\mathcal{S}(\kappa)$ is a weak* limit of sums of rank one positive operators in $\mathcal{S}(\kappa)$.*

Let (X, μ) be an arbitrary σ-finite measure space.

If $k \in L^2(X \times X)$, the Hilbert-Schmidt operator T_k on $L^2(X, \mu)$ with integral kernel k is defined by

$$T_k f(y) = \int_X k(y,x) f(x) d\mu(x) \quad \text{for} \quad f \in L^2(X, \mu),\ y \in X.$$

For any measurable subset $\kappa \subseteq X \times X$, let

$$\mathcal{S}(\kappa) := \overline{\{T_k : k \in L^2(\kappa)\}}^{\mathrm{w}^*}.$$

Let (X, μ) be an arbitrary σ-finite measure space.

If $k \in L^2(X \times X)$, the Hilbert-Schmidt operator T_k on $L^2(X, \mu)$ with integral kernel k is defined by

$$T_k f(y) = \int_X k(y, x) f(x) d\mu(x) \quad \text{for} \quad f \in L^2(X, \mu),\ y \in X.$$

For any measurable subset $\kappa \subseteq X \times X$, let

$$\mathcal{S}(\kappa) := \overline{\{T_k : k \in L^2(\kappa)\}}^{\mathrm{w}^*}.$$

Let (X,μ) be an arbitrary σ-finite measure space.

If $k \in L^2(X \times X)$, the Hilbert-Schmidt operator T_k on $L^2(X,\mu)$ with integral kernel k is defined by

$$T_k f(y) = \int_X k(y,x) f(x) d\mu(x) \quad \text{for} \quad f \in L^2(X,\mu),\ y \in X.$$

For any measurable subset $\kappa \subseteq X \times X$, let

$$\mathcal{S}(\kappa) := \overline{\{T_k : k \in L^2(\kappa)\}}^{\mathrm{w}^*}.$$

ω-topology (Erdos, Katavolos and Shulman, 1998)

Let (X, μ) be a σ-finite measure space.

- A subset $E \subseteq X \times X$ is called *marginally null* if $E \subseteq (M \times X) \cup (X \times M)$, where $M \subseteq X$ is null.
- Two subsets $E, F \subseteq X \times X$ are called *marginally equivalent*, denoted by $E \cong F$, if $E \triangle F$ is marginally null.
- A set $\kappa \subseteq X \times X$ is called a *rectangle* if $\kappa = \alpha \times \beta$, where α, β are measurable subsets of X; it is called a *square* if $\kappa = \alpha \times \alpha$.
- A set $\kappa \subseteq X \times X$ is called *ω-open* if $\kappa \cong \bigcup_i \alpha_i \times \beta_i$, where $\alpha_i, \beta_i \subseteq X$ are measurable.

ω-topology (Erdos, Katavolos and Shulman, 1998)

Let (X, μ) be a σ-finite measure space.

- A subset $E \subseteq X \times X$ is called *marginally null* if $E \subseteq (M \times X) \cup (X \times M)$, where $M \subseteq X$ is null.
- Two subsets $E, F \subseteq X \times X$ are called *marginally equivalent*, denoted by $E \cong F$, if $E \triangle F$ is marginally null.
- A set $\kappa \subseteq X \times X$ is called a *rectangle* if $\kappa = \alpha \times \beta$, where α, β are measurable subsets of X; it is called a *square* if $\kappa = \alpha \times \alpha$.
- A set $\kappa \subseteq X \times X$ is called *ω-open* if $\kappa \cong \bigcup_i \alpha_i \times \beta_i$, where $\alpha_i, \beta_i \subseteq X$ are measurable.

ω-topology (Erdos, Katavolos and Shulman, 1998)

Let (X, μ) be a σ-finite measure space.

- A subset $E \subseteq X \times X$ is called *marginally null* if $E \subseteq (M \times X) \cup (X \times M)$, where $M \subseteq X$ is null.
- Two subsets $E, F \subseteq X \times X$ are called *marginally equivalent*, denoted by $E \cong F$, if $E \bigtriangleup F$ is marginally null.
- A set $\kappa \subseteq X \times X$ is called a *rectangle* if $\kappa = \alpha \times \beta$, where α, β are measurable subsets of X; it is called a *square* if $\kappa = \alpha \times \alpha$.
- A set $\kappa \subseteq X \times X$ is called *ω-open* if $\kappa \cong \bigcup_i \alpha_i \times \beta_i$, where $\alpha_i, \beta_i \subseteq X$ are measurable.

ω-topology (Erdos, Katavolos and Shulman, 1998)

Let (X, μ) be a σ-finite measure space.

- A subset $E \subseteq X \times X$ is called *marginally null* if $E \subseteq (M \times X) \cup (X \times M)$, where $M \subseteq X$ is null.
- Two subsets $E, F \subseteq X \times X$ are called *marginally equivalent*, denoted by $E \cong F$, if $E \bigtriangleup F$ is marginally null.
- A set $\kappa \subseteq X \times X$ is called a *rectangle* if $\kappa = \alpha \times \beta$, where α, β are measurable subsets of X; it is called a *square* if $\kappa = \alpha \times \alpha$.
- A set $\kappa \subseteq X \times X$ is called *ω-open* if $\kappa \cong \bigcup_i \alpha_i \times \beta_i$, where $\alpha_i, \beta_i \subseteq X$ are measurable.

For any measurable ω-closed subset $\kappa \subseteq X \times X$, let

$$\mathcal{S}(\kappa) := \overline{\{T_k : k \in L^2(\kappa)\}}^{\mathrm{w}^*}.$$

Proposition

Let $\kappa \subseteq X \times X$ be ω-closed. The following are equivalent:

1. *$\mathcal{S}(\kappa)$ is an operator system;*
2. *κ is symmetric, i.e., $\kappa \cong \hat{\kappa}$ and $\triangle := \{(x,x) : x \in X\} \subseteq_\omega \kappa$, where $\hat{\kappa} := \{(x,y) \in X \times X : (y,x) \in \kappa\}$.*

For any measurable ω-closed subset $\kappa \subseteq X \times X$, let

$$\mathcal{S}(\kappa) := \overline{\{T_k : k \in L^2(\kappa)\}}^{\mathrm{w}^*}.$$

Proposition

Let $\kappa \subseteq X \times X$ be ω-closed. The following are equivalent:

1. $\mathcal{S}(\kappa)$ *is an operator system;*
2. κ *is symmetric, i.e.,* $\kappa \cong \hat{\kappa}$ *and* $\triangle := \{(x,x) : x \in X\} \subseteq_\omega \kappa$, *where* $\hat{\kappa} := \{(x,y) \in X \times X : (y,x) \in \kappa\}$.

Positivity domains

Definition

An ω-closed set $\kappa \subseteq X \times X$ is called a *positivity domain* if $\kappa \cong \hat{\kappa}$, $\triangle \subseteq_\omega \kappa$ and $\kappa \cong \mathrm{cl}_\omega(\mathrm{int}_\omega(\kappa))$.

Here $\mathrm{int}_\omega(\kappa) = \bigcup_\omega \{\alpha \times \beta : \alpha \times \beta \subseteq \kappa\}$ and $\mathrm{cl}_\omega(\kappa) = \mathrm{int}_\omega(\kappa^c)^c$.

Proposition: For an ω-closed set κ, we have that $\kappa \sim \mathrm{cl}_\omega(\mathrm{int}_\omega(\kappa))$ if and only if $\mathcal{S}(\kappa)$ is the weak* closed span of the rank one operators it contains.

Note: There exists a positivity domain κ such that $\mathcal{S}(\kappa)$ has no positive rank one operators.

Positivity domains

Definition

An ω-closed set $\kappa \subseteq X \times X$ is called a *positivity domain* if $\kappa \cong \hat{\kappa}$, $\triangle \subseteq_\omega \kappa$ and $\kappa \cong \mathrm{cl}_\omega(\mathrm{int}_\omega(\kappa))$.

Here $\mathrm{int}_\omega(\kappa) = \bigcup_\omega \{\alpha \times \beta : \alpha \times \beta \subseteq \kappa\}$ and $\mathrm{cl}_\omega(\kappa) = \mathrm{int}_\omega(\kappa^c)^c$.

Proposition: For an ω-closed set κ, we have that $\kappa \sim \mathrm{cl}_\omega(\mathrm{int}_\omega(\kappa))$ if and only if $\mathcal{S}(\kappa)$ is the weak* closed span of the rank one operators it contains.

Note: There exists a positivity domain κ such that $\mathcal{S}(\kappa)$ has no positive rank one operators.

Positivity domains

Definition

An ω-closed set $\kappa \subseteq X \times X$ is called a *positivity domain* if $\kappa \cong \hat{\kappa}$, $\triangle \subseteq_\omega \kappa$ and $\kappa \cong \mathrm{cl}_\omega(\mathrm{int}_\omega(\kappa))$.

Here $\mathrm{int}_\omega(\kappa) = \bigcup_\omega \{\alpha \times \beta : \alpha \times \beta \subseteq \kappa\}$ and $\mathrm{cl}_\omega(\kappa) = \mathrm{int}_\omega(\kappa^c)^c$.

Proposition: For an ω-closed set κ, we have that $\kappa \sim \mathrm{cl}_\omega(\mathrm{int}_\omega(\kappa))$ if and only if $\mathcal{S}(\kappa)$ is the weak* closed span of the rank one operators it contains.

Note: There exists a positivity domain κ such that $\mathcal{S}(\kappa)$ has no positive rank one operators.

For a positivity domain $\kappa \subseteq X \times X$, let

$$[\mathcal{S}_1^+(\kappa)] = \Big\{ \sum_{i=1}^{n} R_i : R_i \in \mathcal{S}(\kappa)^+ \text{ is of rank one}, 1 \leq i \leq n \Big\}.$$

Theorem

The following are equivalent, for a positivity domain κ:

- *there exists a family $(\alpha_i)_{i \in \mathbb{N}}$ of mutually disjoint measurable subsets of X s.t. $\alpha_i \times \alpha_i \subseteq_\omega \kappa$ for each i and $\bigcup_{i=1}^{\infty} \alpha_i = X$;*
- $I \in \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;
- *κ is generated by squares, i.e. $\kappa \cong \mathrm{cl}_\omega(\mathrm{sqint}_\omega(\kappa))$, where*

$$\mathrm{sqint}_\omega(\kappa) = \bigcup_\omega \{Q : Q \text{ is a square with } Q \subseteq \kappa\}.$$

For a positivity domain $\kappa \subseteq X \times X$, let

$$[\mathcal{S}_1^+(\kappa)] = \left\{ \sum_{i=1}^{n} R_i : R_i \in \mathcal{S}(\kappa)^+ \text{ is of rank one}, 1 \leq i \leq n \right\}.$$

Theorem

The following are equivalent, for a positivity domain κ:

- *there exists a family $(\alpha_i)_{i \in \mathbb{N}}$ of mutually disjoint measurable subsets of X s.t. $\alpha_i \times \alpha_i \subseteq_\omega \kappa$ for each i and $\bigcup_{i=1}^{\infty} \alpha_i = X$;*
- $I \in \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;
- *κ is generated by squares, i.e. $\kappa \cong \mathrm{cl}_\omega(\mathrm{sqint}_\omega(\kappa))$, where*

$$\mathrm{sqint}_\omega(\kappa) = \bigcup_\omega \{Q : Q \text{ is a square with } Q \subseteq \kappa\}.$$

For a positivity domain $\kappa \subseteq X \times X$, let

$$[\mathcal{S}_1^+(\kappa)] = \left\{ \sum_{i=1}^{n} R_i : R_i \in \mathcal{S}(\kappa)^+ \text{ is of rank one}, 1 \leq i \leq n \right\}.$$

Theorem

The following are equivalent, for a positivity domain κ:

- *there exists a family $(\alpha_i)_{i\in\mathbb{N}}$ of mutually disjoint measurable subsets of X s.t. $\alpha_i \times \alpha_i \subseteq_\omega \kappa$ for each i and $\bigcup_{i=1}^{\infty} \alpha_i = X$;*
- $I \in \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;
- *κ is generated by squares, i.e. $\kappa \cong \mathrm{cl}_\omega(\mathrm{sqint}_\omega(\kappa))$, where*

$$\mathrm{sqint}_\omega(\kappa) = \bigcup_\omega \{Q : Q \text{ is a square with } Q \subseteq \kappa\}.$$

For a positivity domain $\kappa \subseteq X \times X$, let

$$[\mathcal{S}_1^+(\kappa)] = \left\{ \sum_{i=1}^{n} R_i : R_i \in \mathcal{S}(\kappa)^+ \text{ is of rank one}, 1 \leq i \leq n \right\}.$$

Theorem

The following are equivalent, for a positivity domain κ:

- *there exists a family $(\alpha_i)_{i\in\mathbb{N}}$ of mutually disjoint measurable subsets of X s.t. $\alpha_i \times \alpha_i \subseteq_\omega \kappa$ for each i and $\bigcup_{i=1}^{\infty} \alpha_i = X$;*
- *$I \in \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;*
- *κ is generated by squares, i.e. $\kappa \cong \mathrm{cl}_\omega(\mathrm{sqint}_\omega(\kappa))$, where*

$$\mathrm{sqint}_\omega(\kappa) = \bigcup_{\omega} \{Q : Q \text{ is a square with } Q \subseteq \kappa\}.$$

Schur multipliers

Let $\kappa \subseteq X \times X$ be a positivity domain. A measurable function $\varphi : \kappa \to \mathbb{C}$ is called *Schur multiplier* if $\exists C > 0$ such that

$$\|T_{\varphi k}\| \leq C\|T_k\| \text{ for every } k \in L^2(\kappa).$$

Proposition

Let $\kappa \subseteq X \times X$ be a positivity domain and let $\varphi : \kappa \to \mathbb{C}$ be a measurable function. The following are equivalent:

- *φ is a Schur multiplier;*
- *$\exists$ a Schur multiplier $\psi : X \times X \to \mathbb{C}$ such that $\psi|_\kappa \sim \varphi$;*
- *$\exists$ a completely bounded weak*-conti. map $\Phi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ such that $\Phi(T_k) = T_{\varphi k}$ for $k \in L^2(\kappa)$.*

Schur multipliers

Let $\kappa \subseteq X \times X$ be a positivity domain. A measurable function $\varphi : \kappa \to \mathbb{C}$ is called *Schur multiplier* if $\exists C > 0$ such that

$$\|T_{\varphi k}\| \leq C\|T_k\| \text{ for every } k \in L^2(\kappa).$$

Proposition

Let $\kappa \subseteq X \times X$ be a positivity domain and let $\varphi : \kappa \to \mathbb{C}$ be a measurable function. The following are equivalent:

- *φ is a Schur multiplier;*
- *$\exists$ a Schur multiplier $\psi : X \times X \to \mathbb{C}$ such that $\psi|_\kappa \sim \varphi$;*
- *$\exists$ a completely bounded weak*-conti. map $\Phi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ such that $\Phi(T_k) = T_{\varphi k}$ for $k \in L^2(\kappa)$.*

A characterisation of partially positive Schur multipliers

Let $\kappa \subseteq X \times X$ be a positivity domain. A Schur multiplier $\varphi : \kappa \to \mathbb{C}$ is called *partially positive* if $\varphi|_{\alpha\times\alpha}$ is a positive Schur multiplier whenever $\alpha \subseteq X$ is a measurable set with $\alpha \times \alpha \subseteq \kappa$.

Proposition

Let κ be a positivity domain. A Schur multiplier $\varphi : \kappa \to \mathbb{C}$ is partially positive if and only if $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$.

A characterisation of partially positive Schur multipliers

Let $\kappa \subseteq X \times X$ be a positivity domain. A Schur multiplier $\varphi : \kappa \to \mathbb{C}$ is called *partially positive* if $\varphi|_{\alpha\times\alpha}$ is a positive Schur multiplier whenever $\alpha \subseteq X$ is a measurable set with $\alpha \times \alpha \subseteq \kappa$.

Proposition

Let κ be a positivity domain. A Schur multiplier $\varphi : \kappa \to \mathbb{C}$ is partially positive if and only if $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$.

A characterisation of partially positive Schur multipliers

Let $\kappa \subseteq X \times X$ be a positivity domain. A Schur multiplier $\varphi : \kappa \to \mathbb{C}$ is called *partially positive* if $\varphi|_{\alpha \times \alpha}$ is a positive Schur multiplier whenever $\alpha \subseteq X$ is a measurable set with $\alpha \times \alpha \subseteq \kappa$.

Proposition

Let κ be a positivity domain. A Schur multiplier $\varphi : \kappa \to \mathbb{C}$ is partially positive if and only if $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$.

Proof:
($\Leftarrow$) Suppose $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$ and $\alpha \times \alpha \subseteq \kappa$ for a measurable set $\alpha \subseteq X$. If T is positive rank one supported by $\alpha \times \alpha$, then $S_\varphi(T) \geq 0$. Since S_φ is weak*-conti. and

$$B(P(\alpha)L^2(X))^+ = \overline{\text{span}\{T : T \text{ positive rank one supported by } \alpha \times \alpha\}}^{w^*}$$

$\Rightarrow \varphi|_{\alpha\times\alpha}$ is positive Schur multiplier.
($\Rightarrow$) Suppose φ is partially positive and $T \in \mathcal{S}(\kappa)$ is a positive rank one operator, say, $T = \eta \otimes \eta^*$ for some $\eta \in L^2(X)$.
If $\operatorname{supp} \eta = \alpha$, then $\alpha \times \alpha \subseteq_\omega \kappa$ (Erdos, Katavolos and Shulman).
Then $S_\varphi(T) \geq 0$ by assumption.

Proof:
($\Leftarrow$) Suppose $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$ and $\alpha \times \alpha \subseteq \kappa$ for a measurable set $\alpha \subseteq X$. If T is positive rank one supported by $\alpha \times \alpha$, then $S_\varphi(T) \geq 0$. Since S_φ is weak*-conti. and

$$B(P(\alpha)L^2(X))^+ = \overline{\text{span}\{T : T \text{ positive rank one supported by } \alpha \times \alpha\}}^{\text{w}^*}$$

$\Rightarrow \varphi|_{\alpha\times\alpha}$ is positive Schur multiplier.
($\Rightarrow$) Suppose φ is partially positive and $T \in \mathcal{S}(\kappa)$ is a positive rank one operator, say, $T = \eta \otimes \eta^*$ for some $\eta \in L^2(X)$.
If $\operatorname{supp} \eta = \alpha$, then $\alpha \times \alpha \subseteq_\omega \kappa$ (Erdos, Katavolos and Shulman).
Then $S_\varphi(T) \geq 0$ by assumption.

Proof:
($\Leftarrow$) Suppose $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$ and $\alpha \times \alpha \subseteq \kappa$ for a measurable set $\alpha \subseteq X$. If T is positive rank one supported by $\alpha \times \alpha$, then $S_\varphi(T) \geq 0$. Since S_φ is weak*-conti. and

$$B(P(\alpha)L^2(X))^+ = \overline{\text{span}\{T : T \text{ positive rank one supported by } \alpha \times \alpha\}}^{\text{w}^*}$$

$\Rightarrow \varphi|_{\alpha\times\alpha}$ is positive Schur multiplier.
($\Rightarrow$) Suppose φ is partially positive and $T \in \mathcal{S}(\kappa)$ is a positive rank one operator, say, $T = \eta \otimes \eta^*$ for some $\eta \in L^2(X)$.
If $\operatorname{supp} \eta = \alpha$, then $\alpha \times \alpha \subseteq_\omega \kappa$ (Erdos, Katavolos and Shulman).
Then $S_\varphi(T) \geq 0$ by assumption.

Proof:
($\Leftarrow$) Suppose $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$ and $\alpha \times \alpha \subseteq \kappa$ for a measurable set $\alpha \subseteq X$. If T is positive rank one supported by $\alpha \times \alpha$, then $S_\varphi(T) \geq 0$. Since S_φ is weak*-conti. and

$$B(P(\alpha)L^2(X))^+ = \overline{\text{span}\{T : T \text{ positive rank one supported by } \alpha \times \alpha\}}^{\text{w}^*}$$

$\Rightarrow \varphi|_{\alpha\times\alpha}$ is positive Schur multiplier.

($\Rightarrow$) Suppose φ is partially positive and $T \in \mathcal{S}(\kappa)$ is a positive rank one operator, say, $T = \eta \otimes \eta^*$ for some $\eta \in L^2(X)$.
If $\operatorname{supp} \eta = \alpha$, then $\alpha \times \alpha \subseteq_\omega \kappa$ (Erdos, Katavolos and Shulman).
Then $S_\varphi(T) \geq 0$ by assumption.

Proof:

($\Leftarrow$) Suppose $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$ and $\alpha \times \alpha \subseteq \kappa$ for a measurable set $\alpha \subseteq X$. If T is positive rank one supported by $\alpha \times \alpha$, then $S_\varphi(T) \geq 0$. Since S_φ is weak*-conti. and

$$B(P(\alpha)L^2(X))^+ = \overline{\text{span}\{T : T \text{ positive rank one supported by } \alpha \times \alpha\}}^{\mathrm{w}^*}$$

$\Rightarrow \varphi|_{\alpha\times\alpha}$ is positive Schur multiplier.

($\Rightarrow$) Suppose φ is partially positive and $T \in \mathcal{S}(\kappa)$ is a positive rank one operator, say, $T = \eta \otimes \eta^*$ for some $\eta \in L^2(X)$.

If $\operatorname{supp} \eta = \alpha$, then $\alpha \times \alpha \subseteq_\omega \kappa$ (Erdos, Katavolos and Shulman).

Then $S_\varphi(T) \geq 0$ by assumption.

Proof:
($\Leftarrow$) Suppose $S_\varphi(\mathcal{S}_1^+(\kappa)) \subseteq B(L^2(X))^+$ and $\alpha \times \alpha \subseteq \kappa$ for a measurable set $\alpha \subseteq X$. If T is positive rank one supported by $\alpha \times \alpha$, then $S_\varphi(T) \geq 0$. Since S_φ is weak*-conti. and

$$B(P(\alpha)L^2(X))^+ = \overline{\text{span}\{T : T \text{ positive rank one supported by } \alpha \times \alpha\}}^{\mathrm{w}^*}$$

$\Rightarrow \varphi|_{\alpha\times\alpha}$ is positive Schur multiplier.
($\Rightarrow$) Suppose φ is partially positive and $T \in \mathcal{S}(\kappa)$ is a positive rank one operator, say, $T = \eta \otimes \eta^*$ for some $\eta \in L^2(X)$.
If $\operatorname{supp} \eta = \alpha$, then $\alpha \times \alpha \subseteq_\omega \kappa$ (Erdos, Katavolos and Shulman).
Then $S_\varphi(T) \geq 0$ by assumption.

Definition

Let $\kappa \subseteq X \times X$ be a positivity domain and $\varphi : \kappa \to \mathbb{C}$ be a Schur multiplier. We say that a measurable function $\psi : X \times X \to \mathbb{C}$ is a *positive extension* of φ if ψ is a positive Schur multiplier and $\psi|_\kappa \sim \varphi$.

Theorem

Let κ be a positivity domain. The following are equivalent for a partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$:

1. *φ has a positive extension;*
2. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is positive;*
3. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is completely positive.*

Definition

Let $\kappa \subseteq X \times X$ be a positivity domain and $\varphi : \kappa \to \mathbb{C}$ be a Schur multiplier. We say that a measurable function $\psi : X \times X \to \mathbb{C}$ is a *positive extension* of φ if ψ is a positive Schur multiplier and $\psi|_\kappa \sim \varphi$.

Theorem

Let κ be a positivity domain. The following are equivalent for a partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$:

1. *φ has a positive extension;*
2. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is positive;*
3. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is completely positive.*

Definition

Let $\kappa \subseteq X \times X$ be a positivity domain and $\varphi : \kappa \to \mathbb{C}$ be a Schur multiplier. We say that a measurable function $\psi : X \times X \to \mathbb{C}$ is a *positive extension* of φ if ψ is a positive Schur multiplier and $\psi|_\kappa \sim \varphi$.

Theorem

Let κ be a positivity domain. The following are equivalent for a partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$:

1. *φ has a positive extension;*
2. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is positive;*
3. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is completely positive.*

Definition

Let $\kappa \subseteq X \times X$ be a positivity domain and $\varphi : \kappa \to \mathbb{C}$ be a Schur multiplier. We say that a measurable function $\psi : X \times X \to \mathbb{C}$ is a *positive extension* of φ if ψ is a positive Schur multiplier and $\psi|_\kappa \sim \varphi$.

Theorem

Let κ be a positivity domain. The following are equivalent for a partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$:

1. *φ has a positive extension;*
2. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is positive;*
3. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is completely positive.*

Definition

Let $\kappa \subseteq X \times X$ be a positivity domain and $\varphi : \kappa \to \mathbb{C}$ be a Schur multiplier. We say that a measurable function $\psi : X \times X \to \mathbb{C}$ is a *positive extension* of φ if ψ is a positive Schur multiplier and $\psi|_\kappa \sim \varphi$.

Theorem

Let κ be a positivity domain. The following are equivalent for a partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$:

1. *φ has a positive extension;*
2. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is positive;*
3. *the map $S_\varphi : \mathcal{S}(\kappa) \to \mathcal{S}(\kappa)$ is completely positive.*

Positive extensions of Schur multipliers

Theorem

Let κ be an positivity domain. The following are equivalent:

1. *every partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$ has a positive extension;*
2. $\mathcal{S}(\kappa)^+ = \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;
3. $\mathcal{S}_0(\kappa)^+ = \overline{[\mathcal{S}_1^+(\kappa)]}^{\|\cdot\|}$,
   where $\mathcal{S}_0(\kappa) = \overline{\{T_k : k \in L^2(\kappa)\}}^{\|\cdot\|}$.

Positive extensions of Schur multipliers

Theorem

Let κ be an positivity domain. The following are equivalent:

1. *every partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$ has a positive extension;*
2. $\mathcal{S}(\kappa)^+ = \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;
3. $\mathcal{S}_0(\kappa)^+ = \overline{[\mathcal{S}_1^+(\kappa)]}^{\|\cdot\|}$,

   where $\mathcal{S}_0(\kappa) = \overline{\{T_k : k \in L^2(\kappa)\}}^{\|\cdot\|}$.

Positive extensions of Schur multipliers

Theorem

Let κ be an positivity domain. The following are equivalent:

1. *every partially positive Schur multiplier $\varphi : \kappa \to \mathbb{C}$ has a positive extension;*
2. $\mathcal{S}(\kappa)^+ = \overline{[\mathcal{S}_1^+(\kappa)]}^{\mathrm{w}^*}$;
3. $\mathcal{S}_0(\kappa)^+ = \overline{[\mathcal{S}_1^+(\kappa)]}^{\|\cdot\|}$,
   where $\mathcal{S}_0(\kappa) = \overline{\{T_k : k \in L^2(\kappa)\}}^{\|\cdot\|}$.

THANK YOU!!